Algebraic General Topology. Volume 2 partial draft

Victor Porton

Email address: porton@narod.ru
URL: http://www.mathematics21.org

2000 *Mathematics Subject Classification.* 54J05, 54A05, 54D99, 54E05, 54E15, 54E17, 54E99

Key words and phrases. algebraic general topology, quasi-uniform spaces, generalizations of proximity spaces, generalizations of nearness spaces, generalizations of uniform spaces

Abstract. Partial rough draft of volume 2 of Algebraic General Topology book. This volume is meant to contain materials which refer to more advanced prerequisites than plain ZFC (such as category theory and classical pointfree topology). This is a **very** rough draft.

Contents

This file is a rough draft.
It is a continuation of [5].

CHAPTER 1

Introduction

I remind some definitions from volume 1 [5].

I denote a set definition like $\left\{ \frac{x \in A}{P(x)} \right\}$ instead of customary $\{x \in A \mid P(x)\}$ (in order to reduce formulas size).

I denote partial order as $\sqsubseteq$. I denote lattice operations as $\bigsqcap$, $\bigsqcup$, $\sqcap$, $\sqcup$.

The following generalizes monovalued morphisms in category **Rel**.

Let Hom-sets be complete lattices.

Definition 1860. A morphism f of a partially ordered category is *metamonovalued* when $(\bigsqcap G) \circ f = \bigsqcap_{g \in G}(g \circ f)$ whenever G is a set of morphisms with a suitable domain and image.

Definition 1861. A morphism f of a partially ordered category is *metainjective* when $f \circ (\bigsqcap G) = \bigsqcap_{g \in G}(f \circ g)$ whenever G is a set of morphisms with a suitable domain and image.

Obvious 1862. Metamonovaluedness and metainjectivity are dual to each other.

Definition 1863. A morphism f of a partially ordered category is *metacomplete* when $f \circ (\bigsqcup G) = \bigsqcup_{g \in G}(f \circ g)$ whenever G is a set of morphisms with a suitable domain and image.

Definition 1864. A morphism f of a partially ordered category is *co-metacomplete* when $(\bigsqcup G) \circ f = \bigsqcup_{g \in G}(g \circ f)$ whenever G is a set of morphisms with a suitable domain and image.

Let now Hom-sets be meet-semilattices.

Definition 1865. A morphism f of a partially ordered category is *weakly metamonovalued* when $(g \sqcap h) \circ f = (g \circ f) \sqcap (h \circ f)$ whenever g and h are morphisms with a suitable domain and image.

Definition 1866. A morphism f of a partially ordered category is *weakly metainjective* when $f \circ (g \sqcap h) = (f \circ g) \sqcap (f \circ h)$ whenever g and h are morphisms with a suitable domain and image.

Let now Hom-sets be join-semilattices.

Definition 1867. A morphism f of a partially ordered category is *weakly metacomplete* when $f \circ (g \sqcup h) = (f \circ g) \sqcup (f \circ h)$ whenever g and h are morphisms with a suitable domain and image.

Definition 1868. A morphism f of a partially ordered category is *weakly co-metacomplete* when $(g \sqcup h) \circ f = (g \circ f) \sqcup (h \circ f)$ whenever g and h are morphisms with a suitable domain and image.

Obvious 1869.

1°. Metamonovalued morphisms are weakly metamonovalued.
2°. Metainjective morphisms are weakly metainjective.

3°. Metacomplete morphisms are weakly metacomplete.
4°. Co-metacomplete morphisms are weakly co-metacomplete.

DEFINITION 1870. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^{\dagger} \sqsubseteq 1_{\mathrm{Dst}\, f}$.

DEFINITION 1871. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^{\dagger} \circ f \sqsupseteq 1_{\mathrm{Src}\, f}$.

DEFINITION 1872. For a partially ordered dagger category I will call *injective* morphism such a morphism f that $f^{\dagger} \circ f \sqsubseteq 1_{\mathrm{Src}\, f}$.

DEFINITION 1873. For a partially ordered dagger category I will call *surjective* morphism such a morphism f that $f \circ f^{\dagger} \sqsupseteq 1_{\mathrm{Dst}\, f}$.

REMARK 1874. It is easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective functions as morphisms of the category **Rel**.

OBVIOUS 1875. “Injective morphism” is a dual of “monovalued morphism” and “surjective morphism” is a dual of “entirely defined morphism”.

CHAPTER 2

Products in dagger categories with complete ordered Hom-sets

FiXme: This is a rough draft. It is not yet checked for errors.

NOTE 1876. What I previously denoted $\prod F$ is now denoted $\prod^{(L)} F$ (and likewise for $\coprod$). The other draft chapters referring to this chapter may be not yet updated.

PROPOSITION 1877. FiXme: Should we move this to volume 1?

1°. Every entirely defined monovalued morphism is metamonovalued and metacomplete.
2°. Every surjective injective morphism is metainjective and co-metacomplete.

PROOF. Let's prove the first (the second follows from duality):
Let f be an entirely defined monovalued morphism.
$(\bigsqcap G) \circ f \sqsubseteq \bigsqcap_{g \in G}(g \circ f)$ by monotonicity of composition.
Using the fact that f is monovalued and entirely defined:
$\left(\bigsqcap_{g \in G}(g \circ f)\right) \circ f^{\dagger} \sqsubseteq \bigsqcap_{g \in G}(g \circ f \circ f^{\dagger}) \sqsubseteq \bigsqcap G$;
$\bigsqcap_{g \in G}(g \circ f) \sqsubseteq \left(\bigsqcap_{g \in G}(g \circ f)\right) \circ f^{\dagger} \circ f \sqsubseteq (\bigsqcap G) \circ f$.
So $(\bigsqcap G) \circ f = \bigsqcap_{g \in G}(g \circ f)$.
Let f be a entirely defined monovalued morphism.
$f \circ (\bigsqcup G) \sqsupseteq \bigsqcup_{g \in G}(f \circ g)$ by monotonicity of composition.
Using the fact that f is entirely defined and monovalued:
$f^{\dagger} \circ \left(\bigsqcup_{g \in G}(f \circ g)\right) \sqsupseteq \bigsqcup_{g \in G}(f^{\dagger} \circ f \circ g) \sqsupseteq \bigsqcap G$;
$\bigsqcup_{g \in G}(f \circ g) \sqsupseteq f \circ f^{\dagger} \circ \bigsqcup_{g \in G}(f \circ g) \sqsupseteq f \circ (\bigsqcup G)$.
So $f \circ (\bigsqcup G) = \bigsqcup_{g \in G}(f \circ g)$. □

1. General product in partially ordered dagger category

To understand the below better, you can restrict your imagination to the case when $\mathcal{C}$ is the category **Rel**.

1.1. Infimum product.

Let $\mathcal{C}$ be a dagger category, each Hom-set of which is a complete lattice (having order agreed with the dagger).

We will designate some morphisms as *principal* and require that principal morphisms are both metacomplete and co-metacomplete. (For a particular example of the category **Rel**, all morphisms are considered principal.)

Let $\prod^{(Q)} X$ be an object for each indexed family X of objects.

Let π be a partial function mapping elements $X \in \operatorname{dom} \pi$ (which consists of small indexed families of objects of $\mathcal{C}$) to indexed families $\prod^{(Q)} X \to X_i$ of principal morphisms (called *projections*) for every $i \in \operatorname{dom} X$.

We will denote particular morphisms as π_i^X.

REMARK 1878. In some important examples the function π is entire, that is $\operatorname{dom}\pi$ is the set of all small indexed families of objects of $\mathcal{C}$. However there are also some important examples where it is partial.

DEFINITION 1879. *Infimum product* $\prod F$ (such that π is defined at $\lambda j \in n : \operatorname{Src} F_j$ and $\lambda j \in n : \operatorname{Dst} F_j$) is defined by the formula

$$\prod^{(L)} F = \bigsqcap_{i \in \operatorname{dom} F} ((\pi_i^{\lambda j \in n:\operatorname{Dst} F_j})^\dagger \circ F_i \circ \pi_i^{\lambda j \in n:\operatorname{Src} F_j}).$$

This formula can be (over)simplified to:

$$\prod^{(L)} F = \bigsqcap_{i \in \operatorname{dom} F} ((\pi_i^{\operatorname{Dst} \circ F})^\dagger \circ F_i \circ \pi_i^{\operatorname{Src} \circ F}).$$

REMARK 1880. $(\pi_i^{\lambda j \in n:\operatorname{Dst} F_j})^\dagger \circ F_i \circ \pi_i^{\lambda j \in n:\operatorname{Src} F_j} \in \operatorname{Hom}\left(\prod^{(Q)}_{j \in n} \operatorname{Src} F_j, \prod^{(Q)}_{j \in n} \operatorname{Dst} F_j\right)$ are properly defined and have the same sources and destination (whenever $i \in \operatorname{dom} F$ is), thus the meet in the formulas is properly defined.

REMARK 1881. Thus

$$F_0 \times^{(L)} F_1 = ((\pi_0^{(\operatorname{Dst} F_0, \operatorname{Dst} F_1)})^\dagger \circ F_0 \circ \pi_0^{(\operatorname{Src} F_0, \operatorname{Src} F_1)}) \sqcap ((\pi_1^{(\operatorname{Dst} F_0, \operatorname{Dst} F_1)})^\dagger \circ F_1 \circ \pi_1^{(\operatorname{Src} F_0, \operatorname{Src} F_1)})$$

that is product is defined by a pure algebraic formula.

PROPOSITION 1882. $\prod^{(L)} F = \max\left\{ \frac{\Phi \in \operatorname{Hom}\left(\prod^{(Q)}_{j \in n} \operatorname{Src} F_j, \prod^{(Q)}_{j \in n} \operatorname{Dst} F_j\right)}{\forall i \in n: \Phi \sqsubseteq (\pi_i^{\lambda j \in n:\operatorname{Dst} F_j})^\dagger \circ F_i \circ \pi_i^{\lambda j \in n:\operatorname{Src} F_j}} \right\}$.

PROOF. By definition of meet on a complete lattice. $\square$

COROLLARY 1883. $\prod^{(L)} F = \bigsqcup\left\{ \frac{\Phi \in \operatorname{Hom}\left(\prod^{(Q)}_{j \in n} \operatorname{Src} F_j, \prod^{(Q)}_{j \in n} \operatorname{Dst} F_j\right)}{\forall i \in n: \Phi \sqsubseteq (\pi_i^{\lambda j \in n:\operatorname{Dst} F_j})^\dagger \circ F_i \circ \pi_i^{\lambda j \in n:\operatorname{Src} F_j}} \right\}$.

THEOREM 1884. Let π_i^X be metamonovalued morphisms. If $S \in \mathscr{P}(\operatorname{Hom}(A_0, B_0) \times \operatorname{Hom}(A_1, B_1))$ for some sets A_0, B_0, A_1, B_1 then

$$\bigsqcap\left\{ \frac{a \times^{(L)} b}{(a, b) \in S} \right\} = \bigsqcap \operatorname{dom} S \times^{(L)} \bigsqcap \operatorname{im} S.$$

PROOF.

$$\begin{aligned}
&\bigsqcap\left\{ \frac{a \times b}{(a, b) \in S} \right\} = \\
&\bigsqcap\left\{ \frac{((\pi_0^{(\operatorname{Dst} a, \operatorname{Dst} b)})^\dagger \circ a \circ \pi_0^{(\operatorname{Src} a, \operatorname{Src} b)}) \sqcap ((\pi_1^{(\operatorname{Dst} a, \operatorname{Dst} b)})^\dagger \circ b \circ \pi_1^{(\operatorname{Src} a, \operatorname{Src} b)})}{(a, b) \in S} \right\} = \\
&\bigsqcap\left\{ \frac{(\pi_0^{(\operatorname{Dst} a, \operatorname{Dst} b)})^\dagger \circ a \circ \pi_0^{(\operatorname{Src} a, \operatorname{Src} b)}}{a \in \operatorname{dom} S} \right\} \sqcap \bigsqcap\left\{ \frac{(\pi_1^{(\operatorname{Dst} a, \operatorname{Dst} b)})^\dagger \circ b \circ \pi_1^{(\operatorname{Src} a, \operatorname{Src} b)}}{b \in \operatorname{im} S} \right\} = \\
&\left((\pi_0^{(\operatorname{Dst} a, \operatorname{Dst} b)})^\dagger \circ \bigsqcap\left\{ \frac{a}{a \in \operatorname{dom} S} \right\} \circ \pi_0^{(\operatorname{Src} a, \operatorname{Src} b)}\right) \sqcap \left((\pi_1^{(\operatorname{Dst} a, \operatorname{Dst} b)})^\dagger \circ \bigsqcap\left\{ \frac{b}{b \in \operatorname{im} S} \right\} \circ \pi_1^{(\operatorname{Src} a, \operatorname{Src} b)}\right) = \\
&\left((\pi_0^{(\operatorname{Dst} a, \operatorname{Dst} b)})^\dagger \circ \left(\bigsqcap \operatorname{dom} S\right) \circ \pi_0^{(\operatorname{Src} a, \operatorname{Src} b)}\right) \sqcap \left((\pi_1^{(\operatorname{Dst} a, \operatorname{Dst} b)})^\dagger \circ \left(\bigsqcap \operatorname{im} S\right) \circ \pi_1^{(\operatorname{Src} a, \operatorname{Src} b)}\right) = \\
&\bigsqcap \operatorname{dom} S \times \bigsqcap \operatorname{im} S.
\end{aligned}$$

$\square$

COROLLARY 1885. $(a_0 \times^{(L)} b_0) \sqcap (a_1 \times^{(L)} b_1) = (a_0 \sqcap a_1) \times^{(L)} (b_0 \sqcap b_1)$.

COROLLARY 1886. $a_0 \times^{(L)} b_0 \not\asymp a_1 \times^{(L)} b_1 \Leftrightarrow a_0 \not\asymp a_1 \wedge b_0 \not\asymp b_1$.

1.2. Infimum product for endomorphisms. Let F is an indexed family of endomorphisms of $\mathcal{C}$.

I will denote $\operatorname{Ob} f$ the object (source and destination) of an endomorphism f.

Let also π_i^X be a monovalued entirely defined morphism (for each $i \in \operatorname{dom} F$).

Then $\prod^{(L)} F = \bigsqcap_{i \in \operatorname{dom} F}((\pi_i^{\lambda j \in n : \operatorname{Ob} F_j})^\dagger \circ F_i \circ \pi_i^{\lambda j \in n : \operatorname{Ob} F_j})$ (if π is defined at $\lambda j \in n : \operatorname{Ob} F_j$).

Abbreviate $\pi_i = \pi_i^{\lambda j \in n : \operatorname{Ob} F_j}$.

So $\prod^{(L)} F = \bigsqcap_{i \in \operatorname{dom} F}((\pi_i)^\dagger \circ F_i \circ \pi_i)$.

$\prod^{(L)} F = \max\left\{ \frac{\Phi \in \operatorname{End}\left(\prod^{(Q)}_{j \in n} \operatorname{Ob} F_j\right)}{\forall i \in n : \Phi \sqsubseteq (\pi_i)^\dagger \circ F_i \circ \pi_i} \right\}$.

Taking into account that π_i is a monovalued entirely defined morphism, we get:

Obvious 1887. $\prod^{(L)} F = \max\left\{ \frac{\Phi \in \operatorname{End}\left(\prod^{(Q)}_{j \in n} \operatorname{Ob} F_j\right)}{\forall i \in n : \pi_i \in \mathrm{C}(\Phi, F_i)} \right\}$.

Remark 1888. The above formula may allow to define the product for non-dagger categories (but only for endomorphisms). In this writing I don't introduce a notation for this, however.

Corollary 1889. $\pi_i \in \mathrm{C}\left(\prod^{(L)} F, F_i\right)$ for every $i \in \operatorname{dom} F$.

1.3. Category of continuous morphisms. Let $\pi_i = \pi_i^X$ (for $i \in \operatorname{dom} F$) be entirely defined monovalued morphisms (we suppose it is defined at X).

Let $\bigotimes$ of an indexed family of morphisms is a morphism; $\pi_i \circ \bigotimes f = f_i$; $\bigotimes_{i \in n}(\pi_i \circ f) = f$.

Definition 1890. The category $\operatorname{cont}(\mathcal{C})$ is defined as follows:

- Objects are endomorphisms of the category $\mathcal{C}$.
- Morphisms are triples (f, a, b) where a and b are objects and $f : \operatorname{Ob} a \to \operatorname{Ob} b$ is an entirely defined monovalue principal morphism of the category $\mathcal{C}$ such that $f \in \mathrm{C}(a, b)$ (in other words, $f \circ a \sqsubseteq b \circ f$).
- Composition of morphisms is defined by the formula $(g, b, c) \circ (f, a, b) = (g \circ f, a, c)$.
- Identity morphisms are $(a, a, 1^{\mathcal{C}}_a)$.

It is really a category:

Proof. We need to prove that: composition of morphisms is a morphism, composition is associative, and identity morphisms can be canceled on the left and on the right.

That composition of morphisms is a morphism by properties of generalized continuity.

That composition is associative is obvious.

That identity morphisms can be canceled on the left and on the right is obvious. □

Remark 1891. The "physical" meaning of this category is:

- Objects (endomorphisms of $\mathcal{C}$) are spaces.
- Morphisms are continuous functions between spaces.
- $f \circ a \sqsubseteq b \circ f$ intuitively means that f combined with an infinitely small is less than infinitely small combined with f (that is f is continuous).

Definition 1892. $\pi_i^{\operatorname{cont}(\mathcal{C})} = \left(\prod^{(L)} F, F_i, \pi_i\right)$.

Proposition 1893. π_i are continuous, that is $\pi^{\operatorname{cont}}(\mathcal{C})_i$ are morphisms.

PROOF. We need to prove $\pi_i \in \mathrm{C}\left(\prod^{(L)} F, F_i\right)$ but that was proved above. □

LEMMA 1894. $f \in \mathrm{Hom}_{\mathrm{cont}(\mathcal{C})}\left(Y, \prod^{(L)} F\right)$ is continuous iff all $\pi_i \circ f$ are continuous.

PROOF.

$\Rightarrow$. Let $f \in \mathrm{Hom}_{\mathrm{cont}(\mathcal{C})}\left(Y, \prod^{(L)} F\right)$. Then $f \circ Y \sqsubseteq \left(\prod^{(L)} F\right) \circ f$; $\pi_i \circ f \circ Y \sqsubseteq \pi_i \circ \left(\prod^{(L)} F\right) \circ f$; $\pi_i \circ f \circ Y \sqsubseteq \left(\prod^{(L)} F\right) \circ \pi_i \circ f$. Thus $\pi_i \circ f$ is continuous.

$\Leftarrow$. Let all $\pi_i \circ f$ be continuous. Then $\pi_i^{\mathrm{cont}(\mathcal{C})} \circ f \in \mathrm{Hom}_{\mathrm{cont}(\mathcal{C})}(Y, F_i)$; $\pi_i^{\mathrm{cont}(\mathcal{C})} \circ f \circ Y \sqsubseteq F_i \circ \pi_i^{\mathrm{cont}(\mathcal{C})} \circ f$. We need to prove $Y \sqsubseteq f^\dagger \circ \left(\prod^{(L)} F\right) \circ f$ that is

$$Y \sqsubseteq f^\dagger \circ \bigsqcap_{i \in n} ((\pi_i)^\dagger \circ F_i \circ \pi_i) \circ f$$

for what is enough (because f is metamonovalued)

$$Y \sqsubseteq \bigsqcap_{i \in n} (f^\dagger \circ (\pi_i)^\dagger \circ F_i \circ \pi_i \circ f)$$

what follows from $Y \sqsubseteq \bigsqcap_{i \in n} (f^\dagger \circ (\pi_i)^\dagger \circ \pi_i \circ f \circ Y)$ what is obvious.

□

THEOREM 1895. $\prod^{(L)}$ together with $\bigotimes$ is a (partial) product in the category $\mathrm{cont}(\mathcal{C})$.

PROOF. Obvious.

Check http://math.stackexchange.com/questions/102632/how-to-check-whether-it-is-a-direct-product/102677#102677 □

2. On duality

We will consider duality where both the category $\mathcal{C}$ and orders on Mor-sets are replaced with their dual. I will denote $A \xleftrightarrow{\text{dual}} B$ when two formulas A and B are dual with this duality.

PROPOSITION 1896. $f \in \mathrm{C}(\mu, \nu) \xleftrightarrow{\text{dual}} f^\dagger \in \mathrm{C}(\nu^\dagger, \mu^\dagger)$.

