Abstract

In this work I introduce and study in details the concepts of funcoids which generalize proximity spaces and reloids which generalize uniform spaces, and generalizations thereof. The concept of funcoid is generalized concept of proximity, the concept of reloid is cleared from superfluous details (generalized) concept of uniformity. Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets).

Also funcoids and reloids can be considered as a generalization of (oriented) graphs, this provides us with a common generalization of calculus and discrete mathematics.

The concept of continuity is defined by an algebraic formula (instead of old messy epsilon-delta notation) for arbitrary morphisms (including funcoids and reloids) of a partially ordered category. In one formula continuity, proximity continuity, and uniform continuity are generalized.

Also I define connectedness for funcoids and reloids.

Then I consider generalizations of funcoids: pointfree funcoids and generalization of pointfree funcoids: multifuncoids. Also I define several kinds of products of funcoids and other morphisms.

Before going to topology, this book studies properties of co-brouwerian lattices and filters.

Keywords: algebraic general topology, quasi-uniform spaces, generalizations of proximity spaces, generalizations of nearness spaces, generalizations of uniform spaces

A.M.S. subject classification: 54J05, 54A05, 54D99, 54E05, 54E15, 54E17, 54E99
# Table of contents

Abstract ...................................................... 3

1 Introduction ................................................ 11
  1.1 Draft status ............................................. 11
  1.2 Intended audience ........................................ 11
  1.3 Reading Order ........................................... 11
  1.4 Our topic and rationale .................................. 11
  1.5 Earlier works ............................................ 12
  1.6 Kinds of continuity ...................................... 13
  1.7 Structure of this book .................................... 13
  1.8 Basic notation .......................................... 13
     1.8.1 Grothendieck universes ............................. 5
     1.8.2 Misc ............................................. 14
  1.9 Unusual notation ........................................ 14

2 Common knowledge, part 1 ................................ 17
  2.1 Order theory ............................................. 17
     2.1.1 Posets ............................................. 17
         2.1.1.1 Intersecting and joining elements ............ 18
     2.1.2 Linear order ....................................... 18
     2.1.3 Meets and joins .................................... 18
     2.1.4 Semilattices ....................................... 20
     2.1.5 Lattices and complete lattices .................... 20
     2.1.6 Distributivity of lattices .......................... 21
     2.1.7 Difference and complement ........................ 21
     2.1.8 Boolean lattices .................................. 21
     2.1.9 Center of a lattice .................................. 22
     2.1.10 Atoms of posets ................................... 23
     2.1.11 Kuratowski’s lemma ................................ 5
     2.1.12 Homomorphisms of posets and lattices ............ 24
     2.1.13 Galois connections ................................ 25
     2.1.14 Co-Brouwerian lattices ............................ 27
     2.1.15 Dual pseudocomplement on co-Heyting lattices .... 29
  2.2 Intro to category theory ................................ 5
  2.3 Intro to group theory ................................... 5

3 More on order theory ....................................... 33
  3.1 Straight maps and separation subsets ................... 33
     3.1.1 Straight maps ....................................... 33
     3.1.2 Separation subsets and full stars ................. 34
     3.1.3 Atomically Separable Lattices ...................... 35
  3.2 Free Stars ................................................ 36
     3.2.1 Starrish posets ..................................... 37
  3.3 Quasidifference and Quasicomplement .................... 37
  3.4 Several equal ways to express pseudodifference ........ 39
  3.5 Partially ordered categories ............................ 40
     3.5.1 Partially ordered categories ....................... 40
3.5.2 Dagger categories ....................................... 40
3.5.2.1 Some special classes of morphisms .................. 41
3.6 Partitioning .............................................. 42
3.7 A proposition about binary relations ...................... 42
3.8 Infinite associativity and ordinated product .............. 43
  3.8.1 Introduction ......................................... 43
  3.8.2 Used notation ........................................ 43
    3.8.2.1 Currying and uncurrying .......................... 43
    The customary definition ................................. 43
    Currying and uncurrying with a dependent variable ...... 44
  3.8.2.2 Functions with ordinal numbers of arguments ...... 44
3.8.3 On sums of ordinals ................................... 44
3.8.4 Ordinated product ..................................... 45
  3.8.4.1 Introduction ...................................... 45
  3.8.4.2 Concatenation .................................... 45
  3.8.4.3 Finite example .................................... 45
  3.8.4.4 The definition .................................... 45
  3.8.4.5 Definition with composition for every multiplier .. 46
  3.8.4.6 Definition with shifting arguments ................ 46
  3.8.4.7 Associativity of ordinated product ................. 48
    Infinite associativity implies associativity ............. 48
    Concatenation is associative ............................ 48

4 Filters and filtrators ....................................... 51
4.1 Introduction to filters and filtrators .................... 51
  4.1.1 Filters on a set .................................... 51
  4.1.2 Intro to filters on a meet-semilattice ............... 52
  4.1.3 Intro to filters on a poset .......................... 52
  4.1.4 Intro to filtrators ................................ 52
4.2 Filtrators ................................................. 52
  4.2.1 Core Part ........................................... 53
  4.2.2 Filtrators with Separable Core ....................... 54
  4.2.3 Intersection and Joining with an Element of the Core 54
  4.2.4 Characterization of Finitely Meet-Closed Filtrators ... 55
  4.2.5 Stars of Elements of Filtrators ....................... 55
  4.2.6 Atomic Elements of a Filtrator ....................... 56
  4.2.7 Prime Filtrator Elements ............................ 56
  4.2.8 Some Criteria ....................................... 57
  4.2.9 Complements and Core Parts ........................ 58
  4.2.10 Core Part and Atomic Elements ...................... 60
  4.2.11 Distributivity of Core Part over Lattice Operations 60
  4.2.12 Co-Separability of Core ........................... 61
  4.2.13 Filtrators over Boolean Lattices .................... 61
    4.2.13.1 Distributivity for an Element of Boolean Core .. 61
4.3 Filters on a poset ......................................... 62
  4.3.1 Filters on posets ................................... 62
  4.3.2 Filters on meet-semilattices ........................ 62
  4.3.3 Order of filters. Principal filters ................. 63
    4.3.3.1 Minimal and maximal filters .................... 63
  4.3.4 Primary filtrator is filtered ....................... 64
  4.3.5 Alignment ........................................... 64
  4.3.6 Co-separability of Core for Primary Filtrators ...... 64
  4.3.7 Core Part ........................................... 64
  4.3.8 Intersecting and Joining with an Element of the Core 64
  4.3.9 Formulas for Meets and Joins of Filters ............ 65
  4.3.10 Separability of Core for Primary Filtrators .......... 66
| 4.3.11 | Distributivity of the Lattice of Filters | 66 |
| 4.3.12 | Filters over Boolean Lattices | 67 |
| 4.3.12.1 | Distributivity for an Element of Boolean Core | 67 |
| 4.3.13 | Generalized Filter Base | 67 |
| 4.3.14 | Stars for filters | 68 |
| 4.3.14.1 | Stars of Filters on Boolean Lattices | 68 |
| 4.3.15 | More about the Lattice of Filters | 70 |
| 4.3.16 | Atomic Filters | 70 |
| 4.3.17 | Some Criteria | 71 |
| 4.3.18 | Filters and a Special Sublattice | 72 |
| 4.3.19 | Core Part and Atomic Elements | 73 |
| 4.3.20 | Complements and Core Parts | 73 |
| 4.3.21 | Complementive Filters and Factoring by a Filter | 74 |
| 4.3.22 | Pseudodifference of filters | 76 |
| 4.4 | Filters on a Set | 76 |
| 4.4.1 | Fréchet Filter | 81 |
| 4.4.2 | Number of Filters on a Set | 82 |
| 4.5 | Some Counter-Examples | 83 |
| 4.5.1 | Weak and Strong Partition | 84 |
| 4.6 | Open problems about filters | 85 |
| 4.6.1 | Partitioning | 85 |
| 4.6.2 | Quasidifference | 87 |
| 4.6.3 | Non-Formal Problems | 87 |

5 Common knowledge, part 2 (topology) | 89

5.1 Metric spaces | 89
| 5.1.1 | Open and closed sets | 89 |
| 5.1.2 | Continuity | 90 |
| 5.2 | Pretopological spaces | 90 |
| 5.2.1 | Pretopology induced by a metric | 91 |
| 5.3 | Topological spaces | 91 |
| 5.3.1 | Relationships between pretopologies and topologies | 92 |
| 5.3.1.1 | Topological space induced by preclosure space | 92 |
| 5.3.1.2 | Preclosure space induced by topological space | 92 |
| 5.3.1.3 | Topology induced by a metric | 93 |
| 5.4 | Proximity spaces | 93 |

6 Funcoids | 95

6.1 Informal introduction into funcoids | 7
| 6.2 | Basic definitions | 96 |
| 6.2.1 | Composition of funcoids | 97 |
| 6.3 | Funcoid as continuation | 98 |
| 6.4 | Lattices of funcoids | 100 |
| 6.5 | More on composition of funcoids | 101 |
| 6.6 | Domain and range of a funcoid | 102 |
| 6.7 | Categories of funcoids | 104 |
| 6.8 | Specifying funcoids by functions or relations on atomic filters | 104 |
| 6.9 | Direct product of filters | 107 |
| 6.10 | Atomic funcoids | 109 |
| 6.11 | Complete funcoids | 111 |
| 6.12 | Funcoids corresponding to pretopologies | 114 |
| 6.13 | Completion of funcoids | 114 |
| 6.13.1 | More on completion of funcoids | 116 |
| 6.13.1.1 | Open maps | 117 |
| 6.14 | Monovalued and injective funcoids | 117 |
| 6.15 | $T_0$, $T_1$, $T_2$, and $T_3$-separable funcoids | 118 |
6.16 Filters closed regarding a funcoid .................................. 119

7 Reloids ................................................................. 121
7.1 Basic definitions .................................................. 121
7.2 Composition of reloids .......................................... 121
7.3 Direct product of filters ........................................ 123
7.4 Restricting reloid to a filter. Domain and image .......... 124
7.5 Categories of reloids ............................................. 126
7.6 Monovalued and injective reloids .............................. 126
7.7 Complete reloids and completion of reloids .................. 127

8 Relationships between funcoids and reloids .................... 131
8.1 Funcoid induced by a reloid .................................... 131
8.2 Reloids induced by a funcoid .................................. 134
8.3 Galois connections between funcoids and reloids ......... 8
8.4 Funcoidal reloids ................................................. 137

9 On distributivity of composition with a principal reloid .... 139
9.1 Decomposition of composition of binary relations .......... 8
9.2 Decomposition of composition of reloids ..................... 8
9.3 Lemmas for the main result .................................... 140
9.4 Proof of the main result ....................................... 140
9.5 Embedding reloids into funcoids .............................. 8

10 Continuous morphisms ........................................... 143
10.1 Traditional definitions of continuity ......................... 143
   10.1.1 Pretopology ............................................ 8
   10.1.2 Proximity spaces ...................................... 8
   10.1.3 Uniform spaces ....................................... 8
10.2 Our three definitions of continuity ......................... 8
10.3 Continuity of a restricted morphism ....................... 8

11 Connectedness regarding funcoids and reloids ............... 147
11.1 Some lemmas .................................................. 147
11.2 Endomorphism series ......................................... 147
11.3 Connectedness regarding binary relations .................. 148
11.4 Connectedness regarding funcoids and reloids ............ 149
11.5 Algebraic properties of $S$ and $S^*$ ....................... 151

12 Total boundness of reloids ..................................... 153
12.1 Thick binary relations ...................................... 153
12.2 Totally bounded endoreloids ................................ 154
12.3 Special case of uniform spaces ............................ 154
12.4 Relationships with other properties ........................ 155
12.5 Additional predicates ....................................... 155

13 Orderings of filters in terms of reloids ....................... 157
13.1 Equivalent filters ............................................ 157
13.2 Ordering of filters ........................................... 158
   13.2.1 Existence of no more than one monovalued injective reloid for a given pair of ultrafilters ....... 164
      13.2.1.1 The lemmas .................................... 164
      13.2.1.2 The main theorem and its consequences ....... 166
13.3 Rudin-Keisler equivalence and Rudin-Keisler order ....... 166
17.13 On products and projections .................................... 230  
 17.13.1 Staroidal product ............................................ 230  
 17.13.2 Cross-composition product of pointfree funcoids ........................................ 230  
 17.13.3 Subatomic product ........................................ 231  
 17.13.4 Other .................................................. 232  
17.14 Relationships between cross-composition and subatomic products ........................................ 232  
17.15 Coordinate-wise continuity ........................................ 10  
17.16 Counter-examples ............................................... 11  
17.17 Conjectures .................................................. 11  
  17.17.1 Informal questions ....................................... 11  

18 Identity staroids .................................................. 11  
  18.1 Additional propositions ....................................... 12  
  18.2 On pseudofuncoids ........................................... 13  
  18.3 Complete staroids and multifuncoids ......................... 13  
    18.3.1 Complete free stars .................................... 13  
    18.3.1.1 Completely starrish posets .......................... 13  
    18.3.2 More on free stars and complete free stars ............ 14  
  18.4 Complete staroids and multifuncoids ....................... 14  
  18.5 Identity staroids and multifuncoids ....................... 17  
    18.5.1 Identity relations .................................... 17  
    18.5.2 Universal definitions of identity staroids ............ 17  
    18.5.3 Identities are staroids ................................ 17  
    18.5.4 Special case of sets and filters ...................... 17  
    18.5.5 Relationships between big and small identity staroids ........................................ 17  
    18.5.6 Identity staroids on principal filters ................ 17  
    18.5.7 Identity staroids represented as meets and joins .... 17  
  18.6 Finite case ................................................ 17  
  18.7 Counter-examples and conjectures ........................... 17  

19 Postface ....................................................... 17  

Bibliography ....................................................... 17  

Index .......................................................... 18
Chapter 1
Introduction

For related materials, articles, research questions, and erratum consult the Web page of the author of the book:

1.1 Draft status
This is a draft.

1.2 Intended audience
This book is suitable for any math student as well as for researchers.

To make this book be understandable even for first grade students, I made a chapter about basic concepts (posets, lattices, topological spaces, etc.), which an already knowledgeable person may skip reading. It is assumed that the reader knows basic set theory.

But it is also valuable for mature researchers, as it contains much original research which you could not find in any other source except of my work.

Knowledge of the basic set theory is expected from the reader.

Despite that this book presents new research, it is well structured and is suitable to be used as a textbook for a college course.

Your comments about this book are welcome to the email porton@narod.ru.

1.3 Reading Order
If you know basic order and lattice theory (including Galois connections and brouwerian lattices) and basics of category theory, you may skip reading the chapter 2 (“Common knowledge, part 1”).

You are recommended to read the rest of this book by the order.

1.4 Our topic and rationale
From [38]: Point-set topology, also called set-theoretic topology or general topology, is the study of the general abstract nature of continuity or "closeness" on spaces. Basic point-set topological notions are ones like continuity, dimension, compactness, and connectedness.

In this work we study a new approach to point-set topology (and pointfree topology).

Traditionally general topology is studied using topological spaces (defined below in the section 5.3). I however argue that the theory of topological spaces is not the best method of studying general topology and introduce an alternative theory, the theory of funcoids. Despite of popularity of the theory of topological spaces it has some drawbacks and is in my opinion not the most appropriate formalism to study most of general topology. Because topological spaces are tailored for study of special sets, so called open and closed sets, studying general topology with topological spaces is a little anti-natural and ugly. In my opinion the theory of funcoids is more elegant than the theory of topological spaces, and it is better to study funcoids than topological spaces. One of the main purposes of this work is to present an alternative General Topology based on funcoids instead of being based on topological spaces as it is customary. In order to study funcoids the prior knowledge of topological spaces is not necessary. Nevertheless in this work I will consider topological spaces and the topic of interrelation of funcoids with topological spaces.
In fact funcoids are a generalization of topological spaces, so the well known theory of topological spaces is a special case of the below presented theory of funcoids.

But probably the most important reason to study funcoids is that funcoids are a generalization of proximity spaces (see section 2.2 for the definition of proximity spaces). Before this work it was written that the theory of proximity spaces was an example of a stalled research, almost nothing interesting was discovered about this theory. It was so because the proper way to research proximity spaces is to research their generalization, funcoids. And so it was stalled until discovery of funcoids. That generalized theory of proximity spaces will bring us yet many interesting results.

In addition to funcoids I research reloids. Using below defined terminology it may be said that reloids are (basically) filters on Cartesian product of sets, and this is a special case of uniform spaces. We don’t need to define uniform spaces in this work, it is enough for the reader just to know that uniform spaces are certain filters on direct product of sets.

Afterward we study some generalizations.

Somebody might ask, why to study it? My approach relates to traditional general topology like complex numbers to real numbers theory. Be sure this will find applications.

This book has a deficiency: It does not properly relate my theory with previous research in general topology and does not consider deeper category theory properties. It is however OK for now, as I am going to do this study in later volumes (continuation of this book).

Many proofs in this book may seem too easy and thus this theory not sophisticated enough. But it is largely a result of a well structured digraph of proofs, where more difficult results are made easy by reducing them to easier lemmas and propositions.

1.5 Earlier works

Some mathematicians were researching generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Proximity structures were introduced by Smirnov in [11].

Some references to predecessors:

- In [14], [15], [24], [2], [33] generalized uniformities and proximities are studied.
- Proximities and uniformities are also studied in [21], [22], [32], [34], [35].

Some works ([31]) about proximity spaces consider relationships of proximities and compact topological spaces. In this work the attempt to define or research their generalization, compactness of funcoids or reloids is not done. It seems potentially productive to attempt to borrow the definitions and procedures from the above mentioned works. I hope to do this study in a separate volume.

[10] studies mappings between proximity structures. (In this volume no attempt to research mappings between funcoids is done.) [25] researches relationships of quasi-uniform spaces and topological spaces. [1] studies how proximity structures can be treated as uniform structures and compactification regarding proximity and uniform spaces.

This book is based partially on my articles [29], [27], [28]. [TODO: Add more references to my articles.]

In [29] I introduced the concept of filter objects. This was probably not a very good idea. In this work I instead use plain filters (not filter objects) and \( \sqcup \) and \( \sqcap \) notation for joins and meets instead of \( \cup \) and \( \cap \), which may be confused with set theoretic operations, for lattices in consideration (and for the lattice of filters the order is reverse to the set theoretic inclusion). Also this work differs from [29] in using in some formulations the lattice of principal filters which is isomorphic to the base poset instead of using the base poset itself (what was possible in [29] thanks to using filter objects). I’ve replaced \( (\mathfrak{g}; \mathfrak{X}) \) notation for primary filtrators with \( (\mathfrak{g}; \mathfrak{X}) \) for consistency of notation among sections.
1.6 Kinds of continuity

A research result based on this book but not fully included in this book (and not yet published) is that the following kinds of continuity are described by the same algebraic (or rather categorical) formulas for different kinds of continuity and have common properties:

- discrete continuity (between digraphs);
- (pre)topological continuity;
- proximal continuity;
- uniform continuity;
- Cauchy continuity;
- (probably other kinds of continuity).

Thus my research justifies using the same word “continuity” for these diverse kinds of continuity. See http://www.mathematics21.org/algebraic-general-topology.html

1.7 Structure of this book

In the chapter 2 “Common knowledge, part 1” some well known definitions and theories are considered. You may skip its reading if you already know it. That chapter contains info about:

- posets;
- lattices and complete lattices;
- Galois connections;
- co-brouwerian lattices;
- a very short intro into category theory (It is very basic, I even don’t define functors as they have no use in my theory);
- a very short introduction to group theory.

Afterward there are my little additions to poset/lattice and category theory. Afterward there is the theory of filters and filtrators. Then there is “Common knowledge, part 2 (topology)”, which considers briefly:

- metric spaces;
- topological spaces;
- pretopological spaces;
- proximity spaces.

Despite of the name “Common knowledge” this second common knowledge chapter is recommended to be read completely even if you know topology well, because it contains some rare theorems not known to most mathematicians and hard to find in literature.

Then the most interesting thing in this book, the theory of funcoids, starts. Afterwards there is the theory of reloids. Then I show relationships between funcoids and reloids. The last I research generalizations of funcoids, pointfree funcoids, staroids and multifuncoids and some different kinds of products of morphisms.

1.8 Basic notation

1.8.1 Grothendieck universes

We will work in ZFC with an infinite and uncountable Grothendieck universe.
A Grothendieck universe is just a set big enough to make all usual set theory inside it. For example if $\mathcal{U}$ is a Grothendieck universe, and sets $X, Y \in \mathcal{U}$, then also $X \cup Y \in \mathcal{U}$, $X \cap Y \in \mathcal{U}$, $X \times Y \in \mathcal{U}$, etc.

A set which is a member of a Grothendieck universe is called a small set (regarding this Grothendieck universe). We can restrict our consideration to small sets in order to get rid troubles with proper classes.

**Definition 1.1.** Grothendieck universe is a set $\mathcal{U}$ such that:

1. If $x \in \mathcal{U}$ and $y \in x$ then $y \in \mathcal{U}$.
2. If $x, y \in \mathcal{U}$, then $\{x, y\} \in \mathcal{U}$.
3. If $x \in \mathcal{U}$ then $\mathcal{U} \times x \in \mathcal{U}$.
4. If $\{x_i \mid i \in I \in \mathcal{U}\}$ is a family of elements of $\mathcal{U}$, then $\bigcup_{i \in I} x_i \in \mathcal{U}$.

One can deduce from this also:

1. If $x \in \mathcal{U}$, then $\{x\} \in \mathcal{U}$.
2. If $x$ is a subset of $y \in \mathcal{U}$, then $x \in \mathcal{U}$.
3. If $x, y \in \mathcal{U}$ then the ordered pair $(x; y) = \{\{x, y\}, x\} \in \mathcal{U}$.
4. If $x, y \in \mathcal{U}$ then $x \cup y$ and $x \times y$ are in $\mathcal{U}$.
5. If $\{x_i \mid i \in I \in \mathcal{U}\}$ is a family of elements of $\mathcal{U}$, then the product $\prod_{i \in I} x_i \in \mathcal{U}$.
6. If $x \in \mathcal{U}$, then the cardinality of $x$ is strictly less than the cardinality of $\mathcal{U}$.

1.8.2 Misc

In this book quantifiers bind tightly. That is $\forall x \in A: P(x) \land Q$ and $\forall x \in A: P(x) \Rightarrow Q$ should be read $\forall x \in A: P(x)) \land Q$ and $\forall x \in A: P(x)) \Rightarrow Q$ not $\forall x \in A: (P(x) \land Q)$ and $\forall x \in A: (P(x) \Rightarrow Q)$.

The set of functions from a set $A$ to a set $B$ is denoted as $B^A$.

I will often skip parentheses and write $fx$ instead of $f(x)$ to denote the result of a function $f$ acting on the argument $x$.

I will denote $\langle f \rangle X = \{f \alpha \mid \alpha \in X\}$ and $X[f]Y \Rightarrow \exists x \in X, y \in Y: x f y$ for sets $X, Y$ and a binary relation $f$. (Note that functions are a special case of binary relations.)

By just $\langle f \rangle$ and $[f]$ I will denote the corresponding function and relation on small sets.

$\lambda x \in D: f(x) = \{\{x; f(x)\} \mid x \in D\}$ for a set $D$ and and a form $f$ depending on the variable $x$.

I will denote source and destination of a morphism $f$ of any category (See “Common knowledge, part 1” chapter for a definition of a category.) as $\text{Src} f$ and $\text{Dst} f$ correspondingly. Note that below defined domain and image of a funcoid are not the same as it source and destination.

I will denote $\text{GR}(A; B; f) = f$ for any morphism $(A; B; f)$ of either Set or Rel.

I will denote $\langle f \rangle = \langle \text{GR} f \rangle$ and $[f] = [\text{GR} f]$ for any morphism $f$ of either Set or Rel.

1.9 Unusual notation

In the chapter 2 (which you may skip reading if you are already knowledgeable) some non-standard notation is defined. I summarize here this notation for the case if you choose to skip reading that chapter:

- Partial order is denoted as $\subseteq$.
- Meets and joins are denoted as $\cap$, $\cup$, $\bigcap$, $\bigcup$.
- I call element $b$ substractive from an elements $a$ (of a distributive lattice $\mathfrak{A}$) when the difference $a \setminus b$ exists. I call $b$ complementive to $a$ when there exist $c \in \mathfrak{A}$ such that $b \cap c = 0$ and $b \cup c = a$. We will prove that $b$ is complementive to $a$ iff $b$ is substractive from $a$ and $b \subseteq a$.

**Definition 1.2.** Call $a$ and $b$ of a poset $\mathfrak{A}$ intersecting, denoted $a \not\perp b$, when there exists a non-least element $c$ such that $c \subseteq a \land c \subseteq b$. 


Definition 1.3. $a \succ b \iff \neg (a \not\succ b)$.

