5°. Cauchy spaces;
6°. completely Cauchy spaces.

Denote $Z(f) = \left\{ \frac{F \sqcup G}{F \in f, G \in f, F \not\asymp G} \right\} \cup \{\bot\}$ for every set f of filters (on some fixed set).

PROPOSITION 2332. $Z(f) \sqsupseteq f$ for every set f of filters.

PROOF. Consider for $F \in f$ both cases $F = \bot$ and $F \neq \bot$. □

LEMMA 2333. For graphs of low spaces f, g (on the same set)

$$Q = \bigcup S \cup Z\Big(\bigcup S\Big) \cup Z\Big(Z\Big(\bigcup S\Big)\Big) \cup \dots$$

is a graph of some almost sub-join space.

PROOF. That it is nonempty and a lower set of filters is obvious. It remains to prove that it is an almost sub-join-semilattice.

Let $\mathcal{A}, \mathcal{B} \in Q$ and $\mathcal{A} \not\asymp \mathcal{B}$. Then

$$\mathcal{A}, \mathcal{B} \in \underbrace{Z \dots Z}_{n \text{ times}}\Big(\bigcup S\Big)$$

for a natural n. Thus

$$\mathcal{A} \sqcup \mathcal{B} \in \underbrace{Z \dots Z}_{n+1 \text{ times}}\Big(\bigcup S\Big)$$

and so $\mathcal{A} \sqcup \mathcal{B} \in Q$. □

PROPOSITION 2334. Join on the lattice of graphs of almost sub-join spaces is described by the formula

$$\bigsqcup^{\mathbf{ASJ}} S = \bigcup S \cup Z\Big(\bigcup S\Big) \cup Z\Big(Z\Big(\bigcup S\Big)\Big) \cup \dots$$

PROOF. The right part of the above formula μ is a graph of an almost sub-join space (lemma).

That μ is an upper bound of S is obvious.

It remains to prove that μ is the least upper bound.

Suppose ν is an upper bound of S. Then $\nu \supseteq \bigcup S$. Thus, because ν is an almost sub-join-semilattice, $Z(\nu) \subseteq \nu$, likewise $Z(Z(\nu)) \subseteq \nu$, etc. Consequently $Z(\bigcup S) \subseteq \nu$, $Z(Z(\bigcup S)) \subseteq \nu$, etc. So we have $\mu \sqsubseteq \nu$. □

PROPOSITION 2335. FiXme: Should be merged with the previous proposition.

$$\bigsqcup^{\mathbf{ASJ}} S = \left\{ \frac{F_0 \sqcup \dots \sqcup F_{n-1}}{F_0, \dots, F_{n-1} \in \bigcup S, F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1} \text{ for } n \in \mathbb{N}} \right\}.$$

REMARK 2336. We take $F_0 \sqcup \dots \sqcup F_{n-1} = \bot$ for $n = 0$.

PROOF. Denote the right part of the above formula as R.

Suppose $F \in R$. Let's prove by induction that $F \in Q$. If $F = \bot$ that's obvious. Suppose we know that $F_0 \sqcup \dots \sqcup F_{n-1} \in Q$ that is for a natural m

$$F_0 \sqcup \dots \sqcup F_{n-1} \in \underbrace{Z \dots Z}_{m \text{ times}}\Big(\bigcup S\Big)$$

for $F_0, \dots, F_{n-1} \in \bigcup S$, $F_0 \not\asymp F_1 \wedge F_1 \not\asymp F_2 \wedge \dots \wedge F_{n-2} \not\asymp F_{n-1}$ and also $F_{n-1} \not\asymp F_n$. Then $F_0 \sqcup \dots \sqcup F_{n-1} \not\asymp F_n$ and thus $F_0 \sqcup \dots \sqcup F_{n-1} \sqcup F_n \in \underbrace{Z \dots Z}_{m+1 \text{ times}} (\bigcup S)$ that is $F_0 \sqcup \dots \sqcup F_{n-1} \sqcup F_n \in Q$. So $F \in Q$ for every $F \in R$.