CHAPTER 2

Unfixed categories

FiXme: This is a draft not thoroughly checked for errors.

1. Axiomatics for unfixed morphisms

DEFINITION 2056. *Category with restricted identities* is defined axiomatically: *Restricted identity* $\mathrm{id}_X^{\mathcal{C}(A,B)}$ and *projection* $A \mapsto [A]$ are described by the axioms:

1°. $\mathcal{C}$ is a category with the set of objects $\mathfrak{Z}$;
2°. every Hom-set $\mathcal{C}(A, B)$ is a lattice;
3°. $\mathfrak{Z}$ and $\mathfrak{A}$ are lattices;
4°. $A \to [A]$ is a lattice embedding from $\mathcal{C}(A, B)$ to $\mathfrak{A}$ whenever A ranges a Hom-set $\mathcal{C}(A, B)$;
5°. $\mathrm{id}_X^{\mathcal{C}(A,B)} \in \mathrm{Hom}_{\mathcal{C}}(A, B)$ whenever $\mathfrak{A} \ni X \sqsubseteq [A] \sqcap [B]$;
6°. $\mathrm{id}_{[A]}^{\mathcal{C}(A,A)} = 1_A^{\mathcal{C}}$;
7°. $\mathrm{id}_Y^{\mathcal{C}(B,C)} \circ \mathrm{id}_X^{\mathcal{C}(A,B)} = \mathrm{id}_{X \sqcap Y}^{\mathcal{C}(A,C)}$ whenever $\mathfrak{A} \ni X \sqsubseteq [A] \sqcap [B]$ and $\mathfrak{A} \ni Y \sqsubseteq [B] \sqcap [C]$;
8°. $\forall A \in \mathfrak{A} \exists B \in \mathfrak{Z} : A \sqsubseteq [B]$.

For a *partially ordered category with restricted identities* introduce additional axiom $X \sqsubseteq Y \Rightarrow \mathrm{id}_X^{\mathcal{C}(A,B)} \sqsubseteq \mathrm{id}_Y^{\mathcal{C}(A,B)}$.

For *dagger categories with restricted identities* introduce additional axiom $\left(\mathrm{id}_X^{\mathcal{C}(A,B)}\right)^\dagger = \mathrm{id}_X^{\mathcal{C}(B,A)}$.

DEFINITION 2057. I call a category with restricted identities *injective* when the axiom $X \neq Y \Rightarrow \mathrm{id}_X^{\mathcal{C}(A,B)} \neq \mathrm{id}_Y^{\mathcal{C}(A,B)}$ whenever $X, Y \sqsubseteq [A] \sqcap [B]$ holds.

DEFINITION 2058. Define $\mathcal{E}_{\mathcal{C}}^{A,B} = \mathrm{id}_{[A] \sqcap [B]}^{\mathcal{C}(A,B)}$.

PROPOSITION 2059.
1°. If $[A] \sqsubseteq [B]$ then $\mathcal{E}_{\mathcal{C}}^{A,B}$ is a monomorphism.
2°. If $[A] \sqsupseteq [B]$ then $\mathcal{E}_{\mathcal{C}}^{A,B}$ is an epimorphism.

PROOF. We'll prove only the first as the second is dual.
Let $\mathcal{E}_{\mathcal{C}}^{A,B} \circ f = \mathcal{E}_{\mathcal{C}}^{A,B} \circ g$. Then $\mathcal{E}_{\mathcal{C}}^{B,A} \circ \mathcal{E}_{\mathcal{C}}^{A,B} \circ f = \mathcal{E}_{\mathcal{C}}^{B,A} \circ \mathcal{E}_{\mathcal{C}}^{A,B} \circ g$; $1^A \circ f = 1^A \circ g$; $f = g$. □

PROPOSITION 2060. $\mathcal{E}_{\mathcal{C}}^{B,C} \circ \mathcal{E}_{\mathcal{C}}^{A,B} = \mathcal{E}_{\mathcal{C}}^{A,C}$ if $B \sqsupseteq A \sqcap C$ (for every sets A, B, C).

PROOF. $\mathcal{E}_{\mathcal{C}}^{B,C} \circ \mathcal{E}_{\mathcal{C}}^{A,B} = \mathcal{E}_{\mathcal{C}}^{A,C}$ is equivalent to:
$\mathrm{id}_{B \sqcap C}^{\mathcal{C}(B,C)} \circ \mathrm{id}_{A \sqcap B}^{\mathcal{C}(A,B)} = \mathrm{id}_{A \sqcap C}^{\mathcal{C}(A,C)}$ what is obviously true. □

2. Rectangular embedding-restriction

DEFINITION 2061. $\iota_{B_0,B_1} f = \mathcal{E}_{\mathcal{C}}^{\mathrm{Dst}\, f, B_1} \circ f \circ \mathcal{E}_{\mathcal{C}}^{B_0, \mathrm{Src}\, f}$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A_0, A_1)$.

For brevity $\iota_B f = \iota_{B,B} f$.