Definition 1.4. I call elements $a$ and $b$ of a poset $\mathfrak{A}$ joining and denote $a \equiv b$ when there are no non-greatest element $c$ such that $c \supseteq a \land c \supseteq b$.

Definition 1.5. $a \not\equiv b \iff \neg (a \equiv b)$.

Obvious 1.6. $a \not\succ b$ iff $a \sqcap b$ is non-least, for every elements $a$, $b$ of a meet-semilattice.

Obvious 1.7. $a \equiv b$ if $a \sqcup b$ is the greatest element, for every elements $a$, $b$ of a join-semilattice.

I extend the definitions of pseudocomplement and dual pseudocomplement to arbitrary posets (not just lattices as it is customary):

Definition 1.8. Let $\mathfrak{A}$ be a poset. Pseudocomplement of $a$ is

$$\max \{ c \in \mathfrak{A} \mid c \succeq a \}.$$ 

If $z$ is the pseudocomplement of $a$ we will denote $z = a^\ast$.

Definition 1.9. Let $\mathfrak{A}$ be a poset. Dual pseudocomplement of $a$ is

$$\min \{ c \in \mathfrak{A} \mid c \preceq a \}.$$ 

If $z$ is the dual pseudocomplement of $a$ we will denote $z = a^+$. 
Chapter 2
Common knowledge, part 1

In this chapter we will consider some well known mathematical theories. If you already know them you may skip reading this chapter (or its parts).

2.1 Order theory

2.1.1 Posets

Definition 2.1. The identity relation on a set $A$ is $\text{id}_A = \{(a; a) \mid a \in A\}$.

Definition 2.2. A preorder on a set $A$ is a binary relation $\sqsubseteq$ which is:
- reflexive on $A$ ($\sqsubseteq \supseteq \text{id}_A$);
- transitive ($\sqsubseteq \circ \sqsubseteq \subseteq \sqsubseteq$).

Definition 2.3. A partial order on a set $A$ is a preorder on $A$ which is antisymmetric ($\preceq \cap \preceq^{-1} \subseteq \equiv$).

The reverse relation is denoted $\sqsupseteq$.

Definition 2.4. $a$ is a subelement of $b$ (or what is the same $a$ is contained in $b$ or $b$ contains $a$) iff $a \sqsubseteq b$.

Obvious 2.5. The reverse of a partial order is also a partial order.

Definition 2.6. A poset is a set $A$ together with a partial order on it is called a partially ordered set (poset for short).

Definition 2.7. Strict partial order $\subsetneq$ corresponding to the partial order $\subseteq$ on a set $A$ is defined by the formula $\subsetneq = (\subseteq) \setminus \text{id}_A$.

Definition 2.8. A partial order on a set $A$ restricted to a set $B \subseteq A$ is $\subseteq \cap (B \times B)$.

Obvious 2.9. A partial order on a set $A$ restricted to a set $B \subseteq A$ is a partial order on $B$.

Definition 2.10.
- The least element $0$ of a poset $\mathfrak{A}$ is defined by the formula $\forall a \in \mathfrak{A}: 0 \subseteq a$.
- The greatest element $1$ of a poset $\mathfrak{A}$ is defined by the formula $\forall a \in \mathfrak{A}: 1 \supseteq a$.

Proposition 2.11. There exist no more than one least element and no more than one greatest element (for a given poset).

Proof. By antisymmetry. $\square$

Definition 2.12. The dual order for $\subseteq$ is $\supseteq$. 

17
Obvious 2.13. Dual of a partial order is a partial order.

Definition 2.14. The dual poset for a poset \((A; \sqsubseteq)\) is the poset \((A; \sqsupseteq)\).

Below we will sometimes use duality that is replacement of the partial order and all related operations and relations with their duals. In other words, it is enough to prove a theorem for an order \(\sqsubseteq\) and the similar theorem for \(\sqsupseteq\) follows by duality.

2.1.1 Intersecting and joining elements

Let \(A\) be a poset.

Definition 2.15. Call a and b of \(A\) intersecting, denoted \(a \not\sqsupseteq b\), when there exists a non-least element \(c\) such that \(c \sqsubseteq a \land c \sqsubseteq b\).

Definition 2.16. \(a \not\sqsupseteq b \overset{\text{def}}{=} \neg(a \not\sqsubseteq b)\).

Obvious 2.17. \(a_0 \not\sqsupseteq b_0 \land a_1 \sqsupseteq a_0 \land b_1 \sqsupseteq b_0 \Rightarrow a_1 \not\sqsupseteq b_1\).

Definition 2.18. I call elements a and b of \(A\) joining and denote \(a \equiv b\) when there are no non-greatest element \(c\) such that \(c \sqsupseteq a \land c \sqsupseteq b\).

Definition 2.19. \(a \not\equiv b \overset{\text{def}}{=} \neg(a \equiv b)\).

Obvious 2.20. Intersecting is the dual of non-joining.

Obvious 2.21. \(a_0 \equiv b_0 \land a_1 \sqsubseteq a_0 \land b_1 \sqsubseteq b_0 \Rightarrow a_1 \equiv b_1\).

2.1.2 Linear order

Definition 2.22. A poset \(A\) is called linearly ordered set (or what is the same, totally ordered set) if \(a \sqsubseteq b \lor b \sqsubseteq a\) for every \(a, b \in A\).

Example 2.23. The set of real numbers with the customary order is a linearly ordered set.

Definition 2.24. A set \(X \in \mathcal{P}A\) where \(A\) is a poset is called a chain if \(A\) restricted to \(X\) is a total order.

2.1.3 Meets and joins

Let \(A\) be a poset.

Definition 2.25. Given a set \(X \in \mathcal{P}A\) the least element (also called minimum and denoted min \(X\)) of \(X\) is such \(a \in X\) that \(\forall x \in X : a \sqsubseteq x\).

Least element does not necessarily exists. But if it exists:

Proposition 2.26. For a given \(X \in \mathcal{P}A\) there exist no more than one least element.

Proof. It follows from anti-symmetry.

Greatest element is the dual of least element:

Definition 2.27. Given a set \(X \in \mathcal{P}A\) the greatest element (also called maximum and denoted max \(X\)) of \(X\) is such \(a \in X\) that \(\forall x \in X : a \sqsupseteq x\).

Remark 2.28. Least and greatest elements of a set \(X\) is a trivial generalization of the above defined least and greatest element for the entire poset.
Definition 2.29.

- A **minimal** element of a set \( X \in \mathcal{P}A \) is such an \( a \in A \) that \( \exists x \in X : (a \sqsupset x \land x \neq a) \).
- A **maximal** element of a set \( X \in \mathcal{P}A \) is such an \( a \in A \) that \( \exists x \in X : (a \sqsubseteq x \land x \neq a) \).

Remark 2.30. Minimal element is not the same as minimum, and maximal element is not the same as maximum.

Obvious 2.31.

1. The least element (if it exists) is a minimal element.
2. The greatest element (if it exists) is a maximal element.

Exercise 2.1. Show that there may be more than one minimal and more than one maximal element for some poset.

Definition 2.32. **Upper bounds** of a set \( X \) is the set \( \{ y \in A \mid \forall x \in X : y \sqsupset x \} \).

The dual notion:

Definition 2.33. **Lower bounds** of a set \( X \) is the set \( \{ y \in A \mid \forall x \in X : y \sqsubseteq x \} \).

Definition 2.34. **Join** \( \bigcup X \) (also called supremum and denoted “sup \( X \)”) of a set \( X \) is the least element of its upper bounds (if it exists).

Definition 2.35. **Meet** \( \bigcap X \) (also called infimum and denoted “inf \( X \)”) of a set \( X \) is the greatest element of its lower bounds (if it exists).

We will write \( b = \bigcup X \) when \( b \in A \) is the join of \( X \) or say that \( \bigcup X \) does not exist if there are no such \( b \in A \). (And dually for meets.)

Exercise 2.2. Provide an example of \( \bigcup X \notin X \) for some set \( X \) on some poset.

I will denote meets and joins for a specific poset \( A \) as \( \bigcap^A \) and \( \bigcup^A \).

Proposition 2.36.

1. If \( b \) is the greatest element of \( X \) then \( \bigcup X = b \).
2. If \( b \) is the least element of \( X \) then \( \bigcap X = b \).

Proof. We will prove only the first as the second is dual.

Let \( b \) be the greatest element of \( X \). Then upper bounds of \( X \) are \( \{ y \in A \mid y \sqsupset b \} \). Obviously \( b \) is the least element of this set, that is the join. \( \square \)

Definition 2.37. **Binary joins and meets** are defined by the formulas

\[
x \sqcup y = \bigcup \{x, y\} \quad \text{and} \quad x \sqcap y = \bigcap \{x, y\}.
\]

Obvious 2.38. \( \sqcup \) and \( \sqcap \) are symmetric operations (whenever these are defined for given \( x \) and \( y \)).

Theorem 2.39.

1. If \( \bigcup X \) exists then \( y \sqsupset \bigcup X \iff \forall x \in X : y \sqsupset x \).
2. If \( \bigcap X \) exists then \( y \sqsubseteq \bigcap X \iff \forall x \in X : y \sqsubseteq x \).

Proof. I will prove only the first as the second follows by duality.

\( y \sqsupset \bigcup X \iff y \) is an upper bound for \( X \iff \forall x \in X : y \sqsupset x \). \( \square \)

Corollary 2.40.

1. If \( a \sqcup b \) exists then \( y \sqsupset a \sqcup b \iff y \sqsupset a \land y \sqsupset b \).
2. If \( a \cap b \) exists then \( y \subseteq a \cap b \iff y \subseteq a \land y \subseteq b \).

### 2.1.4 Semilattices

**Definition 2.41.**

1. A **join-semilattice** is a poset \( \mathfrak{A} \) such that \( a \sqcup b \) is defined for every \( a, b \in \mathfrak{A} \).
2. A **meet-semilattice** is a poset \( \mathfrak{A} \) such that \( a \sqcap b \) is defined for every \( a, b \in \mathfrak{A} \).

**Theorem 2.42.**

1. The operation \( \sqcup \) is associative for any join-semilattice.
2. The operation \( \sqcap \) is associative for any meet-semilattice.

**Proof.** I will prove only the first as the second follows by duality.

We need to prove \( (a \sqcup b) \sqcup c = a \sqcup (b \sqcup c) \) for every \( a, b, c \in \mathfrak{A} \).

Taking into account the definition of join, it is enough to prove that

\[
x \sqsupseteq (a \sqcup b) \sqcup c \iff x \sqsupseteq a \sqcup (b \sqcup c)
\]

for every \( x \in \mathfrak{A} \). Really, this follows from the chain of equivalences:

\[
x \sqsupseteq (a \sqcup b) \sqcup c \iff x \sqsupseteq (a \sqcup b) \land x \sqsupseteq c \iff x \sqsupseteq a \land x \sqsupseteq b \land x \sqsupseteq c \iff x \sqsupseteq a \land x \sqsupseteq b \sqcup c \iff x \sqsupseteq a \sqcup (b \sqcup c).
\]

**Obvious 2.43.** \( a \not\approx b \) if \( a \sqcap b \) is non-least, for every elements \( a, b \) of a meet-semilattice.

**Obvious 2.44.** \( a \equiv b \) if \( a \sqcup b \) is the greatest element, for every elements \( a, b \) of a join-semilattice.

### 2.1.5 Lattices and complete lattices

**Definition 2.45.** A **bounded** poset is a poset having both least and greatest elements.

**Definition 2.46.** **Lattice** is a poset which is both join-semilattice and meet-semilattice.

**Definition 2.47.** A **complete lattice** is a poset \( \mathfrak{A} \) such that for every \( X \in \mathcal{P}\mathfrak{A} \) both \( \bigsqcup X \) and \( \bigsqcap X \) exist.

**Obvious 2.48.** Every complete lattice is a lattice.

**Proposition 2.49.** Every complete lattice is a bounded poset.

**Proof.** \( \bigsqcup \emptyset \) is the least and \( \bigsqcap \emptyset \) is the greatest element. \( \square \)

**Theorem 2.50.** Let \( \mathfrak{A} \) be a poset.

1. If \( \bigsqcup X \) is defined for every \( X \in \mathcal{P}\mathfrak{A} \), then \( \mathfrak{A} \) is a complete lattice.
2. If \( \bigsqcap X \) is defined for every \( X \in \mathcal{P}\mathfrak{A} \), then \( \mathfrak{A} \) is a complete lattice.

**Proof.** See [26] or any lattice theory reference. \( \square \)

**Obvious 2.51.** If \( X \subseteq Y \) for some \( X, Y \in \mathcal{P}\mathfrak{A} \) where \( \mathfrak{A} \) is a complete lattice, then

1. \( \bigsqcup X \subseteq \bigsqcup Y \);
2. \( \bigsqcap X \supseteq \bigsqcap Y \).

**Proposition 2.52.** If \( S \in \mathcal{P}\mathfrak{A} \) then for every complete lattice \( \mathfrak{A} \)

1. \( \bigsqcup S = \bigsqcup \{ \bigsqcup X \mid X \in S \} \);
2. \( \bigsqcap S = \bigsqcap \{ \bigsqcap X \mid X \in S \} \).
2.1 Order theory


\[
\text{Theorem 2.54. For a lattice to be distributive it is enough just one of the conditions:}
\]

1. \(x \cap (y \cup z) = (x \cap y) \cup (x \cap z)\);
2. \(x \cup (y \cap z) = (x \cup y) \cap (x \cup z)\).

\[
\text{Proof.} \quad (x \cup y) \cap (x \cup z) = ((x \cup y) \cap x) \cup ((x \cup y) \cap z) = x \cup ((x \cap z) \cup (y \cap z)) = (x \cup (x \cap z)) \cup (y \cap z) = x \cup (y \cap z) \quad (\text{applied } x \cap (y \cup z) = (x \cap y) \cup (x \cap z) \text{ twice}).
\]

2.1.7 Difference and complement

Definition 2.55. Let \(\mathfrak{A}\) be a distributive lattice with least element 0. The difference (denoted \(a \setminus b\)) of elements \(a\) and \(b\) is such \(c \in \mathfrak{A}\) that \(b \cap c = 0\) and \(a \cup b = b \cup c\). I will call \(b\) substractive from \(a\) when \(a \setminus b\) exists.

Theorem 2.56. If \(\mathfrak{A}\) is a distributive lattice with least element 0, there exists no more than one difference of elements \(a, b\).

\[
\text{Proof.} \quad \text{Let } c \text{ and } d \text{ be both differences } a \setminus b. \text{ Then } b \cap c = b \cap d = 0 \text{ and } a \cup b = b \cup c = b \cup d. \text{ So }
\]

\[
c = c \cap (b \cup c) = c \cap (b \cup d) = (c \cap b) \cup (c \cap d) = 0 \cup (c \cap d) = c \cap d.
\]

Similarly \(d = d \cap c\). Consequently \(c = c \cap d = d \cap c = d\). □

Definition 2.57. I will call \(b\) complementive to \(a\) iff there exists \(c \in \mathfrak{A}\) such that \(b \cap c = 0\) and \(b \cup c = a\).

Proposition 2.58. \(b\) is complementive to \(a\) iff \(b\) is substractive from \(a\) and \(b \subseteq a\).

\[
\text{Proof.}
\]

\[
\Leftrightarrow. \quad \text{Obvious.}
\]

\[
\Rightarrow. \quad \text{We deduce } b \subseteq a \text{ from } b \cup c = a. \text{ Thus } a \cup b = a = b \cup c.
\]

Proposition 2.59. If \(b\) is complementive to \(a\) then \((a \setminus b) \cup b = a\).

\[
\text{Proof.} \quad \text{Because } b \subseteq a \text{ by the previous proposition.}
\]

Definition 2.60. Let \(\mathfrak{A}\) be a bounded distributive lattice. The complement (denoted \(\bar{a}\)) of an element \(a \in \mathfrak{A}\) is such \(b \in \mathfrak{A}\) that \(a \cap b = 0\) and \(a \cup b = 1\).

Proposition 2.61. If \(\mathfrak{A}\) is a bounded distributive lattice then \(\bar{a} = 1 \setminus a\).

\[
\text{Proof.} \quad b = \bar{a} \iff b \cap a = 0 \land b \cup a = 1 \iff b \cap a = 0 \land 1 \cup a = a \cup b \iff b = 1 \setminus a.
\]

Corollary 2.62. If \(\mathfrak{A}\) is a bounded distributive lattice then exists no more than one complement of an element \(a \in \mathfrak{A}\).
**Definition 2.63.** An element of bounded distributive lattice is called *complemented* when its complement exists.

**Definition 2.64.** A distributive lattice is a *complemented lattice* iff every its element is complemented.

**Proposition 2.65.** For a distributive lattice \((a \setminus b) \setminus c = a \setminus (b \sqcup c)\) if \(a \setminus b\) and \((a \setminus b) \setminus c\) are defined.

**Proof.** \((a \setminus b) \setminus c = a \setminus (b \sqcup c)\) if \(a \setminus b\) and \((a \setminus b) \setminus c\) are defined.

We need to prove \(((a \setminus b) \setminus c) \cap (b \sqcup c) = 0\) and \(((a \setminus b) \setminus c) \cup (b \sqcup c) = a \sqcup (b \sqcup c)\).

In fact,
\[
\begin{align*}
((a \setminus b) \setminus c) \cap (b \sqcup c) &= \emptyset \\
(((a \setminus b) \setminus c) \cap b) \sqcup (((a \setminus b) \setminus c) \cap c) &= \emptyset \\
(((a \setminus b) \setminus c) \cap b) \sqcup 0 &= \emptyset \\
((a \setminus b) \setminus c) \cap b &= 0,
\end{align*}
\]
so \(((a \setminus b) \setminus c) \cap (b \sqcup c) = 0;\)
\[
\begin{align*}
((a \setminus b) \setminus c) \cup (b \sqcup c) &= a \sqcup (b \sqcup c) \\
(((a \setminus b) \setminus c) \cup c) \cap b &= (a \setminus b) \sqcup c \cap b \\
((a \setminus b) \setminus b) \cup c &= a \setminus b \cup c.
\end{align*}
\]

\[\square\]

### 2.1.8 Boolean lattices

**Definition 2.66.** A *boolean lattice* is a complemented distributive lattice.

The most important example of a boolean lattice is \(\mathcal{P}A\) where \(A\) is a set, ordered by set inclusion.

**Theorem 2.67.** (De Morgan’s laws) For every elements \(a, b\) of a boolean lattice

1. \(\overline{a \sqcup b} = \overline{a} \cap \overline{b}\);
2. \(\overline{a \sqcap b} = \overline{a} \cup \overline{b}\).

**Proof.** We will prove only the first as the second is dual.

It is enough to prove that \(a \sqcup b\) is a complement of \(\overline{a} \cap \overline{b}\). Really:
\[
\begin{align*}
(a \sqcup b) \cap (\overline{a} \cap \overline{b}) &\subseteq a \cap (\overline{a} \cap \overline{b}) = (a \cap \overline{a}) \cap \overline{b} = 0 \cap \overline{b} = 0; \\
(a \sqcup b) \cup (\overline{a} \cap \overline{b}) &\subseteq ((a \sqcup b) \cup \overline{a}) \cap ((a \sqcup b) \cup \overline{b}) \supseteq (a \sqcup \overline{a}) \cap (b \cup \overline{b}) = 1 \cap 1 = 1.
\end{align*}
\]
Thus \((a \sqcup b) \cap (\overline{a} \cap \overline{b}) = 0\) and \((a \sqcup b) \cap (\overline{a} \cap \overline{b}) = 1\).

**Definition 2.68.** A complete lattice \(A\) is *join infinite distributive* when \(x \sqcup \bigsqcup S = \bigsqcup (x \sqcup )S;\) complete lattice is *meet infinite distributive* when \(x \sqcap \bigsqcup S = \bigsqcup (x \sqcap )S\) for all \(x \in A\) and \(S \in \mathcal{P}A\).

**Definition 2.69.** *Infinite distributive complete lattice* is a complete lattice which is both join infinite distributive and meet infinite distributive.

**Theorem 2.70.** Every complete boolean lattice is both join infinite distributive and meet infinite distributive.

**Proof.** We will prove only join infinitely distributivity, as the other is dual.

Let \(S\) be a subset of a complete boolean lattice.
2.1 Order theory

Denition 2.80.

2.1.10 Atoms of posets

Proof. Theorem 2.79.

So Remark 2.77.

Proof. Theorem 2.76.

Denition 2.75.

Remark 2.74.

Remark 2.73.

Remark 2.72. The center \( Z(\mathfrak{A}) \) of a bounded distributive lattice \( \mathfrak{A} \) is the set of its complemented elements.

Remark 2.71. (infinite De Morgan’s laws) For every subset \( S \) of a complete boolean lattice

1. \( \bigcup S = \bigcap x \in S \bar{a} \)
2. \( \bigcap \bar{S} = \bigcup x \in S \bar{a} \).

Proof. It’s enough to prove that \( \bigcup S \) is a complement of \( \bigcap x \in S \bar{a} \) (the second follows from duality). Really, using the previous theorem:

\[
\bigcup S \bigcup \bigcap x \in S \bar{a} = \bigcap x \in S \bar{a} \bigcup \bigcup x \in S \bar{a} \bigcup x \in S \{ x \cup \bar{a} \mid x \in S \} = 1;
\]

\[
\bigcup S \bigcap y \in S \bar{a} = \bigcap x \in S \bigcap y \in S \{ x \cap y \mid y \in S \} = 0.
\]

So \( \bigcup S \bigcap y \in S \bar{a} = 1 \) and \( \bigcup S \bigcap x \in S \bar{a} = 0. \)

2.1.9 Center of a lattice

Definition 2.72. The center \( Z(\mathfrak{A}) \) of a bounded distributive lattice \( \mathfrak{A} \) is the set of its complemented elements.

Remark 2.73. For a definition of center of non-distributive lattices see [5].

Remark 2.74. In [23] the word center and the notation \( Z(\mathfrak{A}) \) are used in a dierent sense.

Definition 2.75. A sublattice \( K \) of a complete lattice \( L \) is a closed sublattice of \( L \) if \( K \) contains the meet and the join of any its nonempty subset.

Theorem 2.76. Center of an innitely distributive lattice is its closed sublattice.

Proof. See [16].

Remark 2.77. See [17] for a more strong result.

Theorem 2.78. The center of a bounded distributive lattice constitutes its sublattice.

Proof. Let \( \mathfrak{A} \) be a bounded distributive lattice and \( Z(\mathfrak{A}) \) be its center. Let \( a, b \in Z(\mathfrak{A}) \). Consequently \( a, b \in Z(\mathfrak{A}) \). Then \( a \sqcup b \) is the complement of \( a \sqcap b \) because

\[
(a \sqcap b) \sqcap (a \sqcup b) = (a \sqcap b \sqcap a) \cup (a \sqcap b \sqcap b) = 0 \sqcup 0 = 0 \quad \text{and} \\
(a \sqcap b) \cup (a \sqcup b) = (a \sqcup b \sqcup a) \cap (b \sqcup a \sqcup b) = 1 \sqcap 1 = 1.
\]

So \( a \sqcap b \) is complemented. Similarly \( a \sqcup b \) is complemented.

Theorem 2.79. The center of a bounded distributive lattice constitutes a boolean lattice.

Proof. Because it is a distributive complemented lattice.

2.1.10 Atoms of posets

Definition 2.80. An atom of a poset is an element which has no non-least subelements.
Remark 2.81. This definition is valid even for posets without least element.

I will denote \( \text{atoms}^A a \) or just \( \text{atoms} a \) the set of atoms contained in an element \( a \) of a poset \( \mathfrak{A} \). I will denote \( \text{atoms}^A \) the set of all atoms of a poset \( \mathfrak{A} \).

Definition 2.82. A poset \( \mathfrak{A} \) is called \textit{atomic} iff \( \text{atoms} a \neq \emptyset \) for every non-least element \( a \) of the poset \( \mathfrak{A} \).

Definition 2.83. \textit{Atomistic poset} is such a poset that \( a = \bigcup \text{atoms} a \) for every non-least element \( a \) of this poset.

Obvious 2.84. Every atomistic poset is atomic.

Proposition 2.85. Let \( \mathfrak{A} \) be a poset. If \( a \) is an atom of \( \mathfrak{A} \) and \( B \subseteq \mathfrak{A} \) then \( a \sqsubseteq B \leftrightarrow a \neq B \).

Proof.
\[
\Rightarrow. \ a \subseteq B \Rightarrow a \sqsubseteq a \land a \sqsubseteq B, \text{ thus } a \neq B \text{ because } a \text{ is not least.}\\
\Leftarrow. \ a \neq B \text{ implies existence of non-least element } x \text{ such that } x \sqsubseteq B \text{ and } x \sqsubseteq a. \text{ Because } a \text{ is an atom, we have } x = a. \text{ So } a \subseteq B. \quad \square
\]

Theorem 2.86. \( \bigcap S = \bigcap \{\text{atoms}\} S \) whenever \( \bigcap S \) is defined for every \( S \subseteq \mathfrak{P} \mathfrak{A} \) where \( \mathfrak{A} \) is a poset.

