3. Finite boolean lattices

We can assume $\mathfrak{B} = \mathscr{P}B$ for a set B, $\operatorname{card} B = n$. Then

$f \in \mathsf{pFCD}(\mathfrak{A};\mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f\rangle X \not\asymp Y \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in Y : i \in \langle f\rangle X \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X)$.

Having values of $\langle f^{-1}\rangle\{i\}$ we can restore all $\langle f^{-1}\rangle Y$. [need this paragraph?]

Let $T_i = \left\{ \frac{X \in \mathfrak{A}}{i \in \langle f\rangle X} \right\}$.

Let now $T_i \in \mathscr{P}\mathfrak{A}$ be ideals. Can we restore $\langle f\rangle$? Yes, because we know $i \in \langle f\rangle X$ for every $X \in \mathfrak{A}$.

So, it is equivalent to:

$$\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in Y : X \in T_i \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X). \quad (1)$$

LEMMA 2214. The formula (1) is equivalent to:

$$\forall X \in \mathfrak{A}, i \in B : (X \in T_i \Leftrightarrow \langle f^{-1}\rangle\{i\} \not\asymp X). \quad (2)$$

PROOF. (1)⇒(2). Just take $Y = \{i\}$.

(2)⇒(1). Let (2) holds. Let also $X \in \mathfrak{A}, Y \in \mathfrak{B}$. Then $\langle f^{-1}\rangle Y \not\asymp X \Leftrightarrow \bigcup_{i\in Y}\langle f^{-1}\rangle\{i\} \not\asymp X \Leftrightarrow \exists i \in Y : \langle f^{-1}\rangle\{i\} \not\asymp X \Leftrightarrow \exists i \in Y : X \in T_i$.

□

Further transforming: $\forall i \in B : T_i = \partial\langle f^{-1}\rangle\{i\}$.

So $\langle f^{-1}\rangle\{i\}$ are arbitary elements of $\mathfrak{B}$ and T_i are corresponding arbitrary principal ideals.

In other words, $\mathsf{pFCD}(\mathfrak{A};\mathfrak{B}) \cong \mathfrak{A}\Pi \ldots \Pi\mathfrak{A}$ ($\operatorname{card} B$ times). Thus $\mathsf{pFCD}(\mathfrak{A};\mathfrak{B})$ is a boolean lattice.

4. About infinite case

Let $\mathfrak{A}$ be a complete boolean lattice, $\mathfrak{B}$ be an atomistic boolean lattice.

$f \in \mathsf{pFCD}(\mathfrak{A};\mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f\rangle X \not\asymp Y \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in \operatorname{atoms} Y : i \in \operatorname{atoms}\langle f\rangle X \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X)$.

Let $T_i = \left\{ \frac{X \in \mathfrak{A}}{i \in \operatorname{atoms}\langle f\rangle X} \right\}$.

T_i is an ideal: Really: That it's an upper set is obvious. Let $P \cup Q \in \left\{ \frac{X \in \mathfrak{A}}{i \in \operatorname{atoms}\langle f\rangle X} \right\}$. Then $i \in \operatorname{atoms}\langle f\rangle(P \cup Q) = \operatorname{atoms}\langle f\rangle P \cup \operatorname{atoms}\langle f\rangle Q$; $i \in \langle f\rangle P \vee i \in \langle f\rangle Q$.

Let now $T_i \in \mathscr{P}\mathfrak{A}$ be ideals. Can we restore $\langle f\rangle$? Yes, because we know $i \in \operatorname{atoms}\langle f\rangle X$ for every $X \in \mathfrak{A}$ and $\mathfrak{B}$ is atomistic.

So, it is equivalent to:

$$\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\exists i \in \operatorname{atoms} Y : X \in T_i \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X). \quad (3)$$

LEMMA 2215. The formula (3) is equivalent to:

$$\forall X \in \mathfrak{A}, i \in \operatorname{atoms}^{\mathfrak{B}} : (X \in T_i \Leftrightarrow \langle f^{-1}\rangle i \not\asymp X). \quad (4)$$

PROOF. (3)⇒(4). Let (3) holds. Take $Y = i$. Then $\operatorname{atoms} Y = \{i\}$ and thus $X \in T_i \Leftrightarrow \exists i \in \operatorname{atoms} Y : X \in T_i \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X \Leftrightarrow \langle f^{-1}\rangle i \not\asymp X$.

(4)⇒(3). Let (2) holds. Let also $X \in \mathfrak{A}$, $Y \in \mathfrak{B}$. Then $\langle f^{-1}\rangle Y \not\asymp X \Leftrightarrow \langle f^{-1}\rangle \bigsqcup \operatorname{atoms} Y \not\asymp X \Leftrightarrow \bigsqcup_{i\in\operatorname{atoms} Y}\langle f^{-1}\rangle i \not\asymp X \Leftrightarrow \exists i \in \operatorname{atoms} Y : \langle f^{-1}\rangle i \not\asymp X \Leftrightarrow \exists i \in \operatorname{atoms} Y : X \in T_i$.

□

Further equivalently transforming: $\forall i \in \operatorname{atoms}^{\mathfrak{B}} : T_i = \partial\langle f^{-1}\rangle i$.

So $\langle f^{-1}\rangle i$ are arbitary elements of $\mathfrak{B}$ and T_i are corresponding arbitrary principal ideals.