PROOF. $f \in \mathrm{C}(\mu, \nu) \Leftrightarrow f \circ \mu \sqsubseteq \nu \circ f \xleftrightarrow{\text{dual}} \mu^\dagger \circ f^\dagger \sqsupseteq f^\dagger \circ^\dagger \nu^{-1} \Leftrightarrow f^\dagger \in \mathrm{C}(\nu^\dagger, \mu^\dagger)$. □

f is entirely defined $\Leftrightarrow f^\dagger \circ f \sqsupseteq 1_{\mathrm{Src}\, f} \xleftrightarrow{\text{dual}} f^\dagger \circ f \sqsubseteq 1_{\mathrm{Src}\, f} \Leftrightarrow f$ is injective $\Leftrightarrow f^\dagger$ is monovalued.

f is monovalued $\Leftrightarrow f \circ f^\dagger \sqsubseteq 1_{\mathrm{Dst}\, f} \xleftrightarrow{\text{dual}} f \circ f^\dagger \sqsupseteq 1_{\mathrm{Dst}\, f} \Leftrightarrow f$ is surjective $\Leftrightarrow f^\dagger$ is entirely defined.

3. General coproduct in partially ordered dagger category

The below is the dual of the above, proofs are omitted as they are dual.

Let ι_i FiXme: What is ι? are entirely defined monovalued morphisms to an object Z.

Let $\iota_i \xleftrightarrow{\text{dual}} \pi_i$ that is $\iota_i = (\pi_i)^\dagger$. We have the above equivalent to π_i being monovalued and entirely defined.

3.1. Supremum coproduct. Let $\mathcal{C}$ be a dagger category, each Hom-set of which is a complete lattice (having order agreed with the dagger).

We will designate some morphisms as *principal* and require that principal morphisms are both metacomplete and co-metacomplete. (For a particular example of the category **Rel**, all morphisms are considered principal.)

Let $\coprod^{(Q)} X$ be an object for each indexed family X of objects.

Let ι be a partial function mapping elements $X \in \operatorname{dom}\iota$ (which consists of small indexed families of objects of $\mathcal{C}$) to indexed families $X_i \to \coprod^{(Q)} X$ of principal morphisms (called *injections*) for every $i \in \operatorname{dom} X$.

DEFINITION 1897. *Supremum coproduct* $\coprod^{(L)} F$ (such that ι is defined at $\lambda j \in n : \operatorname{Dst} F_j$ and $\lambda j \in n : \operatorname{Src} F_j$) is defined by the formula

$$\coprod^{(L)} F = \bigsqcup_{i \in \operatorname{dom} F} (\iota_i^{\lambda j \in n : \operatorname{Src} F_j} \circ F_i^\dagger \circ (\iota_i^{\lambda j \in n : \operatorname{Dst} F_j})^\dagger).$$

This formula can be (over)simplified to:

$$\coprod^{(L)} F = \bigsqcup_{i \in \operatorname{dom} F} (\iota_i^{\operatorname{Src} \circ F} \circ F_i^\dagger \circ (\iota_i^{\operatorname{Dst} \circ F})^\dagger).$$

REMARK 1898. $\iota_i^{\lambda j \in n : \operatorname{Src} F_j} \circ F_i \circ (\iota_i^{\lambda j \in n : \operatorname{Dst} F_j})^\dagger \in \operatorname{Hom}\left(\coprod^{(Q)}_{j \in n} \operatorname{Src} F_j, \coprod^{(Q)}_{j \in n} \operatorname{Dst} F_j\right)$ are properly defined and have the same sources and destination (whenever $i \in \operatorname{dom} F$ is), thus the meet in the formulas is properly defined.

REMARK 1899. Thus

$$F_0 \amalg^{(L)} F_1 = (\iota_0^{(\operatorname{Src} F_0, \operatorname{Src} F_1)} \circ F_0^\dagger \circ (\iota_0^{(\operatorname{Dst} F_0, \operatorname{Dst} F_1)})^\dagger) \sqcup (\iota_1^{(\operatorname{Src} F_0, \operatorname{Src} F_1)} \circ F_1^\dagger \circ (\iota_1^{(\operatorname{Dst} F_0, \operatorname{Dst} F_1)})^\dagger)$$

that is coproduct is defined by a pure algebraic formula.

PROPOSITION 1900. $\coprod^{(L)} F = \min\left\{ \frac{\Phi \in \operatorname{End}\left(\coprod^{(Q)}_{j \in n} \operatorname{Ob} F_j\right)}{\forall i \in n : \Phi \sqsupseteq \iota_i^{\lambda j \in n : \operatorname{Src} F_j} \circ F_i^\dagger \circ (\iota_i^{\lambda j \in n : \operatorname{Dst} F_j})^\dagger} \right\}$.

PROOF. By definition of meet on a complete lattice. □

COROLLARY 1901. $\coprod^{(L)} F = \bigsqcap\left\{ \frac{\Phi \in \operatorname{End}\left(\coprod^{(Q)}_{j \in n} \operatorname{Ob} F_j\right)}{\forall i \in n : \Phi \sqsupseteq \iota_i^{\lambda j \in n : \operatorname{Src} F_j} \circ F_i^\dagger \circ (\iota_i^{\lambda j \in n : \operatorname{Dst} F_j})^\dagger} \right\}$.

THEOREM 1902. Let π_i^X be metainjective morphisms. If $S \in \mathscr{P}(\operatorname{Hom}(A_0, B_0) \times \operatorname{Hom}(A_1, B_1))$ for some sets A_0, B_0, A_1, B_1 then

$$\bigsqcup \left\{ \frac{a \times^{(L)} b}{(a, b) \in S} \right\} = \bigsqcup \operatorname{dom} S \times^{(L)} \bigsqcup \operatorname{im} S.$$

COROLLARY 1903. $(a_0 \amalg^{(L)} b_0) \sqcup (a_1 \amalg^{(L)} b_1) = (a_0 \sqcap a_1) \amalg^{(L)} (b_0 \sqcap b_1)$.

COROLLARY 1904. $a_0 \amalg^{(L)} b_0 \equiv a_1 \amalg^{(L)} b_1 \Leftrightarrow a_0 \equiv a_1 \wedge b_0 \equiv b_1$.

3.2. Supremum coproduct for endomorphisms. Let F be an indexed family of endomorphisms of $\mathcal{C}$.

I will denote $\operatorname{Ob} f$ the object (source and destination) of an endomorphism f.

Let also ι_i be a monovalued entirely defined morphism (for each $i \in \operatorname{dom} F$).

DEFINITION 1905. $\coprod^{(L)} F = \bigsqcup_{i \in \operatorname{dom} F} (\iota_i^{\lambda j \in n : \operatorname{Ob} F_j} \circ F_i^\dagger \circ (\iota_i^{\lambda j \in n : \operatorname{Ob} F_j})^\dagger)$ (if ι is defined at $\lambda j \in n : \operatorname{Ob} F_j$). (I call it *supremum coproduct*).

Abbreviate $\iota_i = \iota_i^{\lambda j\in n:\operatorname{Ob} F_j}$.

So $\coprod F = \bigsqcup_{i\in\operatorname{dom} F}(\iota_i \circ F_i^\dagger \circ (\iota_i)^\dagger)$.

$\coprod F = \min\left\{\frac{\Phi\in\operatorname{End}\left(\coprod^{(Q)}_{j\in n}\operatorname{Ob} F_j\right)}{\forall i\in n:\Phi\sqsupseteq\iota_i\circ F_i^\dagger\circ(\iota_i)^\dagger}\right\}$.

Taking into account that ι_i is a monovalued entirely defined morphism, we get:

Obvious 1906. $\coprod^{(L)} = \min\left\{\frac{\Phi\in\operatorname{End}\left(\coprod^{(Q)}_{j\in n}\operatorname{Ob} F_j\right)}{\forall i\in n:\iota_i\in\mathrm{C}(F_i^\dagger,\Phi)}\right\}$.

Corollary 1907. $\iota_i \in \mathrm{C}\left(F_i, \coprod^{(L)} F\right)$ for every $i \in \operatorname{dom} F$.

3.3. Category of continuous morphisms. Let ι_i (for $i \in \operatorname{dom} F$) be entirely defined monovalued and metacomplete morphisms.

Let $\bigoplus$ of an indexed family of morphisms is a morphism; $(\bigoplus f) \circ \iota_i = f_i$; $\bigoplus_{i\in n}(f \circ \iota_i) = f$ (a dual of the above).

Let $F_i \in \operatorname{End}\left(\coprod^{(Q)}_{j\in n}\operatorname{Ob} F_j\right)$ for all $i \in n$ (where n is some index set) (a self-dual of the above).

Definition 1908. $\iota_i^{\operatorname{cont}(\mathcal{C})} = \left(\coprod^{(L)} F, F_i^\dagger, \iota_i\right)$.

Proposition 1909. ι_i are continuous, that is $\iota_i^{\operatorname{cont}(\mathcal{C})}$ are morphisms.

Lemma 1910. $f \in \operatorname{Hom}_{\operatorname{cont}(\mathcal{C})}\left(\coprod^{(L)} F, Y\right)$ FiXme: What is Y? is continuous iff all $f \circ \iota^{\operatorname{cont}(\mathcal{C})}$ are continuous.

Theorem 1911. $\coprod^{(L)}$ together with $\bigoplus$ is a (partial) coproduct in the category $\operatorname{cont}(\mathcal{C})$.

4. Applying this to the theory of funcoids and reloids

4.1. Funcoids.

Definition 1912. $\mathbf{Fcd} \stackrel{\text{def}}{=} \operatorname{cont}\mathsf{FCD}$.

Let F be a family of endofuncoids.

The cartesian product $\prod^{(Q)} X \stackrel{\text{def}}{=} \prod X$.

I define $\pi_i = \pi_i^X \in \mathsf{FCD}(\prod X, X_i)$ as the principal funcoid corresponding to the i-th projection. (Here π is entirely defined.)

The disjoint union $\coprod^{(Q)} X \stackrel{\text{def}}{=} \coprod X$.

I define $\iota_i = \iota_i^X \in \mathsf{FCD}(X_i, \coprod X)$ as the principal funcoid corresponding to the i-th canonical injection. (Here ι is entirely defined.)

Let $\bigotimes$ and $\bigoplus$ be defined in the same way as in category **Set**.

Obvious 1913. $\pi_i \circ \bigotimes f = f_i$; $\bigotimes_{i\in n}(\pi_i \circ f) = f$.

Obvious 1914. $(\bigoplus f) \circ \iota_i = f_i$; $\bigoplus_{i\in n}(f \circ \iota_i) = f$.

It is easy to show that π_i is entirely defined monovalued, and ι_i is metacomplete and co-metacomplete.

Thus we are under conditions for both canonical products and canonical coproducts and thus both $\prod^{(L)} F$ and $\coprod^{(L)} F$ are defined.

4.2. Reloids.

DEFINITION 1915. $\mathbf{Rld} \stackrel{\text{def}}{=} \operatorname{cont} \mathsf{RLD}$.

Let F be a family of endoreloids.

The cartesian product $\prod^{(Q)} X \stackrel{\text{def}}{=} \prod X$.

I define $\pi_i = \pi_i^X \in \mathsf{RLD}(\prod X, X_i)$ as the principal reloid corresponding to the i-th projection. (Here π is entirely defined.)

The disjoint union $\coprod^{(Q)} X \stackrel{\text{def}}{=} \coprod X$.

I define $\iota_i = \iota^X \in \mathsf{RLD}(X_i, \coprod X)$ as the principal reloid corresponding to the i-th canonical injection. (Here ι is entirely defined.)

Let $\bigotimes$ and $\bigoplus$ are defined in the same way as in category **Set**.

OBVIOUS 1916. $\pi_i \circ \bigotimes f = f_i$; $\bigotimes_{i\in n}(\pi_i \circ f) = f$.

OBVIOUS 1917. $(\bigoplus f) \circ \iota_i = f_i$; $\bigoplus_{i\in n}(f \circ \iota_i) = f$.

It is easy to show that π_i is entirely defined monovalued, and ι_i is metacomplete and co-metacomplete.

Thus we are under conditions for both canonical products and canonical coproducts and thus both $\prod^{(L)} F$ and $\coprod^{(L)} F$ are defined.

It is trivial that for uniform spaces infimum product of reloids coincides with product uniformilty.

5. Initial and terminal objects

Initial object of **Fcd** is the endofuncoid $\uparrow^{\mathsf{FCD}(\emptyset,\emptyset)} \emptyset$. It is initial because it has precisely one morphism o (the empty set considered as a function) to any object Y. o is a morphism because $o \circ \uparrow^{\mathsf{FCD}(\emptyset,\emptyset)} \emptyset \sqsubseteq Y \circ o$.

PROPOSITION 1918. Terminal objects of **Fcd** are exactly $\uparrow^{\mathscr{F}} \{*\} \times^{\mathsf{FCD}} \uparrow^{\mathscr{F}} \{*\} = \uparrow^{\mathsf{FCD}} \{(*,*)\}$ where $*$ is an arbitrary point.

PROOF. In order for a function $f : X \to \uparrow^{\mathsf{FCD}} \{(*,*)\}$ be a morphism, it is required exactly $f \circ X \sqsubseteq \uparrow^{\mathsf{FCD}} \{(*,*)\} \circ f$

$f \circ X \sqsubseteq (f^{-1} \circ \uparrow^{\mathsf{FCD}} \{(*,*)\})^{-1}$; $f \circ X \sqsubseteq (\{*\} \times^{\mathsf{FCD}} \langle f^{-1}\rangle\{*\})^{-1}$; $f \circ X \sqsubseteq \langle f^{-1}\rangle\{*\} \times^{\mathsf{FCD}} \{*\}$ what true exactly when f is a constant function with the value $*$. $\square$

If $n = \emptyset$ then $Z = \{\emptyset\}$; $\prod^{(L)} \emptyset = \max \mathsf{FCD}(Z,Z) = \uparrow^{\mathscr{F}} \{\emptyset\} \times^{\mathsf{FCD}} \uparrow^{\mathscr{F}} \{\emptyset\} = \uparrow^{\mathsf{FCD}} \{(\emptyset,\emptyset)\}$.

FiXme: Initial and terminal objects of **Rld**.

6. Canonical product and subatomic product

FiXme: Confusion between filters on products and multireloids.

PROPOSITION 1919. $\operatorname{Pr}_i^{\mathsf{RLD}}|_{\mathfrak{F}(Z)} = \langle \pi_i \rangle$ for every index i of a cartesian product Z.

PROOF. If $\mathcal{X} \in \mathfrak{F}(Z)$ then $(\operatorname{Pr}_i^{\mathsf{RLD}}|_{\mathfrak{F}(Z)})\mathcal{X} = \operatorname{Pr}_i^{\mathsf{RLD}} \mathcal{X} = \prod^{\mathscr{F}} \langle \operatorname{Pr}_i \rangle^* \mathcal{X} = \prod \langle \pi_i \rangle \operatorname{up} \mathcal{X} = \langle \pi_i \rangle \mathcal{X}$. $\square$

PROPOSITION 1920. $\prod^{(A)} F = \prod_{i\in n} \left(\left(\pi_i^{\mathsf{FCD}\left(\prod_{i\in n} \operatorname{Dst} F\right)} \right)^{-1} \circ F_i \circ \pi_i^{\mathsf{FCD}\left(\prod_{i\in n} \operatorname{Src} F\right)} \right)$.

PROOF. $a \left[\prod^{(A)} F\right] b \Leftrightarrow \forall i \in \operatorname{dom} F : \Pr_i^{\mathsf{RLD}} a \; [F_i] \; \Pr_i^{\mathsf{RLD}} b \Leftrightarrow$ $\forall i \in \operatorname{dom} F : \left\langle \left(\pi_i^{\mathsf{FCD}\left(\prod_{i\in n} \operatorname{Dst} F\right)}\right)^{-1} \right\rangle [F_i] \left\langle \pi_i^{\mathsf{FCD}\left(\prod_{i\in n} \operatorname{Src} F\right)} \right\rangle \Leftrightarrow$ $\forall i \in \operatorname{dom} F : a \left[\left(\pi_i^{\mathsf{FCD}\left(\prod_{i\in n} \operatorname{Dst} F\right)}\right)^{-1} \circ F_i \circ \pi_i^{\mathsf{FCD}\left(\prod_{i\in n} \operatorname{Src} F\right)} \right] b \Leftrightarrow$ $a \left[\prod_{i\in n} \left(\left(\pi_i^{\mathsf{FCD}\left(\prod_{i\in n} \operatorname{Dst} F\right)}\right)^{-1} \circ F_i \circ \pi_i^{\mathsf{FCD}\left(\prod_{i\in n} \operatorname{Src} F\right)} \right) \right] b$ for ultrafilters a and b. □

COROLLARY 1921. $\prod^{(L)} F = \prod^{(A)} F$ is F is a small indexed family of funcoids.

7. Further plans

Does the formula $\prod^{(L)}_{i\in n}(g_i \circ f_i) = \prod^{(L)} g \circ \prod^{(L)} f$ hold?

Coordinate-wise continuity.

8. Cartesian closedness

We are not only to prove (or maybe disprove) that our categories are cartesian closed, but also to find (if any) explicit formulas for exponential transpose and evaluation.

"Definition" A category is //cartesian closed// iff:

1°. It has finite products.
2°. For each objects A, B is given an object $\operatorname{MOR}(A,B)$ (//exponentiation//) and a morphism $\varepsilon^{\mathbf{Dig}}_{A,B} : \operatorname{MOR}(A,B) \times A \to B$.
3°. For each morphism $f : Z \times A \to B$ there is given a morphism (//exponential transpose//) $\sim f : Z \to \operatorname{MOR}(A,B)$.
4°. $\varepsilon_{B,C} \circ (\sim f \times 1_A) = f$ for $f : A \to B \times C$.
5°. $\sim (\varepsilon_{B,C} \circ (g \times 1_A)) = g$ for $g : A \to \operatorname{MOR}(B,C)$.

We will also denote $f \mapsto (-f)$ the reverse of the bijection $f \mapsto (\sim f)$.

Our purpose is to prove (or disprove) that categories **Dig**, **Fcd**, and **Rld** are cartesian closed. Note that they have finite (and even infinite) products is already proved.

Alternative way to prove: you can prove that the functor $- \times B$ is left adjoint to the exponentiation $-^B$ where the counit is given by the evaluation map.

8.1. Definitions.

Categories **Dig**, **Fcd**, and **Rld** are respectively categories of:

1°. discretely continuous maps between digraphs;
2°. (proximally) continuous maps between endofuncoids;
3°. (uniformly) continuous maps between endoreloids.

"Definition" //Digraph// is an endomorphism of the category **Rel**.

For a digraph A we denote $\operatorname{Ob} A$ the set of vertexes or A and $\operatorname{GR} A$ the set of edges or A.

"Definition" Category **Dig** of digraphs is the category whose objects are digraphs and morphisms are discretely continuous maps between digraphs. That is morphisms from a digraph μ to a digraph ν are functions (or more precisely morphisms of **Set**) f such that $f \circ \mu \sqsubseteq \nu \circ f$ (or equivalently $\mu \sqsubseteq f^{-1} \circ \nu \circ f$ or equivalently $f \circ \mu \circ f^{-1} \sqsubseteq \nu$).

"Remark" Category of digraphs is sometimes defined in an other (non equivalent) way, allowing multiple edges between two given vertices.

8.2. Conjectures.

CONJECTURE 1922. The categories **Fcd** and **Rld** are cartesian closed (actually two conjectures).

http://mathoverflow.net/questions/141615/how-to-prove-that-there-are-no-exponential-object-in-a-categ suggests to investigate colimits to prove that there are no exponential object.

Our purpose is to prove (or disprove) that categories **Dig**, **Fcd**, and **Rld** are cartesian closed. Note that they have finite (and even infinite) products is already proved.

Alternative way to prove: you can prove that the functor $- \times B$ is left adjoint to the exponentiation $-^B$ where the counit is given by the evaluation map.

See http://www.springer.com/us/book/9780387977102 for another way to prove Cartesian closedness.

8.3. Category Dig is cartesian closed.

Category of digraphs is the simplest of our three categories and it is easy to demonstrate that it is cartesian closed. I demonstrate cartesian closedness of **Dig** mainly with the purpose to show a pattern similarly to which we may probably demonstrate our two other categories are cartesian closed.

Let G and H be graphs:

- $\operatorname{Ob}\operatorname{MOR}(G,H) = (\operatorname{Ob} H)^{\operatorname{Ob} G}$;
- $(f,g) \in \operatorname{GR}\operatorname{MOR}(G,H) \Leftrightarrow \forall (v,w) \in \operatorname{GR} G : (f(v), g(w)) \in \operatorname{GR} H$ for every $f, g \in \operatorname{Ob}\operatorname{MOR}(G,H) = (\operatorname{Ob} H)^{\operatorname{Ob} G}$;

$\operatorname{GR} 1_{\operatorname{MOR}(B,C)} = \operatorname{id}_{\operatorname{Ob}\operatorname{MOR}(B,C)} = \operatorname{id}_{(\operatorname{Ob} H)^{\operatorname{Ob} G}}$

Equivalently

$(f,g) \in \operatorname{GR}\operatorname{MOR}(G,H) \Leftrightarrow \forall (v,w) \in \operatorname{GR} G : g \circ \{(v,w)\} \circ f^{-1} \subseteq \operatorname{GR} H$

$(f,g) \in \operatorname{GR}\operatorname{MOR}(G,H) \Leftrightarrow g \circ (\operatorname{GR} G) \circ f^{-1} \subseteq \operatorname{GR} H$

$(f,g) \in \operatorname{GR}\operatorname{MOR}(G,H) \Leftrightarrow \langle f \times^{(C)} g\rangle \operatorname{GR} G \subseteq \operatorname{GR} H$

The transposition (the isomorphism) is uncurrying.