Proof. For any atom \( c \)
\[
\begin{align*}
c \in \text{atoms} & \iff c \subseteq \bigcap S \\
\forall a \in S: c \subseteq a & \iff \\
\forall a \in S: c \in \text{atoms} a & \iff \\
c \in S & \iff c \in \bigcap \{\text{atoms}\} S.
\end{align*}
\]

\( \square \)

Corollary 2.87. \( \text{atoms}(a \cap b) = \text{atoms} a \cap \text{atoms} b \) for an arbitrary meet-semilattice.

Theorem 2.88. A complete boolean lattice is atomic iff it is atomistic.

Proof.
\[
\Leftarrow. \text{ Obvious.}\\
\Rightarrow. \text{ Let } \mathfrak{A} \text{ be an atomic boolean lattice. Let } a \in \mathfrak{A}. \text{ Suppose } b = \bigcup \text{atoms} a \sqsubseteq a. \text{ If } x \in \text{atoms}(a \setminus b) \text{ then } x \subseteq a \setminus b \text{ and so } x \subseteq a \text{ and hence } x \sqsubseteq b. \text{ But we have } x = x \setminus b \subseteq (a \setminus b) \cap b = 0 \text{ what contradicts to our supposition.} \quad \square
\]

2.11 Kuratowski’s lemma

Theorem 2.89. (Kuratowski lemma) Any chain in a poset is contained in a maximal chain (if we order chains by inclusion).

I will skip the proof of Kuratowski lemma as this proof can be found in any set theory or order theory reference.

2.12 Homomorphisms of posets and lattices

Definition 2.90. A \textit{monotone} function (also called \textit{order homomorphism}) from a poset \( \mathfrak{A} \) to a poset \( \mathfrak{B} \) is such a function \( f \) that \( x \sqsubseteq y \Rightarrow fx \sqsubseteq fy. \)
Definition 2.91. *Order embedding* is a monotone injective function whose inverse is also monotone.

Definition 2.92. *Order isomorphism* is a surjective order embedding.

Order isomorphism preserves properties of posets, such as order, joins and meets, etc.

Definition 2.93.
1. *Join semilattice homomorphism* is a function \( f \) from a join semilattice \( \mathbb{A} \) to a join semilattice \( \mathbb{B} \), such that \( f(x \lor y) = fx \lor fy \) for every \( x, y \in \mathbb{A} \).
2. *Meet semilattice homomorphism* is a function \( f \) from a meet semilattice \( \mathbb{A} \) to a meet semilattice \( \mathbb{B} \), such that \( f(x \land y) = fx \land fy \) for every \( x, y \in \mathbb{A} \).

Obvious 2.94.
1. Join semilattice homomorphisms are monotone.
2. Meet semilattice homomorphisms are monotone.

Definition 2.95. A *lattice homomorphism* is a function from a lattice to a lattice, which is both join semilattice homomorphism and meet semilattice homomorphism.

Definition 2.96. *Complete lattice homomorphism* from a complete lattice \( \mathbb{A} \) to a complete lattice \( \mathbb{B} \) is a function \( f \) from \( \mathbb{A} \) to \( \mathbb{B} \) which preserves all meets and joins, that is \( f \bigcup S = \bigcup (f)S \) and \( f \bigcap S = \bigcap (f)S \).

2.1.13 Galois connections


Definition 2.97. Let \( \mathbb{A} \) and \( \mathbb{B} \) be two posets. A *Galois connection* between \( \mathbb{A} \) and \( \mathbb{B} \) is a pair of functions \( f = (f^*; f_*) \) with \( f^*: \mathbb{A} \to \mathbb{B} \) and \( f_*: \mathbb{B} \to \mathbb{A} \) such that:
\[
\forall x \in \mathbb{A}, y \in \mathbb{B}: (f^* x \subseteq y \iff x \subseteq f_* y).
\]

\( f_* \) is called the upper adjoint of \( f^* \) and \( f^* \) is called the lower adjoint of \( f_* \).

Theorem 2.98. A pair \( (f^*; f_*) \) of functions \( f^*: \mathbb{A} \to \mathbb{B} \) and \( f_*: \mathbb{B} \to \mathbb{A} \) is a Galois connection iff both of the following:
1. \( f^* \) and \( f_* \) are monotone.
2. \( x \subseteq f_* f^* x \) and \( f^* f_* y \subseteq y \) for every \( x \in \mathbb{A} \) and \( y \in \mathbb{B} \).

Proof.

\( \Rightarrow \).

1. Let \( a, b \in \mathbb{A} \) and \( a \subseteq b \). Then \( a \subseteq b \subseteq f_* f^* b \). So by definition \( f^* a \subseteq f^* b \) that is \( f^* \) is monotone. Analogously \( f_* \) in monotone.

\( \Leftarrow \).

\( f^* x \subseteq y \Rightarrow f_* f^* x \subseteq f_* y \Rightarrow x \subseteq f_* y \). The other direction is analogous. \( \square \)

Theorem 2.99.

1. \( f^* \circ f_* \circ f^* = f^* \).
2. \( f_* \circ f^* \circ f_* = f_* \).

Proof.

1. Let \( x \in \mathbb{A} \). We have \( x \subseteq f_* f^* x \); consequently \( f^* x \subseteq f^* f_* f^* x \). On the other hand, \( f^* f_* f^* x \subseteq f^* x \). So \( f^* f_* f^* x = f^* x \).
2. Similar.

**Definition 2.100.** A function \( f \) is called *idempotent* iff \( f(f(X)) = f(X) \) for every argument \( X \).

**Proposition 2.101.** \( f^* \circ f_* \) and \( f_* \circ f^* \) are idempotent.

**Proof.** \( f^* \circ f_* \) is idempotent because \( f^* f_* f = f^* f_* \). \( f_* \circ f^* \) is similar. \( \square \)

**Theorem 2.102.** Each of two adjoints is uniquely determined by the other.

**Proof.** Let \( p \) and \( q \) be both upper adjoints of \( f \). We have for all \( x \in \mathfrak{A} \) and \( y \in \mathfrak{B} \):

\[
x \subseteq p(y) \iff f(x) \subseteq y \iff x \subseteq q(y).
\]

For \( x = p(y) \) we obtain \( p(y) \subseteq q(y) \) and for \( x = q(y) \) we obtain \( q(y) \subseteq p(y) \). So \( q(y) = p(y) \). \( \square \)

**Theorem 2.103.** Let \( f \) be a function from a poset \( \mathfrak{A} \) to a poset \( \mathfrak{B} \).

1. Both:
   1. If \( f \) is monotone and \( g(b) = \max \{ x \in \mathfrak{A} \mid fx \subseteq b \} \) is defined for every \( b \in \mathfrak{B} \) then \( g \) is the upper adjoint of \( f \).
   2. If \( g: \mathfrak{B} \to \mathfrak{A} \) is the upper adjoint of \( f \) then \( g(b) = \max \{ x \in \mathfrak{A} \mid fx \subseteq b \} \) for every \( b \in \mathfrak{B} \).

2. Both:
   1. If \( f \) is monotone and \( g(b) = \min \{ x \in \mathfrak{A} \mid fx \supseteq b \} \) is defined for every \( b \in \mathfrak{B} \) then \( g \) is the lower adjoint of \( f \).
   2. If \( g: \mathfrak{B} \to \mathfrak{A} \) is the lower adjoint of \( f \) then \( g(b) = \min \{ x \in \mathfrak{A} \mid fx \supseteq b \} \) for every \( b \in \mathfrak{B} \).

**Proof.** We will prove only the first as the second is its dual.

1. Let \( g(b) = \max \{ x \in \mathfrak{A} \mid fx \subseteq b \} \) for every \( b \in \mathfrak{B} \). Then

\[
x \subseteq gy \iff x \subseteq \max \{ x \in \mathfrak{A} \mid fx \subseteq y \} \Rightarrow fx \subseteq y
\]

(because \( f \) is monotone) and

\[
x \subseteq gy \iff x \subseteq \max \{ x \in \mathfrak{A} \mid fx \subseteq y \} \Leftarrow fx \subseteq y.
\]

So \( fx \subseteq y \iff x \subseteq gy \) that is \( f \) is the lower adjoint of \( g \).

2. We have

\[
g(b) = \max \{ x \in \mathfrak{A} \mid fx \subseteq b \} \iff fg(b) \subseteq b \land \forall x \in \mathfrak{A}: (fx \subseteq b \Rightarrow x \subseteq gb)
\]

what is true by properties of adjoints. \( \square \)

**Theorem 2.104.** Let \( f \) be a function from a poset \( \mathfrak{A} \) to a poset \( \mathfrak{B} \).

1. If \( f \) is an upper adjoint, \( f \) preserves all existing infima in \( \mathfrak{A} \).
2. If \( \mathfrak{A} \) is a complete lattice and \( f \) preserves all infima, then \( f \) is an upper adjoint of a function \( \mathfrak{B} \to \mathfrak{A} \).
3. If \( f \) is a lower adjoint, \( f \) preserves all existing suprema in \( \mathfrak{A} \).
4. If \( \mathfrak{A} \) is a complete lattice and \( f \) preserves all suprema, then \( f \) is a lower adjoint of a function \( \mathfrak{B} \to \mathfrak{A} \).

**Proof.** We will prove only first two items because the rest items are similar.

1. Let \( S \in \mathcal{P}\mathfrak{A} \) and \( \bigcap S \) exists. \( f \bigcap S \) is a lower bound for \( (f)S \) because \( f \) is order-preserving. If \( a \) is a lower bound for \( (f)S \) then \( \forall x \in S: a \subseteq fx \) that is \( \forall x \in S: ga \subseteq x \) where \( g \) is the lower adjoint of \( f \). Thus \( ga \subseteq \bigcap S \) and hence \( f \bigcap S \supseteq a \). So \( f \bigcap S \) is the greatest lower bound for \( (f)S \).
2. Let $\mathfrak{A}$ be a complete lattice and $f$ preserves all infima. Let 
$$g(a) = \bigcap \{ x \in \mathfrak{A} \mid fx \supseteq a \}. $$
Since $f$ preserves infima, we have 
$$f(g(a)) = \bigcap \{ f(x) \mid x \in \mathfrak{A}, fx \supseteq a \} \supseteq a. $$
$$g(f(b)) = \bigcap \{ x \in \mathfrak{A} \mid fx \supseteq fb \} \subseteq b. $$
Obviously $f$ is monotone and thus $g$ is also monotone. So $f$ is the upper adjoint of $g$. 

\[ \Box \]

**Corollary 2.105.** Let $f$ be a function from a complete lattice $\mathfrak{A}$ to a poset $\mathfrak{B}$. Then:
1. $f$ is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff $f$ preserves all infima in $\mathfrak{A}$.
2. $f$ is an lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff $f$ preserves all suprema in $\mathfrak{A}$.

### 2.1.14 Co-Brouwerian lattices

**Definition 2.106.** Let $\mathfrak{A}$ be a poset. *Pseudocomplement* of $a \in \mathfrak{A}$ is
$$\max \{ c \in \mathfrak{A} \mid c \geq a \}. $$
If $z$ is the pseudocomplement of $a$ we will denote $z = a^\ast$.

**Definition 2.107.** Let $\mathfrak{A}$ be a poset. *Dual pseudocomplement* of $a \in \mathfrak{A}$ is
$$\min \{ c \in \mathfrak{A} \mid c \equiv a \}. $$
If $z$ is the dual pseudocomplement of $a$ we will denote $z = a^+$.

**Proposition 2.108.** If $a$ is a complemented element of a bounded distributive lattice, then $\bar{a}$ is both pseudocomplement and dual pseudocomplement of $a$.

**Proof.** Because of duality it is enough to prove that $\bar{a}$ is pseudocomplement of $a$.

We need to prove $c \geq a \Rightarrow c \subseteq \bar{a}$ for every element $c$ of our poset, and $\bar{a} \geq a$. The second is obvious. Let’s prove $c \geq a \Rightarrow c \subseteq \bar{a}$.

Really, let $c \geq a$. Then $c \cap a = 0; \bar{a} \cup (c \cap a) = \bar{a}; (\bar{a} \cup c) \cap (\bar{a} \cup a) = \bar{a}; \bar{a} \cup c = \bar{a}; c \subseteq \bar{a}$. 

**Definition 2.109.** Let $\mathfrak{A}$ be a join-semilattice. Let $a, b \in \mathfrak{A}$. *Pseudodifference* of $a$ and $b$ is
$$\min \{ z \in \mathfrak{A} \mid a \subseteq b \uplus z \}. $$
If $z$ is a pseudodifference of $a$ and $b$ we will denote $z = a \setminus b$.

**Remark 2.110.** I do not require that $a^\ast$ is undefined if there are no pseudocomplement of $a$ and likewise for dual pseudocomplement and pseudodifference. In fact below I will define quasicomplement, dual quasicomplement, and quasidifference which generalize pseudo-* counterparts. I will denote $a^\ast$ the more general case of quasicomplement than of pseudocomplement, and likewise for other notation.

**Obvious 2.111.** Dual pseudocomplement is the dual of pseudocomplement.

**Definition 2.112.** *Co-brouwerian lattice* is a lattice for which pseudodifference of any two its elements is defined.

**Proposition 2.113.** Every non-empty co-brouwerian lattice $\mathfrak{A}$ has least element.

**Proof.** Let $a$ be an arbitrary lattice element. Then
$$a \setminus a = \min \{ z \in \mathfrak{A} \mid a \subseteq a \uplus z \} = \min \mathfrak{A}. $$
So \( \mathfrak{A} \) exists. \( \square \)

**Definition 2.114.** **Co-Heyting lattice** is co-brouwerian lattice with greatest element.

**Theorem 2.115.** For a co-brouwerian lattice \( a \uplus - \) is an upper adjoint of \( - \, \uplus \, a \) for every \( a \in \mathfrak{A} \).

**Proof.** \( g(b) = \min \{ x \in \mathfrak{A} \mid a \uplus x \supseteq b \} = b \, \uplus \, a \) exists for every \( b \in \mathfrak{A} \) and thus is the lower adjoint of \( a \uplus - \). \( \square \)

**Corollary 2.116.** \( \forall a, x, y \in \mathfrak{A} : (x \, \uplus \, a \subseteq y \Leftrightarrow x \subseteq a \uplus y) \) for a co-brouwerian lattice.

**Definition 2.117.** Let \( a, b \in \mathfrak{A} \) where \( \mathfrak{A} \) is a complete lattice. Quasidifference \( a \, \uplus \, b \) is defined by the formula:

\[
 a \, \uplus \, b = \bigwedge \{ z \in \mathfrak{A} \mid a \subseteq b \uplus z \}.
\]

**Remark 2.118.** A more detailed theory of quasidifference (as well as quasicomplement and dual quasicomplement) will be considered below.

**Lemma 2.119.** \( (a \, \uplus \, b) \uplus b = a \uplus b \) for elements \( a, b \) of a meet infinite distributive complete lattice.

**Proof.**

\[
 (a \, \uplus \, b) \uplus b = \bigwedge \{ z \in \mathfrak{A} \mid a \subseteq b \uplus z \} \uplus b = \bigwedge \{ z \uplus b \mid z \in \mathfrak{A}, a \subseteq b \uplus z \} = \bigwedge \{ t \in \mathfrak{A} \mid t \supseteq b, a \subseteq t \} = a \uplus b.
\]

**Theorem 2.120.** The following are equivalent for a complete lattice \( \mathfrak{A} \):

1. \( \mathfrak{A} \) is meet infinite distributive.
2. \( \mathfrak{A} \) is a co-brouwerian lattice.
3. \( \mathfrak{A} \) is a co-Heyting lattice.
4. \( a \uplus - \) has lower adjoint for every \( a \in \mathfrak{A} \).

**Proof.**

(2) \( \Leftrightarrow \) (3). Obvious (taking into account completeness of \( \mathfrak{A} \)).

(4) \( \Rightarrow \) (1). Let \( - \, \uplus \, a \) be the lower adjoint of \( a \uplus - \). Let \( S \in \mathcal{P} \mathfrak{A} \). For every \( y \in S \) we have \( y \supseteq (a \uplus y) \, \uplus \, a \) by properties of Galois connections; consequently \( y \supseteq \bigcap \langle a \uplus \rangle S \, \uplus \, a \);

\[
 \bigcap S \supseteq \bigcap \langle a \uplus \rangle S \, \uplus \, a.
\]

So

\[
 a \uplus \bigcap S \supseteq \bigcap \langle a \uplus \rangle S \, \uplus \, a \supseteq \bigcap \langle a \uplus \rangle S.
\]

But \( a \uplus \bigcap S \subseteq \bigcap \langle a \uplus \rangle S \) is obvious.

(1) \( \Rightarrow \) (2). Let \( a \, \uplus \, b = \bigwedge \{ z \in \mathfrak{A} \mid a \subseteq b \uplus z \} \). To prove that \( \mathfrak{A} \) is a co-brouwerian lattice it is enough to prove \( a \subseteq b \uplus (a \, \uplus \, b) \). But it follows from the lemma.

(2) \( \Rightarrow \) (4). \( a \, \uplus \, b = \min \{ z \in \mathfrak{A} \mid a \subseteq b \uplus z \} \). So \( a \uplus - \) is the upper adjoint of \( - \, \uplus \, a \).

(1) \( \Rightarrow \) (4). Because \( a \uplus - \) preserves all meets. \( \square \)

**Corollary 2.121.** Co-brouwerian lattices are distributive.

The following theorem is essentially borrowed from [18]:
Proposition 2.125. A lattice $\mathfrak{A}$ with least element 0 is co-brouwerian with pseudodifference $\setminus^*$ iff $\setminus^*$ is a binary operation on $\mathfrak{A}$ satisfying the following identities:

1. $a \setminus^* a = 0$;
2. $a \cup (b \setminus^* a) = a \cup b$;
3. $b \cup (b \setminus^* a) = b$;
4. $(b \cup c) \setminus^* a = (b \setminus^* a) \cup (c \setminus^* a)$.

Proof.

$\Leftarrow$. We have

$$c \supseteq b \setminus^* a \Rightarrow c \cup a \supseteq a \cup (b \setminus^* a) = a \cup b \supseteq b;$$

$$c \cup a \supseteq b = c \cup (c \setminus^* a) \supseteq (a \setminus^* a) \cup (c \setminus^* a) = (a \cup c) \setminus^* a \supseteq b \setminus^* a.$$

So $c \supseteq b \setminus^* a$ is an upper adjoint of $\setminus^* a$. By a theorem above our lattice is co-brouwerian. By another theorem above $\setminus^*$ is a pseudodifference.

$\Rightarrow$.

1. Obvious.
2. $$a \cup (b \setminus^* a) = a \cup \bigcap \{ z \in \mathfrak{A} \mid b \subseteq a \cup z \} = \bigcap \{ a \cup z \mid z \in \mathfrak{A}, b \subseteq a \cup z \} = a \cup b.$$
3. $b \cup (b \setminus^* a) = b \cup \bigcap \{ z \in \mathfrak{A} \mid b \subseteq a \cup z \} = \bigcap \{ b \cup z \mid z \in \mathfrak{A}, b \subseteq a \cup z \} = b.$
4. Obviously $(b \cup c) \setminus^* a \supseteq b \setminus^* a$ and $(b \cup c) \setminus^* a \supseteq c \setminus^* a$. Thus $(b \cup c) \setminus^* a \supseteq (b \setminus^* a) \cup (c \setminus^* a)$. We have

$$((b \setminus^* a) \cup (c \setminus^* a)) \cup a = (b \setminus^* a) \cup (c \setminus^* a) = (b \cup c) \cup (c \setminus^* a) = a \cup b \cup c \supseteq b \cup c.$$

From this by definition of adjoints: $(b \setminus^* a) \cup (c \setminus^* a) \supseteq (b \cup c) \setminus^* a$.  

$\Box$

Theorem 2.123. $(\bigcup S) \setminus^* a = \bigcup \{ x \setminus^* a \mid x \in S \}$ for all $a \in \mathfrak{A}$ and $S \in \mathcal{P}\mathfrak{A}$ where $\mathfrak{A}$ is a co-brouwerian lattice and $\bigcup S$ is defined.

Proof. Because lower adjoint preserves all suprema.  

$\Box$

Theorem 2.124. $(a \setminus^* b) \setminus^* c = a \setminus^* (b \cup c)$ for elements $a$, $b$, $c$ of a complete co-brouwerian lattice.

Proof. $a \setminus^* b = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$.

$$(a \setminus^* b) \setminus^* c = \bigcap \{ z \in \mathfrak{A} \mid a \setminus^* b \subseteq c \cup z \}. $$

$$(a \setminus^* (b \cup c)) = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup c \cup z \}. $$

It is left to prove $a \setminus^* b \subseteq c \cup z \Leftrightarrow a \subseteq b \cup c \cup z$. Let $a \setminus^* b \subseteq c \cup z$. Then $a \cup b \subseteq b \cup c \cup z$ by the lemma and consequently $a \subseteq b \cup c \cup z$. Let $a \subseteq b \cup c \cup z$. Then $a \setminus^* b \subseteq (b \cup c \cup z) \setminus^* b \subseteq c \cup z$ by a theorem above.  

$\Box$

2.1.15 Dual pseudocomplement on co-Heyting lattices

Proposition 2.125. For Heyting algebras $1 \setminus^* b = b^+$.
Proof. $1^* b = \min \{ z \in \mathbb{A} \mid 1 \subseteq b \cup z \} = \min \{ z \in \mathbb{A} \mid 1 = b \cup z \} = \min \{ z \in \mathbb{A} \mid b \equiv z \} = b^*$.

**Theorem 2.126.** $(a \cap b)^+ = a^+ \cap b^+$ for every elements $a, b$ of a Heyting algebra.

**Proof.** $a \cup (a \cap b)^+ \supseteq (a \cap b) \cup (a \cap b)^+ \supseteq 1$. So $a \cup (a \cap b)^+ \supseteq 1; (a \cap b)^+ \supseteq 1 \setminus^* a = a^+$. We have $(a \cap b)^+ \supseteq a^+$. Similarly $(a \cap b)^+ \supseteq b^+$. Thus $(a \cap b)^+ \supseteq a^+ \cap b^+$.

On the other hand, $a^+ \cup b^+ \cup (a \cap b) = (a^+ \cup b^+ \cup a) \cap (a^+ \cup b^+ \cup b)$. Obviously $a^+ \cup b^+ \cup a = a^+ \cup b^+ = 1$. So $a^+ \cup b^+ \cup (a \cap b) \supseteq 1$ and thus $a^+ \cup b^+ \supseteq 1 \setminus^* (a \cap b) = (a \cap b)^+$. So $(a \cap b)^+ = a^+ \cap b^+$.

### 2.2 Intro to category theory

I recall that this is a **very** basic introduction to category theory, I even do not define **functors** as them have no use in my theory.

**Definition 2.127.** A **directed multigraph** is:

1. a set $O$ (vertices);
2. a set $M$ (edges);
3. functions $\text{Src}$ and $\text{Dst}$ (source and destination) from $M$ to $O$.

Note that in category theory vertices are called **objects** and edges are called **morphisms**.

**Definition 2.128.** A **precategory** is a directed multigraph together with a partial binary operation $\circ$ on the set $M$ such that $g \circ f$ is defined iff $\text{Dst} f = \text{Src} g$ (for every morphisms $f$ and $g$) such that

1. $\text{Src}(g \circ f) = \text{Src} f$ and $\text{Dst}(g \circ f) = \text{Dst} g$ whenever the composition $g \circ f$ of morphisms $f$ and $g$ is defined.
2. $(h \circ g) \circ f = h \circ (g \circ f)$ whenever compositions in this equation are defined.

**Definition 2.129.** The set $\text{Mor}(A; B)$ (morphisms from an object $A$ to an object $B$) is exactly morphisms which have $A$ as the source and $B$ as the destination.

**Definition 2.130.** **Identity morphism** is such a morphism $e$ that $e \circ f = f$ and $g \circ e = g$ whenever compositions in these formulas are defined.

**Definition 2.131.** A **category** is a precategory with additional requirement that for every object $X$ there exists identity morphism $1_X$.

**Proposition 2.132.** For every object $X$ there exist no more than one identity morphism.

**Proof.** Let $p$ and $q$ be both identity morphisms for a object $X$. Then $p = p \circ q = q$.

**Definition 2.133.** An **isomorphism** is such a morphism $f$ of a category that there exists a morphism $f^{-1}$ (inverse of $f$) such that $f \circ f^{-1} = 1_{\text{Dst} f}$ and $f^{-1} \circ f = 1_{\text{Src} f}$.

**Proposition 2.134.** An isomorphism has exactly one inverse.

**Proof.** Let $g$ and $h$ be both inverses of $f$. Then $h = h \circ 1_{\text{Dst} f} = h \circ f \circ g = 1_{\text{Src} f} \circ g = g$.

