PROOF. $\mathcal{A} \div A$ and $\mathcal{A} \sqcap A$ are determined by each other by the following formulas:

$$\mathcal{A} \div A = (\mathcal{A} \sqcap A) \div A \quad \text{and} \quad \mathcal{A} \sqcap A = (\mathcal{A} \div A) \div \operatorname{Base}(\mathcal{A}).$$

Prove the formulas: $(\mathcal{A} \sqcap A) \div A = \prod \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A} \sqcap A} \right\} = \prod \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A}} \right\} = \mathcal{A} \div A$.

$$(\mathcal{A} \div A) \div \operatorname{Base}(\mathcal{A}) = \prod \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A}} \right\} \div \operatorname{Base}(\mathcal{A}) = \prod \left\{ \frac{\uparrow^{\operatorname{Base}(\mathcal{A})} (Y \cap \operatorname{Base}(\mathcal{A}))}{Y \in \prod \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A}} \right\}} \right\} =$$

(by properties of filter bases) $= \prod \left\{ \frac{\uparrow^{\operatorname{Base}(\mathcal{A})} (X \cap A \cap \operatorname{Base}(\mathcal{A}))}{X \in} \right\} = \prod \left\{ \frac{\uparrow^{\operatorname{Base}(\mathcal{A})} (X \cap A)}{X \in \mathcal{A}} \right\} = \mathcal{A} \sqcap A$.

That this defines a bijection, follows from $\mathcal{A} \div A \sim \mathcal{A} \sqcap A$ what easily follows from the above formulas. □

PROPOSITION 2153. $\left\{ \frac{(\iota_{X,Y} f, \mathrm{id}_Y^{\mathbf{Rel}(\operatorname{Dst} f)} \circ f \circ \mathrm{id}_X^{\mathbf{Rel}(\operatorname{Src} f)})}{f \in \mathbf{Rel}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \operatorname{Src} f$, $Y \subseteq \operatorname{Dst} f$.

PROOF. $\iota_{X,Y} f = (X, Y, \operatorname{GR} f \cap (X \times Y))$; $\mathrm{id}_Y^{\mathbf{Rel}} \circ f \circ \mathrm{id}_X^{\mathbf{Rel}} = (\operatorname{Src} f, \operatorname{Dst} f, \operatorname{GR} f \cap (X \times Y))$. The isomorphism (both order and semigroup) is evident. □

PROPOSITION 2154. $\left\{ \frac{(\iota_{X,Y} f, \mathrm{id}_Y^{\mathsf{FCD}(\operatorname{Dst} f)} \circ f \circ \mathrm{id}_X^{\mathsf{FCD}(\operatorname{Src} f)})}{f \in \mathsf{FCD}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \operatorname{Src} f$, $Y \subseteq \operatorname{Dst} f$.

PROOF. From symmetry it follows that it's enough to prove that $\left\{ \frac{(\mathcal{E}^Y \circ f, \mathrm{id}_Y^{\mathsf{FCD}} \circ f)}{f \in \mathsf{FCD}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for a set $Y \subseteq \operatorname{Dst} f$.

Really, $\left\{ \frac{(\langle \mathcal{E}^Y \rangle x, \langle \mathrm{id}_Y^{\mathsf{FCD}} \rangle x)}{x \in \operatorname{Dst} f} \right\} = \left\{ \frac{(x \div Y, x \sqcap Y)}{x \in \operatorname{Dst} f} \right\}$ is an order isomorphism by proved above. This implies that $\left\{ \frac{(\mathcal{E}^Y \circ f, \mathrm{id}_Y^{\mathsf{FCD}} \circ f)}{f \in \mathsf{FCD}(A,B)} \right\}$ is an isomorphism (both order and semigroup). □

PROPOSITION 2155. $\left\{ \frac{(\iota_{X,Y} f, \mathrm{id}_Y^{\mathsf{RLD}(\operatorname{Dst} f)} \circ f \circ \mathrm{id}_X^{\mathsf{RLD}(\operatorname{Src} f)})}{f \in \mathsf{RLD}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \operatorname{Src} f$, $Y \subseteq \operatorname{Dst} f$.

PROOF. $\iota_{X,Y} f = (X, Y, (\operatorname{up} f) \div (X \times Y))$; $\mathrm{id}_Y^{\mathsf{RLD}} \circ f \circ \mathrm{id}_X^{\mathsf{RLD}} = (\operatorname{Src} f, \operatorname{Dst} f, (\operatorname{up} f) \sqcap (X \times Y))$. They are order isomorphic by proved above.

$\iota_{Y,Z} g \circ \iota_{X,Y} f = \mathcal{E}^{\operatorname{Dst} g, Z} \circ g \circ \mathcal{E}^{Y, \operatorname{Src} g} \circ \mathcal{E}^{\operatorname{Dst} f, Y} \circ f \circ \mathcal{E}^{X, \operatorname{Src} f} = \mathcal{E}^{\operatorname{Dst} g, Z} \circ g \circ \mathrm{id}_Y^{\mathsf{RLD}} \circ \mathrm{id}_Y^{\mathsf{RLD}} \circ f \circ \mathcal{E}^{X, \operatorname{Src} f}$ because $\mathcal{E}^{Y, \operatorname{Src} g} \circ \mathcal{E}^{\operatorname{Dst} f, Y} = \mathrm{id}_Y^{\mathbf{Rel}} = \mathrm{id}_Y^{\mathbf{Rel}} \circ \mathrm{id}_Y^{\mathbf{Rel}}$. Thus by proved above

$$\left\{ \frac{(\iota_{Y,Z} g \circ \iota_{X,Y} f, \mathrm{id}_Z^{\mathsf{RLD}} \circ g \circ \mathrm{id}_Y^{\mathsf{RLD}} \circ \mathrm{id}_Y^{\mathsf{RLD}} \circ f \circ \mathrm{id}_X^{\mathsf{RLD}})}{f \in \mathsf{RLD}(A,B)} \right\}$$

is a bijection. □

Can three last propositions be generalized into one?