$\sim f = \lambda a \in Z \lambda y \in A : f(a,y)$ that is $(\sim f)(a)(y) = f(a,y)$.

$(-f)(a,y) = f(a)(y)$

If $f : A \times B \to C$ then $\sim f : A \to \operatorname{MOR}(B,C)$

"Proposition" Transposition and its inverse are morphisms of **Dig**.

"Proof" It follows from the equivalence $\sim f : A \to \operatorname{MOR}(B,C) \Leftrightarrow \forall x,y : (xAy \Rightarrow (\sim f)x(\operatorname{MOR}(B,C))(\sim f)y) \Leftrightarrow$
$\forall x,y : (xAy \Rightarrow \forall (v,w) \in B : ((\sim f)xv, (\sim f)yw) \in C) \Leftrightarrow$
$\forall x,y,v,w : (xAy \wedge vBw \Rightarrow ((\sim f)xv, (\sim f)yw) \in C) \Leftrightarrow$
$\forall x,y,v,w : ((x,v)(A \times B)(y,w) \Rightarrow (f(x,v), f(y,w)) \in C) \Leftrightarrow f : A \times B \to C$.

Evaluation $\varepsilon : \operatorname{MOR}(G,H) \times G \to H$ is defined by the formula:

Then evaluation is $\varepsilon_{B,C} = -(1_{\operatorname{MOR}(B,C)})$.

So $\varepsilon_{B,C}(p,q) = (-(1_{\operatorname{MOR}(B,C)}))(p,q) = (1_{\operatorname{MOR}(B,C)})(p)(q) = p(q)$.

"Proposition" Evaluation is a morphism of **Dig**.

"Proof" Because $\varepsilon_{B,C}(p,q) = -(1_{\operatorname{MOR}(B,C)})$.

It remains to prove: * $\varepsilon_{B,C} \circ (\sim f \times 1_A) = f$ for $f : A \to B \times C$; * $\sim (\varepsilon_{B,C} \circ (g \times 1_A)) = g$ for $g : A \to \operatorname{MOR}(B,C)$.

"Proof" $\varepsilon_{B,C}(\sim f \times 1_A)(a,p) = \varepsilon_{B,C}((\sim f)a,p) = (\sim f)ap = f(a,p)$. So $\varepsilon_{B,C} \circ (\sim f \times 1_A) = f$.

$\sim (\varepsilon_{B,C} \circ (g \times 1_A))(p)(q) = (\varepsilon_{B,C} \circ (g \times 1_A))(p,q) = \varepsilon_{B,C}(g \times 1_A)(p,q) = \varepsilon_{B,C}(gp,q) = g(p)(q)$. So $\sim (\varepsilon_{B,C} \circ (g \times 1_A)) = g$.

8.4. By analogy with the proof that Dig is cartesian closed.

The most obvious way for proof attempt that **Fcd** is cartesian closed is an analogy with the proof that **Dig** is cartesian closed.

Consider the long formula above. The proof would arise if we replace x and y in this formula with filters and operations and relations on set element with operations and relations on filters.

This proof could be simplified in either of two ways:

- replace x and y with ultrafilters, see [[Proof for Fcd using ultrafilters]];
- replace x and y with sets (principal filter), see [[Proof for Fcd using sets]].

This is not quite easy however, because we need to calculate uncurrying for a entirely defined monovalued principal funcoid (what is essentially the same as a function of a **Set**-morphisms) taking either ultrafilters or principal filters as arguments. Such (generalized) uncurrying is not quite easy.

To sum what we need to prove:

- Transposition is a morphism.
- Evaluation is a morphism.
- $\varepsilon_{B,C} \circ (\sim f \times 1_A) = f$ for $f : A \to B \times C$.
- $\sim (\varepsilon_{B,C} \circ (g \times 1_A)) = g$ for $g : A \to \mathrm{MOR}(B, C)$.

8.5. Attempt to describe exponentials in Fcd.

- Exponential object HOM(A, B) is the following endofuncoid:
-
  - Object $\operatorname{Ob}\mathrm{HOM}(\mathrm{A},\mathrm{B}) = (\operatorname{Ob} B)^{\operatorname{Ob} A}$;
  - Graph is $\operatorname{GR}\mathrm{HOM}(A, B) = \uparrow^{\mathsf{FCD}} \left\{ \frac{(f,g)}{f,g \in \mathrm{Hom}_{\mathbf{Set}}(\operatorname{Ob} A, \operatorname{Ob} B) \wedge \uparrow^{\mathsf{FCD}} g \circ A \circ \uparrow^{\mathsf{FCD}} f^{-1} \sqsubseteq B} \right\}$.
- Transposition is uncurrying.
- Evaluation is $\varepsilon_{A,B} x = \langle \Pr_0^{(A)} x \rangle \Pr_1^{(A)} x$.

We need to prove that the above defined are really an exponential and an evaluation.

Possible ways to prove that **Fcd** is cartesian closed follow:

8.6. Proof for Fcd using sets. Currying for sets is $\langle f \rangle (X \times Y) = \bigcup \langle \langle \sim f \rangle X \rangle Y$ (as it's easy to prove). This simple formula gives hope, but...

It does not work with sets because an analogy for sets of the last equality of the above mentioned long formula would be:

$\forall X, Y, V, W \in \mathscr{P} \operatorname{Ob} A : (X \times V \ [A \times B]^* \ Y \times W \Rightarrow \langle f \rangle (X \times V) \ [C]^* \ \langle f \rangle (Y \times W)) \Rightarrow$

$f : A \times B \to C$

but this implication seems false.

The most obvious way for proof attempt that **Fcd** is cartesian closed is an analogy with the proof that Dig is cartesian closed.

Use the exponential object, transposition, and evaluation as defined in [[this page|Is category Fcd cartesian closed?]]

8.7. Reducing to the fact that Dig is cartesian closed. It is probably a simpler way to prove that **Fcd** is cartesian closed by embedding it into **Dig** (which is [[already known to be cartesian closed|Category Dig is cartesian closed]]).

Fcd can be embedded into **Dig** by the formulas:

- $A \mapsto \langle A \rangle$;
- $f \mapsto \langle f \rangle$.

That this really maps a morphism of **Fcd** into a morphism of **Dig** follows from the fact that $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$.

Obviously this embedding (denote it T) is an injective (both on objects and morphisms) functor.

We will define:

- $\varepsilon^{\mathbf{Fcd}}_{A,B} = T^{-1} \varepsilon^{\mathbf{Dig}}_{TA,TB}$;
- $\sim^{\mathbf{Fcd}} f = T^{-1} \sim^{\mathbf{Dig}} Tf$.

Due to functoriality and injectivity of T it is enough to prove that above defined $\varepsilon^{\mathbf{Fcd}}_{A,B}$ and $\sim^{\mathbf{Fcd}} f$ exist and are morphisms of **Fcd**.

$\varepsilon^{\mathbf{Dig}}_{TA,TB} \neq T\varepsilon^{\mathbf{Fcd}}_{A,B}$ because $\varepsilon^{\mathbf{Dig}}_{TA,TB}$ accepts ordered pairs as the argument and $T\varepsilon^{\mathbf{Fcd}}_{A,B}$ accepts sets as the argument. So this is a dead end. Can the proof idea be salvaged?

9. Is category Rld cartesian closed?

We may attempt to prove that **Rld** is cartesian closed by embedding it into supposedly cartesian closed category **Fcd** by the function ρ:

$\langle \rho f \rangle x = f \circ x \quad \text{and} \quad \langle \rho f^{-1} \rangle y = f^{-1} \circ y$.

TODO: More to write on this topic.

CHAPTER 3

Equalizers and co-Equalizers in Certain Categories

It is a rough draft. Errors are possible.
FiXme: Change notation $\prod \to \prod^{(L)}$.

1. Categories with embeddings

NOTE 1923. This section in not used below, it is just to feed your intuition.

The following generalizes the well known concept of embedding function $A \hookrightarrow B$ for from a set A to a set B where $A \subseteq B$.

I will set that the unique morphism from an object A to an object B of a thin category is equal to the pair (A, B).

DEFINITION 1924. A *category with embeddings of objects* is a dagger category with a preorder of the set of objects together with a functor $\hookrightarrow$ (we will denote applying this functor to the object (A, B) as $A \hookrightarrow B$.) such that:

- $\hookrightarrow$ is an identity on objects.
- Every $A \hookrightarrow B$ is a monomorphism.
- $(A \hookrightarrow B)^\dagger \circ (A \hookrightarrow B) = 1_A$.

OBVIOUS 1925. $A \hookrightarrow B$ is defined when (A, B) is a morphism of the preorder that is when $A \sqsubseteq B$.

OBVIOUS 1926. $A \hookrightarrow B : A \to B$ when $A \sqsubseteq B$.

PROPOSITION 1927. $A \hookrightarrow A = 1_A$.

PROOF. Because (A, A) is an identity morphism and $\hookrightarrow$ preserves identities. □

PROPOSITION 1928. $(B \hookrightarrow C) \circ (A \hookrightarrow B) = A \hookrightarrow C$ whenever $A \sqsubseteq B \sqsubseteq C$.

PROOF. $(B \hookrightarrow C)\circ(A \hookrightarrow B) = \hookrightarrow (B, C)\circ \hookrightarrow (A, B) = \hookrightarrow ((B, C)\circ(A, B)) = \hookrightarrow (A, C) = A \hookrightarrow C$. □

2. Categories under Rel

DEFINITION 1929. The **Rel**-morphism $\mathcal{E}^{A,B}$ (*restriction-embedding*) is defined by the formula: $\mathcal{E}^{A,B} = (A, B, \mathrm{id}_{A\cap B})$.

When A is clear from context, I will denote it just as $\mathcal{E}^B$.

OBVIOUS 1930. If $A \subseteq B$ then $\mathcal{E}^{A,B}$ is an embedding $A \hookrightarrow B = (A, B, \mathrm{id}_A)$.

OBVIOUS 1931. If $A \supseteq B$ then $\mathcal{E}^{A,B} = (A, B, \mathrm{id}_B)$.

OBVIOUS 1932. $\mathcal{E}^{A,A} = 1^{\mathbf{Rel}}_A$.

OBVIOUS 1933. $(\mathcal{E}^{A,B})^{-1} = \mathcal{E}^{B,A}$.

DEFINITION 1934. *Dagger functor* between two dagger categories is a functor between these categories, which commutes with the daggers. FiXme: Clearer wording.

DEFINITION 1935. *Category under* **Rel** is a pair $(C, \uparrow)$ where C is a category whose objects are small sets and $\uparrow$ is an identity-on-objects functor $\mathbf{Rel} \to C$. I call $\uparrow$ *up-arrow functor*.

DEFINITION 1936. *Dagger category under* **Rel** is a pair $(C, \uparrow)$ where C is a dagger category whose objects are small sets and $\uparrow$ is a dagger identity-on-objects functor $\mathbf{Rel} \to C$.

DEFINITION 1937. $\mathcal{E}_{\mathcal{C}}^{A,B} = \uparrow \mathcal{E}^{A,B}$. In other words, $\mathcal{E}_{\mathcal{C}} = \uparrow \circ \mathcal{E}$.
When A is clear from context, I will denote it just as $\mathcal{E}_{\mathcal{C}}^{B}$.

PROPOSITION 1938. $\mathcal{E}_{\mathcal{C}}^{A,A} = 1_{\mathcal{C}}^{A}$.

PROOF. $\mathcal{E}_{\mathcal{C}}^{A,A} = \uparrow \mathcal{E}^{A,A} = \uparrow 1_{\mathbf{Rel}} = 1_{\mathcal{C}}^{A}$. □

PROPOSITION 1939. If $f : X \to Y$ is a **Rel**-morphism and $\operatorname{im} f = A \subseteq Y$ then

$$\mathcal{E}^{A,Y} \circ \mathcal{E}^{Y,A} \circ f = f.$$

PROOF. $\mathcal{E}^{A,Y} \circ \mathcal{E}^{Y,A} \circ f = \mathrm{id}_A \circ f = f$. □

COROLLARY 1940. If $f : X \to Y$ is a morphism of a category under **Rel** and $\operatorname{im} f = A \subseteq Y$, then

$$\mathcal{E}_{\mathcal{C}}^{A,Y} \circ \mathcal{E}_{\mathcal{C}}^{Y,A} \circ \uparrow f = \uparrow f.$$

PROPOSITION 1941.

1°. If $A \subseteq B$ then $\mathcal{E}_{\mathcal{C}}^{A,B}$ is a monomorphism.
2°. If $A \supseteq B$ then $\mathcal{E}_{\mathcal{C}}^{A,B}$ is a epimorphism.

PROOF. We'll prove only the first as the second is dual.
Let $\mathcal{E}_{\mathcal{C}}^{A,B} \circ f = \mathcal{E}^{A,B} \circ g$. Then $\mathcal{E}^{B,A} \circ \mathcal{E}_{\mathcal{C}}^{A,B} \circ f = \mathcal{E}_{\mathcal{C}}^{B,A} \circ \mathcal{E}_{\mathcal{C}}^{A,B} \circ g$; $1^A \circ f = 1^A \circ g$; $f = g$. □

PROPOSITION 1942. $\mathcal{E}^{B,C} \circ \mathcal{E}^{A,B} = \mathcal{E}^{A,C}$ iff $B \supseteq A \cap C$ (for every sets A, B, C).

PROOF. $\mathcal{E}^{B,C} \circ \mathcal{E}^{A,B} = \mathcal{E}^{A,C}$ is equivalent to:
$(B, C, \mathrm{id}_{B \cap C}) \circ (A, B, \mathrm{id}_{A \cap B}) = (A, C, \mathrm{id}_{A \cap C})$;
$(A, C, \mathrm{id}_{A \cap B \cap C}) = (A, C, \mathrm{id}_{A \cap C})$;
$A \cap B \cap C = A \cap C$;
$B \supseteq A \cap C$. □

COROLLARY 1943. $\mathcal{E}^{B,C} \circ \mathcal{E}^{A,B} = \mathcal{E}_{\mathcal{C}}^{A,C}$ if $B \supseteq A \cap C$ (for every sets A, B, C).

DEFINITION 1944. *Partially ordered dagger category under* **Rel** is a category which is both a partially ordered dagger category and a category under **Rel** such that $\uparrow \circ f^{-1} = (\uparrow \circ f)^{\dagger}$ and $A \sqsubseteq B \Rightarrow \uparrow A \sqsubseteq \uparrow B$.

PROPOSITION 1945. $(\mathcal{E}_{\mathcal{C}}^{A,B})^{\dagger} = \mathcal{E}_{\mathcal{C}}^{B,A}$ for a dagger category under **Rel**.

PROOF. $(\mathcal{E}_{\mathcal{C}}^{A,B})^{\dagger} = (\uparrow \mathcal{E}^{A,B})^{\dagger} = \uparrow (\mathcal{E}^{A,B})^{-1} = \uparrow \mathcal{E}^{B,A} = \mathcal{E}_{\mathcal{C}}^{B,A}$. □

PROPOSITION 1946. For a partially ordered dagger category $\mathcal{C}$ under **Rel** we have $\mathcal{E}_{\mathcal{C}}^{A,B}$ is:

1°. monovalued;
2°. injective;
3°. entirely defined if $A \subseteq B$;
4°. surjective if $B \subseteq A$.

PROOF.
1°. $\mathcal{E}^{A,B} \circ \mathcal{E}^{B,A} \sqsubseteq 1_B^{\mathbf{Rel}}$; $\mathcal{E}^{A,B} \circ (\mathcal{E}^{A,B})^{-1} \sqsubseteq 1_B^{\mathbf{Rel}}$; $\mathcal{E}_{\mathcal{C}}^{A,B} \circ (\mathcal{E}_{\mathcal{C}}^{A,B})^{\dagger} \sqsubseteq 1_B^{\mathcal{C}}$.

2°. $\mathcal{E}^{B,A} \circ \mathcal{E}^{A,B} \sqsubseteq 1^{\mathbf{Rel}}_A$; $(\mathcal{E}^{A,B})^{-1} \circ \mathcal{E}^{A,B} \sqsubseteq 1^{\mathbf{Rel}}_A$; $(\mathcal{E}^{A,B}_{\mathcal{C}})^\dagger \circ \mathcal{E}^{A,B} \sqsubseteq 1^{\mathcal{C}}_A$.
3°. $\mathcal{E}^{B,A} \circ \mathcal{E}^{A,B} \sqsupseteq 1^{\mathbf{Rel}}_A$; $(\mathcal{E}^{A,B})^{-1} \circ \mathcal{E}^{A,B} \sqsupseteq 1^{\mathbf{Rel}}_A$; $(\mathcal{E}^{A,B}_{\mathcal{C}})^\dagger \circ \mathcal{E}^{A,B}_{\mathcal{C}} \sqsupseteq 1^{\mathcal{C}}_A$.
4°. $\mathcal{E}^{A,B} \circ \mathcal{E}^{B,A} \sqsupseteq 1^{\mathbf{Rel}}_A$; $\mathcal{E}^{A,B} \circ (\mathcal{E}^{A,B})^{-1} \sqsupseteq 1^{\mathbf{Rel}}_A$; $\mathcal{E}^{A,B}_{\mathcal{C}} \circ (\mathcal{E}^{A,B}_{\mathcal{C}})^\dagger \sqsupseteq 1^{\mathcal{C}}_A$.

□

3. Rectangular embedding-restriction

DEFINITION 1947. $\iota_{B_0,B_1} f = \mathcal{E}^{B_1}_{\mathcal{C}} \circ f \circ (\mathcal{E}^{B_0}_{\mathcal{C}})^{-1}$ for $f \in \mathrm{Hom}_{\mathcal{C}}(A_0, A_1)$.

For brevity $\iota_B f = \iota_{B,B} f$.

PROPOSITION 1948. $\iota_{\mathrm{Src}\, f, \mathrm{Dst}\, f} f = f$.

PROOF. $\iota_{\mathrm{Src}\, f, \mathrm{Dst}\, f} f = \mathcal{E}^{\mathrm{Dst}\, f}_{\mathcal{C}} \circ f \circ \mathcal{E}^{\mathrm{Src}\, f}_{\mathcal{C}} = 1^{\mathrm{Dst}\, f}_{\mathcal{C}} \circ f \circ 1^{\mathrm{Src}\, f}_{\mathcal{C}} = f$. □

PROPOSITION 1949. The function $\iota_{B_0,B_1}|_{f \in \mathrm{Hom}_{\mathcal{C}}(A_0,A_1)}$ is injective, if $A_0 \subseteq B_0 \wedge A_1 \subseteq B_1$.

PROOF. Because $\mathcal{E}^{A_1,B_1}_{\mathcal{C}}$ is a monomorphism and $\mathcal{E}^{A_0,B_0}_{\mathcal{C}}$ is an epimorphism. □

PROPOSITION 1950. $\iota_{C_0,C_1} \iota_{B_0,B_1} f = \iota_{C_0,C_1} f$ for $B_0 \supseteq A_0 \cap C_0$, $B_1 \supseteq A_1 \cap C_1$ and $f : A_0 \to A_1$.

PROOF. $\iota_{C_0,C_1} \iota_{B_0,B_1} f = \mathcal{E}^{B_1,C_1}_{\mathcal{C}} \circ \mathcal{E}^{A_1,B_1}_{\mathcal{C}} \circ f \circ \mathcal{E}^{B_0,A_0}_{\mathcal{C}} \circ \mathcal{E}^{C_0,B_0}_{\mathcal{C}} = \mathcal{E}^{A_1,C_1}_{\mathcal{C}} \circ f \circ \mathcal{E}^{C_0,A_0}_{\mathcal{C}} = \iota_{C_0,C_1} f$. □

PROPOSITION 1951. Let $f : A_0 \to A_1$ and $g : A_1 \to A_2$ and $A_1 \subseteq B_1$. Then $\iota_{B_0,B_2}(g \circ f) = \iota_{B_1,B_1} g \circ \iota_{B_0,B_1} f$.

PROOF. $\iota_{B_0,B_2}(g \circ f) = \mathcal{E}^{A_2,B_2}_{\mathcal{C}} \circ (g \circ f) \circ \mathcal{E}^{B_0,A_0}_{\mathcal{C}} = \mathcal{E}^{A_2,B_2}_{\mathcal{C}} \circ g \circ \mathrm{id}_{A_1} \circ f \circ \mathcal{E}^{B_0,A_0}_{\mathcal{C}} = \mathcal{E}^{A_2,B_2}_{\mathcal{C}} \circ g \circ \mathcal{E}^{B_1,A_1} \circ \mathcal{E}^{A_1,B_1} \circ f \circ \mathcal{E}^{B_0,A_0}_{\mathcal{C}} = \iota_{B_1,B_1} g \circ \iota_{B_0,B_1} f$. □

4. Examples of partially ordered dagger categories under Rel

4.1. Generalized rebase of filters. In [5] I defined *rebase* $\mathcal{A} \div A$ for a set-theoretic filter $\mathcal{A}$ and a set X such that $\exists X \in \mathcal{A} : X \subseteq A$.