**Definition 2.135.** A **groupoid** is a category all of whose morphisms are isomorphisms.

Some important examples of categories:

**Exercise 2.3.** Prove that the below examples of categories are really categories.

**Definition 2.136.** The category $\text{Set}$ is:

- Objects are small sets.
• Morphisms from an object $A$ to an object $B$ are triples $(A; B; f)$ where $f$ is a function from $A$ to $B$.

• Composition of morphism is defined by the formula: $(B; C; g) \circ (A; B; f) = (A; C; g \circ f)$ where $g \circ f$ is function composition.

**Definition 2.137.** The category $\text{Rel}$ is:

• Objects are small sets.

• Morphisms from an object $A$ to an object $B$ are triples $(A; B; f)$ where $f$ is a binary relation between $A$ and $B$.

• Composition of morphism is defined by the formula: $(B; C; g) \circ (A; B; f) = (A; C; g \circ f)$ where $g \circ f$ is relation composition.

I will denote $\text{GR}(A; B; f) = f$ for any morphism $(A; B; f)$ of either Set or Rel.

I will denote $\langle f \rangle = (\text{GR} f)$ and $[f] = [\text{GR} f]$ for any morphism $f$ of either Set or Rel.

**Definition 2.138.** A morphism whose source is the same as destination is called *endomorphism*.

**Definition 2.139.** *Wide subcategory* of a category $(O; M)$ is a category $(O; M')$ where $M' \subseteq M$ and the composition on $(O; M')$ is a restriction of composition of $(O; M)$. (Similarly *wide sub-precategory* can be defined.)

### 2.3 Intro to group theory

**Definition 2.140.** A *semigroup* is a pair of a set $G$ and an associative binary operation $\cdot$ on $G$.

**Definition 2.141.** A *group* is a pair of a set $G$ and a binary operation $\cdot$ on $G$ such that:

1. $(h \cdot g) \cdot f = h \cdot (g \cdot f)$ for every $f, g, h \in G$.
2. There exists an element $e$ (*identity*) of $G$ such that $f \cdot e = e \cdot f = f$ for every $f \in G$.
3. For every element $f$ there exists an element $f^{-1}$ such that $f \cdot f^{-1} = f^{-1} \cdot f = e$.

**Obvious 2.142.** Every group is a semigroup.

**Proposition 2.143.** In every group there exist exactly one identity element.

**Proof.** If $p$ and $q$ are both identities, then $p = p \cdot q = q$. □

**Proposition 2.144.** Every group element has exactly one inverse.

**Proof.** Let $p$ and $q$ be both inverses of $f \in G$. Then $f \cdot p = p \cdot f = e$ and $f \cdot q = q \cdot f = e$. Then $p = p \cdot e = p \cdot f \cdot q = e \cdot q = q$. □

**Proposition 2.145.** $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$ for every group elements $f$ and $g$.

**Proof.** $(f^{-1} \cdot g^{-1}) \cdot (g \cdot f) = f^{-1} \cdot g^{-1} \cdot g \cdot f = f^{-1} \cdot e \cdot f = f^{-1} \cdot f = e$. Similarly $(g \cdot f) \cdot (f^{-1} \cdot g^{-1}) = e$. So $f^{-1} \cdot g^{-1}$ is the inverse of $g \cdot f$. □

**Definition 2.146.** A *permutation group* on a set $D$ is a group whose elements are functions on $D$ and whose composition is function composition.

**Obvious 2.147.** Elements of a permutation group are bijections.

**Definition 2.148.** A *transitive* permutation group on a set $D$ is such a permutation group $G$ on $D$ that for every $x, y \in D$ there exists $r \in G$ such that $y = r(x)$.

A category with single (arbitrarily chosen) object corresponds to every group. The morphisms of this category are elements of the group and the composition of morphisms is the group operation.
Chapter 3
More on order theory

3.1 Straight maps and separation subsets

3.1.1 Straight maps

Definition 3.1. Let $f$ be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. I call $f$ a straight map when

$$\forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow fa = f(a \sqcap b)).$$

Proposition 3.2. The following statements are equivalent for a monotone map $f$:

1. $f$ is a straight map.
2. $\forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow fa \sqsubseteq f(a \sqcap b))$.
3. $\forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow fa \sqsupseteq f(a \sqcap b))$.
4. $\forall a, b \in \mathfrak{A}: (fa \sqsupseteq f(a \sqcap b) \Rightarrow fa \sqsubseteq fb)$.

Proof. (1)$\Leftrightarrow$(2)$\Leftrightarrow$(3). Due $fa \sqsubseteq f(a \sqcap b)$.

(3)$\Leftrightarrow$(4). Obvious. \hfill \square

Remark 3.3. The definition of straight map can be generalized for any poset $\mathfrak{A}$ by the formula

$$\forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow \exists c \in \mathfrak{A}: (c \subseteq a \land c \subseteq b \land fa = fc)).$$

This generalization is not yet researched however.

Proposition 3.4. Let $f$ be a monotone map from a meet-semilattice $\mathfrak{A}$ to a meet-semilattice $\mathfrak{B}$. If

$$\forall a, b \in \mathfrak{A}: f(a \sqcap b) = fa \sqcap fb$$

then $f$ is a straight map.

Proof. Let $fa \subseteq fb$. Then $f(a \sqcap b) = fa \sqcap fb = fa$. \hfill \square

Proposition 3.5. Let $f$ be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. If

$$\forall a, b \in \mathfrak{A}: (fa \subseteq fb \Rightarrow a \subseteq b)$$

then $f$ is a straight map.

Proof. $fa \subseteq fb \Rightarrow a \subseteq b \Rightarrow a \sqcap b \Rightarrow fa = f(a \sqcap b)$. \hfill \square

Theorem 3.6. If $f$ is a straight monotone map from a meet-semilattice $\mathfrak{A}$ then the following statements are equivalent:

1. $f$ is an injection.
2. \( \forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow a \sqsubseteq b) \).
3. \( \forall a, b \in \mathfrak{A}: (a \sqsubseteq b \Rightarrow fa \sqsubseteq fb) \).
4. \( \forall a, b \in \mathfrak{A}: (a \sqsubseteq b \Rightarrow fa \neq fb) \).
5. \( \forall a, b \in \mathfrak{A}: (a \sqsubseteq b \Rightarrow fa \not\sqsupseteq fb) \).
6. \( \forall a, b \in \mathfrak{A}: (fa \sqsubseteq fb \Rightarrow a \not\sqsubseteq b) \).

**Proof.**

(1) \(\Rightarrow\) (3). Let \(a, b \in \mathfrak{A}\). Let \(fa = fb \Rightarrow a = b\). Let \(a \sqsubseteq b\). \(fa \neq fb\) because \(a \neq b\). \(fa \sqsubseteq fb\) because \(a \not\sqsubseteq b\). So \(fa \sqsubseteq fb\).

(2) \(\Rightarrow\) (1). Let \(a, b \in \mathfrak{A}\). Let \(fa \sqsubseteq fb \Rightarrow a \sqsubseteq b\). Let \(fa = fb\). Then \(a \sqsubseteq b\) and \(b \sqsubseteq a\) and consequently \(a = b\).

(3) \(\Rightarrow\) (2). Let \(\forall a, b \in \mathfrak{A}: (a \sqsubseteq b \Rightarrow fa \sqsubseteq fb)\). Let \(a \not\sqsupseteq b\). Then \(a \sqsupseteq f(a \sqcap b)\). If \(fa \sqsubseteq fb\) then \(fa \sqsubseteq f(a \sqcap b)\) what is a contradiction.

(3) \(\Rightarrow\) (5) \(\Rightarrow\) (4). Obvious.

(4) \(\Rightarrow\) (3). Because \(a \sqsubseteq b \Rightarrow a \sqsubseteq b \Rightarrow fa \sqsubseteq fb\).

(5) \(\Leftrightarrow\) (6). Obvious. \(\square\)

### 3.1.2 Separation subsets and full stars

**Definition 3.7.** \(\partial_Y a = \{ x \in Y \mid x \neq a \} \) for an element \(a\) of a poset \(\mathfrak{A}\) and \(Y \in \mathcal{P}\mathfrak{A}\).

**Definition 3.8.** Full star of \(a\) is \(\star a = \partial_\mathfrak{A} a\).

**Proposition 3.9.** If \(\mathfrak{A}\) is a meet-semilattice, then \(\star\) is a straight monotone map.

**Proof.** Monotonicity is obvious. Let \(\star a \not\sqsubseteq (a \sqcap b)\). Then it exists \(x \in \star a\) such that \(x \not\sqsubseteq (a \sqcap b)\). So \(x \sqcap a \not\sqsubseteq b\) but \(x \sqcap a \in \star a\) and consequently \(\star a \not\sqsubseteq b\). \(\square\)

**Definition 3.10.** A separation subset of a poset \(\mathfrak{A}\) is such its subset \(Y\) that

\[ \forall a, b \in \mathfrak{A}: (\partial_Y a = \partial_Y b \Rightarrow a = b). \]

**Definition 3.11.** I call separable such poset that \(\star\) is an injection.

**Obvious 3.12.** A poset is separable iff it has separation subset.

**Definition 3.13.** A poset \(\mathfrak{A}\) has disjunction property of Wallman iff for any \(a, b \in \mathfrak{A}\) either \(b \sqsubseteq a\) or there exists a non-least element \(c \sqsubseteq b\) such that \(a \preceq c\).

**Theorem 3.14.** For a meet-semilattice with least element the following statements are equivalent:

1. \(\mathfrak{A}\) is separable.
2. \(\forall a, b \in \mathfrak{A}: (\star a \sqsubseteq \star b \Rightarrow a \sqsubseteq b)\).
3. \(\forall a, b \in \mathfrak{A}: (a \sqsubseteq b \Rightarrow \star a \sqsubseteq \star b)\).
4. \(\forall a, b \in \mathfrak{A}: (a \sqsubseteq b \Rightarrow \star a \neq \star b)\).
5. \(\forall a, b \in \mathfrak{A}: (a \sqsubseteq b \Rightarrow \star a \not\sqsupseteq \star b)\).
6. \(\forall a, b \in \mathfrak{A}: (\star a \sqsubseteq \star b \Rightarrow a \not\sqsubseteq b)\).
7. \(\mathfrak{A}\) conforms to Wallman’s disjunction property.
8. \(\forall a, b \in \mathfrak{A}: (a \sqsubseteq b \Rightarrow \exists c \in \mathfrak{A} \setminus \{0\}: (c \preceq a \wedge c \sqsubseteq b))\).
Proof.

(1) $\Leftrightarrow$ (2) $\Leftrightarrow$ (3) $\Leftrightarrow$ (4) $\Leftrightarrow$ (5) $\Leftrightarrow$ (6). By the above theorem.

(8) $\Rightarrow$ (4). Let property (8) hold. Let $a \sqsubseteq b$. Then it exists element $c \subseteq b$ such that $c \neq 0$ and $c \cap a = 0$. But $c \cap b \neq 0$. So $a \neq b$.

(2) $\Rightarrow$ (7). Let property (2) hold. Let $a \supseteq b$. Then $\star a \supseteq b$ that is it there exists $c \in \star a$ such that $c \notin b$, in other words $c \cap a \neq 0$ and $c \cap b = 0$. Let $d = c \cap a$. Then $d \subseteq a$ and $d \neq 0$ and $d \cap b = 0$. So disjunction property of Wallman holds.

(7) $\Rightarrow$ (8). Let $a \sqsubseteq b$. Then $a \cap b \supseteq a$ that is it there exists $c \in a$ such that $c \neq 0$, in other words $c \cap a = c \cap b = 0$. Let $d = c \cap a$. Then $d \sqsubseteq a$ and $d \cap b = 0$. So disjunction property of Wallman holds.

3.1.3 Atomically Separable Lattices

Proposition 3.15. Every boolean lattice is separable.

Proof. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a boolean lattice an $a \neq b$. Then $a \cap b \neq 0$ or $a \cap b \neq 0$ because otherwise $a \cap b = 0$ and $a \cup b = 1$ and thus $a = b$. Without loss of generality assume $a \cap b \neq 0$. Then $a \cap c \neq 0$ and $b \cap c = 0$ for $c = a \cap b \neq 0$.

Theorem 3.20. If a lattice with least element is atomic and separable then it is atomistic.

Proof. Suppose the contrary that is $a \sqsubseteq \bigcup \text{atoms } a$. Then, because our lattice is separable, there exists $c \in \mathfrak{A}$ such that $c \cap a \neq 0$ and $c \cap \bigcup \text{atoms } a = 0$. There exists atom $d \subseteq c$ such that $d \subseteq c \cap a$. $d \cap \bigcup \text{atoms } a \subseteq c \cap \bigcup \text{atoms } a = 0$. But $d \in \text{atoms } a$. Contradiction.

Theorem 3.21. Let $\mathfrak{A}$ be an atomic meet-semilattice with least element. Then the following statements are equivalent:

1. $\mathfrak{A}$ is separable.
2. $\mathfrak{A}$ is atomically separable.
3. $\mathfrak{A}$ conforms to Wallman’s disjunction property.
4. \( \forall a, b \in \mathfrak{A}; (a \subseteq b \Rightarrow \exists c \in \mathfrak{A}\setminus\{0\} : (c \neq a \land c \subseteq b)). \)

**Proof.**

(1) \(\Leftrightarrow\) (3) \(\Leftrightarrow\) (4). Proved above.

(2) \(\Rightarrow\) (4). Let our semilattice be atomically separable. Let \( a \sqsubseteq b \). Then atoms \( a \sqsubseteq \text{atoms} \ b \) and so there exists \( c \in \text{atoms} \ b \) such that \( c \notin \text{atoms} \ a \). \( c \neq 0 \) and \( c \sqsubseteq b \), from which (taking into account that \( c \) is an atom) \( c \sqsubseteq b \) and \( c \cap a = 0 \). So our semilattice conforms to the formula (4).

(4) \(\Rightarrow\) (2). Let formula (4) hold. Then for any elements \( a \sqsubseteq b \) there exists \( c \neq 0 \) such that \( c \sqsubseteq b \) and \( c \cap a = 0 \). Because \( \mathfrak{A} \) is atomic there exists atom \( d \sqsubseteq c \). \( d \in \text{atoms} \ b \) and \( d \notin \text{atoms} \ a \). So atoms \( a \neq \text{atoms} \ b \) and atoms \( a \subseteq \text{atoms} \ b \). Consequently atoms \( a \subseteq \text{atoms} \ b \). \( \square \)

**Theorem 3.22.** Any atomistic meet-semilattice with least element is separable.

**Proof.** From the above. \( \square \)

### 3.2 Free Stars

**Definition 3.23.** An upper set is such a set \( F \in \mathcal{P} \mathfrak{A} \) that

\[
\forall X \in F, Y \in \mathfrak{A} : (Y \supseteq X \Rightarrow Y \in F).
\]

**Definition 3.24.** Let \( \mathfrak{A} \) be a poset. Free stars on \( \mathfrak{A} \) are such \( S \in \mathcal{P}\mathfrak{A} \) that the least element (if it exists) is not in \( S \) and for every \( X, Y \in \mathfrak{A} \)

\[
\forall Z \in \mathfrak{A} : (Z \supseteq X \land Z \supseteq Y \Rightarrow Z \in S) \Leftrightarrow X \in S \lor Y \in S.
\]

**Proposition 3.25.** \( S \in \mathcal{P}\mathfrak{A} \) where \( \mathfrak{A} \) is a poset is a free star iff all of the following:

1. The least element (if it exists) is not in \( S \).
2. \( \forall Z \in \mathfrak{A} : (Z \supseteq X \land Z \supseteq Y \Rightarrow Z \in S) \Rightarrow X \in S \lor Y \in S \) for every \( X, Y \in \mathfrak{A} \).
3. \( S \) is an upper set.

**Proof.**

\( \Rightarrow \). (1) and (2) are obvious. Let prove that \( S \) is an upper set. Let \( X \in S \) and \( X \sqsubseteq Y \in \mathfrak{A} \). Then \( X \in S \lor Y \in S \) and thus \( \forall Z \in \mathfrak{A} : (Z \supseteq X \land Z \supseteq X \Rightarrow Z \in S) \) that is \( \forall Z \in \mathfrak{A} : (Z \supseteq X \Rightarrow Z \in S) \), and so \( Y \in S \).

\( \Leftarrow \). We need to prove that

\[
\forall Z \in \mathfrak{A} : (Z \supseteq X \land Z \supseteq Y \Rightarrow Z \in S) \Leftrightarrow X \in S \lor Y \in S.
\]

Let \( X \in S \lor Y \in S \). Then \( Z \supseteq X \land Z \supseteq Y \Rightarrow Z \in S \) for every \( Z \in \mathfrak{A} \) because \( S \) is an upper set. \( \square \)

**Proposition 3.26.** Let \( \mathfrak{A} \) be a join-semilattice. \( S \in \mathcal{P}\mathfrak{A} \) is a free star iff all of the following:

1. The least element (if it exists) is not in \( S \).
2. \( X \sqcup Y \in S \Rightarrow X \in S \lor Y \in S \) for every \( X, Y \in \mathfrak{A} \).
3. \( S \) is an upper set.

**Proof.**

\( \Rightarrow \). We need to prove only \( X \sqcup Y \in S \Rightarrow X \in S \lor Y \in S \). Let \( X \sqcup Y \in S \). Because \( S \) is an upper set, we have \( \forall Z \in \mathfrak{A} : (Z \supseteq X \lor Y \Rightarrow Z \in S) \) and thus \( \forall Z \in \mathfrak{A} : (Z \supseteq X \land Z \supseteq Y \Rightarrow Z \in S) \) from which we conclude \( X \in S \lor Y \in S \).

\( \Leftarrow \). We need to prove \( \forall Z \in \mathfrak{A} : (Z \supseteq X \land Z \supseteq Y \Rightarrow Z \in S) \Leftrightarrow X \in S \lor Y \in S \).

But it trivially follows from that \( S \) is an upper set. \( \square \)
Proposition 3.27. Let $\mathfrak{A}$ be a join-semilattice. $S \in \mathcal{P}\mathfrak{A}$ is a free star if the least element (if it exists) is not in $S$ and for every $X, Y \in \mathfrak{A}$

$$X \sqcup Y \in S \iff X \in S \lor Y \in S.$$ 

Proof.

$\Rightarrow$. We need to prove only $X \sqcup Y \in S \iff X \in S \lor Y \in S$ what follows from that $S$ is an upper set.

$\Leftarrow$. We need to prove only that $S$ is an upper set. Let $X \in S$ and $X \not\subseteq Y \in \mathfrak{A}$. Then $X \in S \Rightarrow X \in S \lor Y \in S \Rightarrow X \sqcup Y \in S \Rightarrow Y \in S$. So $S$ is an upper set.

\[ \square \]

3.2.1 Starrish posets

Definition 3.28. I will call a poset *starrish* when the full star $*a$ is a free star for every element $a$ of this poset.

Proposition 3.29. Every distributive lattice is starrish.

Proof. Let $\mathfrak{A}$ be a distributive lattice, $a \in \mathfrak{A}$. Obviously $0 \notin *a$ (if 0 exists); obviously $*a$ is an upper set. If $x \sqcup y \in *a$, then $(x \sqcup y) \sqcap a$ is non-least that is $(x \sqcap a) \sqcup (y \sqcap a)$ is non-least what is equivalent to $x \sqcap a$ or $y \sqcap a$ being non-least that is $x \in *a \lor y \in *a$.

Theorem 3.30. If $\mathfrak{A}$ is a starrish join-semilattice lattice then

$$\text{atoms}(a \sqcup b) = \text{atoms} a \cup \text{atoms} b.$$ 

for every $a, b \in \mathfrak{A}$.

Proof. For every atom $c$ we have: $c \in \text{atoms}(a \sqcup b) \iff c \neq a \sqcup b \Rightarrow a \sqcup b \in *c \Rightarrow a \in *c \lor b \in *c \Rightarrow c \neq a \lor c \neq b \Rightarrow c \in \text{atoms} a \cap \text{atoms} b$.

\[ \square \]

3.3 Quasidifference and Quasicomplement

I’ve got quasidifference and quasicomplement (and dual quasicomplement) replacing max and min in the definition of pseudodifference and pseudocomplement (and dual pseudocomplement) with $\sqcap$ and $\sqcup$. Thus quasidifference and (dual) quasicomplement are generalizations of their pseudo-counterparts.

Remark 3.31. Pseudocomplements and pseudodifferences are standard terminology. Quasi-counterparts are my neologisms.

Definition 3.32. Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$. Quasicomplement of $a$ is

$$a^* = \bigsqcup \{ c \in \mathfrak{A} \mid c \succ a \}.$$ 

Definition 3.33. Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$. Dual quasicomplement of $a$ is

$$a^+ = \bigsqcap \{ c \in \mathfrak{A} \mid c \preceq a \}.$$ 

I will denote quasicomplement and dual quasicomplement for a specific poset $\mathfrak{A}$ as $a^*(\mathfrak{A})$ and $a^+(\mathfrak{A})$.

Definition 3.34. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice. Quasidifference of $a$ and $b$ is

$$a \setminus^* b = \bigsqcap \{ z \in \mathfrak{A} \mid a \sqsubseteq b \sqcup z \}.$$ 

Definition 3.35. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice. Second quasidifference of $a$ and $b$ is

$$a \# b = \bigsqcup \{ z \in \mathfrak{A} \mid z \sqsubseteq a \land z \succeq b \}.$$
Theorem 3.36. $a \setminus^* b = \bigcap \{ z \in \mathfrak{A} \mid z \subseteq a \land a \subseteq b \cup z \}$ where $\mathfrak{A}$ is a distributive lattice and $a, b \in \mathfrak{A}$.

Proof. Obviously $\{ z \in \mathfrak{A} \mid z \subseteq a \land a \subseteq b \cup z \} \subseteq \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$. Thus $\bigcap \{ z \in \mathfrak{A} \mid z \subseteq a \land a \subseteq b \cup z \} \supseteq a \setminus^* b$.

Let $z \in \mathfrak{A}$ and $z' = z \cap a$.

$a \subseteq b \cup z \Rightarrow a \subseteq (b \cup z) \cap a \Rightarrow a \subseteq (b \cap a) \cup (z \cap a) \Rightarrow a \subseteq b \cap a \cup z' \Rightarrow a \subseteq b \cap z'$ and $a \subseteq b \cup z \Rightarrow a \subseteq b \cap z'$. Thus $a \subseteq b \cup z \Rightarrow a \subseteq b \cap z'$. If $z \in \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$ then $a \subseteq b \cup z$ and thus $z' \in \{ z \in \mathfrak{A} \mid z \subseteq a \land a \subseteq b \cup z \}$. But $z' \subseteq z$ thus having $\bigcap \{ z \in \mathfrak{A} \mid z \subseteq a \land a \subseteq b \cup z \} \subseteq \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$. \hfill \Box

Remark 3.37. If we drop the requirement that $\mathfrak{A}$ is distributive, two formulas for quasidifference (the definition and the last theorem) fork.

Obvious 3.38. Dual quasicomplement is the dual of quasicomplement.

Obvious 3.39.

- Every pseudocomplement is quasidifference.
- Every dual pseudocomplement is dual quasicomplement.
- Every pseudodifference is quasicomplement.

Below we will stick to the more general quasies than pseudos. If needed, one can check that a quasicomplement $a^*$ is a pseudocomplement by the equation $a^* \cong a$ (and analogously with other quasies).

Next we will express quasidifference through quasicomplement.

Proposition 3.40.

1. $a \setminus^* b = a \setminus^* (a \cap b)$ for any distributive lattice;
2. $a \# b = a \# (a \cap b)$ for any distributive lattice with least element.

Proof.

1. $a \subseteq (a \cap b) \cup z \Leftrightarrow a \subseteq (a \cup z) \cap (b \cup z) \Leftrightarrow a \subseteq a \cup z \land a \subseteq b \cap z \Leftrightarrow a \subseteq b \cup z$. Thus $a \setminus^* (a \cap b) = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq (a \cap b) \cup z \} = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \} = a \setminus^* b$.

2. $a \# (a \cap b) = \bigcup \{ z \in \mathfrak{A} \mid z \subseteq a \land a \cap b = 0 \} = \bigcup \{ z \in \mathfrak{A} \mid z \subseteq a \land (z \cap a) \cap a \cap b = 0 \} = \bigcup \{ z \cap a \mid z \in \mathfrak{A}, z \cap a \cap b = 0 \} = \bigcup \{ z \in \mathfrak{A} \mid z \subseteq a, z \cap b = 0 \} = a \# b$. \hfill \Box

I will denote $Da$ the lattice $\{ x \in \mathfrak{A} \mid x \subseteq a \}$.

Theorem 3.41. For $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice with least element

1. $a \setminus^* b = (a \cap b)^+(Da)$;
2. $a \# b = (a \cap b)^*(Da)$.

