DEFINITION 2317. A *transitive* low space is such low space f that $(\mathsf{RLD})_{\mathrm{Low}} f \circ (\mathsf{RLD})_{\mathrm{Low}} f = (\mathsf{RLD})_{\mathrm{Low}} f$.

REMARK 2318. The composition $(\mathsf{RLD})_{\mathrm{Low}} f \circ (\mathsf{RLD})_{\mathrm{Low}} f$ may be inequal to $(\mathsf{RLD})_{\mathrm{Low}} \mu$ for all low spaces μ (exercise!). Thus usefulness of the concept of transitive low spaces is questionable.

REMARK 2319. Every low space is "symmetric" in the sense that it corresponds to a symmetric reloid.

THEOREM 2320. (Low) is the upper adjoint of $(\mathsf{RLD})_{\mathrm{Low}}$.

PROOF. We will prove $(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}} f \sqsupseteq f$ and $(\mathsf{RLD})_{\mathrm{Low}}(\mathrm{Low})g \sqsubseteq g$ (that (Low) and $(\mathsf{RLD})_{\mathrm{Low}}$ are monotone is obvious).

Really:

$$\mathrm{GR}(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}} f = \mathrm{GR}(\mathrm{Low}) \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathrm{GR} f} \right\} = \left\{ \frac{\mathcal{Y} \in \mathscr{F} \operatorname{Ob}(f)}{\mathcal{Y} \times^{\mathsf{RLD}} \mathcal{Y} \sqsubseteq \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathrm{GR} f} \right\}} \right\} \supseteq \left\{ \frac{\mathcal{Y} \in \mathscr{F} \operatorname{Ob}(f)}{\mathcal{Y} \in \mathrm{GR} f} \right\} = \mathrm{GR} f;$$

$$(\mathsf{RLD})_{\mathrm{Low}}(\mathrm{Low})g = \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathrm{GR}(\mathrm{Low})g} \right\} = \bigsqcup \left\{ \frac{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}}{\mathcal{X} \in \mathscr{F}(\operatorname{Ob} g), \mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq g} \right\} \sqsubseteq g.$$

□

COROLLARY 2321.

1°. $(\mathsf{RLD})_{\mathrm{Low}} \bigsqcup S = \bigsqcup \langle (\mathsf{RLD})_{\mathrm{Low}} \rangle^* S$;
2°. $(\mathrm{Low}) \bigsqcap S = \bigsqcap \langle (\mathrm{Low}) \rangle^* S$.

Below it's proved that (Low) and $(\mathsf{RLD})_{\mathrm{Low}}$ can be restricted to completely almost sub-join spaces and symmetrically transitive reloids. Thus they preserve joins of (completely) almost sub-join spaces and meets of symmetrically transitive reloids. FiXme: Check. FiXme: Move it to be below the definition.

6. Lattices of low spaces

PROPOSITION 2322. $\mu \sqsubseteq \nu \Leftrightarrow \forall \mathcal{X} \in \mathrm{GR}\, \mu \exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} \sqsubseteq \mathcal{Y}$ for low filter spaces (on the same set U).

PROOF.

$\Rightarrow$. $\mu \sqsubseteq \nu \Leftrightarrow \mathrm{GR}\, \mu \subseteq \mathrm{GR}\, \nu \Rightarrow \forall \mathcal{X} \in \mathrm{GR}\, \mu \exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} = \mathcal{Y} \Rightarrow \forall \mathcal{X} \in \mathrm{GR}\, \mu \exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} \sqsubseteq \mathcal{Y}$.

$\Leftarrow$. Let $\forall \mathcal{X} \in \mathrm{GR}\, \mu \exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} \sqsubseteq \mathcal{Y}$. Take $\mathcal{X} \in \mathrm{GR}\, \mu$. Then $\exists \mathcal{Y} \in \mathrm{GR}\, \nu : \mathcal{X} \sqsubseteq \mathcal{Y}$. Thus $\mathcal{X} \in \mathrm{GR}\, \nu$. So $\mathrm{GR}\, \mu \subseteq \mathrm{GR}\, \nu$ that is $\mu \sqsubseteq \nu$.

□

OBVIOUS 2323.

1°. $(\mathsf{RLD})_{\mathrm{Low}}$ is an order embedding.
2°. (Low) is an order homomorphism.

I will denote $\bigsqcup$, $\bigsqcap$, $\sqcup$, $\sqcap$ the lattice operations on low spaces or graphs of low spaces.

PROPOSITION 2324. $\bigsqcup S = \bigcup S$ for every set S of graphs of low spaces on some set.

PROOF. It's enough to prove that there is a low space μ such that $\mathrm{GR}\, \mu = \bigcup S$. In other words, it's enough to prove that $\bigcup S$ is a nonempty lower set, but that's obvious. FiXme: A little more detailed proof. □