Now define a generalized rebase for every set-theoretic filter $\mathcal{A}$ and every set A:

DEFINITION 1952. $\mathcal{A} \div A = \bigsqcap \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A}} \right\}$.

PROPOSITION 1953. These two definitions coincide.

PROOF. It is proved in [5] $\left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \cap A \subseteq X} \right\}$ is a filter.

If $P \in \left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \cap A \subseteq X} \right\}$ then $P \in \mathscr{P} A$ and $Y \cap A \subseteq P$ for some $Y \in \mathcal{A}$. Thus $P \supseteq Y \cap A \in \bigsqcap \left\{ \frac{\uparrow^A (Y \cap A)}{Y \in \mathcal{A}} \right\}$.

If $P \in \bigsqcap \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A}} \right\}$ then by properties of generalized filter bases, there exists $X \in \mathcal{A}$ such that $P \supseteq X \cap A$. Also $P \in \mathscr{P} A$. Thus $P \in \left\{ \frac{X \in \mathscr{P} A}{\exists Y \in \mathcal{A} : Y \cap A \subseteq X} \right\}$.

FiXme: Clear this proof: wording, consistent use of letters. □

PROPOSITION 1954. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \subseteq A$.

PROOF. $(\mathcal{X} \div A) \div B = \bigsqcap \left\{ \frac{\uparrow^B (Y \cap B)}{Y \in \bigsqcap \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{X}} \right\}} \right\} = \bigsqcap \left\{ \frac{\uparrow^B (X \cap A)}{X \in \mathcal{X}} \right\} \sqcap \uparrow^B B = \bigsqcap \left\{ \frac{\uparrow^B (X \cap A \cap B)}{X \in \mathcal{X}} \right\} = \mathcal{X} \div (A \cap B) = \mathcal{X} \div B$. □

4.2. Category Rel. Category **Rel** with the identity up-arrow functor to itself and "reverse relation" as the dagger is an obvious example of a partially ordered dagger category under **Rel**.

PROPOSITION 1955. $\iota_{A,B}f = (A, B, \operatorname{GR} f \cap (A \times B))$.

PROOF. $\iota_{A,B}f = \mathcal{E}^B \circ f \circ (\mathcal{E}^A)^{-1} = (A, B, \operatorname{GR} f \cap (A \times B))$. □

4.3. Category FCD. Category FCD with the up-arrow functor $\uparrow^{\mathsf{FCD}}$ and "reverse funcoid" as the dagger is a partially ordered dagger category under **Rel**.

PROPOSITION 1956. $\mathcal{E}^{A,B}_{\mathsf{FCD}} = (A, B, \lambda \mathcal{X} \in \mathfrak{F}(A) : \mathcal{X} \div B, \lambda \mathcal{Y} \in \mathfrak{F}(B) : \mathcal{Y} \div A)$ for objects $A \subseteq B$ of FCD.

PROOF. $\langle \mathcal{E}^{A,B}_{\mathsf{FCD}} \rangle \mathcal{X} = \bigsqcap \left\{ \frac{\langle \mathcal{E}^{A,B}_{\mathsf{FCD}} \rangle^* X}{X \in \mathcal{X}} \right\} = \bigsqcap \left\{ \frac{\uparrow^B \langle \mathcal{E}^{A,B} \rangle X}{X \in \mathcal{X}} \right\} = \bigsqcap \left\{ \frac{\uparrow^B (X \cap A \cap B)}{X \in \mathcal{X}} \right\} = \bigsqcap \left\{ \frac{\uparrow^B (X \cap B)}{X \in \mathcal{X}} \right\} = \mathcal{X} \div B$.
Rest follows from symmetry. □

PROPOSITION 1957.
1°. $\langle \mathcal{E}^{A,B}_{\mathsf{FCD}} \rangle^* X = \uparrow^B X$ for every $X \in \mathscr{P} A$ if $A \subseteq B$.
2°. $\langle \mathcal{E}^{B,A}_{\mathsf{FCD}} \rangle^* Y = \uparrow^A (Y \cap A)$ for every $Y \in \mathscr{P} B$ if $A \subseteq B$.

PROOF. By definition of principal funcoid. □

4.4. Category RLD. Category RLD with the up-arrow functor $\uparrow^{\mathsf{RLD}}$ and "reverse reloid" as the dagger is a partially ordered dagger category under **Rel**.

OBVIOUS 1958. $\mathcal{E}^{A,B}_{\mathsf{RLD}} = \uparrow^{\mathsf{RLD}(A,B)} \operatorname{id}_{A \cap B}$.

DEFINITION 1959. $f \div (A \times B) = (A, B, (\operatorname{GR} f) \div (A \times B))$ for every reloid f.

PROPOSITION 1960. $\iota_{A,B} f = f \div (A \times B)$.

PROOF. $\iota_{A,B} f = \mathcal{E}^B_{\mathsf{RLD}} \circ f \circ (\mathcal{E}^A_{\mathsf{RLD}})^{-1} = \bigsqcap \left\{ \frac{\uparrow^{\mathsf{RLD}} (\mathcal{E}^B \circ F \circ (\mathcal{E}^A)^{-1})}{F \in \operatorname{GR} f} \right\} = \bigsqcap \left\{ \frac{\uparrow^{\mathsf{RLD}} (F \cap (A \times B))}{F \in \operatorname{GR} f} \right\} = f \div (A \times B)$.
FiXme: Filters on cartesian products vs reloids. □

4.5. Some isomorphisms.

PROPOSITION 1961. $\left\{ \frac{(\mathcal{A} \div A, \mathcal{A} \sqcap A)}{\mathcal{A} \in \mathfrak{F}(U)} \right\}$ is a function and moreover is an order isomorphism for a set $A \subseteq U$.

PROOF. $\mathcal{A} \div A$ and $\mathcal{A} \sqcap A$ are determined by each other by the following formulas:

$$\mathcal{A} \div A = (\mathcal{A} \sqcap A) \div A \quad \text{and} \quad \mathcal{A} \sqcap A = (\mathcal{A} \div A) \div \operatorname{Base}(\mathcal{A}).$$

Prove the formulas: $(\mathcal{A} \sqcap A) \div A = \bigsqcap \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A} \sqcap A} \right\} = \bigsqcap \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A}} \right\} = \mathcal{A} \div A$.

$(\mathcal{A} \div A) \div \operatorname{Base}(\mathcal{A}) = \bigsqcap \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A}} \right\} \div \operatorname{Base}(\mathcal{A}) = \bigsqcap \left\{ \frac{\uparrow^{\operatorname{Base}(\mathcal{A})} (Y \cap \operatorname{Base}(\mathcal{A}))}{Y \in \bigsqcap \left\{ \frac{\uparrow^A (X \cap A)}{X \in \mathcal{A}} \right\}} \right\} =$ (by properties of filter bases) $= \bigsqcap \left\{ \frac{\uparrow^{\operatorname{Base}(\mathcal{A})} (X \cap A \cap \operatorname{Base}(\mathcal{A}))}{X \in} \right\} = \bigsqcap \left\{ \frac{\uparrow^{\operatorname{Base}(\mathcal{A})} (X \cap A)}{X \in \mathcal{A}} \right\} = \mathcal{A} \sqcap A$.

That this defines a bijection, follows from $\mathcal{A} \div A \sim \mathcal{A} \sqcap A$ what easily follows from the above formulas. □

PROPOSITION 1962. $\left\{ \frac{(\iota_{X,Y} f, \operatorname{id}^{\mathbf{Rel}}_Y \circ f \circ \operatorname{id}^{\mathbf{Rel}}_X)}{f \in \mathbf{Rel}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \operatorname{Src} f$, $Y \subseteq \operatorname{Dst} f$.

Proof. $\iota_{X,Y} f = (X, Y, \operatorname{GR} f \cap (X \times Y))$; $\mathrm{id}^{\mathbf{Rel}}_Y \circ f \circ \mathrm{id}^{\mathbf{Rel}}_X = (\operatorname{Src} f, \operatorname{Dst} f, \operatorname{GR} f \cap (X \times Y))$. The isomorphism (both order and semigroup) is evident. □

Proposition 1963. $\left\{ \frac{(\iota_{X,Y} f, \mathrm{id}^{\mathsf{FCD}}_Y \circ f \circ \mathrm{id}^{\mathsf{FCD}}_X)}{f \in \mathsf{FCD}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \operatorname{Src} f$, $Y \subseteq \operatorname{Dst} f$.

Proof. From symmetry it follows that it's enough to prove that $\left\{ \frac{(\mathcal{E}^Y \circ f, \mathrm{id}^{\mathsf{FCD}}_Y \circ f)}{f \in \mathsf{FCD}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for a set $Y \subseteq \operatorname{Dst} f$.

Really, $\left\{ \frac{(\langle \mathcal{E}^Y \rangle x, \langle \mathrm{id}^{\mathsf{FCD}}_Y \rangle x)}{x \in \operatorname{Dst} f} \right\} = \left\{ \frac{(x \div Y, x \sqcap Y)}{x \in \operatorname{Dst} f} \right\}$ is an order isomorphism by proved above. This implies that $\left\{ \frac{(\mathcal{E}^Y \circ f, \mathrm{id}^{\mathsf{FCD}}_Y \circ f)}{f \in \mathsf{FCD}(A,B)} \right\}$ is an isomorphism (both order and semigroup). □

Proposition 1964. $\left\{ \frac{(\iota_{X,Y} f, \mathrm{id}^{\mathsf{RLD}}_Y \circ f \circ \mathrm{id}^{\mathsf{RLD}}_X)}{f \in \mathsf{RLD}(A,B)} \right\}$ is a function and moreover is an (order and semigroup) isomorphism, for sets $X \subseteq \operatorname{Src} f$, $Y \subseteq \operatorname{Dst} f$.

Proof. $\iota_{X,Y} f = (X, Y, (\operatorname{up} f) \div (X \times Y))$; $\mathrm{id}^{\mathsf{RLD}}_Y \circ f \circ \mathrm{id}^{\mathsf{RLD}}_X = (\operatorname{Src} f, \operatorname{Dst} f, (\operatorname{up} f) \sqcap (X \times Y))$. They are order isomorphic by proved above.

$\iota_{Y,Z} g \circ \iota_{X,Y} f = \mathcal{E}^Z \circ g \circ (\mathcal{E}^Y)^{-1} \circ \mathcal{E}^Y \circ f \circ (\mathcal{E}^X)^{-1} = \mathcal{E}^Z \circ g \circ \mathrm{id}^{\mathsf{RLD}}_Y \circ \mathrm{id}^{\mathsf{RLD}}_Y \circ f \circ (\mathcal{E}^X)^{-1}$ because $(\mathcal{E}^Y)^{-1} \circ \mathcal{E}^Y = \mathrm{id}^{\mathbf{Rel}}_Y = \mathrm{id}^{\mathbf{Rel}}_Y \circ \mathrm{id}^{\mathbf{Rel}}_Y$. Thus by proved above

$$\left\{ \frac{(\iota_{Y,Z} g \circ \iota_{X,Y} f, \mathrm{id}^{\mathsf{RLD}}_Z \circ g \circ \mathrm{id}^{\mathsf{RLD}}_Y \circ \mathrm{id}^{\mathsf{RLD}}_Y \circ f \circ \mathrm{id}^{\mathsf{RLD}}_X)}{f \in \mathsf{RLD}(A,B)} \right\}$$

is a bijection. □

5. Equalizers

Categories $\operatorname{cont}(\mathcal{C})$ are defined above.

I will denote W the forgetful functor from $\operatorname{cont}(\mathcal{C})$ to $\mathcal{C}$.

In the definition of the category $\operatorname{cont}(\mathcal{C})$ take values of $\uparrow$ as principal morphisms. FiXme: Wording.

Lemma 1965. Let $f : X \to Y$ be a morphism of the category $\operatorname{cont}(\mathcal{C})$ where $\mathcal{C}$ is a concrete category (so $W f = \uparrow \varphi$ for a **Rel**-morphism φ because f is principal) and $\operatorname{im} \varphi = A \subseteq \operatorname{Ob} Y$. Factor it $\varphi = \mathcal{E}^{\operatorname{Ob} Y} \circ u$ where $u : \operatorname{Ob} X \to A$ using properties of **Set**. Then u is a morphism of $\operatorname{cont}(\mathcal{C})$ (that is a continuous function $X \to \iota_A Y$).

Proof. $(\mathcal{E}^{\operatorname{Ob} Y})^{-1} \circ \varphi = (\mathcal{E}^{\operatorname{Ob} Y})^{-1} \circ \mathcal{E}^{\operatorname{Ob} Y} \circ u$;
$(\mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}})^{-1} \circ \uparrow \varphi = (\mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}})^{-1} \circ \mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}} \circ \uparrow u$;
$(\mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}})^{-1} \circ \uparrow \varphi = \uparrow u$;
$X \sqsubseteq (\uparrow u)^{-1} \circ \pi_A Y \circ \uparrow u \Leftrightarrow X \sqsubseteq (\uparrow \varphi)^{-1} \circ \mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}} \circ \pi_A Y \circ (\mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}})^{-1} \circ \uparrow \varphi \Leftrightarrow X \sqsubseteq (\uparrow \varphi)^{-1} \circ \mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}} \circ (\mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}})^{-1} \circ Y \circ \mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}} \circ (\mathcal{E}^{\operatorname{Ob} Y}_{\mathcal{C}})^{-1} \circ \uparrow \varphi \Leftrightarrow X \sqsubseteq (\uparrow \varphi)^{-1} \circ Y \circ \uparrow \varphi \Leftrightarrow X \sqsubseteq (W f)^{-1} \circ Y \circ W f$ what is true by definition of continuity. □

Equational definition of equalizers:
http://nforum.mathforge.org/comments.php?DiscussionID=5328/

Theorem 1966. The following is an equalizer of parallel morphisms $f, g : A \to B$ of category $\operatorname{cont}(\mathcal{C})$:

- the object $X = \iota_{\left\{ \frac{x \in \operatorname{Ob} A}{f x = g x} \right\}} A$;
- the morphism $\mathcal{E}^{\operatorname{Ob} X, \operatorname{Ob} A}$ considered as a morphism $X \to A$.

PROOF. Denote $e = \mathcal{E}^{\operatorname{Ob} X, \operatorname{Ob} A}$.

Let $f \circ z = g \circ z$ for some morphism z.

Let's prove $e \circ u = z$ for some $u : \operatorname{Src} z \to X$. Really, as a morphism of **Set** it exists and is unique.

Consider z as as a generalized element.

$f(z) = g(z)$. So $z \in X$ (that is $\operatorname{Dst} z \in X$). Thus $z = e \circ u$ for some u (by properties of **Set**). The generalized element u is a $\operatorname{cont}(\mathcal{C})$-morphism because of the lemma above. It is unique by properties of **Set**. □

We can (over)simplify the above theorem by the obvious below:

OBVIOUS 1967. $\left\{ \frac{x \in \operatorname{Ob} A}{fx = gx} \right\} = \operatorname{dom}(f \cap g)$.

6. Co-equalizers

http://math.stackexchange.com/questions/539717/how-to-construct-co-equalizers-in-mathbftop

Let $\sim$ be an equivalence relation. Let's denote π its canonical projection.

DEFINITION 1968. $f/\sim = \uparrow \pi \circ f \circ \uparrow \pi^{-1}$ for every morphism f.

OBVIOUS 1969. $\operatorname{Ob}(f/\sim) = (\operatorname{Ob} f)/r$.

OBVIOUS 1970. $f/\sim = \langle \uparrow^{\mathsf{FCD}} \pi \times^{(C)} \uparrow^{\mathsf{FCD}} \pi \rangle f$ for every morphism f.

To define co-equalizers of morphisms f and g let $\sim$ be is the smallest equivalence relation such that $fx = gx$.

LEMMA 1971. Let $f : X \to Y$ be a morphism of the category $\operatorname{cont}(\mathcal{C})$ where $\mathcal{C}$ is a concrete category (so $Wf = \uparrow \varphi$ for a **Rel**-morphism φ because f is principal) such that φ respects $\sim$. Factor it $\varphi = u \circ \pi$ where $u : \operatorname{Ob}(X/\sim) \to \operatorname{Ob} Y$ using properties of **Set**. Then u is a morphism of $\operatorname{cont}(\mathcal{C})$ (that is a continuous function $X/\sim \to Y$).

PROOF. $f \circ X \circ f^{-1} \sqsubseteq Y$; $\uparrow u \circ \uparrow \pi \circ X \circ \uparrow \pi^{-1} \circ \uparrow u^{-1} \sqsubseteq Y$; $\uparrow u \in \mathrm{C}(\uparrow \pi \circ X \circ \uparrow \pi^{-1}, Y) = \mathrm{C}(X/\sim, Y)$. □

THEOREM 1972. The following is a co-equalizer of parallel morphisms $f, g : A \to B$ of category $\operatorname{cont}(\mathcal{C})$:

- the object $Y = f/\sim$;
- the morphism π considered as a morphism $B \to Y$.

PROOF. Let $z \circ f = z \circ g$ for some morphism z.

Let's prove $u \circ \pi = z$ for some $u : Y \to \operatorname{Dst} z$. Really, as a morphism of **Set** it exists and is unique.

$\operatorname{Src} z \in Y$. Thus $z = u \circ \pi$ for some u (by properties of **Set**). The function u is a $\operatorname{cont}(\mathcal{C})$-morphism because of the lemma above. It is unique by properties of **Set** (π obviously respects equivalence classes). □

7. Rest

THEOREM 1973. The categories $\operatorname{cont}(\mathcal{C})$ (for example in **Fcd** and **Rld**) are complete. FiXme: Note that small complete category is a preorder!

PROOF. They have products and equalizers. □

THEOREM 1974. The categories $\operatorname{cont}(\mathcal{C})$ (for example in **Fcd** and **Rld**) are co-complete.

PROOF. They have co-products and co-equalizers. □

Definition 1975. I call morphisms f and g of a category with embeddings *equivalent* ($f \sim g$) when there exist a morphism p such that $\operatorname{Src} p \sqsubseteq \operatorname{Src} f$, $\operatorname{Src} p \sqsubseteq \operatorname{Src} g$, $\operatorname{Dst} p \sqsubseteq \operatorname{Dst} f$, $\operatorname{Dst} p \sqsubseteq \operatorname{Dst} g$ and $\iota_{\operatorname{Src} f, \operatorname{Dst} f} p = f$ and $\iota_{\operatorname{Src} g, \operatorname{Dst} g} p = g$.

Problem 1976. Find under which conditions:

1°. Equivalence of morphisms is an equivalence relation.
2°. Equivalence of morphisms is a congruence for our category.

CHAPTER 4

Categories of filters

In [1] two categories, whose objects are related with filters on sets, are defined and researched.

Accordingly [1] infinite product is defined just in the first (denoted $\mathscr{F}$ there) of these two categories. So we will for now consider the first category. (Usefulness of the second category for our research is questionable.)

Let $f : A \to B$ be a function, $\mathcal{A}$ be a filter on A.

Proposition 1977. $\left\{ \frac{Y \in \mathscr{P}B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$ is a filter.

Proof. That it is an upper set is obvious.

Let $Y_0, Y_1 \in \left\{ \frac{Y \in \mathscr{P}B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$. Then $\left\langle f^{-1} \right\rangle^* Y_0 \in \mathcal{A}$ and $\left\langle f^{-1} \right\rangle^* Y_1 \in \mathcal{A}$. We have

$$\left\langle f^{-1} \right\rangle^* (Y_0 \cap Y_1) = \left\langle f^{-1} \right\rangle^* Y_0 \cap \left\langle f^{-1} \right\rangle^* Y_1 \in \mathcal{A}$$

since f is monovalued. Thus $Y_0 \cap Y_1 \in \left\{ \frac{Y \in \mathscr{P}B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$. □

Theorem 1978. FiXme: Should be moved above in the book. $\left\{ \frac{Y \in \mathscr{P}B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$ is equal to the filter generated by the filter base $\left\langle \langle f \rangle^* \right\rangle^* \mathcal{A}$, for every filter $\mathcal{A}$.

Proof. Denote $\mathcal{B} = \left\{ \frac{Y \in \mathscr{P}B}{\langle f^{-1} \rangle^* Y \in \mathcal{A}} \right\}$, $\mathcal{C} = \left\langle \langle f \rangle^* \right\rangle^* \mathcal{A}$.