Proof.

1. 

\[
(a \cap b)^+(Da) = \\
\bigcap \{ c \in Da \mid c \cup (a \cap b) = a \} = \\
\bigcap \{ c \in Da \mid c \cup (a \cap b) \supseteq a \} = \\
\bigcap \{ c \in Da \mid (c \cup a) \cap (c \cup b) \supseteq a \} = \\
\bigcap \{ c \in Da \mid c \subseteq a \land c \cup b \supseteq a \} = \\
a \setminus^* b.
\]
2.

\[(a \cap b)^*(D_a) = \bigsqcup \{ c \in D_a \mid c \cap a \cap b = 0 \} = \bigsqcup \{ c \in A \mid c \subseteq a \land c \cap a \cap b = 0 \} = \bigsqcup \{ c \in A \mid c \subseteq a \land c \cap b = 0 \} = a \neq b.\]

\[\square\]

**Proposition 3.42.** \((a \cup b) \setminus b \subseteq a\) for an arbitrary complete lattice.

**Proof.** \((a \cup b) \setminus b = \bigsqcup \{ z \in A \mid a \subseteq b \cup z \}.\)

But \(a \subseteq z \Rightarrow a \cup b \subseteq b \cup z.\) So \(\{ z \in A \mid a \cup b \subseteq b \cup z \} \supseteq \{ z \in A \mid a \subseteq z \}.\)

Consequently, \((a \cup b) \setminus b \subseteq \{ z \in A \mid a \subseteq z \} = a.\)

\[\square\]

### 3.4 Several equal ways to express pseudodifference

**Theorem 3.43.** For an atomistic co-Brouwerian lattice \(A\) and \(a, b \in A\) the following expressions are always equal:

1. \(a \setminus b = \bigsqcup \{ z \in A \mid a \subseteq b \cup z \}\) (quasidifference of \(a\) and \(b\));
2. \(a \# b = \bigsqcup \{ z \in A \mid z \subseteq a \land z \cap b = 0 \}\) (second quasidifference of \(a\) and \(b\));
3. \(\bigsqcup (a \setminus \text{atoms } b)\).

**Proof.** Proof of (1)=(3):

\(a \setminus b = \bigsqcup \{ z \in A \mid a \subseteq b \cup z \},\) so it’s enough to prove that

\[a \setminus b = \bigsqcup (a \setminus \text{atoms } b).\]

Really:

\[a \setminus b = \left( \bigsqcup \text{atoms } a \right) \setminus b = \left( \text{theorem 2.123} \right) \]

\[\bigsqcup \{ A \setminus b \mid A \in \text{atoms } a \} =
\bigsqcup \left\{ \begin{cases} A & \text{if } A \notin \text{atoms } b \\ 0 & \text{if } A \in \text{atoms } b \end{cases} \right\} \mid A \in \text{atoms } a \} =
\bigsqcup \{ A \mid A \in \text{atoms } a, A \notin \text{atoms } b \} =
\bigsqcup (a \setminus \text{atoms } b).

**Proof of (2)=(3):**

\(a \setminus b\) is defined because our lattice is co-Brouwerian. Taking the above into account, we have

\[a \setminus b = \bigsqcup (a \setminus \text{atoms } b) = \bigsqcup \{ z \in \text{atoms } a \mid z \cap b = 0 \}.\]

So \(\bigsqcup \{ z \in \text{atoms } a \mid z \cap b = 0 \}\) is defined.

If \(z \subseteq a \land z \cap b = 0\) then \(z' = \bigsqcup \{ x \in \text{atoms } z \mid x \cap b = 0 \}\) is defined. \(z'\) is a lower bound for \(\{ z \in \text{atoms } a \mid z \cap b = 0 \}\).

Thus \(z' \subseteq \{ z \in A \mid z \subseteq a \land z \cap b = 0 \}\) and so \(\bigsqcup \{ z \in \text{atoms } a \mid z \cap b = 0 \}\) is an upper bound of \(\{ z \in A \mid z \subseteq a \land z \cap b = 0 \}\).

If \(y\) is above every \(z' \in \{ z \in A \mid z \subseteq a \land z \cap b = 0 \}\) then \(y\) is above every \(z \in \text{atoms } a\) such that \(z \cap b = 0\) and thus \(y\) is above \(\bigsqcup \{ z \in \text{atoms } a \mid z \cap b = 0 \}\).
Thus \( \bigsqcup \{ z \in \text{atoms } a \mid z \cap b = 0^a \} \) is least upper bound of
\( \{ z \in \mathfrak{A} \mid z \subseteq a \land z \cap b = 0^a \} \),
that is
\[
\bigsqcup \{ z \in \mathfrak{A} \mid z \subseteq a \land z \cap b = 0^a \} = \bigsqcup \{ z \in \text{atoms } a \mid z \cap b = 0^a \} = \bigsqcup (\text{atoms } a \setminus \text{atoms } b).
\]

### 3.5 Partially ordered categories

#### 3.5.1 Partially ordered categories

**Definition 3.44.** I will call a partially ordered (pre)category a (pre)category together with partial order \( \subseteq \) on each of its Mor-sets with the additional requirement that
\[
f_1 \subseteq f_2 \land g_1 \subseteq g_2 \Rightarrow g_1 \circ f_1 \subseteq g_2 \circ f_2
\]
for every morphisms \( f_1, g_1, f_2, g_2 \) such that \( \text{Src } f_1 = \text{Src } f_2 \land \text{Dst } f_1 = \text{Dst } f_2 = \text{Src } g_1 = \text{Src } g_2 \land \text{Dst } g_1 = \text{Dst } g_2 \).

#### 3.5.2 Dagger categories

**Definition 3.45.** I will call a dagger precategory a precategory together with an involutive contravariant identity-on-objects prefunctor \( x \mapsto x^\dagger \).

In other words, a dagger precategory is a precategory equipped with a function \( x \mapsto x^\dagger \) on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms \( f \) and \( g \):

1. \( f^{\dagger\dagger} = f \);
2. \( (g \circ f)^\dagger = f^\dagger \circ g^\dagger \).

**Definition 3.46.** I will call a dagger category a category together with an involutive contravariant identity-on-objects functor \( x \mapsto x^\dagger \).

In other words, a dagger category is a category equipped with a function \( x \mapsto x^\dagger \) on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms \( f \) and \( g \) and object \( A \):

1. \( f^{\dagger\dagger} = f \);
2. \( (g \circ f)^\dagger = f^\dagger \circ g^\dagger \);
3. \( (1_A)^\dagger = 1_A \).

**Theorem 3.47.** If a category is a dagger precategory then it is a dagger category.

**Proof.** We need to prove only that \( (1_A)^\dagger = 1_A \). Really,
\[
(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A.
\]

For a partially ordered dagger (pre)category I will additionally require (for every morphisms \( f \) and \( g \) with the same source and destination)
\[
f^\dagger \subseteq g^\dagger \Leftrightarrow f \subseteq g.
\]

An example of dagger category is the category \( \text{Rel} \) whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with \( f^\dagger = f^{-1} \).

**Definition 3.48.** A morphism \( f \) of a dagger category is called unitary when it is an isomorphism and \( f^\dagger = f^{-1} \).
Definition 3.49. Symmetric (endo)morphism of a dagger precategory is such a morphism \( f = f^\dagger \).

Definition 3.50. Transitive (endo)morphism of a precategory is such a morphism \( f = f \circ f \).

Theorem 3.51. The following conditions are equivalent for a morphism \( f \) of a dagger precategory:

1. \( f \) is symmetric and transitive.
2. \( f = f^\dagger \circ f \).

Proof.

(1) \(\Rightarrow\) (2). If \( f \) is symmetric and transitive then \( f^\dagger \circ f = f \circ f = f \).

(2) \(\Rightarrow\) (1). \( f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^\dagger = f^\dagger \circ f = f \), so \( f \) is symmetric. \( f = f^\dagger \circ f = f \circ f \), so \( f \) is transitive. \( \square \)

3.5.2.1 Some special classes of morphisms

Definition 3.52. For a partially ordered dagger category I will call monovalued morphism such a morphism \( f \) that \( f \circ f^\dagger \subseteq 1_{\textrm{Dst} f} \).

Definition 3.53. For a partially ordered dagger category I will call entirely defined morphism such a morphism \( f \) that \( f^\dagger \circ f \subseteq 1_{\textrm{Src} f} \).

Definition 3.54. For a partially ordered dagger category I will call injective morphism such a morphism \( f \) that \( f^\dagger \circ f \subseteq 1_{\textrm{Src} f} \).

Definition 3.55. For a partially ordered dagger category I will call surjective morphism such a morphism \( f \) that \( f \circ f^\dagger \subseteq 1_{\textrm{Dst} f} \).

Remark 3.56. It is easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective functions as morphisms of the category \( \text{Rel} \).

Obvious 3.57. “Injective morphism” is a dual of “monovalued morphism” and ”surjective morphism” is a dual of ”entirely defined morphism”.

Definition 3.58. For a given partially ordered dagger category \( C \) the category of monovalued (entirely defined, injective, surjective) morphisms of \( C \) is the category with the same set of objects as of \( C \) and the set of morphisms being the set of monovalued (entirely defined, injective, surjective) morphisms of \( C \) with the composition of morphisms the same as in \( C \).

We need to prove that these are really categories, that is that composition of monovalued (entirely defined, injective, surjective) morphisms is monovalued (entirely defined, injective, surjective) and that identity morphisms are monovalued, entirely defined, injective, and surjective.

Proof. We will prove only for monovalued morphisms and entirely defined morphisms, as injective and surjective morphisms are their duals.

Monovalued. Let \( f \) and \( g \) be monovalued morphisms, \( \textrm{Dst} f = \textrm{Src} g \). \( (g \circ f) \circ (g \circ f)^\dagger = g \circ f \circ f^\dagger \circ g^\dagger \subseteq g \circ 1_{\textrm{Dst} f} \circ f^\dagger \circ g^\dagger = g \circ g^\dagger = g \circ g^\dagger \subseteq 1_{\textrm{Dst} g} = 1_{\textrm{Dst} (g \circ f)} \). So \( g \circ f \) is monovalued.

That identity morphisms are monovalued follows from the following: \( 1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A \circ 1_{\textrm{Dst} 1_A} \subseteq 1_{\textrm{Dst} 1_A} \).

Entirely defined. Let \( f \) and \( g \) be entirely defined morphisms, \( \textrm{Dst} f = \textrm{Src} g \). \( (g \circ f)^\dagger \circ (g \circ f) = f^\dagger \circ g^\dagger \circ g \circ f \supseteq f^\dagger \circ 1_{\textrm{Src} g} \circ f = f^\dagger \circ 1_{\textrm{Dst} f} \circ f = f^\dagger \circ f \supseteq 1_{\textrm{Src} (g \circ f)} \). So \( g \circ f \) is entirely defined.

That identity morphisms are entirely defined follows from the following: \( (1_A)^\dagger \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\textrm{Src} 1_A} \subseteq 1_{\textrm{Src} 1_A} \). \( \square \)

Definition 3.59. I will call a bijective morphism a morphism which is entirely defined, monovalued, injective, and surjective.
**Proposition 3.60.** If a morphism is bijective then it is an isomorphism.

**Proof.** Let $f$ be bijective. Then $f \circ f^\dagger \subseteq 1_{\text{Dst } f}$, $f^\dagger \circ f \supseteq 1_{\text{Src } f}$, $f \circ f^\dagger \subseteq 1_{\text{Src } f}$, $f \circ f^\dagger \supseteq 1_{\text{Dst } f}$. Thus $f \circ f^\dagger = 1_{\text{Dst } f}$ and $f^\dagger \circ f = 1_{\text{Src } f}$ that is $f^\dagger$ is an inverse of $f$. $$\square$$

**Definition 3.61.** A morphism $f$ of a partially ordered category is *metamonovaled* when $(\bigsqcap G) \circ f = \bigsqcap_{g \in G} (g \circ f)$ whenever $G$ is a set of morphisms with a suitable domain and image.

**Definition 3.62.** A morphism $f$ of a partially ordered category is *metainjective* when $f \circ (\bigsqcup G) = \bigsqcup_{g \in G} (f \circ g)$ whenever $G$ is a set of morphisms with a suitable domain and image.

**Obvious 3.63.** Metamonovaledness and metainjectivity are dual to each other.

**Definition 3.64.** A morphism $f$ of a partially ordered category is *metacomplete* when $f \circ (\bigsqcap G) = \bigsqcap_{g \in G} (f \circ g)$ whenever $G$ is a set of morphisms with a suitable domain and image.

**Definition 3.65.** A morphism $f$ of a partially ordered category is *co-metacomplete* when $(\bigsqcup G) \circ f = \bigsqcup_{g \in G} (f \circ g)$ whenever $G$ is a set of morphisms with a suitable domain and image.

### 3.6 Partitioning

**Definition 3.66.** Let $\mathfrak{A}$ be a complete lattice. *Torning of an element* $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{0\}$ such that

\[ \bigsqcup S = a \quad \text{and} \quad \forall x, y \in S: (x \not= y \Rightarrow x \succeq y). \]

**Definition 3.67.** Let $\mathfrak{A}$ be a complete lattice. *Weak partition* of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{0\}$ such that

\[ \bigsqcup S = a \quad \text{and} \quad \forall x \in S: x \succeq \bigsqcup (S \setminus \{x\}). \]

**Definition 3.68.** Let $\mathfrak{A}$ be a complete lattice. *Strong partition* of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{0\}$ such that

\[ \bigsqcup S = a \quad \text{and} \quad \forall A, B \in \mathcal{P}\mathfrak{A}: (A \succeq B \Rightarrow \bigsqcup A \succeq \bigsqcup B). \]

**Obvious 3.69.**

1. Every strong partition is a weak partition.
2. Every weak partition is a torning.

### 3.7 A proposition about binary relations

**Proposition 3.70.** Let $f$, $g$, $h$ be binary relations. Then $g \circ f \not= h \iff g \not= h \circ f^{-1}$. 

**Proof.**

\[
g \circ f \not= h \iff \\
\exists a, c: a (\langle g \circ f \rangle \cap h) c \iff \\
\exists a, c: (a (g \circ f) c \land a h c) \iff \\
\exists a, b, c: (a f b \land b g c \land a h c) \iff \\
\exists b, c: (b g c \land (h \circ f^{-1}) c) \iff \\
\exists b, c: (b (g \cap (h \circ f^{-1})) c) \iff \\
g \not= h \circ f^{-1}.
\]
3.8 Infinite associativity and ordinated product

3.8.1 Introduction
We will consider some function \( f \) which takes an arbitrary ordinal number of arguments. That is \( f \) can be taken for arbitrary (small, if to be precise) ordinal number of arguments. More formally: Let \( x = x_{i \in n} \) be a family indexed by an ordinal \( n \). Then \( f(x) \) can be taken. The same function \( f \) can take different number of arguments. (See below for the exact definition.)

Some of such functions \( f \) are associative in the sense defined below. If a function is associative in the below defined sense, then the binary operation induced by this function is associative in the usual meaning of the word “associativity” as defined in basic algebra.

I also introduce and research an important example of infinitely associative function, which I call *ordinated product*. Note that my searching about infinite associativity and ordinals in Internet has provided no useful results. As such there is a reason to assume that my research of generalized associativity in terms of ordinals is novel.

3.8.2 Used notation
We identify natural numbers with finite Von Neumann’s ordinals (further just *ordinals* or *ordinal numbers*).

For simplicity we will deal with small sets (members of a Grothendieck universe). We will denote the Grothendieck universe (aka *universal set*) as \( \mathbb{F} \).

I will denote a tuple of \( n \) elements like \( [a_0; \ldots; a_{n-1}] \). By definition
\[
[a_0; \ldots; a_{n-1}] = [(0; a_0),\ldots, (n-1; a_{n-1})].
\]

Note that an ordered pair \( (a; b) \) is not the same as the tuple \([a; b]\) of two elements.

**Definition 3.71.** An anchored relation is a tuple \([n; \mathcal{R}]\) where \( n \) is an index set and \( \mathcal{R} \) is an \( n \)-ary relation.

For an anchored relation arity \([n; \mathcal{R}] = n \). The graph \( \mathcal{G}[n; \mathcal{R}] = \mathcal{R} \).

**Definition 3.72.** \( \text{Pr}_i \) is a function defined by the formula
\[
\text{Pr}_i f = \{x_i \mid x \in f\}
\]
for every small \( n \)-ary relation \( f \) where \( n \) is an ordinal number and \( i \in n \). Particularly for every \( n \)-ary relation \( f \) and \( i \in n \) where \( n \in \mathbb{N} \).
\[
\text{Pr}_i f = \{x_i \mid [x_0, \ldots, x_{n-1}] \in f\}.
\]

Recall that Cartesian product is defined as follows:
\[
\prod a = \{z \in (\bigcup \text{im} a)^{\text{dom} a} \mid \forall i \in \text{dom} a: z(i) \in a_i\}.
\]

**Obvious 3.73.** If \( a \) is a small function, then \( \prod a = \{z \in \bigcup \text{dom} a \mid \forall i \in \text{dom} a: z(i) \in a_i\} \).

3.8.2.1 Currying and uncurrying

The customary definition
Let \( X, Y, Z \) be sets.

We will consider variables \( x \in X \) and \( y \in Y \).

Let a function \( f \in Z^{X \times Y} \). Then \( \text{curry}(f) \in (Z^Y)^X \) is the function defined by the formula
\[
\text{curry}(f)(x) = f(x; y).
\]

Let now \( f \in (Z^Y)^X \). Then \( \text{uncurry}(f) \in Z^{X \times Y} \) is the function defined by the formula
\[
\text{uncurry}(f)(x; y) = (fx) y.
\]

3.1. It is unrelated with graph theory.
Obvious 3.74.
1. uncurry$(\text{curry}(f)) = f$ for every $f \in Z^{X \times Y}$.
2. curry$(\text{uncurry}(f)) = f$ for every $f \in (Z^Y)^X$.

Currying and uncurrying with a dependent variable

Let $X$, $Z$ be sets and $Y$ be a function with the domain $X$. (Vaguely saying, $Y$ is a variable dependent on $X$.)

The disjoint union $\bigsqcup_{i \in X} Y = \bigcup \{(i) \times Y_i \mid i \in \text{dom } Y\} = \{(i; x) \mid i \in \text{dom } Y, x \in Y_i\}.$

We will consider variables $x \in X$ and $y \in Y_x.$

Let a function $f \in \bigsqcup_{i \in X} Y_i$ (or equivalently $f \in \prod_{i \in X} Y_i$). Then curry$(f) \in \prod_{i \in X} Z^{Y_i}$ is the function defined by the formula $\text{curry}(f)(x) = f(x; y).$

Let now $f \in \prod_{i \in X} Z^{Y_i}$. Then uncurry$(f) \in \bigsqcup_{i \in X} Y_i$ is the function defined by the formula uncurry$(f)(x; y) = (fx)_y$.

Obvious 3.75.
1. uncurry$(\text{curry}(f)) = f$ for every $f \in \bigsqcup_{i \in X} Z^{Y_i}$.
2. curry$(\text{uncurry}(f)) = f$ for every $f \in \prod_{i \in X} Z^{Y_i}$.

3.8.2.2 Functions with ordinal numbers of arguments

Let Ord be the set of small ordinal numbers. If

- $\exists R \in \text{ord}$ (or equivalently $\exists n \in \text{ord}$).
- The set of functions taking $n$ arguments on the set $X$ and returning a value in $Y$ is $X^n.$
- The set of all small functions taking ordinal numbers of arguments is $Y^{\bigcup_{n \in \text{ord}} X^n}.$

I will denote $\text{OrdVar}(X) = \bigcup_{n \in \text{ord}} X^n$ and call it ordinal variadic. ("Var" in this notation is taken from the word variadic in the collocation variadic function used in computer science.)

3.8.3 On sums of ordinals

Let $a$ be an ordinal-indexed family of ordinals.

**Proposition 3.76.** $\bigsqcup a$ with lexicographic order is a well-ordered set.

**Proof.** Let $S$ be non-empty subset of $\bigsqcup a$.

Take $i_0 = \min \text{Pr}_0 S$ and $x_0 = \min \{\text{Pr}_1 y \mid y \in S, y(0) = i_0\}$ (these exist by properties of ordinals). Then $(i_0; x_0)$ is the least element of $S$.

**Definition 3.77.** $\sum a$ is the unique ordinal order-isomorphic to $\bigsqcup a$.

This ordinal exists and is unique because our set is well-ordered.

**Remark 3.78.** An infinite sum of ordinals is not customary defined.

The structured sum $\bigoplus a$ of $a$ is an order isomorphism from lexicographically ordered set $\bigsqcup a$ into $\sum a$.

There exists (for a given $a$) exactly one structured sum, by properties of well-ordered sets.

Obvious 3.79. $\sum a = \text{im} \bigoplus a$.

**Theorem 3.80.** $(\bigoplus a)(n; x) = \sum_{i \in n} a_i + x$.

**Proof.** We need to prove that it is an order isomorphism. Let’s prove it is an injection that is $m > n \Rightarrow \sum_{i \in m} a_i + x > \sum_{i \in n} a_i + x$ and $y > x \Rightarrow \sum_{i \in n} a_i + y > \sum_{i \in n} a_i + x$.

Really, if $m > n$ then $\sum_{i \in m} a_i + x \geq \sum_{i \in n} a_i + x \geq \sum_{i \in n} a_i + x = \sum_{i \in n} a_i + x$. The second formula is true by properties of ordinals.

Let’s prove that it is a surjection. Let $r \in \sum a$. There exist $n \in \text{dom } a$ and $x \in a_n$ such that $r = (\bigoplus a)(n; x)$. Thus $r = (\bigoplus a)(n; 0) + x = \sum_{i \in n} a_i + x$ because $(\bigoplus a)(n; 0) = \sum_{i \in n} a_i$ since $(n; 0)$ has $\sum_{i \in n} a_i$ predecessors.
3.8.4 Ordinated product

3.8.4.1 Introduction

Ordinated product defined below is a variation of Cartesian product, but is associative unlike Cartesian product. However, ordinated product unlike Cartesian product is defined not for arbitrary sets, but only for relations having ordinal numbers of arguments.

Let $F$ indexed by an ordinal number be a small family of anchored relations.

3.8.4.2 Concatenation

Definition 3.81. Let $z$ be an indexed by an ordinal number family of functions each taking an ordinal number of arguments. The concatenation of $z$ is

$$\text{concat } z = \text{uncurry}(z) \circ \left( \bigoplus \text{ (dom } o z \text{)} \right)^{-1}.$$ 

Obvious 3.82. If $z$ is a finite family of finitary functions, it is concatenation of dom $z$ tuples in the usual sense (as it is commonly used in computer science).

Proposition 3.83. If $z \in \prod (\text{GR } o F)$ then $\text{concat } z = \text{uncurry}(z) \circ \left( \bigoplus \text{ (arity } o F) \right)^{-1}$.

Proof. If $z \in \prod (\text{GR } o F)$ then $\text{dom } (i) = \text{dom } (\text{GR } o F)_i = \text{dom } F_1 = \text{arity } F_1$ for every $i \in \text{dom } F$.

Thus $\text{dom } z = \text{arity } o F$.

Proposition 3.84. $\text{dom } \text{concat } z = \sum_{i \in \text{dom } z} \text{dom } z_i$.

Proof. Because $\text{dom } \left( \bigoplus \text{ (dom } o z) \right)^{-1} = \sum_{i \in \text{dom } F} \text{dom } z_i$, it is enough to prove that $\text{dom uncurry}(z) = \bigoplus \text{ (dom } o z)$.

Really,

$$\text{dom uncurry}(z) = \bigoplus \text{ (dom } o z) = \{(i; x) \mid i \in \text{dom } (\text{dom } o z), x \in \text{dom } z_i\} = \prod_{i \in X} z_i = \prod z,$$

and $\text{dom uncurry}(z) = \prod_{i \in X} z_i = \prod z$.

3.8.4.3 Finite example

If $F$ is a finite family (indexed by a natural number dom $F$) of anchored finitary relations, then by definition $\text{GR } \prod \text{ (ord)} F = \{[a_0, 0; 0; \ldots; a_0, \text{arity } F_0 - 1; \ldots; a_0, \text{arity } F_0 - 1; \ldots; a_0, \text{arity } F_0 - 1] \in \text{GR } F_0 \wedge \ldots \wedge [a_0, \text{arity } F_{\text{dom } F - 1} - 1] \in \text{GR } F_{\text{dom } F - 1} \}

\text{(ord)}$

and

$$\text{arity } \prod \text{ (ord)} F = \text{arity } F_0 + \ldots + \text{arity } F_{\text{dom } F - 1}.$$ 

The above formula can be shortened to

$$\text{GR } \prod \text{ (ord)} F = \{\text{concat } z \mid z \in \prod \text{ (GR } o F)\}.$$ 

3.8.4.4 The definition

Definition 3.85. The anchored relation (which I call ordinated product) $\prod \text{ (ord)} F$ is defined by the formulas:

$$\text{arity } \prod \text{ (ord)} F = \sum \text{ (arity } o F);$$

$$\text{GR } \prod \text{ (ord)} F = \{\text{concat } z \mid z \in \prod \text{ (GR } o F)\}.$$ 

Proposition 3.86. $\prod \text{ (ord)} F$ is a properly defined anchored relation.
46

More on order theory

Proof. dom concat z =

P

i2dom F

dom zi =

P

i2dom F

arity Fi =

P

(arity  F ).



