DEFINITION 2196. $W(d) = \prod^{\mathsf{RLD}} \left\{ \frac{R(d,\phi)}{\phi\in\mathbb{R},\phi>0} \right\} \sqcap \prod^{\mathsf{RLD}}_{r>0} B_r(0)$. FiXme: This is defined for infinite dimensional case. FiXme: Consider also FCD instead of RLD.

PROPOSITION 2197. For finite dimensional case $\mathbb{R}^n$ we have $W(d) = \prod^{\mathsf{RLD}} \left\{ \frac{R(d,\phi)}{\phi\in\mathbb{R},\phi>0} \right\} \sqcap \Delta^{(\mathsf{RLD})n}$ where

$$\Delta^{(\mathsf{RLD})n} = \underbrace{\Delta \times^{\mathsf{RLD}} \cdots \times^{\mathsf{RLD}} \Delta}_{n \text{ times}}.$$

PROOF. ?? □

Finally our funcoids are the complete funcoids Q_+ and Q_- described by the formulas

$$\langle Q_+\rangle^* @\{p\} = \langle p+\rangle W(d(p)) \quad \text{and} \quad \langle Q_-\rangle^* @\{p\} = \langle p+\rangle W(-d(p)).$$

Here Δ is taken from the "counter-examples" section.

In other words,

$$Q_+ = \bigsqcup_{p\in\mathbb{R}} (@\{p\} \times^{\mathsf{FCD}} \langle p+\rangle W(d(p))); \quad Q_- = \bigsqcup_{p\in\mathbb{R}} (@\{p\} \times^{\mathsf{FCD}} \langle p+\rangle W(-d(p))).$$

That is $\langle Q_+\rangle^* @\{p\}$ and $\langle Q_-\rangle^* @\{p\}$ are something like infinitely small spherical sectors (with infinitely small aperture and infinitely small radius).

FiXme: Describe the co-complete funcoids reverse to these complete funcoids.

THEOREM 2198. A D^1 curve f is an reparametrized integral curve for a direction field d iff $f \in \mathrm{C}(\iota_D|\mathbb{R}|_>, Q_+) \cap \mathrm{C}(\iota_D|\mathbb{R}|_<, Q_-)$.

PROOF. Equivalently transform $f \in \mathrm{C}(\iota_D|\mathbb{R}|, Q_+)$; $f \circ \iota_D|\mathbb{R}| \sqsubseteq Q_+ \circ f$; $\langle f \circ \iota_D|\mathbb{R}|\rangle^* @\{t\} \sqsubseteq \langle Q_+ \circ f\rangle^* @\{t\}$; $\langle f\rangle^* \Delta_>(t) \sqcap D \sqsubseteq \langle Q_+\rangle^* f(t)$; $\langle f\rangle^* \Delta_>(t) \sqsubseteq \langle Q_+\rangle^* f(t)$; $\langle f\rangle^* \Delta_>(t) \sqsubseteq f(t) + W(D(f(t)))$; $\langle f\rangle^* \Delta_>(t) - f(t) \sqsubseteq W(D(f(t)))$;

$$\forall r>0, \phi>0 \exists \delta>0 : \langle f\rangle^* (]t; t+\delta[) - f(t) \subseteq R(d(f(t)),\phi) \cap B_r(f(t));$$

$$\forall r>0, \phi>0 \exists \delta>0 \forall 0<\gamma<\delta : \langle f\rangle^* (]t; t+\gamma[) - f(t) \subseteq R(d(f(t)),\phi) \cap B_r(f(t));$$

$$\forall r>0, \phi>0 \exists \delta>0 \forall 0<\gamma<\delta : \frac{\langle f\rangle^* (]t; t+\gamma[) - f(t)}{\gamma} \subseteq R(d(f(t)),\phi) \cap B_{r/\delta}(f(t));$$

$$\forall r>0, \phi>0 \exists \delta>0 : \partial_+ f(t) \subseteq R(d(f(t)),\phi) \cap B_{r/\delta}(f(t));$$

$$\forall r>0, \phi>0 : \partial_+ f(t) \subseteq R(d(f(t)),\phi);$$

$$\partial_+ f(t) \upuparrows d(f(t))$$

where ∂_+ is the right derivative.

In the same way we derive that $\mathrm{C}(|\mathbb{R}|_<, Q_-) \Leftrightarrow \partial_- f(t) \upuparrows d(f(t))$.

Thus $f'(t) \upuparrows d(f(t))$ iff $f \in \mathrm{C}(|\mathbb{R}|_>, Q_+) \cap \mathrm{C}(|\mathbb{R}|_<, Q_-)$. □

6.4. C^n curves. We consider the differential equation $f'(t) = d(f(t))$.

We can consider this equation in any topological vector space V (https://en.wikipedia.org/wiki/Frechet_derivative), see also https://math.stackexchange.com/q/2977274/4876. Note that I am not an expert in topological vector spaces and thus my naive generalizations may be wrong in details.

n-th derivative $f^{(n)}(t) = d_n(f(t))$; $f^{(n+1)}(t) = d'_n(f(t)) \circ f'(t) = d'_n(f(t)) \circ d(f(t))$. So $d_{n+1}(y) = d'_n(y) \circ d(y)$.

Given a point $y \in V$ define

$$R^n(y) = \left\{ \frac{v \in V}{\widehat{v d_0(y)} < \frac{d_1}{1!}(y)|v| + \frac{d_2(y)}{2!}|v|^2 + \cdots + \frac{d_{n-1}(y)}{(n-1)!}|v|^{n-1} + O(|v|^n), v \neq 0} \right\}$$

for $d_0(y) \neq 0$ and $R^n = \{0\}$ if $d_0(y) = 0$.