Let $Y \in \mathcal{C}$. Then $Y = \langle f \rangle^* A$ where $A \in \mathcal{A}$. Then $\left\langle f^{-1} \right\rangle^* \langle f \rangle^* A \supseteq A$ and so $\left\langle f^{-1} \right\rangle^* \langle f \rangle^* A \in \mathcal{A}$. This proves $\langle f \rangle^* A \in \mathcal{B}$, that is $Y \in \mathcal{B}$.

Let now $Y \in \mathcal{B}$. Then $\langle f \rangle^* \left\langle f^{-1} \right\rangle^* Y \subseteq Y$. Since $\left\langle f^{-1} \right\rangle^* Y \in \mathcal{A}$, we have that Y is a supset of some set of the form $\langle f \rangle^* A$, so $Y \in \mathcal{C}$. □

Corollary 1979. $\operatorname{up} \langle f \rangle \mathcal{A} = \left\{ \frac{Y \in \mathscr{P}B}{\langle f^{-1} \rangle^* Y \in \operatorname{up} \mathcal{A}} \right\}$.

Definition 1980. The *category of filtered sets* **Filt** is the category defined as follows:

1°. Objects are pairs $(A, \mathcal{A})$ where A is a (small) set and $\mathcal{A}$ is a filter on A.
2°. Morphisms from $(A, \mathcal{A})$ to $(B, \mathcal{B})$ are functions $f : A \to B$ such that $\langle f \rangle \mathcal{A} \sqsubseteq \mathcal{B}$.
3°. Identities are identity functions.

To verify that it is a category is straightforward.

It is the same category as $\mathscr{F}$ in [1], as follows from an above proposition.

We will prove that starred reloidal product is a categorical product in this category. First we will prove the special case that binary reloidal product is a categorical product in this category.

Theorem 1981. $\times^{\mathsf{RLD}}$ (together with projections $\Pr_0$ and $\Pr_1$) is a categorical product in **Filt**.

Proof. Let our objects be $\mathcal{A}$, $\mathcal{B}$.

Denote p the left projection from $\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})$ to $\operatorname{Base}(\mathcal{A})$.

We need to check that p is a **Filt**-morphism that is $p(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \sqsubseteq \mathcal{A}$ what is obvious.

Similarly for the right projection q.

It remains to check the universal property: Let $\mathcal{C}$ be a filter and $f : \mathcal{C} \to \mathcal{A}$, $g : \mathcal{C} \to \mathcal{B}$. We need to prove that there are a unique $u : \mathcal{C} \to \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ such that $f = p \circ u$ and $g = q \circ u$. Denote $h(z) = (f(z), g(z))$.

h is the unique function $\operatorname{Base}(\mathcal{C}) \to \operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})$ such that $f = p \circ h$ and $g = q \circ h$, so it remains to check that h is a morphism of **Filt** that is $\langle h \rangle \mathcal{C} \sqsubseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$, what obviously follows from $\langle f \rangle \mathcal{C} \sqsubseteq \mathcal{A}$ and $\langle g \rangle \mathcal{C} \sqsubseteq \mathcal{B}$. □

THEOREM 1982. $\prod^{\mathsf{RLD}*}$ together with projections Pr_k is a categorical product in **Filt**.

PROOF. Consider an indexed family $\mathcal{A}$ of objects.

Denote p_k the k-th projection from $\prod_{i \in \operatorname{dom} \mathcal{A}} \operatorname{Base}(\mathcal{A}_i)$.

We need to check that p_k s a **Filt**-morphism that is $p_k \left(\prod^{\mathsf{RLD}*} \mathcal{A} \right) \sqsubseteq \mathcal{A}_k$ what is obvious.

It remains to check the universal property: Let $\mathcal{C}$ be a filter and $f_k : \mathcal{C} \to \mathcal{A}_k$. We need to prove that there are a unique $u : \mathcal{C} \to \prod^{\mathsf{RLD}*} \mathcal{A}$ such that $f_k = p_k \circ u$. Denote $h(z) = \lambda i \in \operatorname{dom} \mathcal{A} : f_i z$.

h is the unique function $\operatorname{Base}(\mathcal{C}) \to \prod_{i \in \operatorname{dom} \mathcal{A}} \operatorname{Base}(\mathcal{A}_i)$ such that $f_k = p_k \circ h$, so it remains to check that h is a morphism of **Filt** that is $\langle h \rangle \mathcal{C} \sqsubseteq \prod^{\mathsf{RLD}*} \mathcal{A}$. It follows from

$$\overset{\mathsf{RLD}}{\underset{i}{\operatorname{Pr}}} \langle h \rangle \mathcal{C} = \bigsqcap \langle Pr_i \rangle^* \langle h \rangle^* \operatorname{up} \mathcal{C} = \bigsqcap \langle \underset{i}{\operatorname{Pr}} \circ h \rangle^* \operatorname{up} \mathcal{C} = \bigsqcap \langle f_i \rangle^* \operatorname{up} \mathcal{C} = \langle f_i \rangle \mathcal{C} \sqsubseteq \mathcal{A}_i.$$

□

CHAPTER 5

Power of filters

1. Germs of functions

Definition 1983. Functions $f, g \in \mathbf{Rel}(\operatorname{Ob}\mathcal{X}, B)$ *are of the same $\mathcal{X}$-germ* for a filter object $\mathcal{X}$ iff there exists $X \in \operatorname{up}\mathcal{X}$ such that $f|_X = g|_X$.

Proposition 1984. Being of the same germ is an equivalence relation.

Proof.
Reflexivity. Take arbitrary $X \in \operatorname{up}\mathcal{X}$.
Symmetry. Obvious.
Transitivity. Let $f|_X = g|_X$ and $g|_Y = h|_Y$. Then $f|_{X\cap Y} = h|_{X\cap Y}$.

□

Definition 1985. A *germ* is an equivalence class of being the same germ.

Obvious 1986. Every germ is a filter on **Set**.

Theorem 1987. Let A, B be sets.
The following are mutually inverse bijections between monovalued reloids $f : A \to B$ with $\operatorname{dom} f = \mathcal{X}$ and $\mathcal{X}$-germs S of functions $A \to B$ for $\mathcal{X} \in \mathscr{F}A$:
1°. $f \mapsto \operatorname{up}^{\mathbf{Set}} f$;
2°. $S \mapsto s|_{\mathcal{X}}$ if $s \in S$.

The second bijection can also be written as $S \mapsto \left(\prod^{\mathsf{RLD}} S\right)|_{\mathcal{X}}$ or if $\operatorname{card} B \neq 1$ as $S \mapsto \prod^{\mathsf{RLD}} S$.

Remark 1988. $s|_{\mathcal{X}}$ is always defined because S is nonempty (it is an equivalence class).

Proof. First prove that $\operatorname{up}^{\mathbf{Set}} f$ is an $\mathcal{X}$-germ. Really, $F \in \operatorname{up}^{\mathbf{Set}} f \Leftrightarrow F \sqsupseteq f \Leftrightarrow F|_{\mathcal{X}} = f \Leftrightarrow \exists X \in \operatorname{up}\mathcal{X} : F|_X \sqsupseteq f$; thus $F, G \in \operatorname{up}^{\mathbf{Set}} f \Rightarrow \exists X \in \operatorname{up}\mathcal{X} : F|_X \sqsupseteq f \wedge \exists Y \in \operatorname{up}\mathcal{X} : G|_Y \sqsupseteq f \Rightarrow \exists X \in \operatorname{up}\mathcal{X} : F|_{X\cap Y} \sqsupseteq f \wedge \exists Y \in \operatorname{up}\mathcal{X} : G|_{X\cap Y} \sqsupseteq f \Rightarrow \exists Z \in \operatorname{up}\mathcal{X} : (F|_Z \sqsupseteq f \wedge G|_Z \sqsupseteq f) \Rightarrow \exists Z \in \operatorname{up}\mathcal{X} : (F \sqcap G)|_Z \sqsupseteq f$ and $F \in \operatorname{up}^{\mathbf{Set}} f \wedge \exists X \in \operatorname{up}\mathcal{X} : F|_X = G|_X \Rightarrow F \sqsupseteq f \wedge F|_{\mathcal{X}} = G|_{\mathcal{X}} \Rightarrow G|_{\mathcal{X}} \sqsupseteq f \Rightarrow G \in \operatorname{up}^{\mathbf{Set}} f$. We have proved that $\operatorname{up}^{\mathbf{Set}} f$ is an equivalence class of the suitable equivalence relation, that is $\operatorname{up}^{\mathbf{Set}} f$ is an $\mathcal{X}$-germ.

That $\prod^{\mathsf{RLD}} S$ is a monovalued reloid is obvious. Also $\operatorname{im}\prod^{\mathsf{RLD}} S = \mathcal{X}$ is obvious.

Now prove that our correspondences are mutually inverse.

Let $f_0 : A \to B$ be a monovalued reloid and $\operatorname{dom} f = \mathcal{X}$. Let $S = \operatorname{up}^{\mathbf{Set}} f_0$ and $f_1 = s|_{\mathcal{X}}$ for an $s \in S$. We need to prove $f_1 = f_0$. Really, $f_1 = F|_{\mathcal{X}}$ for an $F \in \operatorname{up}^{\mathbf{Set}} f_0$; thus $f_1 = f_0$.

Let S_0 be an $\mathcal{X}$-germ of functions $A \to B$. Let $f = s|_{\mathcal{X}}$ for an $s \in S_0$ and $S_1 = \operatorname{up}^{\mathbf{Set}} f$. We need to prove $S_1 = S_0$. Really,

$$S_1 = \operatorname{up}^{\mathbf{Set}}(s|_{\mathcal{X}}) = \left\{ \frac{F \in \mathbf{Set}}{F \sqsupseteq s|_{\mathcal{X}}} \right\} = \left\{ \frac{F \in \mathbf{Set}}{\exists X \in \operatorname{up}\mathcal{X} : F|_X \sqsupseteq s|_X} \right\} = \left\{ \frac{F \in \mathbf{Set}}{\exists X \in \operatorname{up}\mathcal{X} : F|_X = s|_X} \right\} = S_0.$$

$\left(\prod^{\mathsf{RLD}} S\right)|_{\mathcal{X}} = \prod^{\mathsf{RLD}}_{s\in S} s|_{\mathcal{X}} = s|_{\mathcal{X}}$ for every choice of $s \in S$.

We can assume that $B \neq \emptyset$ because otherwise the theorem is obvious. Thus we can assume $\operatorname{card} B > 1$.

If $\mathcal{X} = X$ then obviously S has just one element F and $\operatorname{im} \prod^{\mathsf{RLD}} S = \operatorname{im} F = X = \mathcal{X}$. Otherwise for every $X \in \operatorname{up} \mathcal{X}$ there are elements F, G of S such that $\operatorname{dom}(F \sqcap G) \sqsubseteq X$ (using $\operatorname{card} B > 1$).

By properties of generalized filter bases $X \times \top \sqsupseteq \prod^{\mathsf{RLD}} S \Leftrightarrow \exists F, G \in S : X \times \top \sqsupseteq F \sqcap G \Leftrightarrow X \sqsupseteq \mathcal{X}$. Thus $\operatorname{im} \prod^{\mathsf{RLD}} S = \mathcal{X}$. □

2. Power of filters

Let's define $\mathcal{Y}^{\mathcal{X}}$ for filters $\mathcal{X}$, $\mathcal{Y}$:

First define $Y^{\mathcal{X}}$ for a set Y:

$$Y^{\mathcal{X}} = \left\{ \frac{f \in \mathsf{RLD}(\operatorname{Ob} \mathcal{X}, Y)}{\operatorname{dom} f = \mathcal{X} \wedge f \text{ is monovalued}} \right\}.$$

Now $\mathcal{Y}^{\mathcal{X}} = \prod^{\mathsf{RLD}}_{Y \in \operatorname{up} \mathcal{Y}} Y^{\mathcal{X}}$.

[1] defines an isomorphic to this way to define "exponentiation" of filters.

TODO: Check $\mathcal{Y}^1 \cong \mathcal{Y}$; $\mathcal{Z}^{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}} \cong (\mathcal{Z}^{\mathcal{X}})^{\mathcal{Y}}$; $\mathcal{Z}^{\mathcal{X} \amalg \mathcal{Y}} \cong \mathcal{Z}^{\mathcal{X}} \times^{\mathsf{RLD}} \mathcal{Z}^{\mathcal{Y}}$; $\mathcal{Y}^2 \cong \mathcal{Y} \times^{\mathsf{RLD}} \mathcal{Y}$; $\mathcal{Y}^0 \cong 1$; $\mathcal{Y}^N \cong \prod^{\mathsf{RLD}}_{n \in N} \mathcal{Y}$. More formulas at https://en.wikipedia.org/wiki/Cartesian_closed_category.

Andreas Blass says in a private email that it is not cartesian closed: "Unfortunately, the two categories of filters in my paper are not cartesian closed. This is mentioned in a parenthetical comment near the bottom of page 141. The operation of cartesian product with the cofinite filter on the natural numbers has no right adjoint, because it does not preserve infinite coproducts." about [1].

But it is probably a braided closed monoidal category?

See [1] for more categorical properties of filters.

CHAPTER 6

Matters related to tensor product

These consideration on (possibly infinite) indexed families of join-semilattices is based on [7] (for the finite case).

Let $\mathfrak{A}$ be an indexed family of join-semilattices with least elements. Let T also be a join-semilattice.

Let $F(X)$ mean free join-semilattice for a set X.

DEFINITION 1989. $\mathbf{SepJoin}(\prod \mathfrak{A}, T)$ is the set of maps from $\prod \mathfrak{A}$ to T, preserving joins in every argument $i \in \operatorname{dom} \mathfrak{A}$.

OBVIOUS 1990. The set of free join-semilattices $F(X)$ is order-isomorphic to the set of subsets X of a "universal" set $\mho$.

Let $i : \prod \mathfrak{A} \to F(\prod \mathfrak{A})$ be the universal embedding.

Let $\sim$ be defined as the smallest equivalence relation on $F(\prod \mathfrak{A})$ that for every $k \in \operatorname{dom} \mathfrak{A}$, $L \in \prod_{i \in (\operatorname{dom} \mathfrak{A}) \setminus \{k\}} \mathfrak{A}_i$:

1°. $i(L \cup \{(k, g \sqcup h)\}) \sim i(L \cup \{(k, g)\}) \sqcup i(L \cup \{(k, h)\})$;
2°. $\perp \sim i(L \cup \{(k, \perp)\})$;
3°. $x \sim y \wedge x' \sim y' \Rightarrow x \sqcup x' \sim y \sqcup y'$ for all $x, y, x', y' \in F(\prod \mathfrak{A})$.

OBVIOUS 1991. Some function $h : X \to Y$ induces a well defined map $\psi : X/E \to Y$ on equivalence classes, if $E \subseteq F$ where $x \mathrel{F} y \Leftrightarrow hx = hy$.

LEMMA 1992. The set of join-homomorphisms $\psi : F(\prod \mathfrak{A})/\sim \to T$ is isomorphic to the set of maps $\phi : \prod \mathfrak{A} \to T$ preserving finite joins in separate arguments.

PROOF. The quotient map $q : F(\prod \mathfrak{A}) \to F(\prod \mathfrak{A})/\sim$ which takes an element x to its equivalence class $[x]$ map is well defined because

$$x \sim y \wedge x' \sim y' \Rightarrow x \sqcup x' \sim y \sqcup y'.$$

The map q preserves join. $F(\prod \mathfrak{A})/\sim$ is associative, commutative, and idempotent since it is so on $F(\prod \mathfrak{A})$ and thus is a join-semilattice.

Let join-preserving map $\psi : F(\prod \mathfrak{A})/\sim \to T$. It is easy to show that $\psi \circ q \circ i$ preserves joins in separate arguments.

Let now $\phi : \prod \mathfrak{A} \to T$ preserves joins in separate arguments. There is a unique join-preserving map $\tilde{\phi} : F(\prod \mathfrak{A}) \to T$ such that $\tilde{\phi} \circ i = \phi$. We must show that this induces a well-defined join-preserving map $\psi : F(\prod \mathfrak{A})/\sim \to T$ such that $\psi(q(x)) = \tilde{\phi}(x)$ for all $x \in F(\prod \mathfrak{A})$ (clearly at most one function ψ can satisfy this equation since q is surjective). This will show that ψ bijectively correspond to $\tilde{\phi}$ and thus bijectively correspond to ϕ. (This will finish the proof as that this bijection is monotone is obvious.)

Using the "obvious" above, it's enough (taking into account that $\sim$ is the minimal equivalence relation subject to the above formulas) to prove that:

1°. $\tilde{\phi}(i(L \cup \{(k, g \sqcup h)\})) = \tilde{\phi}(i(L \cup \{(k, g)\}) \sqcup i(L \cup \{(k, h)\}))$;
2°. $\tilde{\phi}(\perp) = \tilde{\phi}(i(L \cup \{(k, \perp)\}))$;
3°. $\tilde{\phi}(x) = \tilde{\phi}(y) \wedge \tilde{\phi}(x') = \tilde{\phi}(y') \Rightarrow \tilde{\phi}(x \sqcup x') = \tilde{\phi}(y \sqcup y')$

The first easily follows from $\tilde{\phi} \circ i = \phi$ and the fact that $\tilde{\phi}$ preserves binary joins.

The second easily follows from $\tilde{\phi} \circ i = \phi$ and that ϕ preserves $\bot$.

The third follows from the fact that $\tilde{\phi}$ preserves joins. □

COROLLARY 1993. The poset of prestaroids $\mathsf{preStrd}(\mathfrak{A})$ is isomorphic to an ideal (on a join-semilattice), provided that $\mathfrak{A}$ is an indexed family of join-semilattices.

PROOF. $\mathsf{preStrd}(\mathfrak{A}) \cong \mathbf{SepJoin}(\mathfrak{A}, 2) \cong F(\prod \mathfrak{A}) / \sim \rightarrow 2 \cong \mathfrak{I}(F(\prod \mathfrak{A}) / \sim)$. □

FiXme: Check below (especially posets vs dual posets) for errors.

COROLLARY 1994. $\mathsf{preStrd}$ is a complete lattice.

PROOF. Corollary 506. □

COROLLARY 1995. $\mathsf{preStrd}$ is a filtered filtrator.

PROOF. Theorem 521. □

FiXme: Try to prove that $\mathsf{preStrd}$ is atomic and moreover atomistic (under certain conditions). Other properties?

CHAPTER 7

Mappings between endofuncoids and topological spaces

Oreder topologies reversely to set-theoretic inclusion. That is for topologies t and s we set $t \sqsubseteq s \Leftrightarrow t \supseteq s$. (Intuitively: The less is the topology, the lesser are its open sets.)

Let's study mappings between topological spaces and endofuncoids.

Definition 1996. Let t be a topology.

1°. $F^\star t = \bigsqcup_{x \in \operatorname{Ob} t}\left(\{x\} \times \prod^{\mathscr{F}}\left\{\frac{E \in t}{x \in E}\right\}\right)$;

2°. $(F_\star t)E = \bigcap\left\{\frac{D \in t}{E \subseteq D}\right\}$.

Proposition 1997. Let t be a topology.

1°. $F^\star t$ is complete, reflexive, transitive funcoid.
2°. $F_\star t$ is co-complete, reflexive, transitive funcoid.
3°. $F^\star$ and $F_\star$ are injections.
4°. $F_\star t = (F^\star t)^{-1}$.

Proof. By theorem 708. □

Definition 1998. Let f be an endofuncoid.

$$Tf = \left\{\frac{E \in \mathscr{P} \operatorname{Ob} f}{\forall x \in E : \langle f \rangle \{x\} \sqsubseteq E}\right\}.$$

Proposition 1999. Tf is a topology.

Proof. Union of open sets is open. $S \subseteq Tf \Rightarrow \forall E \in S \forall x \in E : \langle f \rangle x \sqsubseteq E \Rightarrow \forall x \in \bigcup S : \langle f \rangle x \sqsubseteq \bigcup S$

Intersection of two open sets is open. Let $X, Y \in Tf$. Then $\forall x \in X : \langle f \rangle x \sqsubseteq X$ and $\forall x \in Y : \langle f \rangle x \sqsubseteq Y$. So if $x \in X \cap Y$ then $\langle f \rangle x \sqsubseteq X$ and $\langle f \rangle x \sqsubseteq Y$, so $\langle f \rangle x \sqsubseteq X \cap Y$. So $X \cap Y \in Tf$.

$\operatorname{Ob} f$ is an open set. Obvious.

□

Obvious 2000. $Tf = \left\{\frac{E \in \mathscr{P} \operatorname{Ob} f}{\langle \operatorname{Compl} f \rangle E \sqsubseteq E}\right\}$.

