5. Binary product

DEFINITION 2091. The category *with binary product morphism* is a category with restricted identities and additional axioms

1°. $\mathrm{id}_Y^{\mathcal{C}(B,B)} \circ f \circ \mathrm{id}_X^{\mathcal{C}(A,A)} = f \sqcap (X \times_{A,B} Y)$ (holding for every $A, B \in \mathfrak{Z}$, $\mathfrak{A} \ni X \sqsubseteq [A]$, $\mathfrak{A} \ni Y \sqsubseteq [B]$, $X \times_{A,B} Y \in \mathcal{C}(A,B)$ and morphism $f \in \mathcal{C}(A,B)$);
2°. $\iota_{A_1,B_1}(X \times_{A_0,B_0} Y) = X \times_{A_1,B_1} Y$ whenever $X \sqsubseteq [A_0] \sqcap [A_1]$ and $Y \sqsubseteq [B_0] \sqcap [B_1]$.

PROPOSITION 2092. The second axiom is equivalent to the following axiom:

1°. $f \sim X \times_{A_0,B_0} Y \Leftrightarrow f = X \times_{A_1,B_1} Y$ whenever $X \sqsubseteq [A_0] \sqcap [A_1]$ and $Y \sqsubseteq [B_0] \sqcap [B_1]$, $f : A_1 \to B_1$.

PROOF.

$\Leftarrow$. Obvious.

$\Rightarrow$. $f \sim X \times_{A_0,B_0} Y \Leftarrow f = X \times_{A_1,B_1} Y$ because $\iota_{A_1,B_1}(X \times_{A_0,B_0} Y) = X \times_{A_1,B_1} Y$ and $\iota_{A_0,B_0}(X \times_{A_1,B_1} Y) = X \times_{A_0,B_0} Y$.

Let's prove $f \sim X \times_{A_0,B_0} Y \Rightarrow f = X \times_{A_1,B_1} Y$. Really, if $f \sim X \times_{A_0,B_0} Y$ then $f = \iota_{A_1,B_1} f \sim \iota_{A_1,B_1}(X \times_{A_0,B_0} Y) = X \times_{A_1,B_1} Y$ and thus $f = X \times_{A_1,B_1} Y$. □

PROPOSITION 2093. $[A] \times_{A,B} [B]$ is the greatest morphism $\top^{\mathcal{C}(A,B)} : A \to B$.

PROOF. It's enough to prove $f \sqcap ([A] \times_{A,B} [B]) = f$ for every $f : A \to B$. Really, $f \sqcap ([A] \times_{A,B} [B]) = \mathrm{id}_B^{\mathcal{C}(B,B)} \circ f \circ \mathrm{id}_A^{\mathcal{C}(A,A)} = 1^B \circ f \circ 1^A = f$. □

PROPOSITION 2094. For every category with binary product morphism

$$X \times_{A,B} Y = \mathrm{id}_Y^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_X^{\mathcal{C}(A,A)}$$

PROOF. $X \times_{A,B} Y \sqsupseteq \mathrm{id}_Y^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_X^{\mathcal{C}(A,A)}$ because $\mathrm{id}_Y^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_X^{\mathcal{C}(A,A)} = \top^{\mathcal{C}(A,B)} \sqcap (X \times_{A,B} Y)$.

$\mathrm{id}_Y^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_X^{\mathcal{C}(A,A)} \sqsupseteq \mathrm{id}_Y^{\mathcal{C}(B,B)} \circ (X \times_{A,B} Y) \circ \mathrm{id}_X^{\mathcal{C}(A,A)} = (X \times_{A,B} Y) \sqcap (X \times_{A,B} Y) = X \times_{A,B} Y$. □

PROPOSITION 2095. $\iota_{A,B}(f \sqcap g) = \iota_{A,B} f \sqcap \iota_{A,B} g$ for every parallel morphisms f and g and objects A and B, whenever all $\mathcal{E}^{X,Y}$ are metamonovalued and metainjective.

PROOF. $\iota_{A,B}(f \sqcap g) = \mathcal{E}^{\mathrm{Dst}\, f,B} \circ (f \sqcap g) \circ \mathcal{E}^{A,\mathrm{Src}\, f} = (\mathcal{E}^{\mathrm{Dst}\, f,B} \circ f \circ \mathcal{E}^{A,\mathrm{Src}\, f}) \sqcap (\mathcal{E}^{\mathrm{Dst}\, f,B} \circ g \circ \mathcal{E}^{A,\mathrm{Src}\, f}) = \iota_{A,B} f \sqcap \iota_{A,B} g$. □

PROPOSITION 2096. $(X_0 \times_{A,B} Y_0) \sqcap (X_1 \times_{A,B} Y_1) = (X_0 \sqcap X_1) \times_{A,B} (Y_0 \sqcap Y_1)$.

PROOF. $(X_0 \times_{A,B} Y_0) \sqcap (X_1 \times_{A,B} Y_1) = \mathrm{id}_{Y_1}^{\mathcal{C}(B,B)} \circ (X_0 \times_{A,B} Y_0) \circ \mathrm{id}_{X_1}^{\mathcal{C}(A,A)} = \mathrm{id}_{Y_1}^{\mathcal{C}(B,B)} \circ \mathrm{id}_{Y_0}^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_{X_1}^{\mathcal{C}(A,A)} \circ \mathrm{id}_{X_0}^{\mathcal{C}(A,A)} = \mathrm{id}_{Y_0 \sqcap Y_1}^{\mathcal{C}(B,B)} \circ \top^{\mathcal{C}(A,B)} \circ \mathrm{id}_{X_0 \sqcap X_1}^{\mathcal{C}(A,A)} = (X_0 \sqcap X_1) \times_{A,B} (Y_0 \sqcap Y_1)$. □

PROPOSITION 2097. For a category with binary product morphism $\mathrm{Im}\, f$, $\mathrm{Dom}\, f$, $\mathrm{IM}\, f$, and $\mathrm{DOM}\, f$ are filters.

PROOF. That they are upper sets was proved above.

To prove that $\mathrm{Im}\, f$ is a filter it remains to show $A, B \in \mathrm{Im}\, f \Leftrightarrow A \sqcap B \in \mathrm{Im}\, f$. Really, $A, B \in \mathrm{Im}\, f \Leftrightarrow \top \times A \sqsupseteq f \wedge \top \times B \sqsupseteq f \Rightarrow \top \times (A \sqcap B) \sqsupseteq f \Leftrightarrow A \sqcap B \in \mathrm{Im}\, f$.

$\mathrm{Dom}\, f$ is similar.

The thesis for $\mathrm{IM}\, f$, $\mathrm{DOM}\, f$ follows from above proved for $\mathrm{Im}\, f$, $\mathrm{Dom}\, f$. □