NOTE 2098. For example for below defined category of funcoids (with binary product morphism), these filters are filters on filters on sets not filters of sets and thus are not the same as im and dom.

6. Operations on the set of unfixed morphisms

6.1. Semigroup of unfixed morphisms.

PROPOSITION 2099. Let $f : A_0 \to A_1$ and $g : A_1 \to A_2$ and $A_1 \sqsubseteq B_1$. Then $\iota_{B_0,B_2}(g \circ f) = \iota_{B_1,B_2} g \circ \iota_{B_0,B_1} f$.

PROOF. $\iota_{B_0,B_2}(g \circ f) = \mathcal{E}_{\mathcal{C}}^{A_2,B_2} \circ g \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0,A_0} = \mathcal{E}_{\mathcal{C}}^{A_2,B_2} \circ g \circ 1^{A_1} \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0,A_0} = \mathcal{E}_{\mathcal{C}}^{A_2,B_2} \circ g \circ \mathrm{id}_{A_1}^{\mathcal{C}(\mathrm{Dst}\, f, \mathrm{Src}\, g)} \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0,A_0} = \mathcal{E}_{\mathcal{C}}^{A_2,B_2} \circ g \circ \mathcal{E}^{B_1,A_1} \circ \mathcal{E}^{A_1,B_1} \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0,A_0} = \iota_{B_1,B_2} g \circ \iota_{B_0,B_1} f$. □

DEFINITION 2100. We will turn the category $\mathcal{C}$ into a semigroup $\mathcal{UC}$ (*the semigroup of unfixed morphisms*) by the formula $[g] \circ [f] = [g \circ f]$ whenever f and g are composable morphisms.

We need to prove that $[g] \circ [f]$ does not depend on choice of f and g (provided that f and g are composable). We also need to prove that $[g] \circ [f]$ is always defined for every morphisms (not necessarily composable) f and g. That the resulting structure is a semigroup (that is, $\circ$ is associative) is then obvious.

PROOF. That $[g] \circ [f]$ is defined in at least one way for every morphisms f and g is simple to prove. Just consider the morphisms $f' = \iota_{\mathrm{Src}\, f, \mathrm{Dst}\, f \sqcup \mathrm{Src}\, g} f \sim f$ and $g' = \iota_{\mathrm{Dst}\, f \sqcup \mathrm{Src}\, g, \mathrm{Dst}\, g} g \sim g$. Then we can take $[g] \circ [f] = [g' \circ f']$.

It remains to prove that $[g] \circ [f]$ does not depend on choice of f and g. Really, take arbitrary composable pairs of morphisms $(f_0 : A_0 \to B_0, g_0 : B_0 \to C_0)$ and $(f_1 : A_1 \to B_1, g_1 : B_1 \to C_1)$ such that $f_0 \sim f_1$ and $g_0 \sim g_1$. It remains to prove that $g_0 \circ f_0 \sim g_1 \circ f_1$. We have

$$\iota_{B_0 \sqcup B_1, C_0 \sqcup C_1} g_0 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f_0 = (\text{proposition 2099}) = \mathcal{E}_{\mathcal{C}}^{C_0, C_0 \sqcup C_1} \circ g_0 \circ f_0 \circ \mathcal{E}_{\mathcal{C}}^{A_0 \sqcup A_1, B_0} = \iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_0 \circ f_0).$$

Similarly

$$\iota_{B_0 \sqcup B_1, C_0 \sqcup C_1} g_1 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f_1 = \iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_1 \circ f_1).$$

But

$$\iota_{B_0 \sqcup B_1, C_0 \sqcup C_1} g_0 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f_0 = \iota_{B_0 \sqcup B_1, C_0 \sqcup C_1} g_1 \circ \iota_{A_0 \sqcup A_1, B_0 \sqcup B_1} f_1$$

thus having $\iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_0 \circ f_0) = \iota_{A_0 \sqcup A_1, C_0 \sqcup C_1}(g_1 \circ f_1)$ and so $g_0 \circ f_0 \sim g_1 \circ f_1$. □

6.2. Restricted identities.

DEFINITION 2101. *Restricted identity* for unfixed morphisms is defined as: $\mathrm{id}_X = [\mathrm{id}_X^{\mathcal{C}(A,B)}]$ for an $X \sqsubseteq [A] \sqcap [B]$.

We need to prove that it does not depend on the choice of A and B.

PROOF. Let $\mathfrak{A} \ni X \sqsubseteq [A_0] \sqcap [B_0]$ and $\mathfrak{A} \ni X \sqsubseteq [A_1] \sqcap [B_1]$ for $A_0, B_0, A_1, B_1 \in \mathfrak{Z}$. We need to prove $\mathrm{id}_X^{\mathcal{C}(A_0,B_0)} \sim \mathrm{id}_X^{\mathcal{C}(A_1,B_1)}$.

Really, $\iota_{A_1,B_1} \mathrm{id}_X^{\mathcal{C}(A_0,B_0)} = \mathcal{E}^{B_0,B_1} \circ \mathrm{id}_X^{\mathcal{C}(A_0,B_0)} \circ \mathcal{E}^{A_1,A_0} = \mathrm{id}_{[B_0] \sqcap [B_1]}^{\mathcal{C}(B_0,B_1)} \circ \mathrm{id}_X^{\mathcal{C}(A_0,B_0)} \circ \mathrm{id}_{[A_0] \sqcap [A_1]}^{\mathcal{C}(A_1,A_0)} = \mathrm{id}_{[A_0] \sqcap [A_1] \sqcap [B_0] \sqcap [B_1] \sqcap X}^{\mathcal{C}(A_1,B_1)} = \mathrm{id}_X^{\mathcal{C}(A_1,B_1)}$. Similarly $\iota_{A_0,B_0} \mathrm{id}_X^{\mathcal{C}(A_1,B_1)} = \mathrm{id}_X^{\mathcal{C}(A_0,B_0)}$.

So $\mathrm{id}_X^{\mathcal{C}(A_0,B_0)} \sim \mathrm{id}_X^{\mathcal{C}(A_1,B_1)}$. □

PROPOSITION 2102. $\mathrm{id}_Y \circ \mathrm{id}_X = \mathrm{id}_{X \sqcap Y}$ for every $X, Y \in \mathfrak{A}$.