

- 1°. f is monotone and $f \in C(A, B)$ iff $f \in C(A, B) \cap C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$
 iff $f \in C(A, B) \cap C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ iff $f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \cap C(\iota_A|\mathbb{R}|_{\leq}, \iota_B|\mathbb{R}|_{\leq})$.
- 2°. f is strictly monotone and $f \in C(A, B)$ iff $f \in C(A, B) \cap C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$
 iff $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>}) \cap C(\iota_A|\mathbb{R}|_{<}, \iota_B|\mathbb{R}|_{<})$.

FiXme: Generalize for arbitrary posets. **FiXme:** Generalize for f being a funcoïd.

PROOF. Because f is continuous, we have $\langle f \circ \iota_A|\mathbb{R}| \rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}| \circ f \rangle^* \{x\}$ that is $\langle f \rangle^* \Delta(x) \sqsubseteq \Delta(f(x))$ for every x .

If f is monotone, we have $\langle f \rangle^* \Delta_{\geq}(x) \sqsubseteq [f(x); \infty[$. Thus $\langle f \rangle^* \Delta_{\geq}(x) \sqsubseteq \Delta_{\geq}(f(x))$, that is $\langle f \circ \iota_A|\mathbb{R}|_{\geq} \rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{\geq} \circ f \rangle^* \{x\}$, thus $f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$.

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Let now $f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$.

Take any $a \in A$ and let $c = \left\{ \frac{b \in B}{b \geq a, \forall x \in [a; b[: f(x) \geq f(a)} \right\}$. It's enough to prove that c is the right endpoint (finite or infinite) of A .

Indeed by continuity $f(a) \leq f(c)$ and if c is not already the right endpoint of A , then there is $b' > c$ such that $\forall x \in [c; b'[: f(x) \geq f(c)$. So we have $\forall x \in [a; b'[: f(x) \geq f(c)$ what contradicts to the above.

So f is monotone on the entire A .

$f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \Rightarrow f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ is obvious. Reversely $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq}) \Rightarrow f \circ \iota_A|\mathbb{R}|_{>} \sqsubseteq \iota_B|\mathbb{R}|_{\geq} \circ f \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \langle \iota_A|\mathbb{R}|_{>} \rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{\geq} \rangle^* \langle f \rangle^* \{x\} \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \Delta_{>}(x) \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \Delta_{>}(x) \sqcup \{f(x)\} \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \Delta_{>}(x) \sqcup \{x\} \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \Delta_{>}(x) \sqcup \{x\} \sqsubseteq \Delta_{\geq} f(x) \Leftrightarrow \forall x \in \mathbb{R} : \langle f \rangle \langle \iota_A|\mathbb{R}|_{\geq} \rangle^* \{x\} \sqsubseteq \langle \iota_B|\mathbb{R}|_{\geq} \rangle^* \langle f \rangle^* \{x\} \Leftrightarrow \forall x \in \mathbb{R} : f \circ \iota_A|\mathbb{R}|_{\geq} \sqsubseteq \iota_B|\mathbb{R}|_{\geq} \circ f \Leftrightarrow f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$.

Let $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$. Then $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ and thus it is monotone. We need to prove that f is strictly monotone. Suppose the contrary. Then there is a nonempty interval $[p; q] \subseteq A$ such that f is constant on this interval. But this is impossible because $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$.

Prove that $f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \cap C(\iota_A|\mathbb{R}|_{\leq}, \iota_B|\mathbb{R}|_{\leq})$ implies $f \in C(A, B)$. Really, it implies $\langle f \rangle \Delta_{\leq}(x) \sqsubseteq \Delta_{\leq}(f(x))$ and $\langle f \rangle \Delta_{\geq}(x) \sqsubseteq \Delta_{\geq}(f(x))$ thus $\langle f \rangle \Delta(x) = \langle f \rangle (\Delta_{\leq}(x) \sqcup \{x\} \sqcup \Delta_{\geq}(x)) \sqsubseteq \Delta_{\leq} f(x) \sqcup \{f(x)\} \sqcup \Delta_{\geq} f(x) = \Delta(f(x))$.

Prove that $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>}) \cap C(\iota_A|\mathbb{R}|_{<}, \iota_B|\mathbb{R}|_{<})$ implies $f \in C(A, B)$. Really, it implies $\langle f \rangle \Delta_{<}(x) \sqsubseteq \Delta_{<}(f(x))$ and $\langle f \rangle \Delta_{>}(x) \sqsubseteq \Delta_{>}(f(x))$ thus $\langle f \rangle \Delta(x) = \langle f \rangle (\Delta_{<}(x) \sqcup \{x\} \sqcup \Delta_{>}(x)) \sqsubseteq \Delta_{<} f(x) \sqcup \{f(x)\} \sqcup \Delta_{>} f(x) = \Delta(f(x))$. \square

THEOREM 2177. Let function $f : A \rightarrow B$ where $A, B \in \mathcal{P}\mathbb{R}$.

- 1°. f is locally monotone and $f \in C(A, B)$ iff $f \in C(A, B) \cap C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq})$
 iff $f \in C(A, B) \cap C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{\geq})$ iff $f \in C(\iota_A|\mathbb{R}|_{\geq}, \iota_B|\mathbb{R}|_{\geq}) \cap C(\iota_A|\mathbb{R}|_{\leq}, \iota_B|\mathbb{R}|_{\leq})$.
- 2°. f is locally strictly monotone and $f \in C(A, B)$ iff $f \in C(A, B) \cap C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>})$ iff $f \in C(\iota_A|\mathbb{R}|_{>}, \iota_B|\mathbb{R}|_{>}) \cap C(\iota_A|\mathbb{R}|_{<}, \iota_B|\mathbb{R}|_{<})$.

PROOF. By the lemma it is (strictly) monotone on each connected component. \square

See also related math.SE questions:

- 1°. <http://math.stackexchange.com/q/1473668/4876>
 2°. <http://math.stackexchange.com/a/1872906/4876>
 3°. <http://math.stackexchange.com/q/1875975/4876>