3.8.4.5 Denition with composition for every multiplier
L
def
q(F )i = (curry(
(arity  F )))i.
Theorem 3.87. GR

Q (ord)

P

F = L 2 f (arityF ) j 8i 2 dom F : L  q(F )i 2 GR Fi .

Q (ord)

P

S
F = L 2 ( im(GR  F )) (arityF ) j 8i 2 dom F : L  q(F )i 2 GR Fi .

Q
Q
Proof. GR (ord) F =fconcat z j z 2
(GR  F )g;

Q (ord)
L
Q
GR
F = uncurry(z)  (
(arity  F ))¡1 j z 2 i2dom F farity Fi ; 8i 2 dom F : z(i) 2
GR Fi ;
Let L = uncurry(z). Then z = curry(L).

Q (ord)
L
Q
GR
F = L(
(arity  F ))¡1 j curry(L) 2 i2dom F farity Fi ; 8i 2 dom F : curry(L)i 2
GR Fi ;
n
o
`
Q (ord)
L
arity Fi
GR
F = L(
(arity  F ))¡1 j L 2 f i2dom F
; 8i 2 dom F : curry(L)i 2 GR Fi ;
P

Q
L
GR (ord) F = L 2 f (arityF ) j 8i 2 dom F : curry(L 
(arity  F ))i 2 GR Fi ;
L
L
(curry(L 
(arity  F ))i)x = L((curry(
(arity  F ))i)x) = L(q(F )ix) = (L  q(F )i)x;
L
curry(L 
(arity  F ))i = L  q(F )i;
P

Q (ord)
GR
F = L 2 f (arityF ) j 8i 2 dom F : L  q(F )i 2 GR Fi .

Corollary 3.88. GR

Corollary 3.89. GR

Q (ord)

F is small if F is small.

3.8.4.6 Denition with shifting arguments
Let Fi0 = fL  Pr1jfigarity Fi j L 2 GR Fi g.
Proposition 3.90. Fi0 = fL  Pr1jfigf j L 2 GR Fi g.
Proof. If L 2 GR Fi then dom L = arity Fi. Thus
L  Pr1jfigarity Fi=L  Pr1jfigdom L=L  Pr1jfigf:



Proposition 3.91. Fi0 is an (fig  arity Fi)-ary relation.
Proof. We need to prove that dom(L  Pr1jfigarity Fi) = fig  arity Fi for L 2 GR Fi, but that's
obvious.

Obvious 3.92.

`

(arity  F ) =

Lemma 3.93. P 2

Q

where Pi 2 f figarity Fi

S

i2dom F

(fig  arity Fi) =

S

i2dom F

dom Fi0.

S
Q
Fi0 , curry( im P ) 2
(GR  F ) for a dom F indexed family P
`
for every i 2 dom F , that is for P 2 i2dom F f figarity Fi.

i2dom F

`
Proof. For every P 2 i2dom F f figarity Fi we have:
Q
P 2 i2dom F Fi0 , P 2 fz 2 fdom F j 8i 2 dom F : z(i) 2 Fi0g , P 2 fdom F ^ 8i 2 dom F :
P (i) 2 Fi0 , P 2 fdom F ^ 8i 2 dom F 9L 2 GR Fi: Pi = L  (Pr1jfigf) , P 2 fdom F ^
¡

8i 2 dom F 9L 2 GR Fi: Pi 2 f figarity Fi ^ 8x 2 arity Fi: Pi(i; x) = L x , P 2 fdom F ^
¡

8i 2 dom F 9L 2 GR Fi: Pi 2 f figarity Fi ^ curry(Pi)i = L , P 2 fdom F ^ 8i 2 dom F :
¡

Pi 2 f figarity Fi ^ curry(Pi)i 2 GR Fi , 8i 2 dom F 9Qi 2 (farity Fi) fig: (Pi = uncurry(Qi) ^
¡
¡S

(Qi)i 2 farity Fi ^ Qi i 2 GR Fi) , 8i 2 dom F 9Qi 2 (farity Fi) fig Pi = uncurry(Qi) ^
i2dom F Qi i 2

¡
S
Q
GR Fi , 8i 2 dom F 9Qi 2 (farity Fi) fig Pi = uncurry(Qi) ^ i2dom F Qi 2
(GR  F ) ,
¡S

S
Q
Q
8i 2 dom F : i2dom F curry(Pi) 2
(GR  F ) , curry i2dom F Pi 2
(GR  F ) ,
S
Q
curry( im P ) 2
(GR  F ).



Lemma 3.94. \( \{ \text{curry}(f) \circ \bigoplus \ (\text{arity} \circ F) \mid f \in \text{GR} \prod^{\text{(ord)}} F \} = \prod (\text{GR} \circ F). \)

**Proof.** First \( \text{GR} \prod^{\text{(ord)}} F = \{ \text{uncurry}(z) \circ (\bigoplus (\text{dom} \circ z))^{-1} \mid z \in \prod (\text{GR} \circ F) \} \), that is
\[
\left\{ f \mid f \in \text{GR} \prod^{\text{(ord)}} F \right\} = \{ \text{uncurry}(z) \circ (\bigoplus (\text{arity} \circ F))^{-1} \mid z \in \prod (\text{GR} \circ F) \}.
\]
Since \( \bigoplus (\text{arity} \circ F) \) is a bijection, we have
\[
\left\{ f \circ \bigoplus (\text{arity} \circ F) \mid f \in \text{GR} \prod^{\text{(ord)}} F \right\} = \{ \text{uncurry}(z) \mid z \in \prod (\text{GR} \circ F) \} \text{ what is equivalent to } \left\{ \text{curry}(f) \circ \bigoplus (\text{arity} \circ F) \mid f \in \text{GR} \prod^{\text{(ord)}} F \right\} = \{ z \mid z \in \prod (\text{GR} \circ F) \} \text{ that is } \left\{ \text{curry}(f) \circ \bigoplus (\text{arity} \circ F) \mid f \in \text{GR} \prod^{\text{(ord)}} F \right\} = \prod (\text{GR} \circ F). \]

\[
\square \]

Lemma 3.95. \( \{ \bigcup \text{ im } P \mid P \in \prod_{i \in \text{dom} F} \bigcup^{(i) \times \text{arity} F_i} \land \text{curry}(\bigcup \text{ im } P) \in \prod (\text{GR} \circ F) \} = \{ L \in \bigcup_{i \in \text{arity} F_i} \text{curry}(L) \in \prod (\text{GR} \circ F) \} \).

**Proof.** Let \( L' \in \{ L \in \bigcup_{i \in \text{arity} F_i} \mid \text{curry}(L) \in \prod (\text{GR} \circ F) \} \). Then \( L' \in \bigcup_{i \in \text{arity} F_i} \text{curry}(L) \) and \( \text{curry}(L') \in \prod (\text{GR} \circ F) \).

Let \( P = \lambda i : \text{dom } F : L' \cap^{(i) \times \text{arity} F_i} \). Then \( P \in \prod_{i \in \text{dom} F} \bigcup^{(i) \times \text{arity} F_i} \land \text{curry}(\bigcup \text{ im } P) \in \prod (\text{GR} \circ F) \). Let now \( L' \in \{ \bigcup \text{ im } P \mid P \in \prod_{i \in \text{dom} F} \bigcup^{(i) \times \text{arity} F_i} \land \text{curry}(\bigcup \text{ im } P) \in \prod (\text{GR} \circ F) \} \). Then there exists \( P \in \prod_{i \in \text{dom} F} \bigcup^{(i) \times \text{arity} F_i} \), such that \( L' = \bigcup \text{ im } P \) and \( \text{curry}(L') \in \prod (\text{GR} \circ F) \). Evidently \( L' \in \bigcup_{i \in \text{arity} F_i} \text{curry}(L) \). So \( L' \in \{ \bigcup \text{ im } P \mid P \in \prod_{i \in \text{dom} F} \bigcup^{(i) \times \text{arity} F_i} \land \text{curry}(\bigcup \text{ im } P) \in \prod (\text{GR} \circ F) \} \).

**Theorem 3.97.** \( \text{GR} \prod^{\text{(ord)}} F = \{ (\bigcup \text{ im } P) \circ (\bigoplus (\text{arity} \circ F))^{-1} \mid P \in \prod_{i \in \text{dom } F} F_i' \} \).

**Proof.** From the lemma, because \( \bigoplus (\text{arity} \circ F) \) is a bijection.

**Theorem 3.98.** \( \text{GR} \prod^{\text{(ord)}} F = \left\{ \bigcup_{i \in \text{dom } F} (P_i \circ (\bigoplus (\text{arity} \circ F))^{-1}) \mid P \in \prod_{i \in \text{dom } F} F_i' \right\} \).

**Proof.** From the previous theorem.

**Theorem 3.99.** \( \text{GR} \prod^{\text{(ord)}} F = \left\{ \bigcup \text{ im } P \mid P \in \prod_{i \in \text{dom } F} \{ f \circ (\bigoplus (\text{arity} \circ F))^{-1} \mid f \in F_i' \} \right\} \).
Proof. From the previous.

Remark 3.100. Note that the above formulas contain both \( \bigcup_{i \in \text{dom} F} F_i \) and \( \bigcup_{i \in \text{dom} F} F_i' \). These forms are similar but different.

3.8.4.7 Associativity of ordinated product

Let \( f \) be an ordinal variadic function.

Let \( S \) be an ordinal indexed family of functions of ordinal indexed families of functions each taking an ordinal number of arguments in a set \( X \).

I call \( f \) infinite associative when

1. \( f(f \circ S) = f(\text{concat} S) \) for every \( S \);
2. \( f([x]) = x \) for \( x \in X \).

Infinite associativity implies associativity

Proposition 3.101. Let \( f \) be an infinitely associative function taking an ordinal number of arguments in a set \( X \). Define \( x \ast y = f([x; y]) \) for \( x, y \in X \). Then the binary operation \( \ast \) is associative.

Proof. Let \( x, y, z \in X \). Then \( (x \ast y) \ast z = f([f([x; y]; z)] = f(f([x; y]; [z])) = f([x; y; z]) \). Similarly \( x \ast (y \ast z) = f([x; y; z]) \). So \( x \ast (y \ast z) = (x \ast y) \ast z \).

Concatenation is associative

First we will prove some lemmas.

Let \( a \) and \( b \) be functions on a poset. Let \( a \sim b \) iff there exist an order isomorphism \( f \) such that \( a = b \circ f \). Evidently \( \sim \) is an equivalence relation.

Obvious 3.102. \( \text{concat} a = \text{concat} b \iff \text{uncurry}(a) \sim \text{uncurry}(b) \) for every ordinal indexed families \( a \) and \( b \) of functions taking an ordinal number of arguments.

Thank to the above, we can reduce properties of concat to properties of uncurry.

Lemma 3.103. \( a \sim b \Rightarrow \text{uncurry} a \sim \text{uncurry} b \) for every ordinal indexed families \( a \) and \( b \) of functions taking an ordinal number of arguments.

Proof. There exist an order isomorphism \( f \) such that \( a = b \circ f \).

\[
\text{uncurry}(a)(x; y) = (ax)y = (bfx)y = \text{uncurry}(b)(fx; y) = \text{uncurry}(b)g(x; y) \text{ where } g(x; y) = (fx; y).
\]

\( g \) is an order isomorphism because \( g(x_0; y_0) \geq g(x_1; y_1) \Leftrightarrow (x_0; y_0) \geq (x_1; y_1) \). (Injectivity and surjectivity are obvious.)

Lemma 3.104. Let \( a_i \sim b_i \) for some \( f_i \), for every \( i \). Then \( \text{uncurry} a \sim \text{uncurry} b \) for every ordinal indexed families \( a \) and \( b \) of ordinal indexed families of functions taking an ordinal number of arguments.

Proof. Let \( a_i = b_i \circ f_i \) where \( f_i \) is an order isomorphism for every \( i \).

\[
\text{uncurry}(a)(i; y) = a_i y = b_i y = \text{uncurry}(b)(i; f_i y) = \text{uncurry}(b)g(i; y) = (\text{uncurry}(b) \circ g)(i; y) \text{ where } g(i; y) = (i; f_i y).
\]

\( g \) is an order isomorphism because \( g(i; y_0) \geq g(i; y_1) \Leftrightarrow f_i y_0 \geq f_i y_1 \Leftrightarrow y_0 \geq y_1 \) and \( i_0 > i_1 \Rightarrow g(i_0; y_0) > g(i_1; y_1) \). (Injectivity and surjectivity are obvious.)

Let now \( S \) be an ordinal indexed family of ordinal indexed families of functions taking an ordinal number of arguments.

Lemma 3.105. \( \text{uncurry}(\text{uncurry} \circ S) \sim \text{uncurry}(\text{uncurry} S) \).

Proof. \( \text{uncurry} \circ S = \lambda i \in S : \text{uncurry}(S_i) \);

\[
\text{uncurry}(\text{uncurry} \circ S)(i; [x; y]) = (\text{uncurry} S_i)(x; y) = (S_i x)y;
\]

\[
\text{uncurry}(\text{uncurry} S)(i; [x; y]) = ((\text{uncurry} S)(i; x)y) = (S_i x)y.
\]
Thus \(\text{uncurry}(\text{uncurry} \circ S)(i; (x; y)) = (\text{uncurry}(\text{uncurry} S))(i; x; y)\) and thus evidently \(\text{uncurry}(\text{uncurry} \circ S) \sim \text{uncurry}(\text{uncurry} S)\).

**Theorem 3.106.** \(\text{concat}\) is an infinitely associative function.

**Proof.** \(\text{concat}([[]]) = x\) for a function \(x\) taking an ordinal number of argument is obvious. It is remained to prove

\[
\text{concat}(\text{concat} \circ S) = \text{concat}(\text{concat} S);
\]

We have, using the lemmas, \(\text{concat}(\text{concat} \circ S) \sim \text{uncurry}(\text{concat} \circ S) \sim (\text{by lemma 3.104}) \sim \text{uncurry}(\text{uncurry}(\text{concat} \circ S)) \sim \text{concat}(\text{concat} S)\).

**Corollary 3.107.** Ordinated product is an infinitely associative function.
Chapter 4
Filters and filtrators

This chapter is based on my article [29].

This chapter is grouped in the following way:

- First it goes a short introduction in pedagogical order (first less general stuff and examples, last the most general stuff):
  - filters on a set;
  - filters on a meet-semilattice;
  - filters on a poset;
  - filtrators.

- Then it goes the formal part in the order from the most general to the least general:
  - filtrators;
  - filters on a poset;
  - filters on a set.

Most theorems about filtrators (and also some theorems about filters on posets) have the form $A \Rightarrow B$ where $A$ is the specific theorem condition and $B$ is the main theorem statement. To most such theorems correspond simple $B$ when we restrict to consideration only to the filtrator of filters on a fixed set. In some sense only $B$ here is important, $A$ here is a technical condition. So reading theorems about filtrators concentrate on the theorem statement rather than on theorem conditions.

4.1 Introduction to filters and filtrators

4.1.1 Filters on a set

We sometimes want to define something resembling an infinitely small (or infinitely big) set, for example the infinitely small interval near 0 on the real line. Of course there are no such set, just like as there are no natural number which is the difference $2 - 3$. To overcome this shortcoming we introduce whole numbers, and $2 - 3$ becomes well defined. In the same way to consider things which are like infinitely small (or infinitely big) sets we introduce filters.

An example of a filter is the infinitely small interval near 0 on the real line. To come to infinitely small, we consider all intervals $(-\varepsilon; \varepsilon)$ for all $\varepsilon > 0$. This filter consists of all intervals $(-\varepsilon; \varepsilon)$ for all $\varepsilon > 0$ and also all subsets of $\mathbb{R}$ containing such intervals as subsets. Informally speaking, this is the greatest filter contained in every interval $(-\varepsilon; \varepsilon)$ for all $\varepsilon > 0$.

**Definition 4.1.** A filter on a set $\mathcal{U}$ is a $\mathcal{F} \in \mathcal{P}(\mathcal{U})$ such that:

1. $\forall A, B \in \mathcal{F}: A \cap B \in \mathcal{F}$;
2. $\forall A, B \in \mathcal{P}(\mathcal{U}):(A \in \mathcal{F} \wedge B \supseteq A \Rightarrow B \in \mathcal{F})$.

**Exercise 4.1.** Verify that the above introduced infinitely small interval near 0 on the real line is a filter on $\mathbb{R}$.  

51
**Exercise 4.2.** Describe “the neighborhood of positive infinity” filter on $\mathbb{R}$.

**Definition 4.2.** A filter not containing empty set is called a *proper filter*.

**Obvious 4.3.** The non-proper filter is $\mathcal{P}\emptyset$.

**Remark 4.4.** Some other authors require that all filters are proper. This is a stupid idea and we allow non-proper filters, in the same way as we allow to use the number 0.

### 4.1.2 Intro to filters on a meet-semilattice

A trivial generalization of the above:

**Definition 4.5.** A filter on a meet-semilattice $\mathcal{Z}$ is a $\mathcal{F} \subseteq \mathcal{P}\mathcal{Z}$ such that:

1. $\forall A, B \in \mathcal{F}: A \cap B \in \mathcal{F}$;
2. $\forall A, B \in \mathcal{Z}: (A \in \mathcal{F} \land B \supseteq A \Rightarrow B \in \mathcal{F})$.

### 4.1.3 Intro to filters on a poset

**Definition 4.6.** A filter on a poset $\mathfrak{A}$ is a $\mathcal{F} \subseteq \mathcal{P}\mathcal{Z}$ such that:

1. $\forall A, B \in \mathcal{F}: \exists C \in \mathcal{F}: C \supseteq A, B$;
2. $\forall A, B \in \mathcal{Z}: (A \in \mathcal{F} \land B \supseteq A \Rightarrow B \in \mathcal{F})$.

It is easy to show (and there is a proof of it somewhere below) that this coincides with the above definition in the case if $\mathcal{Z}$ is a meet-semilattice.

### 4.1.4 Intro to filtrators

**Definition 4.7.** Filter $\uparrow x = \{c \in \mathcal{Z} \mid c \supseteq x\}$ for an element $x$ of a filtrator.

I denote $\mathcal{P}\mathfrak{A}$ the set of all principal filters (for a given poset $\mathcal{Z}$).

Now let (only in this paragraph) $\mathfrak{A}$ is an arbitrary poset and $\mathcal{P}\mathfrak{A}$ is its subset. I call pairs $(\mathfrak{A}; \mathcal{P}\mathfrak{A})$ of a poset with its subset *filtrators*. And when $\mathcal{Z}$ is the set of filters and $\mathcal{P}\mathfrak{A}$ is the set of principal filters on some poset I call them *primary filtrators*.

Filtrators are a more general case than the special case of filtrators on powersets.

### 4.2 Filtrators

**Definition 4.8.** I will call a filtrator a pair $(\mathfrak{A}; \mathcal{Z})$ of a poset $\mathfrak{A}$ and its subset $\mathcal{Z} \subseteq \mathfrak{A}$. I call $\mathfrak{A}$ the *base* of the filtrator and $\mathcal{Z}$ the *core* of the filtrator. I will also say that $(\mathfrak{A}; \mathcal{Z})$ is a filtrator over poset $\mathcal{Z}$.

**Definition 4.9.** I will call a lattice filtrator a pair $(\mathfrak{A}; \mathcal{Z})$ of $\mathfrak{A}$ a lattice and its subset $\mathcal{Z} \subseteq \mathfrak{A}$.

**Definition 4.10.** I will call a complete lattice filtrator a pair $(\mathfrak{A}; \mathcal{Z})$ of $\mathfrak{A}$ a complete lattice and its subset $\mathcal{Z} \subseteq \mathfrak{A}$.

**Definition 4.11.** I will call a central filtrator a filtrator $(\mathfrak{A}; Z(\mathfrak{A}))$ where $Z(\mathfrak{A})$ is the center of a bounded lattice $\mathfrak{A}$.

**Definition 4.12.** I will call element of a filtrator an element of its base.

**Definition 4.13.** $\mathrm{up\,a} = \{c \in \mathcal{Z} \mid c \supseteq a\}$ for an element $a$ of a filtrator.
**Definition 4.14.** \(\downarrow a = \{c \in \mathfrak{A} \mid c \subseteq a\}\) for an element \(a\) of a filtrator.

**Obvious 4.15.** “up” and “down” are dual.

Our main purpose here is knowing properties of the core of a filtrator to infer properties of the base of the filtrator, specifically properties of \(\up\) \(a\) for every element \(a\).

**Definition 4.16.** I call a filtrator with join-closed core such a filtrator \((\mathfrak{A}; \mathfrak{J})\) that \(\bigcup^3 S = \bigcup^3 S\) whenever \(\bigcup^3 S\) exists for \(S \in \mathcal{P}\mathfrak{A}\).

**Definition 4.17.** I call a filtrator with meet-closed core such a filtrator \((\mathfrak{A}; \mathfrak{J})\) that \(\bigcap^3 S = \bigcap^3 S\) whenever \(\bigcap^3 S\) exists for \(S \in \mathcal{P}\mathfrak{A}\).

**Definition 4.18.** I call a filtrator with finitely join-closed core such a filtrator \((\mathfrak{A}; \mathfrak{J})\) that \(a \sqcup^3 b = a \sqcup^3 b\) whenever \(a \sqcup^3 b\) exists for \(a, b \in \mathfrak{A}\).

**Definition 4.19.** I call a filtrator with finitely meet-closed core such a filtrator \((\mathfrak{A}; \mathfrak{J})\) that \(a \sqcap^3 b = a \sqcap^3 b\) whenever \(a \sqcap^3 b\) exists for \(a, b \in \mathfrak{A}\).

**Definition 4.20.** Filtered filtrator is a filtrator \((\mathfrak{A}; \mathfrak{J})\) such that \(\forall a \in \mathfrak{A}: a = \bigcap^3 \up a\).

**Definition 4.21.** Prefiltered filtrator is a filtrator \((\mathfrak{A}; \mathfrak{J})\) such that “up” is injective.

**Definition 4.22.** Semifiltered filtrator is a filtrator \((\mathfrak{A}; \mathfrak{J})\) such that
\[\forall a, b \in \mathfrak{A}: (\up a \supseteq \up b \Rightarrow a \subseteq b).\]

**Obvious 4.23.**
- Every filtered filtrator is semifiltered.
- Every semifiltered filtrator is prefILTERED.

**Obvious 4.24.** “up” is a strict map from \(\mathfrak{A}\) to the dual of the poset \(\mathcal{P}\mathfrak{A}\) if \((\mathfrak{A}; \mathfrak{J})\) is a semifiltered filtrator.

**Theorem 4.25.** Each semifiltered filtrator is a filtrator with join-closed core.

**Proof.** Let \((\mathfrak{A}; \mathfrak{J})\) be a semifiltered filtrator. Let \(S \in \mathcal{P}\mathfrak{A}\) and \(\bigcup^3 S\) is defined. We need to prove \(\bigcup^3 S = \bigcup^3 S\). That \(\bigcup^3 S\) is an upper bound for \(S\) is obvious. Let \(a \in \mathfrak{A}\) be an upper bound for \(S\). It’s enough to prove that \(\bigcup^3 S \subseteq a\). Really,
\[c \in \up a \Rightarrow c \supseteq a \Rightarrow \forall x \in S: c \supseteq x \Rightarrow c \supseteq \bigcup^3 S \Rightarrow c \subseteq \up \bigcup^3 S;\]
so \(\up a \subseteq \up \bigcup^3 S\) and thus \(a \supseteq \bigcup^3 S\) because it is semifiltered.

### 4.2.1 Core Part

**Definition 4.26.** The core part of an element \(a \in \mathfrak{A}\) is \(\text{Cor} a = \bigcap^3 \up a\).

**Definition 4.27.** The dual core part of an element \(a \in \mathfrak{A}\) is \(\text{Cor}' a = \bigcup^3 \down a\).

**Obvious 4.28.** \(\text{Cor}'\) is dual of Cor.

**Theorem 4.29.** \(\text{Cor} a \subseteq a\) whenever \(\text{Cor} a\) exists for any element \(a\) of a filtered filtrator.

**Proof.** \(\text{Cor} a = \bigcap^3 \up a \subseteq \bigcap^3 \up a = a\).

**Corollary 4.30.** \(\text{Cor} a \in \down a\) whenever \(\text{Cor} a\) exists for any element \(a\) of a filtered filtrator.
Theorem 4.31. Cor’ \( a \subseteq a \) whenever Cor’ \( a \) exists for any element \( a \) of a filtrator with join-closed core.

