Now suppose $F \in Q$ that is for a natural m

$$F \in \underbrace{Z \ldots Z}_{m \text{ times}}\left(\bigcup S\right).$$

Let's prove by induction that $F = F_0 \sqcup \dots \sqcup F_{n-1}$ for some $F_0, \ldots, F_{n-1} \in \bigcup S$ such that $F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1}$. If $m = 0$ then $F \in \bigcup S$ and our promise is obvious. Let our statement holds for a natural m. Prove that it holds for

$$F' \in \underbrace{Z \ldots Z}_{m+1 \text{ times}}\left(\bigcup S\right).$$

We have $F' = Z(F)$ for some $F = F_0 \sqcup \dots \sqcup F_{n-1}$ where $F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1}$. The case $F' = \bot$ is easy. So we can assume $F' = A \sqcup B$ where $A, B \in F$ and $A \not\asymp B$. By the statement of induction $A = A_0 \sqcup \dots \sqcup A_{p-1}$, $B = B_0 \sqcup \dots \sqcup B_{q-1}$ for natural p and q, where $A_0 \not\asymp A_1 \wedge A_1 \not\asymp A_2 \wedge \dots \wedge A_{p-2} \not\asymp A_{p-1}$, $B_0 \not\asymp B_1 \wedge B_1 \not\asymp B_2 \wedge \dots \wedge B_{n-2} \not\asymp B_{n-1}$. Take j such that $A \not\asymp B_j$ and then take i such that $A_i \not\asymp B_j$. Then (using symmetry of the relation $\not\asymp$) we have $(A_0 \not\asymp A_1 \wedge A_1 \not\asymp A_2 \wedge \dots \wedge A_{p-2} \not\asymp A_{p-1}) \wedge (A_{p-1} \not\asymp A_{p-2} \not\asymp \ldots A_{i+1} \not\asymp A_i) \wedge A_i \not\asymp B_j \wedge (B_j \not\asymp B_{j-1} \wedge \dots \wedge B_1 \not\asymp B_0) \wedge (B_0 \not\asymp B_1 \wedge B_1 \not\asymp B_2 \wedge \dots \wedge B_{q-2} \not\asymp B_{q-1})$. So $F' = A \sqcup B$ is representable as the join of a finite sequence of filters with each adjacent pair of filters in this sequence being intersecting. That is $F' \in Q$. □

PROPOSITION 2337. The lattice of Cauchy spaces (on some set) is a complete sublattice of the lattice of almost sub-join spaces.

PROOF. It's obvious, taking into account obvious 2301. □

Denote $Z_\infty(f) = \left\{ \frac{\bigsqcup T}{T \in \mathscr{P} f \wedge \bigsqcap T \neq \bot} \right\} \cup \{\bot\}$.

PROPOSITION 2338. $Z_\infty(f) \sqsupseteq f$.

PROOF. Consider for $F \in f$ both cases $F = \bot$ and $F \neq \bot$. □

LEMMA 2339. If S is a set of graphs of low spaces, then

$$Q = \bigcup S \cup Z_\infty\left(\bigcup S\right) \cup Z_\infty\left(Z_\infty\left(\bigcup S\right)\right) \cup \ldots$$

is a graph of a completely Cauchy space.

PROOF. That it is nonempty and a lower set of filters is obvious. It remains to prove that it is a completely almost sub-join-semilattice.

Let $T \in \mathscr{P} Q$ and $\bigsqcap T \neq \bot$. Then

$$T \in \mathscr{P} \underbrace{Z_\infty \ldots Z_\infty}_{n \text{ times}}\left(\bigcup S\right)$$

for a natural n. Thus

$$T \in \mathscr{P} \underbrace{Z_\infty \ldots Z_\infty}_{n+1 \text{ times}}\left(\bigcup S\right)$$

and so $\bigsqcup T \in Q$. □

PROPOSITION 2340. The lattice of completely Cauchy spaces (on some set) is a complete sublattice of the lattice of completely almost sub-join spaces.

PROOF. It's obvious, taking into account obvious 2301. □

PROPOSITION 2341. Join of a set S on the lattice of graphs of completely almost sub-join-semilattice is described by the formula:

$$\bigsqcup^{\mathbf{CASJ}} S = \bigcup S \cup Z_\infty\left(\bigcup S\right) \cup Z_\infty\left(Z_\infty\left(\bigcup S\right)\right) \cup \ldots$$