In some reason when starting this research I assumed that the following funcoid (for every endofuncoid f) is a Kuratowski closure:

$$1 \sqcup \operatorname{CoCompl} f \sqcup (\operatorname{CoCompl} f)^2 \sqcup \ldots.$$

It is not true:

Example 2001. There exists such a co-complete endofuncoid f that $1 \sqcup f \sqcup f^2 \sqcup \ldots$ is not transitive that is

$$(1 \sqcup f \sqcup f^2 \sqcup \ldots) \circ (1 \sqcup f \sqcup f^2 \sqcup \ldots) \neq 1 \sqcup f \sqcup f^2 \sqcup \ldots$$

Proof. Take $f = \operatorname{cl} \circ g$ where g is the principal funcoid which maps every real number a into the closed interval $\left[\frac{-1-|a|}{2}; \frac{1+|a|}{2}\right]$.

Take $X = \left[-\frac{1}{2}; \frac{1}{2}\right]$. $\langle f^n \rangle^* X = \left[-1 + \frac{1}{2^{n+1}}; 1 - \frac{1}{2^{n+1}}\right]$.

We have $\langle 1 \sqcup f \sqcup f^2 \sqcup \ldots \rangle^* X =]-1; 1[$;

$\langle 1 \sqcup f \sqcup f^2 \sqcup \ldots \rangle^* \langle 1 \sqcup f \sqcup f^2 \sqcup \ldots \rangle^* X = [-1; 1]$.

Thus follows our inequality. □

That $F^\star$ and $F_\star$ are functors (if we map morphisms to themselves except of changing the objects) follows from conjecture 1095.

Theorem 2002. T (if we map morphisms to themselves except of changing the objects) is a functor.

Proof. Based on https://math.stackexchange.com/a/2792239/4876

Let $f : \mu \to \nu$ that is $f \circ \mu \sqsubseteq \nu \circ f$. We need to prove $f : T\mu \to T\nu$ that is $E \in T\nu \Rightarrow \langle f^{-1} \rangle^* E \in T\mu$.

Suppose $E \in T\nu$ that is $\langle \nu \rangle^* E \sqsubseteq E$. We will prove $\langle \mu \rangle^* \langle f^{-1} \rangle^* E \sqsubseteq \langle f^{-1} \rangle^* E$.

FiXme: Can we use arbitrary filters rather than atoms?

Really, let atom $y \sqsubseteq \langle \mu \rangle^* \langle f^{-1} \rangle^* E$. Then there exists atom $x \sqsubseteq \langle f^{-1} \rangle^* E$ such that $x \, [\mu]^* \, y$.

$x \, [f \circ \mu]^* \, \langle f \rangle y$ and thus $x \, [\nu \circ f]^* \, \langle f \rangle y$, so $\langle f \rangle x \, [\nu]^* \, \langle f \rangle y$. But $\langle f \rangle x \sqsubseteq E$, so $\langle f \rangle y \sqsubseteq \langle \nu \rangle^* E \sqsubseteq E$, that is $\langle \mu \rangle^* \langle f^{-1} \rangle^* E \sqsubseteq E$. □

Proposition 2003. $f \in \mathrm{C}(\mu, \nu) \Rightarrow f \in \mathrm{C}(\mu^n, \nu^n)$ for every endofuncoids μ and ν and positive natural number n. FiXme: Move this proposition to the book.

Proof. $f \circ \mu \sqsubseteq \nu \circ f$; $f \circ \mu \circ \mu \sqsubseteq \nu \circ f \circ \mu$; $f \circ \mu^2 \sqsubseteq \nu^2 \circ f$; $f \circ \mu^3 \sqsubseteq \nu^3 \circ f \ldots$ □

Proposition 2004. For every endofuncoid μ:

1°. $F_\star T\mu \sqsupseteq \operatorname{Compl} \mu$;
2°. $F^\star T\mu \sqsupseteq \operatorname{Compl} \mu$;
3°. $F_\star T\mu \sqsupseteq \operatorname{CoCompl} \mu$;
4°. $F^\star T\mu \sqsupseteq \operatorname{CoCompl} \mu$;

Proof. We will prove only the first two as the rest are dual.

$\langle F_\star T\mu \rangle^* E = \bigcap \left\{ \frac{D \in T\mu}{D \sqsupseteq E} \right\} = \bigcap \left\{ \frac{D \in \mathscr{P} \operatorname{Ob} \mu}{\langle \operatorname{Compl} \mu \rangle^* D \sqsubseteq D \wedge D \sqsupseteq E} \right\} \sqsupseteq \bigcap \left\{ \frac{\langle \operatorname{Compl} \mu \rangle^* D}{D \in \mathscr{P} \operatorname{Ob} \mu, \langle \operatorname{Compl} \mu \rangle^* D \sqsubseteq D \wedge D \sqsupseteq E} \right\} \sqsupseteq \langle \operatorname{Compl} \mu \rangle^* E$.

$\langle F^\star T\mu \rangle^* \{x\} = \bigsqcap^{\mathscr{F}} \left\{ \frac{E \in T\mu}{x \in E} \right\} = \bigsqcap^{\mathscr{F}} \left\{ \frac{E \in \operatorname{Ob} \mu}{x \in E, \langle \operatorname{Compl} \mu \rangle^* E \sqsubseteq E} \right\} \sqsupseteq \bigsqcap^{\mathscr{F}} \left\{ \frac{\langle \operatorname{Compl} \mu \rangle^* E}{E \in \operatorname{Ob} \mu, x \in E, \langle \operatorname{Compl} \mu \rangle^* E \sqsubseteq E} \right\} \sqsupseteq \langle \operatorname{Compl} \mu \rangle^* \{x\}$. □

Lemma 2005. For every endofuncoid μ:

1°. $F_\star T\mu \sqsubseteq 1 \sqcup \operatorname{Compl} \mu \sqcup (\operatorname{Compl} \mu)^2 \sqcup \ldots$;
2°. $F^\star T\mu \sqsubseteq 1 \sqcup \operatorname{CoCompl} \mu \sqcup (\operatorname{CoCompl} \mu)^2 \sqcup \ldots$

Proof. We will prove only the first as the second is dual.

$\langle 1 \sqcup \operatorname{Compl} \mu \sqcup (\operatorname{Compl} \mu)^2 \sqcup \ldots \rangle^* E = E \sqcup \langle \operatorname{Compl} \mu \rangle^* E \sqcup \langle (\operatorname{Compl} \mu)^2 \rangle^* E \sqcup \ldots$

Take $D = E \sqcup \langle \operatorname{Compl} \mu \rangle^* E \sqcup \langle (\operatorname{Compl} \mu)^2 \rangle^* E \sqcup \ldots$ We have $\langle \operatorname{Compl} \mu \rangle^* D \sqsubseteq \langle \operatorname{Compl} \mu \rangle^* E \sqcup \langle (\operatorname{Compl} \mu)^2 \rangle^* E \sqcup \ldots \sqsubseteq D$. So

$\bigcap \left\{ \frac{D \in \mathscr{P} \operatorname{Ob} \mu}{\langle \operatorname{Compl} \mu \rangle^* D \sqsubseteq D \wedge D \sqsupseteq E} \right\} \subseteq D \sqsubseteq \langle 1 \sqcup \operatorname{Compl} \mu \sqcup (\operatorname{Compl} \mu)^2 \sqcup \ldots \rangle^* E$. □

Theorem 2006. If we restrict the functor T only to complete endofuncoids (= complete endoreloids), then T is a left adjoint of both $F_\star$ and $F^\star$.

PROOF. We will prove only for $F_\star$ as the other is dual.

We will disprove $f \in \mathrm{C}(T\mu, s) \Leftrightarrow f \in \mathrm{C}(\mu, F_\star s)$ what is equivalent (because $F_\star$ is full and faithful) to

$$f \in \mathrm{C}(F_\star T\mu, F_\star s) \Leftrightarrow f \in \mathrm{C}(\mu, F_\star s);$$

$F_\star T\mu \sqsubseteq f^{-1} \circ F_\star s \circ f \Leftrightarrow \mu \sqsubseteq f^{-1} \circ F_\star s \circ f$.

$F_\star T\mu \sqsubseteq f^{-1} \circ F_\star s \circ f \Rightarrow \mu \sqsubseteq f^{-1} \circ F_\star s \circ f$ because $F_\star T\mu \sqsupseteq \mu$.

If $\mu \sqsubseteq f^{-1} \circ F_\star s \circ f$ then $\mu^n \sqsubseteq f^{-1} \circ (F_\star s)^n \circ f = f^{-1} \circ F_\star s \circ f$. Also obviously $1 \sqsubseteq f^{-1} \circ F_\star s \circ f$. Thus

$$1 \sqcup \mu \sqcup \mu^2 \sqcup \ldots \sqsubseteq f^{-1} \circ F_\star s \circ f$$

and so $1 \sqcup \operatorname{Compl} \mu \sqcup (\operatorname{Compl} \mu)^2 \sqcup \ldots \sqsubseteq f^{-1} \circ F_\star s \circ f$. So $F_\star T\mu \sqsubseteq f^{-1} \circ F_\star s \circ F$. □

FiXme: *F and T are also a Galois connection, isn't it?*

EXAMPLE 2007. T is a not left adjoint of both $F_\star$ and $F^\star$, with bijection which preserves the "function" part of the morphism.

PROOF. We will disprove only from $F_\star$ as the other is dual.

We will disprove $f \in \mathrm{C}(T\mu, s) \Leftrightarrow f \in \mathrm{C}(\mu, F_\star s)$ what is equivalent (because $F_\star$ is full and faithful) to

$$f \in \mathrm{C}(F_\star T\mu, F_\star s) \Leftrightarrow f \in \mathrm{C}(\mu, F_\star s);$$

$F_\star T\mu \sqsubseteq f^{-1} \circ F_\star s \circ f \Leftrightarrow \mu \sqsubseteq f^{-1} \circ F_\star s \circ f$.

This equivalence does not hold: Take s the discrete space on $\mathbb{R}$, $f = \mathrm{id}_\mathbb{R}$, and $\langle\mu\rangle^* X = X$ for finite sets X and $\langle\mu\rangle^* X = \top$ for infinite X. □

CHAPTER 8

Funcoids as closed sets

Idea [6] by TODD TRIMBLE.

FiXme: https://ncatlab.org/toddtrimble/published/topogeny and https://math.stackexchange.com/q/2681502/4876

FiXme: What about the infinite products?

THEOREM 2008. The set of staroids $\mathscr{P}X_1 \times \cdots \times \mathscr{P}X_n \to 2$ is order isomorphic to co-frame of closed subsets of topological product $\beta X_1 \times \cdots \times \beta X_n$.

PROOF. $\mathscr{P}X_1 \times \cdots \times \mathscr{P}X_n \to 2$ can be order-embedded to the frame of ideals $\mathfrak{J}(\mathscr{P}X_1 \times \cdots \times \mathscr{P}X_n)$ what is dual (check!) to the frame of ideals of the distributive lattice $\mathscr{P}X_1 \otimes \cdots \otimes \mathscr{P}X_n$. This by ?? is the coproduct $\sum_i \mathscr{P}X_i$ in the category of boolean algebras. By Stone duality it is isomorphic to the topology of it spectrum $\beta X_1 \times \cdots \times \beta X_n$. □

Elements of $\beta X_1 \times \cdots \times \beta X_n$ are closed subsets. So every n-staroid one-to-one corresponds to a closed set of $\beta X_1 \times \cdots \times \beta X_n$.

Note that $\beta X_1 \times \cdots \times \beta X_n$ is a compact Hausdorff space (as a product of compact Hausdorff spaces).

It seems that there is an easy way to describe the above order embedding in both directions:

$$f \mapsto \left\{ \frac{(x_1, \dots, x_n)}{x_1, \dots, x_n \in \operatorname{atoms}^{\mathscr{F}}, x_1 \times^{\mathsf{FCD}} \dots \times^{\mathsf{FCD}} x_n \sqsubseteq f} \right\};$$

$$X \mapsto \bigsqcup \left\{ \frac{p_1 \times^{\mathsf{FCD}} \dots \times^{\mathsf{FCD}} p_n}{p \in X} \right\}.$$

FiXme: Try to prove that composition of funcoids is isomorphic to composition of relations $\beta A \times \beta B$.

CHAPTER 9

Categories related with funcoids

I consider some categories related with pointfree funcoids.

1. Draft status

This is a rough partial draft.

2. Topic of this article

In this article are considered some categories related to *pointfree funcoids.*

3. Category of continuous morphisms

I will denote $\operatorname{Ob} f$ the object (source and destination) of an endomorphism f.

DEFINITION 2009. Let C is a partially ordered category. The category $\operatorname{cont}(C)$ (which I call *the category of continuous morphism* over C) is:

- Objects are endomorphisms of category C.
- Morphisms are triples (f, a, b) where a and b are objects and $f : \operatorname{Ob} a \to \operatorname{Ob} b$ is a morphism of the category C such that $f \circ a \sqsubseteq b \circ f$.
- Composition of morphisms is defined by the formula $(g, b, c) \circ (f, a, b) = (g \circ f, a, c)$.
- Identity morphisms are $(a, a, 1^C_a)$.

It is really a category:

PROOF. We need to prove that: composition of morphisms is a morphism, composition is associative, and identity morphisms can be canceled on the left and on the right.

That composition of morphisms is a morphism follows from these implications:

$$f \circ a \sqsubseteq b \circ f \wedge g \circ b \sqsubseteq c \circ g \Rightarrow g \circ f \circ a \sqsubseteq g \circ b \circ f \sqsubseteq c \circ g \circ f.$$

That composition is associative is obvious.

That identity morphisms can be canceled on the left and on the right is obvious. □

REMARK 2010. The "physical" meaning of this category is:

- Objects (endomorphisms of C) are spaces.
- Morphisms are continuous functions between spaces.
- $f \circ a \sqsubseteq b \circ f$ intuitively means that f combined with an infinitely small is less than infinitely small combined with f (that is f is continuous).

REMARK 2011. Every $\operatorname{Hom}(\mathfrak{A}, \mathfrak{B})$ of **Pos** is partially ordered by the formula $a \leqslant b \Leftrightarrow \forall x \in \mathfrak{A} : a(x) \leqslant b(x)$. So $\operatorname{cont}(\mathbf{Pos})$ is defined.

DEFINITION 2012. I call a **Pos**-morphism *monovalued* when it maps atoms to atoms or least element.

DEFINITION 2013. I call a **Pos**-morphism *entirely defined* when its value is non-least on every non-least element.

OBVIOUS 2014. A morphism is both monovalued and entirely defined iff it maps atoms into atoms.

FiXme: Show how it relates with dagger categories.

DEFINITION 2015. **mePos** is the subcategory of **Pos** with only monovalued and entirely defined morphisms.

OBVIOUS 2016. This is a well defined category.

DEFINITION 2017. **mefp**FCD is the subcategory of **fp**FCD with only monovalued and entirely defined morphisms.

REMARK 2018. In the two above definitions different definitions of monovaluedness and entire definedness from different articles.

4. Definition of the categories

DEFINITION 2019. A *(pointfree) endo-funcoid* is a (pointfree) funcoid with the same source and destination (an endomorphism of the category of (pointfree) funcoids). I will denote $\operatorname{Ob} f$ the object of an endomorphism f.

OBVIOUS 2020. The *category of continuous pointfree funcoids* $\operatorname{cont}(\mathbf{fp}\mathsf{FCD})$ is:

- Objects are small pointfree endo-funcoids.
- Morphisms from an object a to an object b are triples (f, a, b) where f is a pointfree funcoid from $\operatorname{Ob} a$ to $\operatorname{Ob} b$ such that f is a continuous morphism from a to b (that is $f \circ a \sqsubseteq b \circ f$, or equivalently $a \sqsubseteq f^{-1} \circ b \circ f$, or equivalently $f \circ a \circ f^{-1} \sqsubseteq f$).
- Composition is the composition of pointfree funcoids.
- Identity for an object a is $(I^{\mathsf{FCD}}_{\operatorname{Ob} a}, a, a)$.

5. Isomorphisms

THEOREM 2021. If f is an isomorphism $a \to b$ of the category $\operatorname{cont}(\mathbf{fp}\mathsf{FCD})$, then:

1°. $f \circ a = b \circ f$;
2°. $a = f^{-1} \circ b \circ f$;
3°. $f \circ a \circ f^{-1} = b$.

PROOF. Note that f is monovalued and entirely defined.

1. We have $f \circ a \sqsubseteq b \circ f$ and $f^{-1} \circ b \sqsubseteq a \circ f^{-1}$. Consequently $f^{-1} \circ f \circ a \sqsubseteq f^{-1} \circ b \circ f$; $a \sqsubseteq f^{-1} \circ b \circ f$; $a \circ f^{-1} \sqsubseteq f^{-1} \circ b \circ f \circ f^{-1}$; $a \circ f^{-1} \sqsubseteq f^{-1} \circ b$. Similarly $b \circ f \sqsubseteq f \circ a$. So $f \circ a = b \circ f$.

2 and 3. Follow from the definition of isomorphism. □

Isomorphisms are meant to preserve structure of objects. I will show that (under certain conditions) isomorphisms of $\operatorname{cont}(\mathbf{fp}\mathsf{FCD})$ really preserve structure of objects.

First we will consider an isomorphism between objects a and b which are funcoids (not the general case of pointfree funcoids). In this case a map which preserves structure of objects is a *bijection*. It is really a bijection as the following theorem says:

THEOREM 2022. If f is an isomorphism of the category of funcoids then f is a discrete funcoid (so, it is essentially a bijection). FiXme: Split it into two propositions: about completeness and co-completeness.

PROOF. $\langle f\rangle^* A \sqcap \langle f\rangle^*((\operatorname{Src} f) \setminus A) = 0^{\operatorname{Dst} f}$ because f is monovalued.
$\langle f\rangle^* A \sqcup \langle f\rangle^*((\operatorname{Src} f) \setminus A) = 1^{\operatorname{Dst} f}$.
Therefore $\langle f\rangle^* A$ is a principal filter (theorem 49 in [4]). So f is co-complete.
That f is complete follows from symmetry. □

For wider class of pointfree funcoids the concept of bijection does not make sense. Instead we would want a structure preserving map to be *order isomorphism*.

Actually, for mapping between $\mathscr{P}A$ and $\mathscr{P}B$ where A and B are some sets (including the above considered case of funcoids from A to B) bijection and order isomorphism are essentially the same:

PROPOSITION 2023. Bijections F between sets A and B bijectively correspond to order isomorphisms f between $\mathscr{P}A$ and $\mathscr{P}B$ by the formula $f = \langle F\rangle$.

PROOF. Let F is a bijection. Then $X \subseteq Y \Rightarrow \langle F\rangle X \subseteq \langle F\rangle Y$ and $\langle F^{-1}\rangle\langle F\rangle X = X$ for every sets $X, Y \in \mathscr{P}A$. Thus $f = \langle F\rangle$ is an order isomorphism.

Let now f is an order isomorphism between $\mathscr{P}A$ and $\mathscr{P}B$. Then $f(\{x\})$ is a singleton for every $x \in A$. Take $F(x)$ to the unique y such that $f(\{x\}) = \{y\}$. Obviously f is a bijection and $f = \langle F\rangle$. □

For arbitrary pointfree funcoids isomorphisms do not necessarily preserve structure. It holds only for *increasing pointfree funcoids*:

DEFINITION 2024. I call a pointfree funcoid f *increasing* iff $\langle f\rangle$ and $\langle f^{-1}\rangle$ are monotone functions.

PROPOSITION 2025. If f is an increasing isomorphism of the category of pointfree funcoids then $\langle f\rangle$ is an order isomorphism.

PROOF. We have: $\langle f\rangle \circ \langle f^{-1}\rangle = \langle f \circ f^{-1}\rangle = \langle \mathrm{id}^{\mathsf{FCD}}_{\mathfrak{B}}\rangle = \mathrm{id}_{\mathfrak{B}}$ and $\langle f^{-1}\rangle \circ \langle f\rangle = \langle f^{-1} \circ f\rangle = \langle \mathrm{id}^{\mathsf{FCD}}_{\mathfrak{A}}\rangle = \mathrm{id}_{\mathfrak{A}}$. Thus $\langle f\rangle$ is a bijection.
$\langle f\rangle$ is increasing and bijective. □

REMARK 2026. Non-increasing isomorphisms of the category of pointfree funcoids are against sound mind, they don't preserve the structure of the source, that is for them $\langle f\rangle$ or $\langle f^{-1}\rangle$ are not order isomorphisms.

OBVIOUS 2027. Isomorphisms of $\operatorname{cont}(\mathbf{Pos})$ and $\operatorname{cont}(\mathbf{mePos})$ are order isomorphisms.

6. Direct products

FiXme: Now this section is a complete mess. Clean it up.

Consider the category **contFcd** which is the full subcategory $\operatorname{cont}(\mathbf{mePos})$ restricted to objects which are essentially increasing pointfree funcoids.

Let $f_1 : Y \to X_1$ and $f_2 : Y \to X_2$ are morphisms of **contFcd**.

The product object is $X_1 \times^{(C)} X_2$ (cross composition product of funcoids used). It is easy to see that $X_1 \times^{(C)} X_2$ is an object of **contFcd** that is an endo-funcoid.