Proof. Cor’ \( a = \bigsqcup^3 \) down \( a = \bigsqcup^3 \) down \( a \subseteq a \).

Corollary 4.32. Cor’ \( a \in \) down \( a \) whenever Cor’ \( a \) exists for any element \( a \) of a filtrator with join-closed core.

Proposition 4.33. Cor’ \( a \subseteq \) Cor \( a \) whenever both Cor \( a \) and Cor’ \( a \) exist for any element \( a \) of a filtrator with join-closed core.

Proof. Cor \( a \) = \( \bigsqcup^3 \) up \( a \) \( \not\subseteq \) Cor’ \( a \) because \( \forall A \in \text{up} \ a: \text{Cor' } a \subseteq A \).

Theorem 4.34. Cor’ \( a = \) Cor \( a \) whenever both Cor \( a \) and Cor’ \( a \) exist for any element of a filtered filtrator.

Proof. It is with join-closed core because it is semifiltered. So Cor’ \( a \subseteq \) Cor \( a \). Cor \( a \in \) down \( a \). So Cor \( a \subseteq \bigsqcup^3 \) down \( a = \) Cor’ \( a \).

Obvious 4.35. Cor’ \( a = \) max \( \) down \( a \) for an element \( a \) of a filtrator with join-closed core.

4.2.2 Filtrators with Separable Core

Definition 4.36. Let \( (\mathfrak{A}; \mathfrak{3}) \) be a filtrator. It is a filtrator with separable core when
\[ \forall x, y \in \mathfrak{A}: (x \succeq^\mathfrak{A} y \Rightarrow \exists X \in \text{up} \ x: X \succeq^\mathfrak{A} y). \]

Proposition 4.37. Let \( (\mathfrak{A}; \mathfrak{3}) \) be a filtrator. It is a filtrator with separable core iff
\[ \forall x, y \in \mathfrak{A}: (x \succeq^\mathfrak{A} y \Rightarrow \exists X \in \text{up} \ x, Y \in \text{up} \ y: X \succeq^\mathfrak{A} Y). \]

Proof.
\[ \Rightarrow. \] Apply the definition twice.
\[ \Leftarrow. \] Obvious.

Definition 4.38. Let \( (\mathfrak{A}; \mathfrak{3}) \) be a filtrator. It is a filtrator with co-separable core when
\[ \forall x, y \in \mathfrak{A}: (x \preceq^\mathfrak{A} y \Rightarrow \exists X \in \text{down} \ x: X \preceq^\mathfrak{A} y). \]

Obvious 4.39. Co-separability is the dual of separability.

Proposition 4.40. Let \( (\mathfrak{A}; \mathfrak{3}) \) be a filtrator. It is a filtrator with co-separable core iff
\[ \forall x, y \in \mathfrak{A}: (x \preceq^\mathfrak{A} y \Rightarrow \exists X \in \text{down} \ x, Y \in \text{down} \ y: X \preceq^\mathfrak{A} Y). \]

Proof. By duality.

4.2.3 Intersection and Joining with an Element of the Core

Definition 4.41. I call down-aligned filtrator such a filtrator \( (\mathfrak{A}; \mathfrak{3}) \) that \( \mathfrak{A} \) and \( \mathfrak{3} \) have common least element. (Let’s denote it 0.)

Definition 4.42. I call up-aligned filtrator such a filtrator \( (\mathfrak{A}; \mathfrak{3}) \) that \( \mathfrak{A} \) and \( \mathfrak{3} \) have common greatest element. (Let’s denote it 1.)

Theorem 4.43. For a filtrator \( (\mathfrak{A}; \mathfrak{3}) \) where \( \mathfrak{3} \) is a boolean lattice, for every \( B \in \mathfrak{3}, A \in \mathfrak{A}:
1. \( B \sqsubseteq^\mathfrak{3} A \Leftrightarrow \overline{B} \sqsupseteq \overline{A} \) if it is down-aligned, with finitely meet-closed and separable core;
2. $B \supseteq^\exists A \Leftrightarrow \overline{B} \subseteq A$ if it is up-aligned, with finitely join-closed and co-separable core.

**Proof.** We will prove only the first as the second is dual.

\[
B \supseteq^\exists A \Leftrightarrow \exists A \in \text{up } A : B \supseteq^\exists A \Leftrightarrow \\
\exists A \in \text{up } A : B \cap^\exists A = 0 \Leftrightarrow \\
\exists A \in \text{up } A : B \cap^\exists A = 0 \Leftrightarrow \\
\exists A \in \text{up } A : \overline{B} \supseteq A \Leftrightarrow \\
\overline{B} \in \text{up } A \Leftrightarrow \\
\overline{B} \supseteq A.
\]

\[\Box\]

### 4.2.4 Characterization of Finitely Meet-Closed Filtrators

**Theorem 4.44.** The following are equivalent for a filtrator $(\mathfrak{A}; \mathfrak{S})$ whose core is a meet semilattice such that $\forall a \in \mathfrak{A}: \text{up } a \neq \emptyset$:

1. The filtrator is finitely meet-closed.
2. $\text{up } a$ is a filter for every $a \in \mathfrak{A}$.

**Proof.**

(1)$\Rightarrow$(2). Let $X, Y \in \text{up } a$. Then $X \cap^3 Y = X \cap^3 Y \supseteq a$. That $\text{up } a$ is an upper set is obvious. So taking into account that $\text{up } a \neq \emptyset$, $\text{up } a$ is a filter.

(2)$\Rightarrow$(1). It is enough to prove that $a \subseteq A, B \Rightarrow a \subseteq A \cap^3 B$ for every $A, B \in \mathfrak{A}$. Really:

\[
a \subseteq A, B \Rightarrow A, B \in \text{up } a \Rightarrow A \cap^3 B \in \text{up } a \Rightarrow a \subseteq A \cap^3 B. \]

\[\Box\]

### 4.2.5 Stars of Elements of Filtrators

**Definition 4.45.** Let $(\mathfrak{A}; \mathfrak{S})$ be a filtrator. **Core star** of an element $a$ of a filtrator is

\[
\partial a = \{ x \in \mathfrak{S} \mid x \nsubseteq^A a\}.
\]

**Proposition 4.46.** $\text{up } a \subseteq \partial a$ for any non-least element $a$ of a filtrator.

**Proof.** For any element $X \in \mathfrak{S}$

\[
X \in \text{up } a \Rightarrow a \subseteq X \wedge a \subseteq a \Rightarrow X \nsubseteq^A a \Rightarrow X \in \partial a. \]

\[\Box\]

**Theorem 4.47.** Let $(\mathfrak{A}; \mathfrak{S})$ be a distributive lattice filtrator with least element and finitely join-closed core which is a join semilattice. Then $\partial a$ is a free star for each $a \in \mathfrak{A}$.

**Proof.** For every $A, B \in \mathfrak{S}$

\[
A \cup^3 B \in \partial a \Leftrightarrow \\
A \cup^3 B \in \partial a \Leftrightarrow \\
(A \cup^3 B) \cap^3 a \neq 0^A \Leftrightarrow \\
(A \cap^3 a) \cup^3 (B \cap^3 a) \neq 0^A \Leftrightarrow \\
A \cap^3 a \neq 0^A \vee B \cap^3 a \neq 0^A \Leftrightarrow \\
A \in \partial a \vee B \in \partial a.
\]

That $\partial a$ doesn’t contain $0^\mathfrak{A}$ is obvious.  

\[\Box\]
Definition 4.48. I call a filtrator star-separable when its core is a separation subset of its base.

4.2.6 Atomic Elements of a Filtrator


Theorem 4.49. Let \((\mathfrak{A}; \mathfrak{Z})\) be a semifiltered down-aligned filtrator with finitely meet-closed core \(\mathfrak{Z}\) which is a meet-semilattice. Then \(a\) is an atom of \(\mathfrak{Z}\) iff \(a \in \mathfrak{Z}\) and \(a\) is an atom of \(\mathfrak{A}\).

Proof. Obvious.

We need to prove that if \(a\) is an atom of \(\mathfrak{Z}\) then \(a\) is an atom of \(\mathfrak{A}\). Suppose the contrary that \(a\) is not an atom of \(\mathfrak{A}\). Then there exists \(x \in \mathfrak{A}\) such that \(0 \neq x \sqsubset a\). Because “up” is a straight monotone map to the dual of the poset \(P\mathfrak{Z}\) (obvious 4.24), \(up a \subseteq up x\). So there exists \(K \in up x\) such that \(K \notin up a\). Also \(a \in up x\). We have \(K \cap ^{\mathfrak{Z}} a = K \cap ^{\mathfrak{A}} a \subseteq up x; K \cap ^{\mathfrak{Z}} a \neq 0\) and \(K \cap ^{\mathfrak{Z}} a \sqsubset a\). So \(a\) is not an atom of \(\mathfrak{Z}\).

Theorem 4.50. Let \((\mathfrak{A}; \mathfrak{Z})\) be a semifiltered down-aligned filtrator and \(\mathfrak{A}\) is a meet-semilattice. Then \(a \in \mathfrak{A}\) is an atom of \(\mathfrak{A}\) iff \(up a = \partial a\).

Proof. Let \(a\) be an atom of \(\mathfrak{A}\). \(up a \supseteq \partial a\) because \(a \neq 0\). \(up a \subseteq \partial a\) because for any \(K \in \mathfrak{A}\)

\[ K \in up a \Leftrightarrow K \supseteq a \Leftrightarrow K \cap ^{\mathfrak{A}} a \neq 0 \Leftrightarrow K \in \partial a. \]

\(\Leftarrow\). Let \(up a = \partial a\). Then \(a \neq 0\). Consequently for every \(x \in \mathfrak{A}\) we have

\[ 0 \sqsubset x \sqsubset a \Rightarrow \]
\[ x \cap ^{\mathfrak{A}} a \neq 0 \Rightarrow \]
\[ \forall K \in up x: K \in \partial a \Rightarrow \]
\[ \forall K \in up x: K \subseteq up a \Rightarrow \]
\[ up x \subseteq up a \Rightarrow \]
\[ x \supseteq a. \]

So \(a\) is an atom of \(\mathfrak{A}\).

4.2.7 Prime Filtrator Elements

Definition 4.51. Let \((\mathfrak{A}; \mathfrak{Z})\) be a down-aligned filtrator. Prime filtrator elements are such \(a \in \mathfrak{A}\) that \(up a\) is a free star.

Proposition 4.52. Let \((\mathfrak{A}; \mathfrak{Z})\) be a down-aligned filtrator with finitely join-closed core, where \(\mathfrak{A}\) is a starrish join-semilattice and \(\mathfrak{Z}\) is a join-semilattice. Then atomic elements of this filtrator are prime.

Proof. Let \(a\) be an atom of the lattice \(\mathfrak{A}\). We have for every \(X, Y \in \mathfrak{Z}\)

\[ X \cup ^{\mathfrak{Z}} Y \in up a \Leftrightarrow \]
\[ X \cup ^{\mathfrak{A}} Y \in up a \Leftrightarrow \]
\[ X \cup ^{\mathfrak{A}} Y \sqsubseteq a \Leftrightarrow \]
\[ X \cup ^{\mathfrak{A}} Y \neq ^{\mathfrak{A}} a \Leftrightarrow \]
\[ X \neq ^{\mathfrak{A}} a \lor Y \neq ^{\mathfrak{A}} a \Leftrightarrow \]
\[ X \supseteq a \lor Y \supseteq a \Leftrightarrow \]
\[ X \in up a \lor Y \in up a. \]

\(\square\)
4.2 Filtrators

4.2.8 Some Criteria

**Theorem 4.53.** For a semifiltered, star-separable, down-aligned filtrator \((\mathfrak{A}; \mathfrak{3})\) with finitely meet closed and separable core where \(\mathfrak{3}\) is a complete boolean lattice and both \(\mathfrak{3}\) and \(\mathfrak{A}\) are atomistic lattices the following conditions are equivalent for any \(F \in \mathfrak{A}\):

1. \(F \in \mathfrak{3}\);
2. \(\forall S \in \mathcal{P}\mathfrak{3}: (F \cap \mathfrak{3} S \neq \emptyset \Rightarrow \exists K \in S: F \cap \mathfrak{3} K \neq 0)\);
3. \(\forall S \in \mathcal{P}\mathfrak{3}: (F \cap \mathfrak{3} S \neq \emptyset \Rightarrow \exists K \in S: F \cap \mathfrak{3} K \neq 0)\).

**Proof.** Our filtrator is with join-closed core (theorem 4.25).

(1) \(\Rightarrow\) (2). Let \(F \in \mathfrak{3}\). Then (taking into account the proposition 4.43)

\[F \cap \mathfrak{3} S \neq \emptyset \Leftrightarrow \exists K \in S: F \neq \emptyset K \Leftrightarrow \exists K \in S: F \cap \mathfrak{3} K \neq 0.\]

(2) \(\Rightarrow\) (3). Obvious.

(3) \(\Rightarrow\) (1). Let the formula (3) be true. Then for \(L \in \mathfrak{3}\) and \(S = \text{atoms}^3 L\) it takes the form

\[F \cap \mathfrak{3} \text{atoms}^3 L \neq 0 \Rightarrow \exists K \in S: F \cap \mathfrak{3} K \neq 0\]

that is \(F \cap \mathfrak{3} L \neq 0 \Rightarrow \exists K \in S: F \cap \mathfrak{3} K \neq 0\) because \(\bigcup^3 \text{atoms}^3 L = \bigcup^3 \text{atoms}^3 L = L\). That is \(F \cap \mathfrak{3} L \neq 0 \Rightarrow F \cap \mathfrak{3} K_L \neq 0\) where \(K_L \in S\). Thus \(K_L\) is an atom of both \(\mathfrak{A}\) and \(\mathfrak{3}\) (see the theorem 4.49), so having \(F \cap \mathfrak{3} L \neq 0 \Rightarrow F \subseteq K_L\). Let

\[F = \bigcup^3 \{K_L \mid L \in \mathfrak{3}, F \cap \mathfrak{3} L \neq 0\}.\]

Then

\[F = \bigcup^3 \{K_L \mid L \in \mathfrak{3}, F \cap \mathfrak{3} L \neq 0\}.\]

Obviously \(F \subseteq F\). We have \(L \cap \mathfrak{3} F \neq 0 \Rightarrow K_L \cap \mathfrak{3} F \neq 0 \Rightarrow L \cap \mathfrak{3} F \neq 0 \Rightarrow L \cap \mathfrak{3} F \neq 0\), thus by star separability of our filtrator \(F \subseteq F\) and so \(F = F \in \mathfrak{3}\). \(\Box\)

**Definition 4.54.** Let \(S\) be a subset of a meet-semilattice. The filter base generated by \(S\) is the set

\([S]_{\cap} = \{a_0 \cap \ldots \cap a_n \mid a_i \in S, i = 0, 1, \ldots\}\).

**Lemma 4.55.** The set of all finite subsets of an infinite set \(A\) has the same cardinality as \(A\).

**Proof.** Let denote the number of \(n\)-element subsets of \(A\) as \(s_n\). Obviously \(s_n \leq \text{card} A^n = \text{card} A\). Then the number \(S\) of all finite subsets of \(A\) is equal to \(s_0 + s_1 + \ldots \leq \text{card} A \cdot \text{card} A + \ldots = \text{card} A\). That \(S \geq \text{card} A\) is obvious. So \(S = \text{card} A\). \(\Box\)

**Lemma 4.56.** A filter base generated by an infinite set has the same cardinality as that set.

**Proof.** From the previous lemma. \(\Box\)

**Definition 4.57.** Let \(\mathfrak{A}\) be a complete lattice. A set \(S \in \mathcal{P}\mathfrak{A}\) is filter-closed when for every filter base \(T \in \mathcal{F} S\) we have \(\bigcap T \in S\).

**Theorem 4.58.** A subset \(S\) of a complete lattice is filter-closed iff for every nonempty chain \(T \in \mathcal{F} S\) we have \(\bigcap T \in S\).

**Proof.** (proof sketch by Joel David Hamkins)

\(\Rightarrow\). Because every nonempty chain is a filter base.
We will assume that cardinality of a set is an ordinal defined by von Neumann cardinal assignment (what is a standard practice in ZFC). Recall that $\alpha < \beta \iff \alpha \in \beta$ for ordinals $\alpha, \beta$.

We will take it as given that for every nonempty chain $T \in \mathcal{P}S$ we have $\bigcap T \in S$.

We will prove the following statement: If $\text{card } S = n$ then $S$ is filter closed, for any cardinal $n$.

Instead we will prove it not only for cardinals but for wider class of ordinals: If $\text{card } S = n$ then $S$ is filter closed, for any ordinal $n$.

We will prove it using transfinite induction by $n$.

For finite $n$ we have $\prod T \in S$ because $T \subseteq S$ has minimal element.

Let $T = n$ be an infinite ordinal.

Let the assumption hold for every $m \in \text{card } T$.

We can assign $T = \{a_{\alpha} \mid \alpha \in \text{card } T\}$ for some $a_{\alpha}$ because $\text{card } T = \text{card } T$.

Consider $\beta \in \text{card } T$.

Let $P_{\beta} = \{a_{\alpha} \mid \alpha \in \beta\}$. Let $b_{\beta} = \bigcap P_{\beta}$. Obviously $b_{\beta} = \bigcap \{P_{\beta} | n\}$. We have

$$\text{card } P_{\beta} \cap = \text{card } P_{\beta} = \text{card } \beta < \text{card } T$$

(used the lemma and von Neumann cardinal assignment). By the assumption of induction $b_{\beta} \subseteq S$.

$\forall \beta \in \text{card } T: P_{\beta} \subseteq T$ and thus $b_{\beta} \supseteq \prod T$.

It is easy to see that the set $\{P_{\beta} \mid \beta \in \text{card } T\}$ is a chain. Consequently $\{b_{\beta} \mid \beta \in \text{card } T\}$ is a chain.

By the theorem conditions $b = \bigcap \{b_{\beta} \mid \beta \in \text{card } T\} \in S$ (taken into account that $b_{\beta} \in S$ by the assumption of induction).

Obviously $b \supseteq \prod T$.

$b \subseteq b_{\beta}$ and so $\forall \beta \in \text{card } T, a_{\alpha} \in \beta; b \subseteq a_{\alpha}$. Let $\alpha \in \text{card } T$. Then (because card $T$ is limit ordinal, see [41]) there exists $\beta \in \text{card } T$ such that $\alpha \in \beta \in \text{card } T$. So $b \subseteq a_{\alpha}$ for every $\alpha \in \text{card } T$. Thus $b \subseteq \prod T$.

Finally $\prod T = b \subseteq S$.

4.2.9 Complements and Core Parts

Lemma 4.59. If $(\mathfrak{A}, \mathfrak{F})$ is a filtered, up-aligned filtrator with co-separable core which is a complete lattice, then for any $a, c \in \mathfrak{A}$

$$c \equiv^\mathfrak{A} a \iff c \equiv^\mathfrak{A} \text{ Cor } a.$$ 

Proof.

$\Rightarrow$. If $c \equiv^\mathfrak{A} a$ then by co-separability of the core exists $K \in \text{down } a$ such that $c \equiv^\mathfrak{A} K$. To finish the proof we will show that $K \subseteq \text{ Cor } a$. To show this is enough to show that $\forall X \in \text{up } a: K \subseteq X$ what is obvious.

$\Leftarrow$. Cor $a \subseteq a$ (by the theorem 4.29 using that our filtrator is filtered).

Theorem 4.60. If $(\mathfrak{A}, \mathfrak{F})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+ = \text{ Cor } a$ for every $a \in \mathfrak{A}$.

Proof. Our filtrator is with join-closed core (theorem 4.25).

$$a^+ = \bigcap \{c \in \mathfrak{A} \mid c \cup^\mathfrak{A} a = 1^\mathfrak{A}\} = \bigcap \{c \in \mathfrak{A} \mid c \cup^\mathfrak{A} \text{ Cor } a = 1^\mathfrak{A}\} = \bigcap \{c \in \mathfrak{A} \mid c \supseteq \text{ Cor } a\} = \text{ Cor } a$$
Corollary 4.61. If $(\mathfrak{A}; \mathfrak{3})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^* \in \mathfrak{3}$ for every $a \in \mathfrak{A}$.

Theorem 4.62. If $(\mathfrak{A}; \mathfrak{3})$ is a filtered complete lattice filtrator with down-aligned, finitely meet-closed, separable core which is a complete boolean lattice, then $a^* = \overline{\text{Cor } a} = \overline{\text{Cor}' a}$ for every $a \in \mathfrak{A}$.

Proof. Our filtrator is with join-closed core (Theorem 4.25). $a^* = \bigcup_{a} \{ c \in \mathfrak{A} \mid c \cap a = 0^\mathfrak{A} \}$. But $c \cap a = 0^\mathfrak{A} \Rightarrow \exists C \in \text{up } c: C \cap a = 0^\mathfrak{A}$. So

$$a^* = \bigcup_{a} \{ C \in \mathfrak{3} \mid C \cap a = 0^\mathfrak{A} \} = \bigcup_{a} \{ C \in \mathfrak{3} \mid a \subseteq \overline{C} \} = \bigcup_{a} \{ C \in \mathfrak{3}, a \subseteq C \} = \bigcup_{3} \{ C \mid C \cap a = 0^\mathfrak{A} \} = \bigcup_{3} \{ C \mid C \subseteq \text{up } a \} = \bigcup_{\text{Cor } a} \text{up } a$$

(used theorem 4.43).

Cor $a = \text{Cor}' a$ by the theorem 4.34.

Corollary 4.63. If $(\mathfrak{A}; \mathfrak{3})$ is a filtered down-aligned and up-aligned complete lattice filtrator with finitely meet-closed, separable and co-separable core which is a complete boolean lattice, then $a^* = a^+$ for every $a \in \mathfrak{A}$.

Proof. Comparing two last theorems.

Theorem 4.64. If $(\mathfrak{A}; \mathfrak{3})$ is a complete lattice filtrator with join-closed separable core which is a complete lattice, then $a^* \in \mathfrak{3}$ for every $a \in \mathfrak{A}$.

Proof. \{ $c \in \mathfrak{A} \mid c \cap a = 0^\mathfrak{A}$ \} $\supseteq$ \{ $A \in \mathfrak{3} \mid A \cap a = 0^\mathfrak{A}$ \}; consequently $a^* \supseteq \bigcup_{a} \{ A \in \mathfrak{3} \mid A \cap a = 0^\mathfrak{A} \}$.

But if $c \in \{ c \in \mathfrak{A} \mid c \cap a = 0^\mathfrak{A} \}$ then there exists $A \in \mathfrak{3}$ such that $A \supseteq c$ and $A \cap a = 0^\mathfrak{A}$ that is $A \in \{ A \in \mathfrak{3} \mid A \cap a = 0^\mathfrak{A} \}$. Consequently $a^* \supseteq \bigcup_{a} \{ A \in \mathfrak{3} \mid A \cap a = 0^\mathfrak{A} \}$. We have $a^* = \bigcup_{a} \{ A \in \mathfrak{3} \mid A \cap a = 0^\mathfrak{A} \} = \bigcup_{a} \{ A \in \mathfrak{3} \mid A \cap a = 0^\mathfrak{A} \} \in \mathfrak{3}$.

Theorem 4.65. If $(\mathfrak{A}; \mathfrak{3})$ is an up-aligned filtered complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+$ is dual pseudocomplement of $a$, that is

$$a^+ = \min \{ c \in \mathfrak{A} \mid c \cup a = 1^\mathfrak{A} \}$$

for every $a \in \mathfrak{A}$.

Proof. Our filtrator is with join-closed core (Theorem 4.25). It’s enough to prove that $a^+ \cup a = 1^\mathfrak{A}$. But $a^+ \cup a = \overline{\text{Cor } a} \cup \overline{a} \supseteq \overline{\text{Cor } a} \cup \overline{a} \supseteq \text{Cor } a = \overline{\text{Cor } a} \cup 1^\mathfrak{A} \supseteq 1^\mathfrak{A}$ (used the theorem 4.29 and the fact that our filtrator is filtered).

Definition 4.66. The edge part of an element $a \in \mathfrak{A}$ is $\text{Edg } a = a \setminus \text{Cor } a$, the dual edge part is $\text{Edg}' a = a \setminus \text{Cor}' a$. 

