CHAPTER 5

Boolean funcoids

1. One-element boolean lattice

Let $\mathfrak{A}$ be a boolean lattice and $\mathfrak{B} = \mathscr{P}0$. It's sole element is $\bot$.

$f \in \mathsf{pFCD}(\mathfrak{A};\mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A} : (\langle f\rangle X \not\asymp \bot \Leftrightarrow \langle f^{-1}\rangle \bot \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A} : (0 \Leftrightarrow \langle f^{-1}\rangle \bot \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A} : \langle f^{-1}\rangle \bot \asymp X \Leftrightarrow \forall X \in \mathfrak{A} : \langle f^{-1}\rangle \bot = \bot^{\mathfrak{A}} \Leftrightarrow \langle f^{-1}\rangle \bot = \bot^{\mathfrak{A}} \Leftrightarrow \langle f^{-1}\rangle = \{(\bot; \bot^{\mathfrak{A}})\}$.

Thus $\operatorname{card} \mathsf{pFCD}(\mathfrak{A}; \mathscr{P}0) = 1$.

2. Two-element boolean lattice

Consider the two-element boolean lattice $\mathfrak{B} = \mathscr{P}1$.

Let f be a pointfree protofuncoid from $\mathfrak{A}$ to $\mathfrak{B}$ (that is $(\mathfrak{A};\mathfrak{B};\alpha;\beta)$ where $\alpha \in \mathfrak{B}^{\mathfrak{A}}$, $\beta \in \mathfrak{A}^{\mathfrak{B}}$).

$f \in \mathsf{pFCD}(\mathfrak{A};\mathfrak{B}) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : (\langle f\rangle X \not\asymp Y \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X) \Leftrightarrow \forall X \in \mathfrak{A}, Y \in \mathfrak{B} : ((0 \in \langle f\rangle X \wedge 0 \in Y) \vee (1 \in \langle f\rangle X \wedge 1 \in Y) \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X)$.

$T = \left\{ \frac{X \in \mathfrak{A}}{0 \in \langle f\rangle X} \right\}$ is an ideal. Really: That it's an upper set is obvious. Let $P \cup Q \in \left\{ \frac{X \in \mathfrak{A}}{0 \in \langle f\rangle X} \right\}$. Then $0 \in \langle f\rangle (P \cup Q) = \langle f\rangle P \cup \langle f\rangle Q$; $0 \in \langle f\rangle P \vee 0 \in \langle f\rangle Q$.

Similarly $S = \left\{ \frac{X \in \mathfrak{A}}{1 \in \langle f\rangle X} \right\}$ is an ideal.

Let now $T, S \in \mathscr{P}\mathfrak{A}$ be ideals. Can we restore $\langle f\rangle$? Yes, because we know $0 \in \langle f\rangle X$ and $1 \in \langle f\rangle X$ for every $X \in \mathfrak{A}$.

So it is equivalent to $\forall X \in \mathfrak{A}, Y \in \mathfrak{B} : ((X \in T \wedge 0 \in Y) \vee (X \in S \wedge 1 \in Y) \Leftrightarrow \langle f^{-1}\rangle Y \not\asymp X)$.

$f \in \mathsf{pFCD}(\mathfrak{A};\mathfrak{B})$ is equivalent to conjunction of all rows of this table:

Y	equality
$\emptyset$	$\langle f^{-1}\rangle \emptyset = \emptyset$
$\{0\}$	$X \in T \Leftrightarrow \langle f^{-1}\rangle \{0\} \not\asymp X$
$\{1\}$	$X \in S \Leftrightarrow \langle f^{-1}\rangle \{1\} \not\asymp X$
$\{0,1\}$	$X \in T \vee X \in S \Leftrightarrow \langle f^{-1}\rangle \{0,1\} \not\asymp X$

Simplified:

Y	equality
$\emptyset$	$\langle f^{-1}\rangle \emptyset = \emptyset$
$\{0\}$	$T = \partial \langle f^{-1}\rangle \{0\}$
$\{1\}$	$S = \partial \langle f^{-1}\rangle \{1\}$
$\{0,1\}$	$T \cup S = \partial \langle f^{-1}\rangle \{0,1\}$

From the last table it follows that T and S are principal ideals.

So we can take arbitrary either $\langle f^{-1}\rangle \{0\}$, $\langle f^{-1}\rangle \{1\}$ or principal ideals T and S.

In other words, we take $\langle f^{-1}\rangle \{0\}$, $\langle f^{-1}\rangle \{1\}$ arbitrary and independently. So we have $\mathsf{pFCD}(\mathfrak{A};\mathfrak{B})$ equivalent to product of two instances of $\mathfrak{A}$. So it a boolean lattice. FiXme: I messed product with disjoint union below.)