The morphism $f_1 \times^{(D)} f_2 : Y \to X_1 \times^{(C)} X_2$ is defined by the formula $(f_1 \times^{(D)} f_2)y = f_1 y \times^{\mathsf{FCD}} f_2 y$.

$f_1 \times^{(D)} f_2$ is monovalued and entirely defined because so are f_1 and f_2.

$$(f_1 \times^{(D2)} f_2)y = \bigcup\left\{ f_1 Y \times^{\mathsf{FCD}} f_2 Y \quad \middle| \quad Y \in \operatorname{atoms}^{\mathfrak{A}} y \right\}.$$

FiXme: Is $(f_1 \times^{(D2)} f_2)$ a pointfree funcoid?

To prove that it is really a morphism we need to show

$$(f_1 \times^{(D)} f_2) \circ Y \sqsubseteq (X_1 \times^{(C)} X_2) \circ (f_1 \times^{(D)} f_2)$$

that is (for every y)

$$(f_1 \times^{(D)} f_2)Yy \sqsubseteq (X_1 \times^{(C)} X_2)(f_1 \times^{(D)} f_2)y.$$

Really, $(f_1 \times^{(D)} f_2)Yy = f_1Yy \times^{\mathsf{FCD}} f_2Yy$;

$(X_1 \times^{(C)} X_2)(f_1 \times^{(D)} f_2)y = (X_1 \times^{(C)} X_2)(f_1y \times^{\mathsf{FCD}} f_2y) = X_1f_1y \times^{\mathsf{FCD}} X_2f_2y$;

but it is easy to show $f_1Yy \times^{\mathsf{FCD}} f_2Yy \sqsubseteq X_1f_1y \times^{\mathsf{FCD}} X_2f_2y$.

??

I define ??

FiXme: Prove that it is a direct product in **contFcd**.

CHAPTER 10

Product of funcoids over a filter

The following definition is inspired by the usual definition of Tychonoff product of topological spaces.

Definition 2028. Let f be an indexed family of funcoids. Let $\mathcal{F}$ be a filter on $\operatorname{dom} f$.

$$a\left[\prod^{[\mathcal{F}]} f\right] b \Leftrightarrow \exists N \in \mathcal{F} \forall i \in N : \Pr_i^{\mathsf{RLD}} a \, [f_i] \, \Pr_i^{\mathsf{RLD}} b$$

for atomic reloids a and b.

Remark 2029. We are especially interested in the special case when $\mathcal{F}$ is the cofinite filter. In this case $a\left[\prod^{[\mathcal{F}]} f\right] b$ is defined by the condition that $\Pr_i^{\mathsf{RLD}} a \, [f_i]$ $\Pr_i^{\mathsf{RLD}}$ for an infinite number of indexes i.

Obvious 2030. $a\left[\prod^{\left[\top^{\mathscr{F}(\operatorname{dom} f)}\right]} f\right] b \Leftrightarrow a\left[\prod^{(A)} f\right] b$.

Proposition 2031. $\neg(\mathcal{X} \, [f] \, \mathcal{Y})$ implies $\neg(X \, [f] \, Y)$ for some $X \in \operatorname{up} \mathcal{X}$, $Y \in \operatorname{up} \mathcal{Y}$.

Proof. Suppose $\neg(\mathcal{X} \, [f] \, \mathcal{Y})$. Then $\mathcal{Y} \asymp \langle f \rangle \mathcal{X}$. Thus by separability of core for filters $Y \asymp \langle f \rangle \mathcal{X}$ for some $Y \in \operatorname{up} \mathcal{Y}$, that is $\neg(\mathcal{X} \, [f] \, Y)$. Apply this result twice. □

Lemma 2032.

$$\forall X \in \prod_{i \in D} \operatorname{up} a_i, Y \in \prod_{i \in D} \operatorname{up} b_i \exists x \in \prod_{i \in D} \operatorname{atoms} \uparrow X_i, y \in \prod_{i \in D} \operatorname{atoms} \uparrow Y_i \exists N \in \mathcal{F} \forall j \in N : x_j \, [f_j] \, y_j$$

implies $\exists N \in \mathcal{F} \forall i \in N : a_i \, [f_i] \, b_i$.

Proof. Suppose for the contrary $\neg(a_i \, [f_i] \, b_i)$ for all $i \in N$ where $N \in \mathcal{F}$ (i.e. for an infinite number of indexes if $\mathcal{F}$ is the cofinite filter). Then (lemma above) there are $X_i \in \operatorname{up} a_i$ and $Y_i \in \operatorname{up} b_i$ such that $\neg(X_i \, [f_j]^* \, Y_i)$ for $i \in N$. Thus $\neg(x_i \, [f_i] \, y_i)$ for $i \in N$, contrary to the condition. □

Proposition 2033. The funcoid $\prod^{[\mathcal{F}]} f$ exists.

Proof. We need to prove that

$$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} Y : x \left[\prod^{(A2)} f\right] y$$

implies $a\left[\prod^{[\mathcal{F}]} f\right] b$.

Equivalently transforming it: FiXme: More detailed proof.

$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} Y$
$\exists N \in \mathcal{F} \forall i \in N : \Pr_i^{\mathsf{RLD}} x \, [f_i] \, \Pr_i^{\mathsf{RLD}} y$;
$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \prod_{i \in \operatorname{dom} f} \operatorname{atoms} \uparrow^{\mathsf{RLD}} X_i, y \in \prod_{i \in \operatorname{dom} f} \operatorname{atoms} \uparrow^{\mathsf{RLD}} Y_i$
$\exists N \in \mathcal{F} \forall i \in N : x_i \, [f_i] \, y_i$;

$$\forall X \in \prod_{i\in D} \operatorname{up} a_i, Y \in \prod_{i\in D} \operatorname{up} b_i \exists x \in \prod_{i\in D} \operatorname{atoms}\uparrow X_i, y \in \prod_{i\in D} \operatorname{atoms}\uparrow Y_i \exists N \in \mathcal{F} \forall j \in N : x_j \; [f_j] \; y_j$$

where $D = \operatorname{dom} f$.

Thus by the lemma $\exists N \in \mathcal{F} \forall i \in N : a_i \; [f_i] \; b_i$, that is $a \left[\prod^{[\mathcal{F}]} f\right] b$. □

FiXme: TODO: when $\Pr_j \prod^{[\mathcal{F}]}_{i\in D} a_i = a_j$?

1. More on product of reloids

FiXme: Move this to a more appropriate place.

DEFINITION 2034. $\prod^{(Y)}_{i\in\operatorname{dom} f} f = \prod^{(A)}_{i\in\operatorname{dom} f} (\mathsf{FCD}) f$ for an indexed family f of reloids.

PROPOSITION 2035.

$$a \left[\prod^{(Y)}_{i\in\operatorname{dom} f} f\right] b \Leftrightarrow \forall i \in \operatorname{dom} f : f_i \not\asymp \Pr^{\mathsf{RLD}}_i a \times^{\mathsf{RLD}} \Pr^{\mathsf{RLD}}_i b.$$

PROOF. $f_i \not\asymp \Pr^{\mathsf{RLD}}_i a \times^{\mathsf{FCD}} \Pr^{\mathsf{RLD}}_i b \Leftrightarrow (\mathsf{FCD}) f_i \sqsupseteq \Pr^{\mathsf{RLD}}_i a \times^{\mathsf{FCD}} \Pr^{\mathsf{RLD}}_i b \Leftrightarrow a \; [(\mathsf{FCD}) f_i] \; b$. □

EXAMPLE 2036. The funcoid p described by the formula (for atomic reloids a and b)

$$a \; p \; b \Leftrightarrow \forall i \in \operatorname{dom} f : f_i \sqsupseteq \Pr^{\mathsf{RLD}}_i a \times^{\mathsf{RLD}} \Pr^{\mathsf{RLD}}_i b$$

does not exist (in general), even if we restrict to 2-indexed families only.

PROOF. For the case if $f = \llbracket v, w \rrbracket$ is a 2-indexed family of reloids, the formula which we need to disprove takes the form:

$$a \; p \; b \Leftrightarrow v \sqsupseteq \operatorname{dom} a \times^{\mathsf{RLD}} \operatorname{dom} b \land w \sqsupseteq \operatorname{im} a \times^{\mathsf{RLD}} \operatorname{im} b.$$

Take $v = w = 1^{\mathbf{Rel}}$ on an infinite set. Suppose for the contrary p exists and is a funcoid. Then

$$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms}\uparrow X, y \in \operatorname{atoms}\uparrow Y : x \; p \; y \Rightarrow a \; p \; b.$$

For a counter-example take $a = b$ to be a nontrivial ultrafilter. Then for every $X \in \operatorname{up} a$, $Y \in \operatorname{up} b$ take $x = y$ to be singletons on $X \cap Y$. We have $x \; p \; y$, but not $a \; p \; b$. □

CHAPTER 11

Compact funcoids

Compact funcoids are defined. Attempted to prove that under certain conditions the reloid corresponding to a compact funcoid is the neighborhood of the diagonal of the product funcoid.

This is a rough partial draft. The proofs are with errors.

FiXme: The below examples also show that subatomic product does not coincide with Tychonoff product.

1. The rest

Definition 2037. A funcoid f is *directly compact* iff

$$\forall \mathcal{F} \in \mathfrak{F} : (\langle f \rangle \mathcal{F} \neq \bot \Rightarrow \operatorname{Cor}\langle f \rangle \mathcal{F} \neq \bot).$$

Obvious 2038. A funcoid f is directly compact iff $\forall a \in \operatorname{atoms} \operatorname{dom} f$: $\operatorname{Cor}\langle f \rangle a \neq \bot$.

Obvious 2039. A reflexive funcoid f is directly compact iff

$$\forall \mathcal{F} \in \mathfrak{F} : (\mathcal{F} \neq \bot \Rightarrow \operatorname{Cor}\langle f \rangle \mathcal{F} \neq \bot).$$

Definition 2040. A funcoid f is *reversely compact* iff f^{-1} is directly compact.

Definition 2041. A funcoid is *compact* iff it is both directly compact and reversely compact.

Proposition 2042. $\prod^{\mathsf{RLD}} a = \uparrow^{\mathsf{RLD}} \prod_{i \in \operatorname{dom} a} (\uparrow^{\mathsf{RLD}})^{-1} a_i$ for every indexed family a of principal filters.

Proof. Because $\prod_{i \in \operatorname{dom} a} (\uparrow^{\mathsf{RLD}})^{-1} a_i \in \operatorname{up} \prod^{\mathsf{RLD}} a$. FiXme: More detailed proof. □

Lemma 2043. $\prod^{\mathsf{RLD}}_{i \in \operatorname{dom} a} \operatorname{Cor} a_i = \operatorname{Cor} \prod^{\mathsf{RLD}} a$.

Proof. $\operatorname{Cor} \prod^{\mathsf{RLD}} a = \bigsqcap \{ \uparrow^{\mathsf{RLD}} \prod A \mid A \in \operatorname{up} a \} = \uparrow^{\mathsf{RLD}} \bigcap \{ \prod A \mid A \in \operatorname{up} a \} = \uparrow^{\mathsf{RLD}} \bigcap \{ \prod A \mid A \in \mathscr{P} \prod \mathfrak{U}, \forall i \in \operatorname{dom} a : A_i \in \operatorname{up} a_i \} = \uparrow^{\mathsf{RLD}} \bigcap \{ \prod \bigcap K_i \mid K \in \mathscr{P}\mathscr{P} \prod \mathfrak{U}, \forall i \in \operatorname{dom} a : K_i \in \mathscr{P} \operatorname{up} a_i \} = \uparrow^{\mathsf{RLD}} \bigcap \{ \prod (\uparrow^{\mathsf{RLD}})^{-1} \operatorname{Cor} a_i \mid i \in \operatorname{dom} a \} = \uparrow^{\mathsf{RLD}} \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} a} \operatorname{Cor} a_i$.

FiXme: Check for little errors. □

Corollary 2044. $\prod^{\mathsf{RLD}}_{i \in n} \langle \operatorname{CoCompl} f_i \rangle \mathcal{X}_i = \left\langle \operatorname{CoCompl} \prod^{(A)} f \right\rangle \prod^{\mathsf{RLD}} \mathcal{X}$ for every n-indexed families f of funcoids and $\mathcal{X}$ of filters on the same set (with $\operatorname{Src} f_i = \operatorname{Base}(\mathcal{X}_i)$ for every $i \in n$).

PROOF.

$$\prod_{i\in n}^{\mathsf{RLD}}\langle\operatorname{CoCompl} f_i\rangle\mathcal{X}_i = \prod_{i\in n}^{\mathsf{RLD}}\operatorname{Cor}\langle f_i\rangle\mathcal{X}_i = \operatorname{Cor}\prod_{i\in n}^{\mathsf{RLD}}\langle f_i\rangle\mathcal{X}_i = (*)\ \operatorname{Cor}\prod_{i\in n}^{\mathsf{RLD}}\langle f_i\rangle\operatorname{Pr}_i^{\mathsf{RLD}}\left(\prod^{\mathsf{RLD}}\mathcal{X}\right) = \operatorname{Cor}\left\langle\prod^{(A)} f\right\rangle\prod^{\mathsf{RLD}}\mathcal{X} = \left\langle\operatorname{CoCompl}\prod^{(A)} f\right\rangle\prod^{\mathsf{RLD}}\mathcal{X}.$$

(*) You should verify the special case when $\mathcal{X}_i = \bot^{\mathfrak{F}}$ for some i. □

THEOREM 2045. Let f be an indexed family of funcoids. FiXme: Reverse theorem (for non-least funcoids).

1°. $\prod f$ is directly compact if every f_i is directly compact.
2°. $\prod f$ is reversely compact if every f_i is reversely compact.
3°. $\prod f$ is compact if every f_i is compact.

PROOF. It is enough to prove only the first statement.

Let each f_i is directly compact.

Let $\langle\prod f\rangle a\neq\bot$. Then $\langle\prod f\rangle a = \left\langle\prod^{(A)} f\right\rangle a = \prod^{\mathsf{RLD}}_{i\in\operatorname{dom} f}\langle f_i\rangle\operatorname{Pr}_i^{\mathsf{RLD}} a$. Thus every $\langle f_i\rangle\operatorname{Pr}_i^{\mathsf{RLD}} a\neq\bot$. Consequently by compactness $\operatorname{Cor}\langle f_i\rangle\operatorname{Pr}_i^{\mathsf{RLD}} a\neq\bot$; $\prod_{i\in\operatorname{dom} f}\operatorname{Cor}\langle f_i\rangle\operatorname{Pr}_i^{\mathsf{RLD}} a\neq\bot$; $\operatorname{Cor}\prod_{i\in\operatorname{dom} f}\langle f_i\rangle\operatorname{Pr}_i^{\mathsf{RLD}} a\neq\bot$; $\operatorname{Cor}\langle\prod f\rangle a\neq\bot$.

So $\prod f$ is directly compact. □

PROPOSITION 2046. The following expressions are pairwise equal:

1°. $\langle f\times^{(A)} f\rangle^* 1^{\mathsf{RLD}}$;
2°. $\bigsqcup\left\{\frac{\langle f\times^{(A)} f\rangle p}{p\in\operatorname{atoms}1^{\mathsf{RLD}}}\right\}$;
3°. $\bigsqcup\left\{\frac{\langle f\rangle x\times^{\mathsf{RLD}}\langle f\rangle x}{x\in\operatorname{atoms}^{\mathscr{F}}}\right\}$;

PROOF.

1°⇔2°. Theorem 795.
2°⇔3°. $\bigsqcup\left\{\frac{\langle f\times^{(A)} f\rangle p}{p\in\operatorname{atoms}1^{\mathsf{RLD}}}\right\} = \bigsqcup\left\{\frac{\langle f\rangle\operatorname{dom}p\times^{\mathsf{RLD}}\langle f\rangle\operatorname{im}p}{p\in\operatorname{atoms}1^{\mathsf{RLD}}}\right\} = \bigsqcup\left\{\frac{\langle f\rangle x\times^{\mathsf{RLD}}\langle f\rangle x}{x\in\operatorname{atoms}^{\mathscr{F}}}\right\}$.

□

PROPOSITION 2047. Let g be a reloid and $f = (\mathsf{FCD})g$ and $f = f\circ f^{-1}$. Then $\langle f\times^{(A)} f\rangle^* 1^{\mathsf{RLD}}\sqsupseteq g$.

PROOF. $\langle f\times^{(A)} f\rangle^* 1^{\mathsf{RLD}}\not\asymp{\uparrow^{\mathsf{RLD}}} Y\Leftrightarrow{\uparrow^{\mathsf{RLD}}} 1^{\mathsf{RLD}}\,[f\times^{(A)} f]\,{\uparrow^{\mathsf{RLD}}} Y\Leftrightarrow{\uparrow^{\mathsf{FCD}}} 1^{\mathsf{RLD}}\,[f\times^{(C)} f]\,{\uparrow^{\mathsf{FCD}}} Y\Leftrightarrow f\circ{\uparrow^{\mathsf{FCD}}} 1^{\mathsf{RLD}}\circ f^{-1}\not\asymp{\uparrow^{\mathsf{FCD}}} Y\Leftrightarrow f\circ f^{-1}\not\asymp{\uparrow^{\mathsf{FCD}}} Y\Leftrightarrow f\not\asymp{\uparrow^{\mathsf{FCD}}} Y\Leftrightarrow f\sqcap{\uparrow^{\mathsf{FCD}}} Y\neq\bot\Leftarrow(\mathsf{RLD})_{\mathrm{in}}(f\sqcap{\uparrow^{\mathsf{FCD}}} Y)\neq\bot\Leftrightarrow(\mathsf{RLD})_{\mathrm{in}} f\sqcap(\mathsf{RLD})_{\mathrm{in}}{\uparrow^{\mathsf{FCD}}} Y\neq\bot\Leftarrow(\mathsf{RLD})_{\mathrm{in}} f\sqcap(\mathsf{RLD})_{\mathrm{out}}{\uparrow^{\mathsf{FCD}}} Y\neq\bot\Leftrightarrow(\mathsf{RLD})_{\mathrm{in}} f\sqcap{\uparrow^{\mathsf{RLD}}} Y\neq\bot\Leftrightarrow(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})g\sqcap{\uparrow^{\mathsf{RLD}}} Y\neq\bot\Leftarrow g\sqcap{\uparrow^{\mathsf{RLD}}} Y\neq\bot\Leftrightarrow g\not\asymp{\uparrow^{\mathsf{RLD}}} Y$. □

PROPOSITION 2048. Let g be a reloid and $f = (\mathsf{FCD})g$ and $f = f \circ f^{-1}$. Then $\langle f \times^{\mathrm{in}} f\rangle^* 1^{\mathsf{RLD}} \sqsupseteq g$.

PROOF. $\langle f \times^{\mathrm{in}} f\rangle^* 1^{\mathsf{RLD}} = \langle (\mathsf{RLD})_{\mathrm{in}} f \circ^{(C)} (\mathsf{RLD})_{\mathrm{in}} f\rangle^* 1^{\mathsf{RLD}} = (\mathsf{RLD})_{\mathrm{in}} f \circ 1^{\mathsf{RLD}} \circ (\mathsf{RLD})_{\mathrm{in}} f^{-1} = (\mathsf{RLD})_{\mathrm{in}} f \circ (\mathsf{RLD})_{\mathrm{in}} f^{-1} = (\mathsf{RLD})_{\mathrm{in}} (f \circ f^{-1}) = (\mathsf{RLD})_{\mathrm{in}} f = (\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) g \sqsupseteq g$. □

LEMMA 2049. $\operatorname{Cor}\langle f \times^{(A)} f\rangle^* g \sqsubseteq 1^{\mathsf{RLD}}$ if $(\mathsf{FCD})g = f$ for a T_1-separable reloid g.

1.1. Propositions from [2] which do not hold for our products. In this subsection I present counter-examples against modified propositions from [2] in which I replace Tychonoff product with our subatomic or cross-inner products.

TODO: Consider as a counter-example the non-transitive compact funcoid $\left\{ \frac{(x,y)}{x,y\in[0;1],|x-y|<\frac{1}{3}} \right\}$.

EXAMPLE 2050. $\langle 1^{\mathbf{Rel}} \times^{(A)} 1^{\mathbf{Rel}}\rangle^* 1^{\mathbf{Rel}} \sqsupset 1^{\mathbf{Rel}}$.

PROOF. $\langle 1^{\mathbf{Rel}} \times^{(A)} 1^{\mathbf{Rel}}\rangle^* 1^{\mathbf{Rel}} = \bigsqcup_{x\in\operatorname{atoms} 1^{\mathbf{Rel}}} \langle 1^{\mathbf{Rel}} \times^{(A)} 1^{\mathbf{Rel}}\rangle^* x = \bigsqcup_{x\in\operatorname{atoms} 1^{\mathbf{Rel}}} \left(\langle 1^{\mathbf{Rel}}\rangle^* \operatorname{dom} x \times^{\mathsf{RLD}} \langle 1^{\mathbf{Rel}}\rangle^* \operatorname{im} x \right) = \bigsqcup_{x\in\operatorname{atoms} 1^{\mathbf{Rel}}} \left(\operatorname{dom} x \times^{\mathsf{RLD}} \operatorname{im} x \right) = \bigsqcup_{x\in\operatorname{atoms} \mathscr{F}} (x \times^{\mathsf{RLD}} x) \sqsupset 1^{\mathbf{Rel}}$. □

Statement 2 on page 172 of [2] does not survive modification:

EXAMPLE 2051.

