CHAPTER 8

Systems of sides

Now we will consider a common generalization of (some of pointfree) funcoids and (some of) Galois connections. The main purpose of this is general theorem 2274 below.

First consider some properties of Galois connections:

1. More on Galois connections

Here I will denote $\langle f \rangle$ the lower adjoint of a Galois connection f. FiXme: Switch to this notation in the book?

Let GAL be the category of Galois connections. FiXme: Need to decide whether use $\mathsf{GAL}(A, B)$ or $A \otimes B$.

I will denote $(f, g)^{-1} = (g, f)$ for a Galois connection (f, g).

We will order Galois connections by the formula

$$f \sqsubseteq g \Leftrightarrow \langle f \rangle \sqsubseteq \langle g \rangle \Leftrightarrow \langle f^{-1} \rangle \sqsupseteq \langle g^{-1} \rangle.$$

Obvious 2227. This defines a partial order on the set of Galois connections between any two (fixed) posets.

Proposition 2228. If f and g are Galois connections (between a join-semilattice $\mathfrak{A}$ and a meet-semilattice $\mathfrak{B}$), then there exists a Galois connection $f \sqcup g$ determined by the formula $\langle f \sqcup g \rangle x = \langle f \rangle x \sqcup \langle g \rangle x$.

Proof. It is enough to prove that

$$(x \mapsto \langle f \rangle x \sqcup \langle g \rangle x, y \mapsto \langle f^{-1} \rangle y \sqcap \langle g^{-1} \rangle y)$$

is a Galois connection that is that

$$\langle f \rangle x \sqcup \langle g \rangle x \sqsubseteq y \Leftrightarrow x \sqsubseteq \langle f^{-1} \rangle y \sqcap \langle g^{-1} \rangle y$$

for all relevant x and y.

Really,

$$\langle f \rangle x \sqcup \langle g \rangle x \sqsubseteq y \Leftrightarrow \langle f \rangle x \sqsubseteq y \wedge \langle g \rangle x \sqsubseteq y \Leftrightarrow \\ x \sqsubseteq \langle f^{-1} \rangle y \wedge x \sqsubseteq \langle g^{-1} \rangle y \Leftrightarrow x \sqsubseteq \langle f^{-1} \rangle y \sqcap \langle g^{-1} \rangle y.$$

□

FiXme: Describe infinite join of Galois connections.

Proposition 2229. If $\mathfrak{A}$ is a poset with least element, then $\langle a \rangle \bot = \bot$.

Proof. $\langle a \rangle \bot \sqsubseteq y \Leftrightarrow \bot \sqsubseteq \langle a^{-1} \rangle y \Leftrightarrow 1$. Thus $\langle a \rangle \bot$ is the least element. □

Proposition 2230. $(\mathfrak{A} \times \{\bot^{\mathfrak{B}}\}, \mathfrak{B} \times \{\top^{\mathfrak{A}}\})$ is the least Galois connection from a poset $\mathfrak{A}$ with greatest element to a poset $\mathfrak{B}$ with least element.

Proof. Let's prove that it is a Galois connection. We need to prove

$$(\mathfrak{A} \times \{\bot^{\mathfrak{B}}\}) x \sqsubseteq y \Leftrightarrow x \sqsubseteq (\mathfrak{B} \times \{\top^{\mathfrak{A}}\}) y.$$

But this is trivially equivalent to $1 \Leftrightarrow 1$. Thus it's a Galois connection.

That it the least is obvious. □