4.1. Directed topological spaces. Directed topological spaces are defined at
http://ncatlab.org/nlab/show/directed+topological+space

Definition 2178. A *directed topological space* (or *d-space* for short) is a pair (X, d) of a topological space X and a set $d \subseteq \mathrm{C}([0;1], X)$ (called *directed paths* or *d-paths*) of paths in X such that

1°. (constant paths) every constant map $[0;1] \to X$ is directed;
2°. (reparameterization) d is closed under composition with increasing continuous maps $[0;1] \to [0;1]$;
3°. (concatenation) d is closed under path-concatenation: if the d-paths a, b are consecutive in X ($a(1) = b(0)$), then their ordinary concatenation $a + b$ is also a d-path

$$(a+b)(t) = a(2t), \text{ if } 0 \leq t \leq \frac{1}{2},$$
$$(a+b)(t) = b(2t-1), \text{ if } \frac{1}{2} \leq t \leq 1.$$

I propose a new way to construct a directed topological space. My way is more geometric/topological as it does not involve dealing with particular paths.

Definition 2179. Let T be the complete endofuncoid corresponding to a topological space and $\nu \sqsubseteq T$ be its "subfuncoid". The d-space $(\mathrm{dir})(T, \nu)$ induced by the pair (T, ν) consists of T and paths $f \in \mathrm{C}([0;1], T) \cap \mathrm{C}(|[0;1]|_{\geq}, \nu)$ such that $f(0) = f(1)$.

Proposition 2180. It is really a d-space.

Proof. Every d-path is continuous.

Constant path are d-paths because ν is reflexive.

Every reparameterization is a d-path because they are $\mathrm{C}(|[0;1]|_{\geq}, \nu)$ and we can apply the theorem about composition of continuous functions.

Every concatenation is a d-path. Denote $f_0 = \lambda t \in [0; \frac{1}{2}] : a(2t)$ and $f_1 = \lambda t \in [\frac{1}{2}; 1] : b(2t-1)$. Obviously $f_0, f_1 \in \mathrm{C}([0;1], \mu) \cap \mathrm{C}(|[0;1]|_{\geq}, \nu)$. Then we conclude that $a + b = f_1 \sqcup f_1$ is in $f_0, f_1 \in \mathrm{C}([0;1], \mu) \cap \mathrm{C}(|[0;1]|_{\geq}, \nu)$ using the fact that the operation $\circ$ is distributive over $\sqcup$. □

Below we show that not every d-space is induced by a pair of an endofuncoid and its subfuncoid. But are d-spaces not represented this way good anything except counterexamples?

Let now we have a d-space (X, d). Define funcoid ν corresponding to the d-space by the formula $\nu = \bigsqcup_{a \in d}(a \circ |\mathbb{R}|_{\geq} \circ a^{-1})$.

Example 2181. The two directed topological spaces, constructed from a fixed topological space and two different reflexive funcoids, are the same.

Proof. Consider the indiscrete topology T on $\mathbb{R}$ and the funcoids $1^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}$ and $1^{\mathsf{FCD}(\mathbb{R},\mathbb{R})} \sqcup (\{0\} \times^{\mathsf{FCD}} \Delta_{\geq})$. The only d-paths in both these settings are constant functions. □

Example 2182. A d-space is not determined by the induced funcoid.

Proof. The following a d-space induces the same funcoid as the d-space of all paths on the plane.

Consider a plane $\mathbb{R}^2$ with the usual topology. Let d-paths be paths lying inside a polygonal chain (in the plane). □