1°. There is a funcoid f and $V \in \operatorname{up} f$ such that $V \circ M \circ V^{-1} \notin \operatorname{up}\langle f \times^{(A)} f\rangle^* M$.
2°. $\langle f \times^{(A)} f\rangle^* M \sqsupset g \circ {\uparrow^{\mathsf{RLD}}} M \circ g^{-1}$ for some reloid g, binary relation M and the funcoid $f = (\mathsf{FCD})g$.

PROOF.

1°. Take $f = M = V = 1^{\mathbf{Rel}}$ and use the example above.
2°. Take $f = g = M = 1^{\mathbf{Rel}}$ and use the example above.

□

COROLLARY 2052. $\langle f \times^{(A)} f\rangle^* M \sqsubseteq \langle f \times^{(C)} f\rangle^* M$.

COROLLARY 2053. $V \circ V^{-1} \in \operatorname{up}\langle f \times^{(A)} f\rangle^* 1^{\mathsf{RLD}}$; $f \circ f^{-1} \sqsupseteq \langle f \times^{(A)} f\rangle^* 1^{\mathsf{RLD}}$.

PROOF. ?? □

REMARK 2054. I attempted to generalize the below theorem more than the standard general topology theorem about correspondence of compact and uniform spaces, but haven't really succeeded much, as it appears to be needed that the reloid in question is reflexive, symmetric, and transitive, that is just a uniform space as in the standard general topology.

Does the reverse inequality hold, that is $g \sqsupseteq \langle f \times^{(A)} f\rangle^* 1^{\mathsf{RLD}}$ and/or $g \sqsupseteq \langle f \times^{\mathrm{in}} f\rangle^* 1^{\mathsf{RLD}}$ (for compact $f = (\mathsf{FCD})g$)?

THEOREM 2055. $g \sqsubseteq \langle f \times^{(A)} f\rangle^* 1^{\mathsf{RLD}}$ for compact $f = (\mathsf{FCD})g$. (We have already proved this in an easier way, and not only for compact funcoids.)

Suppose there is $U \in \operatorname{up}\langle f \times^{(A)} f\rangle^* 1^{\mathsf{RLD}}$ such that $U \notin \operatorname{up} g$.
Then $\left\{ \frac{V\setminus U}{V\in\operatorname{up} g} \right\} = g \setminus U$ would be a proper filter.
Thus by reflexivity $\langle f \times^{(A)} f\rangle^* (g \setminus U) \neq \bot$.
By compactness of $f \times^{(A)} f$, $\operatorname{Cor}\langle f \times^{(A)} f\rangle^* (g \setminus U) \neq \bot$.

Suppose $\uparrow\{(x,x)\} \sqsubseteq \langle f \times^{(A)} f\rangle^*(g \setminus U)$; then $g \setminus U \not\asymp \langle f^{-1} \times^{(A)} f^{-1}\rangle\{(x,x)\}$; $U \sqsubset \langle f^{-1} \times^{(A)} f^{-1}\rangle\{(x,x)\} \sqsubseteq \langle f^{-1} \times^{(A)} f^{-1}\rangle 1^{\mathsf{RLD}}$ what is impossible.

Thus there exist $x \neq y$ such that $\{(x,y)\} \sqsubseteq \operatorname{Cor}\langle f \times^{(A)} f\rangle^*(g \setminus U)$. Thus $\{(x,y)\} \sqsubseteq \langle f \times^{(A)} f\rangle^* g$.

Thus by the lemma $\{(x,y)\} \sqsubseteq 1^{\mathsf{RLD}}$ what is impossible. So $U \in \operatorname{up} g$.

We have $\operatorname{up}\langle f \times^{(A)} f\rangle^* 1^{\mathsf{RLD}} \subseteq \operatorname{up} g$; $\langle f \times^{(A)} f\rangle^* 1^{\mathsf{RLD}} \sqsupseteq g$.

Corollary 2056. Let f is a T_1-separable (the same as T_2 for symmetric transitive) compact funcoid and g is a uniform space (reflexive, symmetric, and transitive endoreloid) such that $(\mathsf{FCD})g = f$. Then $g = \langle f \times^{(A)} f\rangle^* 1^{\mathsf{RLD}}$.

An (incomplete) attempt to prove one more theorem follows:

Theorem 2057. Let μ and ν be uniform spaces, $(\mathsf{FCD})\mu$ be a compact funcoid. Then a map f is a continuous map from $(\mathsf{FCD})\mu$ to $(\mathsf{FCD})\nu$ iff f is a (uniformly) continuous map from μ to ν.

Proof. FiXme: errors in this proof.

http://math.stackexchange.com/questions/665202/bourbaki-on-the-fact-that-continuous-function-on-a-compact-is-uniformly-continuo/670956?iemail=1&noredirect=1#670956

We have $\mu = \langle (\mathsf{FCD})\mu \times (\mathsf{FCD})\mu\rangle \uparrow^{\mathsf{RLD}} 1^{\mathsf{RLD}}$

$f \in \mathrm{C}_?((\mathsf{FCD})\mu, (\mathsf{FCD})\nu)$. Then

$$f \times^{(A)} f \in \mathrm{C}_?((\mathsf{FCD})(\mu \times^{(A)} \mu), (\mathsf{FCD})(\nu \times^{(A)} \nu))$$

$(f \times^{(A)} f) \circ (\mathsf{FCD})(\mu \times^{(A)} \mu) \sqsubseteq (\mathsf{FCD})(\nu \times^{(A)} \nu) \circ (f \times^{(A)} f)$

For every $V \in \operatorname{up}(\nu \times^{(A)} \nu)$ we have $\langle g^{-1}\rangle V \in \langle (\mathsf{FCD})(\mu \times^{(A)} \mu)\rangle\{y\}$ for some y.

$\langle g^{-1}\rangle V \in \langle (\mathsf{FCD})\mu \times^{(A)} (\mathsf{FCD})\mu\rangle \uparrow^{\mathsf{RLD}} 1^{\mathsf{RLD}} = \operatorname{up}\mu$

$\langle g\rangle\langle g^{-1}\rangle V \sqsubseteq V$

We need to prove $f \in \mathrm{C}(\mu,\nu)$ that is $\forall p \in \operatorname{up}\nu \exists q \in \operatorname{up}\mu : \langle f\rangle q \sqsubseteq p$. But this follows from the above. □

FiXme: A space is compact if and only if it is both, complete and totally bounded.

http://math.stackexchange.com/questions/1101995/non-symmetric-version-of-compact-totally-bounded-complete

CHAPTER 12

Pointfree funcoids as a generalization of frames

I define an injection from the set of frames to the set of pointfree endo-funcoids. This article is a rough partial draft of a future longer writing.

1. Definitions

1.1. Pointfree funcoid induced by a co-frame. Let $\mathfrak{L}$ is a co-frame.

We will define pointfree funcoid $\Uparrow \mathfrak{L}$.

Let $\mathcal{B}(\mathfrak{L})$ is a boolean lattice whose co-subframe $\mathfrak{L}$ is. (That this mapping exists follows from [3], page 53.) There may be probably more than one such mapping, but we just choose one $\mathcal{B}$ arbitrarily.

Define $\operatorname{cl}(A) = \bigsqcap\{X \in \mathfrak{L} \mid X \sqsupseteq A\}$.

Here $\bigsqcap$ can be taken on either $\mathfrak{L}$ or $\mathcal{B}(\mathfrak{L})$ as they are the same.

Obvious 2058. $\operatorname{cl} \in \mathfrak{L}^{\mathcal{B}(\mathfrak{L})}$.

$\operatorname{cl}(A \sqcup B) = \bigsqcap\{X \in \mathfrak{L} \mid X \sqsupseteq A \sqcup B\} = \bigsqcap\{X \in \mathfrak{L} \mid X \sqsupseteq A, X \sqsupseteq B\} = \bigsqcap\{X_1 \sqcup X_2 \mid X_1 \sqsupseteq A, X_2 \sqsupseteq B\} = \bigsqcap\{X_1 \mid X_1 \sqsupseteq A\} \sqcup \bigsqcap\{X_2 \mid X_2 \sqsupseteq B\} = \operatorname{cl} A \sqcup \operatorname{cl} B$.

$\operatorname{cl} 0 = 0$ is obvious.

Hence we are under conditions of the theorem 14.26 in my book.

So there exists a unique pointfree endo-funcoid $\Uparrow \mathfrak{L} \in \mathsf{FCD}(\mathfrak{F}(\mathcal{B}(\mathfrak{L})), \mathfrak{F}(\mathcal{B}(\mathfrak{L})))$ such that

$$\langle \Uparrow \mathfrak{L}\rangle \mathcal{X} = \bigsqcap^{\mathfrak{F}(\mathcal{B}(\mathfrak{L}))} \langle \operatorname{cl}\rangle \operatorname{up}^{(\mathfrak{F}(\mathcal{B}(\mathfrak{L})), \mathfrak{P}(\mathcal{B}(\mathfrak{L})))} \mathcal{X}$$

for every filter $\mathcal{X} \in \mathfrak{F}(\mathcal{B}(\mathfrak{L}))$.

1.2. Co-frame induced by a pointfree funcoid. The co-frame $\Downarrow f$ for some pointfree endo-funcoids f will be defined to be the reverse of $\Uparrow$. See below for exact meaning of being reverse.

Let restore the co-frame $\mathfrak{L}$ from the pointfree funcoid $\Uparrow \mathfrak{L}$.

Let poset $\Downarrow f$ for every pointfree funcoid f is defined by the formula:

$$\Downarrow f = \{X \in Z(\operatorname{Ob} f) \mid \langle f\rangle X = X\}.$$

Remark 2059. It seems that $\Downarrow$ is *not* a monovalued function from pFCD to $\operatorname{Ob}(\mathbf{Frm})$.

1.3. Isomorphism of co-frames through pointfree funcoids.

Remark 2060. $\mathfrak{P}(\mathcal{B}(\mathfrak{L})) = Z(\mathfrak{F}(\mathcal{B}(\mathfrak{L})))$ (theorem 4.137 in [5]).

Theorem 2061. $\mathfrak{L} \mapsto \Downarrow\Uparrow \mathfrak{L}$ (where $\mathfrak{L}$ ranges all small frames) is an order isomorphism.

Proof. Let $A' \in \Downarrow\Uparrow \mathfrak{L}$. Then there exists $A \in \mathcal{B}(\mathfrak{L})$ such that $A' = \uparrow^{\mathcal{B}(\mathfrak{L})} A$.

$\langle f\rangle A' = \uparrow^{\mathcal{B}(\mathfrak{L})} \operatorname{cl} A$.

$\langle f\rangle A' = A'$ that is $\uparrow^{\mathcal{B}(\mathfrak{L})} \operatorname{cl} A = A' = \uparrow^{\mathcal{B}(\mathfrak{L})} A$. So $\operatorname{cl} A = A$ and thus $A \in \mathfrak{L}$.

Let now $A \in \mathfrak{L}$. Then take $A' = \uparrow^{\mathcal{B}(\mathfrak{L})} A$. We have $\langle f\rangle A' = \operatorname{cl} A = \uparrow^{\mathcal{B}(\mathfrak{L})} A = A'$. So $A' \in \Downarrow\Uparrow \mathfrak{L}$.

We have proved that it is a bijection.

Because A and A' are related by the equation $A' = \uparrow^{\mathcal{B}(\mathfrak{L})} A$ it is obvious that this is an order embedding. $\square$

2. Postface

Pointfree funcoids are a **massive** generalization of locales and frames: They don't only require the lattice of filters to be boolean but these can be even not lattices of filters at all but just arbitrary posets. I think a new era in pointfree topology starts.

Much work is yet needed to relate different properties of frames and locales with corresponding properties of pointfree funcoids.

CHAPTER 13

Singularities

Very rough draft.

1. Singularities funcoids: some special cases

We attempt to prove that $\operatorname{up} z$ is closed regarding finite intersections.

For consideration of this, let's consider two special cases (first of which is a specialization of the second).

Let $\mu = \nu$ be the natural proximity on real numbers $\mathbb{R}$.

Let Δ is the entourage filter of zero.

1. $z = \Delta \times^{\mathsf{FCD}} \Delta$.

2. $z = \nu \circ (\uparrow^{\mathsf{FCD}} f)|_{\Delta}$ for an arbitrary function $f : \mathbb{R} \to \mathbb{R}$.

(1) is [[also formulated in elementary terms|http://math.stackexchange.com/questions/568513/is-a-set-closed-under-finite-intersections-about-filters]] (without using funcoids).

These two above conjectures are shown to be false by a counter-example in [[this blog post|http://portonmath.wordpress.com/2013/12/18/a-negative-result-on-a-conjecture/]]. It is a discouraging result as it seems from it the plain funcoids can't be used for the multilevel theory of singularities.

2. Using plain funcoids

This way if we succeed is the best way to create metasingular numbers because, it (if we succeed) involves just funcoids not some fancy generalization of funcoids.

Approximate definition of "singularity level": //Singularity level// is a transitive, T_2-separable endofuncoid.

Now define the funcoid $\nu_{i+1} = \operatorname{SLA}(\nu_i)$:

$\operatorname{Ob}(\nu_{i+1})$ is defined as the set of all generalized limits (having fixed μ, ν, and G).

$X \; [\nu_{i+1}]^* \; Y \Leftrightarrow \exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists x \in \bigcup X, y \in \bigcup Y : x, y \sqsubseteq K.$

The trouble is to prove that the funcoid ν_{i+1} exists (is really a funcoid).

$\neg (X \; [\nu_{i+1}]^* \; \emptyset)$ and $\neg (\emptyset \; [\nu_{i+1}]^* \; Y)$ are obvious. We need to prove

$$I \cup J \; [\nu_{i+1}]^* \; Y \Leftrightarrow I \; [\nu_{i+1}]^* \; Y \vee J \; [\nu_{i+1}]^* \; Y$$

and

$$X \; [\nu_{i+1}]^* \; I \cup J \Leftrightarrow X \; [\nu_{i+1}]^* \; I \vee X \; [\nu_{i+1}]^* \; J.$$

Let's attempt to prove the first of the above equations (the second is dual).

$I \cup J \; [\operatorname{SLA}(\nu)]^* \; Y \Leftrightarrow$

$\exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists x \in \bigcup I \cup \bigcup J, y \in \bigcup Y : x, y \sqsubseteq K \Leftrightarrow$

$\exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z : (\exists x \in \bigcup I \cup \bigcup J : x \sqsubseteq K \wedge \exists y \in \bigcup Y : y \sqsubseteq K) \Leftrightarrow$

$\exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists x \in \bigcup I \cup \bigcup J : x \sqsubseteq K \wedge$

$\exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists y \in \bigcup Y : y \sqsubseteq K \Leftrightarrow$

??

$\exists z \in \bigcup \operatorname{Ob} \nu : (\forall K \in \operatorname{up} z \exists x \in \bigcup I : x \sqsubseteq K \vee$

$\forall K \in \operatorname{up} z \exists x \in \bigcup J : x \sqsubseteq K) \wedge \exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists y \in \bigcup Y : y \sqsubseteq K \Leftrightarrow$

$(\exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists x \in \bigcup I : x \sqsubseteq K \vee$

$\exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists x \in \bigcup J : x \sqsubseteq K) \wedge \exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists y \in \bigcup Y : y \sqsubseteq$

$K \Leftrightarrow$
$(\exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists x \in \bigcup I : x \sqsubseteq K \wedge \exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists y \in \bigcup Y : y \sqsubseteq K) \vee$
$(\exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists x \in \bigcup J : x \sqsubseteq K \wedge \exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z \exists y \in \bigcup Y : y \sqsubseteq K) \Leftrightarrow$
$(\exists z \in \bigcup \operatorname{Ob} \nu : (\forall K \in \operatorname{up} z \exists x \in \bigcup I : x \sqsubseteq K \wedge \forall K \in \operatorname{up} z \exists y \in \bigcup Y : y \sqsubseteq K)) \vee$
$(\exists z \in \bigcup \operatorname{Ob} \nu : (\forall K \in \operatorname{up} z \exists x \in \bigcup J : x \sqsubseteq K \wedge \forall K \in \operatorname{up} z \exists y \in \bigcup Y : y \sqsubseteq K)) \Leftrightarrow$
$(\exists z \in \bigcup \operatorname{Ob} \nu : (\forall K \in \operatorname{up} z : (\exists x \in \bigcup I : x \sqsubseteq K \wedge \exists y \in \bigcup Y : y \sqsubseteq K))) \vee$
$\exists z \in \bigcup \operatorname{Ob} \nu \forall K \in \operatorname{up} z : (\exists x \in \bigcup J : x \sqsubseteq K \wedge \exists y \in \bigcup Y : y \sqsubseteq K) \Leftrightarrow$
$I \, [\mathrm{SLA}(\nu)]^* \, Y \vee J \, [\mathrm{SLA}(\nu)]^* \, Y$.

To finish the proof we need to fulfill ?? in the above formula. For this it's enough to prove

$\forall K \in \operatorname{up} z \exists x \in \bigcup I \cup \bigcup J : x \sqsubseteq K \Rightarrow$
$\forall K \in \operatorname{up} z \exists x \in \bigcup I : x \sqsubseteq K \vee \forall K \in \operatorname{up} z \exists x \in \bigcup J : x \sqsubseteq K$.

If $z = \uparrow Z$ is a principal funcoid, then

$\forall K \in \operatorname{up} z \exists x \in \bigcup I \cup \bigcup J : x \sqsubseteq K \Rightarrow$
$\exists x \in \bigcup I \cup \bigcup J : x \sqsubseteq z \Rightarrow$
$\exists x \in \bigcup I : x \sqsubseteq z \vee \exists x \in \bigcup J : x \sqsubseteq z \Rightarrow$
$\forall K \in \operatorname{up} z \exists x \in \bigcup I : x \sqsubseteq K \vee \forall K \in \operatorname{up} z \exists x \in \bigcup J : x \sqsubseteq K$.

Following the idea of [[the proof in this math.stackexchange.com question|http://math.stackexchange.com/questions/562908/an-implication-involving-filters#562974]] it is easy to show that our implication is true if $\operatorname{up} z$ is closed regarding finite meets. See [[this page|Singularities funcoids: some special cases]] for attempts to set it true. The question is whether our statement holds for non-principal funcoids. Or is there a counterexampe?

3. Singularities funcoids: special cases proof attempts

To prove that $\mathrm{GR}(\Delta \times^{\mathsf{FCD}} \Delta)$ is closed under finite intersections, it's enough to prove that for every $f \in \mathrm{GR}(\Delta \times^{\mathsf{FCD}} \Delta)$ there is a positive ε such that $\forall x \in]-\varepsilon; \varepsilon[: fx \in \Delta$.

Really, under this assumption:

For $g \in \mathrm{GR}(\Delta \times^{\mathsf{FCD}} \Delta)$ exists $\zeta > 0$ such that $\forall x \in]-\zeta; \zeta[: gx \in \Delta$. Let $\eta = \min\{\varepsilon, \zeta\}$. So $\forall x \in]-\eta; \eta[: (\langle f \rangle x \in \Delta \wedge \langle g \rangle x \in \Delta)$ and so $\forall x \in]-\eta; \eta[: \langle f \cap g \rangle x \in \Delta$ that is $\forall x \in]-\eta; \eta[: \langle \uparrow^{\mathsf{FCD}} (f \cap g) \rangle^* \{x\} \sqsupseteq \Delta$ and consequently $f \cap g \in \mathrm{GR}(\Delta \times^{\mathsf{FCD}} \Delta)$.

TODO: not yet written

Bibliography

[1] Andreas Blass. Two closed categories of filters. *Fundamenta Mathematicae*, 94(2):129–143, 1977.

[2] Nicolas Bourbaki. *Elements of Mathematics. General Topology. Part 1*. Addison-Wesley, 1966.

[3] Peter T. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

[4] Victor Porton. Filters on posets and generalizations. *International Journal of Pure and Applied Mathematics*, 74(1):55–119, 2012. http://www.mathematics21.org/binaries/filters.pdf.

[5] Victor Porton. *Algebraic General Topology. Volume 1*. 2014.

[6] Todd Trimble. https://ncatlab.org/toddtrimble/published/topogeny. nLab wiki. https://ncatlab.org/toddtrimble/published/topogeny.

[7] Todd Trimble et al. Please help with a proof that a category is monoidal. nForum. http://nforum.ncatlab.org/discussion/6765/please-help-with-a-proof-that-a-category-is-monoidal/.