Knowing core part and edge part or dual core part and dual edge part of a filter, the filter can be restored by the formulas:

\[ a = \text{Cor} a \sqcup \text{Edg} a \quad \text{and} \quad a = \text{Cor}' a \sqcup \text{Edg}' a. \]

### 4.2.10 Core Part and Atomic Elements

**Proposition 4.67.** Let \((\mathfrak{A}; \mathfrak{Z})\) be a filtrator with join-closed core and \(\mathfrak{Z}\) is an atomistic lattice. Then for every \(a \in \mathfrak{A}\) such that \(\text{Cor}^0 a\) exists we have

\[ \text{Cor}' a = \bigcup \left\{ x \mid x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a \right\}. \]

**Proof.**

\[ \text{Cor}' a = \bigcup \left\{ A \in \mathfrak{Z} \mid A \sqsubseteq a \right\} = \bigcup \left\{ \bigcup \left\{ \text{atoms}^3 A \mid A \in \mathfrak{Z}, A \sqsubseteq a \right\} \right\} = \bigcup \left\{ \text{atoms}^3 A \mid A \in \mathfrak{Z}, A \sqsubseteq a \right\} = \bigcup \left\{ x \mid x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a \right\}. \]

### 4.2.11 Distributivity of Core Part over Lattice Operations

**Theorem 4.68.** If \((\mathfrak{A}; \mathfrak{Z})\) is a join-closed filtrator and \(\mathfrak{A}\) is a meet-semilattice and \(\mathfrak{Z}\) is a complete lattice, then for every \(a, b \in \mathfrak{A}\)

\[ \text{Cor}' (a \sqcap \mathfrak{A} b) = \text{Cor}' a \sqcap \mathfrak{Z} \text{Cor}' b. \]

**Proof.** From theorem conditions it follows that \(\text{Cor}' (a \sqcap \mathfrak{A} b)\) exists.

We have \(\text{Cor}' p \sqsubseteq p\) for every \(p \in \mathfrak{A}\) because our filtrator is with join-closed core.

Obviously \(\text{Cor}' (a \sqcap \mathfrak{A} b) \sqsubseteq \text{Cor}' a\) and \(\text{Cor}' (a \sqcap \mathfrak{A} b) \sqsubseteq \text{Cor}' b\).

If \(x \sqsubseteq \text{Cor}' a\) and \(x \sqsubseteq \text{Cor}' b\) for some \(x \in \mathfrak{Z}\) then \(x \sqsubseteq a\) and \(x \sqsubseteq b\), thus \(x \sqsubseteq a \sqcap \mathfrak{A} b\) and \(x \sqsubseteq \text{Cor}' (a \sqcap \mathfrak{A} b)\).

**Theorem 4.69.** If \((\mathfrak{A}; \mathfrak{Z})\) is a join-closed filtrator and both \(\mathfrak{A}\) and \(\mathfrak{Z}\) are complete lattices, then for every \(S \in \mathfrak{A}\)

\[ \text{Cor}' \bigcap \mathfrak{A} S = \bigcap \left\{ \text{Cor}' S \right\}. \]

**Proof.** From theorem conditions it follows that \(\text{Cor}' \bigcap \mathfrak{A} S\) exists.

We have \(\text{Cor}' p \sqsubseteq p\) for every \(p \in \mathfrak{A}\) because our filtrator is with join-closed core.

Obviously \(\text{Cor}' \bigcap \mathfrak{A} S \sqsubseteq \text{Cor}' a\) for every \(a \in S\).

If \(x \sqsubseteq \text{Cor}' a\) for every \(a \in S\) for some \(x \in \mathfrak{Z}\) then \(x \sqsubseteq a\), thus \(x \sqsubseteq \bigcap \mathfrak{A} S\) and \(x \sqsubseteq \text{Cor}' \bigcap \mathfrak{A} S\).

**Corollary 4.70.** If \((\mathfrak{A}; \mathfrak{Z})\) is a join-closed filtrator and both \(\mathfrak{A}\) and \(\mathfrak{Z}\) are complete lattices, then for every \(S \in \mathfrak{Z}\)

\[ \text{Cor}' \bigcap \mathfrak{Z} S = \bigcap S. \]
Theorem 4.71. Let $(\mathcal{A}; 3)$ be a semifiltered down-aligned filtrator with finitely meet-closed core $\mathfrak{3}$ which is a complete atomistic lattice and $\mathcal{A}$ is a complete starshaped lattice, then $\text{Cor}^\prime(a \uplus^3 b) = \text{Cor}^\prime a \uplus^3 \text{Cor}^\prime b$ for every $a, b \in \mathcal{A}$.

**Proof.** From theorem conditions it follows that $\text{Cor}^\prime(a \uplus^3 b)$ exists.

$\text{Cor}^\prime(a \uplus^3 b) = \bigcup^3 \{ x \mid x \text{ is an atom of } \mathfrak{3}, x \subseteq a \uplus^3 b \}$ (used proposition 4.67).

By theorem 4.49 we have $\text{Cor}^\prime(a \uplus^3 b) = \bigcup^3 ((\text{atoms}^\mathfrak{3} a \uplus \text{atoms}^\mathfrak{3} b) \cap 3) = \bigcup^3 ((\text{atoms}^\mathfrak{3} a \cap 3) \cup (\text{atoms}^\mathfrak{3} b \cap 3)) = \bigcup^3 (\text{atoms}^\mathfrak{3} a \cap 3) \cup \bigcup^3 (\text{atoms}^\mathfrak{3} b \cap 3)$ (used the theorem 3.30). Again using theorem 4.49, we get

$\text{Cor}^\prime(a \uplus^3 b) = \bigcup^3 \{ x \mid x \text{ is an atom of } \mathfrak{3}, x \subseteq a \} \cup \bigcup^3 \{ x \mid x \text{ is an atom of } \mathfrak{3}, x \subseteq b \} = \text{Cor}^\prime a \uplus^3 \text{Cor}^\prime b$ (again used proposition 4.67).

Theorem 4.72. Let $(\mathcal{A}; 3)$ be a filtered starshaped down-aligned complete filtrator with finitely meet-closed, separable core which is a complete atomistic boolean lattice. Then $(a \uplus^3 b)^* = a^* \cap^3 b^*$.

**Proof.** $(a \uplus^3 b)^* = \text{Cor}^\prime(a \uplus^3 b) = \text{Cor}^\prime a \uplus^3 \text{Cor}^\prime b = \text{Cor}^\prime a \cap^3 \text{Cor}^\prime b = a^* \cap^3 b^*$ (used theorem 4.62).

### 4.2.12 Co-Separability of Core

Theorem 4.73. Let $(\mathcal{A}; 3)$ be an up-aligned filtered filtrator whose core is a meet infinite distributive complete lattice. Then this filtrator is with co-separable core.

**Proof.** Our filtrator is with join-closed core (theorem 4.25).

Let $a, b \in \mathcal{A}$. Cor $a$ and Cor $b$ exist since $\mathfrak{3}$ is a complete lattice.

Cor $a \in \text{down} a$ and Cor $b \in \text{down} b$ by the corollary 4.30 since our filtrator is filtered. So we have

$\exists x \in \text{down} a, y \in \text{down} b : x \uplus^\mathcal{A} y = 1 \iff$

Cor $a \uplus^3 \text{Cor} b = 1 \iff$ (by finite join-closedness of the core)

Cor $a \cap^3 \text{Cor} b = 1 \iff$

$\bigcup^3 \text{up} a \cap^3 \bigcup^3 \text{up} b = 1 \iff$ (by infinite distributivity)

$\bigcup^3 \{ x \mid x \in \text{up} a, y \in \text{up} b \} = 1 \iff$

$\forall x \in \text{up} a, y \in \text{up} b : x \cap^3 y = 1 \iff$ (by finite join-closedness of the core)

$\forall x \in \text{up} a, y \in \text{up} b : x \cap^3 y = 1 \iff$

$a \cap^3 b = 1$.

4.2.13 Filtrators over Boolean Lattices

Proposition 4.74. Let $(\mathcal{A}; 3)$ be a down-aligned and up-aligned finitely meet-closed and finitely join-closed distributive lattice filtrator and $\mathfrak{3}$ is a boolean lattice. Then $a \setminus^\mathfrak{3} B = a \cap^\mathfrak{3} B$ for every $a \in \mathfrak{3}$, $B \in \mathfrak{3}$.

**Proof.** $(a \cap^\mathfrak{3} B) \setminus^\mathfrak{3} B = (a \cap^\mathfrak{3} B) \cap^\mathfrak{3} (B \setminus^\mathfrak{3} B) = (a \cap^\mathfrak{3} B) \cap^\mathfrak{3} (B \setminus^3 B) = (a \cap^\mathfrak{3} B) \cap^\mathfrak{3} 1 = a \cap^\mathfrak{3} B$.

$(a \cap^\mathfrak{3} B) \cap^\mathfrak{3} B = a \cap^\mathfrak{3} (B \cap^\mathfrak{3} B) = a \cap^\mathfrak{3} (B \cap^3 B) = a \cap^\mathfrak{3} 0 = 0$.

So $a \cap^\mathfrak{3} B$ is the difference of $a$ and $B$.

### 4.2.13.1 Distributivity for an Element of Boolean Core

Lemma 4.75. Let $(\mathcal{A}; 3)$ be an up-aligned finitely join-closed and finitely meet-closed distributive lattice filtrator over a boolean lattice. Then $\mathcal{A} \cap^\mathfrak{3} B$ is a lower adjoint of $\mathcal{A} \cap^\mathfrak{3}$ for every $A \in \mathfrak{3}$.

**Proof.** We will use the theorem 2.98.
That $A \cap \mathbb{A}$ and $A \cup \mathbb{A}$ are monotone is obvious. We need to prove (for every $x, y \in \mathbb{A}$) that
\[ x \subseteq A \cup \mathbb{A} (A \cap \mathbb{A} x) \quad \text{and} \quad A \cap \mathbb{A} (A \cup \mathbb{A} y) \subseteq y. \]
Really, $A \cup \mathbb{A} (A \cap \mathbb{A} x) = (A \cup \mathbb{A} A) \cap \mathbb{A} (A \cup \mathbb{A} x) = 1 \cap \mathbb{A} (A \cup \mathbb{A} x) = A \cup \mathbb{A} x \supseteq x$ and $A \cap \mathbb{A} (A \cup \mathbb{A} y) = (A \cap \mathbb{A} A) \cup \mathbb{A} (A \cap \mathbb{A} y) = 1 \cup \mathbb{A} (A \cap \mathbb{A} y) = A \cap \mathbb{A} y \subseteq y$. \( \square \)

**Theorem 4.76.** Let $(\mathbb{A}; 3)$ be an up-aligned finitely join-closed and finitely meet-closed distributive lattice filtrator over a boolean lattice. Then $A \cap \mathbb{A} \bigcup_{A \in 3} S = \bigcup_{A \in 3} (A \cap \mathbb{A})S$ for every $A \in 3$ and every set $S \in \mathcal{P}A$.

**Proof.** Direct consequence of the lemma. \( \square \)

## 4.3 Filters on a poset

### 4.3.1 Filters on posets

Let $3$ be a poset.

**Definition 4.77.** Filter base is a nonempty subset $F$ of $3$ such that
\[ \forall X, Y \in F \exists Z \in F: (Z \subseteq X \wedge Z \subseteq Y). \]

**Obvious 4.78.** A nonempty chain is a filter base.

**Definition 4.79.** Filter is a subset of $3$ which is both a filter base and an upper set.

I will denote the set of filters (for a given or implied poset $3$) as $\mathfrak{F}$ and call $\mathfrak{F}$ the set of filters over the poset $3$.

**Proposition 4.80.** If $1$ is the maximal element of $3$ then $1 \in F$ for every filter $F$.

**Proof.** If $1 \notin F$ then $\forall K \in 3: K \notin F$ and so $F$ is empty what is impossible. \( \square \)

**Proposition 4.81.** Let $S$ be a filter base on a poset. If $A_0, ..., A_n \in S$ ($n \in \mathbb{N}$), then
\[ \exists C \in S: (C \subseteq A_0 \wedge ... \wedge C \subseteq A_n). \]

**Proof.** It can be easily proved by induction. \( \square \)

Dual of filters is called ideals. We do not use ideals in this work however.

### 4.3.2 Filters on meet-semilattices

**Theorem 4.82.** If $3$ is a meet-semilattice and $F$ is a nonempty subset of $3$ then the following conditions are equivalent:
1. $F$ is a filter.
2. $\forall X, Y \in F: X \cap Y \in F$ and $F$ is an upper set.
3. $\forall X, Y \in 3: (X, Y \in F \Leftrightarrow X \cap Y \in F)$.

**Proof.**

$(1) \Rightarrow (2)$. Let $F$ be a filter. Then $F$ is an upper set. If $X, Y \in F$ then $Z \subseteq X \wedge Z \subseteq Y$ for some $Z \in F$. Because $F$ is an upper set and $Z \subseteq X \cap Y$ then $X \cap Y \in F$.
(2)⇒(1). Let \( \forall X, Y \in F: X \cap Y \in F \) and \( F \) be an upper set. We need to prove that \( F \) is a filter base. But it is obvious taking \( Z = X \cap Y \) (we have also taken into account that \( F \neq \emptyset \)).

(2)⇒(3). Let \( \forall X, Y \in F: X \cap Y \in F \) and \( F \) be an upper set. Then

\[
\forall X, Y \in \mathfrak{Z}: (X, Y \in F \Rightarrow X \cap Y \in F).
\]

Let \( X \cap Y \in F \); then \( X, Y \in F \) because \( F \) is an upper set.

(3)⇒(2). Let

\[
\forall X, Y \in \mathfrak{Z}: (X, Y \in F \Leftrightarrow X \cap Y \in F).
\]

Then \( \forall X, Y \in F: X \cap Y \in F \). Let \( X \in F \) and \( X \subseteq Y \in \mathfrak{Z} \). Then \( X \cap Y = X \in F \). Consequently \( X, Y \in F \). So \( F \) is an upper set.

**Proposition 4.83.** Let \( S \) be a filter base on a meet-semilattice. If \( A_0, ..., A_n \in S \ (n \in \mathbb{N}) \), then

\[
\exists C \in S: C \subseteq A_0 \cap ... \cap A_n.
\]

**Proof.** It can be easily proved by induction. \( \Box \)

**Proposition 4.84.** If \( \mathfrak{Z} \) is a meet-semilattice and \( S \) is a filter base on it, \( A \in \mathfrak{Z} \), then \( \langle A \cap \rangle S \) is also a filter base.

**Proof.** \( \langle A \cap \rangle S \neq \emptyset \) because \( S \neq \emptyset \).

Let \( X, Y \in \langle A \cap \rangle S \). Then \( X = A \cap X' \) and \( Y = A \cap Y' \) where \( X', Y' \in S \). There exists \( Z' \in S \) such that \( Z' \subseteq X' \cap Y' \). So \( X \cap Y = A \cap X' \cap Y' \subseteq A \cap Z' \in \langle A \cap \rangle S \). \( \Box \)

### 4.3.3 Order of filters. Principal filters

I will make the set of filters \( \mathfrak{F} \) into a poset by the order defined by the formula: \( a \subseteq b \Leftrightarrow a \supseteq b \).

**Definition 4.85.** The principal filter corresponding to an element \( a \in \mathfrak{Z} \) is

\[
\uparrow a = \{ x \in \mathfrak{Z} | x \supseteq a \}.
\]

Elements of \( \mathfrak{P} = \langle \uparrow \rangle \mathfrak{Z} \) are called principal filters.

**Obvious 4.86.** Principal filters are filters.

**Obvious 4.87.** \( \uparrow \) is an order embedding from \( \mathfrak{Z} \) to \( \mathfrak{F} \).

**Corollary 4.88.** \( \uparrow \) is an order isomorphism between \( \mathfrak{Z} \) and \( \mathfrak{P} \).

**Definition 4.89.** For every poset \( \mathfrak{Z} \) I call \( \langle \mathfrak{Z}; \mathfrak{P} \rangle \) the primary filtrator (for the base \( \mathfrak{Z} \)).

**Proposition 4.90.** \( \uparrow K \sqsupseteq A \Leftrightarrow K \in A \).

**Proof.** \( \uparrow K \sqsupseteq A \Leftrightarrow \uparrow K \subseteq A \Leftrightarrow K \in A \). \( \Box \)

**Proposition 4.91.** \( \uparrow a = \langle \uparrow \rangle a \) for an element \( a \) of a primary filtrator.

**Proof.** For every \( L \in \mathfrak{P} \) we have \( L = \uparrow K \) for some \( K \in \mathfrak{Z} \) and \( L \in \uparrow a \Leftrightarrow L \supseteq a \Leftrightarrow \uparrow K \supseteq a \Leftrightarrow K \in a \Leftrightarrow L \in \langle \uparrow \rangle a \). \( \Box \)

### 4.3.3.1 Minimal and maximal filters

**Obvious 4.92.** The filter \( 0^\mathfrak{Z} = \mathfrak{Z} \) (equal to the principal filter for the least element of \( \mathfrak{Z} \) if it exists) is the least element of the poset of filters.

**Proposition 4.93.** If there exists greatest element \( 1^\mathfrak{Z} \) of the poset \( \mathfrak{Z} \) then \( 1^\mathfrak{Z} = \{1^\mathfrak{Z}\} \) is the greatest element of \( \mathfrak{Z} \).
Proof. Take into account that filters are nonempty.

4.3.4 Primary filtrator is filtered

Theorem 4.94. \( \mathcal{A}_\mathbb{\mathcal{F}} \mathcal{A} \) for every filter \( \mathcal{A} \) on a poset.

Proof. \( \mathcal{A} \) is obviously a lower bound for \( \mathcal{A} \). Let \( \mathcal{B} \) be a lower bound for \( \mathcal{A} \) that is
\[
\forall K \in \mathcal{A} : K \subseteq \uparrow K
\]
that is \( \forall K \in \mathcal{A} : K \subseteq \mathcal{B} \) that is \( \mathcal{B} \subseteq \mathcal{A} \). So \( \mathcal{A} \) is the greatest lower bound for \( \mathcal{A} \).

Corollary 4.95. Every primary filtrator is filtered.

Corollary 4.96. Every primary filtrator is with join-closed core.

Proof. Theorem 4.25.

Proposition 4.97. The filtrator \( \mathcal{G} \mathcal{A} \) is with finitely meet-closed core if \( \mathcal{G} \) is a meet-semilattice.

Proof. Theorem 4.44.

4.3.5 Alignment

Obvious 4.98.

1. If \( \mathcal{G} \) has least element, the primary filtrator is down-aligned.
2. If \( \mathcal{G} \) has greatest element, the primary filtrator is up-aligned.

4.3.6 Co-separability of Core for Primary Filtrators

Proposition 4.99. Every primary filtrator over a meet infinite distributive complete lattice is with co-separable core.

Proof. It is up-aligned, filtered. So we can apply the theorem 4.73.

4.3.7 Core Part

Proposition 4.100. \( \text{Cor} a = \text{Cor} a \) for every filter \( a \) on a complete lattice.

Proof. By the theorem 4.34 and corollary 4.95.

Proposition 4.101. \( \text{Cor} a \subseteq a \) for every filter \( a \) on a complete lattice.

Proof. By the theorem 4.29 and corollary 4.95.

Proposition 4.102. \( \text{Cor} a = \text{max} \text{down} a \) for every filter \( a \) on a complete lattice.


4.3.8 Intersecting and Joining with an Element of the Core

Theorem 4.103. For a filtrator \( \mathcal{G} \mathcal{A} \mathcal{F} \) where \( \mathcal{G} \) is a boolean lattice, for every \( B \in \mathcal{F} \), \( A \in \mathcal{G} \):

1. \( B \subseteq A \Leftrightarrow B \supseteq A \);
2. \( B \supseteq A \Leftrightarrow B \subseteq A \) if \( \mathcal{G} \) is a complete lattice.
Proof.
2. Using theorem 4.43, obvious 4.98, corollary 4.96, theorem 4.73.

4.3.9 Formulas for Meets and Joins of Filters

Lemma 4.104. If \( f \) is an order embedding from a poset \( \mathfrak{A} \) to a complete lattice \( \mathfrak{B} \) and \( S \in \mathcal{P}\mathfrak{A} \) and there exists such \( F \in \mathfrak{A} \) that \( fF = \bigcup^{\mathfrak{B}} \langle f \rangle S \), then \( \bigcup^{\mathfrak{B}} S \) exists and \( f\bigcup^{\mathfrak{B}} S = \bigcup^{\mathfrak{B}} \langle f \rangle S \).

Proof. \( f \) is an order isomorphism from \( \mathfrak{A} \) to \( \mathfrak{B}_{\langle f \rangle} \). Consequently, \( \bigcup^{\mathfrak{B}} \langle f \rangle S \in \mathfrak{B}_{\langle f \rangle} \) and \( \bigcup^{\mathfrak{B}} \langle f \rangle S = \bigcup^{\mathfrak{B}} \langle f \rangle S \).

Combining, \( f\bigcup^{\mathfrak{B}} S = \bigcup^{\mathfrak{B}} \langle f \rangle S \).

Corollary 4.105. If \( \mathfrak{B} \) is a complete lattice and \( \mathfrak{A} \) is its subset and \( S \in \mathcal{P}\mathfrak{A} \) and \( \bigcup^{\mathfrak{B}} S \in \mathfrak{A} \), then \( \bigcup^{\mathfrak{B}} S \) exists and \( \bigcup^{\mathfrak{B}} S = \bigcup^{\mathfrak{B}} \langle f \rangle S \).

Theorem 4.106. If \( \mathfrak{F} \) is a meet-semilattice with greatest element \( 1 \) then \( \bigcap^{\mathfrak{F}} S \) exists and

\[
\bigcap^{\mathfrak{F}} S = \bigcap S
\]

for every \( S \in \mathcal{P}\mathfrak{F} \).

Proof. Taking into account the corollary of the lemma, it is enough to prove that there exists \( F \in \mathfrak{F} \) such that \( F = \bigcap S \), that is that \( \bigcap S \) is a filter.

\( R \) is nonempty because \( 1 \in R \). Let \( A, B \in R \); then \( \forall F \in S: A, B \in F \), consequently \( \forall F \in S: A \cap B \in F \). Consequently \( A \cap B \in \bigcap S = R \). So \( R \) is a filter base. Let \( X \in R \) and \( X \subseteq Y \in \mathfrak{F} \); then \( \forall F \in S: X \in F \); \( \forall F \in S: Y \in F \); \( Y \in R \). So \( R \) is an upper set.

Corollary 4.107. If \( \mathfrak{F} \) is a meet-semilattice with greatest element then \( \mathfrak{F} \) is a complete lattice.

Corollary 4.108. If \( \mathfrak{F} \) is a meet-semilattice with greatest element then for any \( A, B \in \mathfrak{F} \)

\[
\mathfrak{F} \cap^{\mathfrak{F}} B = \mathfrak{F} \cap B.
\]

We will denote meets and joins on the lattice of filters just as \( \cap \) and \( \cup \).

Theorem 4.109. If \( \mathfrak{F} \) is a join-semilattice then \( \mathfrak{F} \) is a join-semilattice and for any \( A, B \in \mathfrak{F} \)

\[
\mathfrak{F} \cup^{\mathfrak{F}} B = \mathfrak{F} \cup B.
\]

Proof. Taking into account the corollary of the lemma, it is enough to prove \( R = \mathfrak{F} \cap B \) is a filter.

\( R \) is nonempty because there exists \( X \in A \) and \( Y \in B \) and \( X \cup^{\mathfrak{F}} Y \).

Let \( A, B \in R \). Then \( A, B \in \mathfrak{F} \); so there exists \( C \subseteq A \) such that \( C \subseteq A \cap C \subseteq B \). Analogously there exists \( D \subseteq B \) such that \( D \subseteq A \cap D \subseteq B \). Let \( E = C \cup^{\mathfrak{F}} D \). Then \( E \in \mathfrak{F} \) and \( E \in B \); \( E \in R \) and \( E \subseteq A \cap E \subseteq B \). So \( R \) is a filter base.

That \( R \) is an upper set is obvious.

Theorem 4.110. If \( \mathfrak{F} \) is a distributive lattice then for \( S \in \mathcal{P}\mathfrak{F} \) \( \setminus \{0\} \)

\[
\bigcap^{\mathfrak{F}} S = \{ K_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} K_n \ | \ K_i \in \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N} \}.
\]

Proof. Let’s denote the right part of the equality to be proven as \( R \). First we will prove that \( R \) is a filter. \( R \) is nonempty because \( S \) is nonempty.

Let \( A, B \in R \). Then \( A = X_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} X_k, B = Y_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} Y_l \) where \( X_i, Y_j \in \bigcup S \). So

\[
A \cap^{\mathfrak{F}} B = X_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} X_k \cap^{\mathfrak{F}} Y_0 \cap^{\mathfrak{F}} \ldots \cap^{\mathfrak{F}} Y_l \in R.
\]
Let \( R \supseteq C \supseteq A \). Consequently (distributivity used)
\[
C = C \sqcup^3 A = (C \sqcup^3 X_0) \cap^3 \ldots \cap^3 (C \sqcup^3 X_k).
\]
\( X_i \in P_i \) for some \( P_i \in S; C \sqcup^3 X_i \in P_i \); \( C \sqcup^3 X_i \in \bigcup S \); consequently \( C \in R \).

We have proved that that \( R \) is a filter base and an upper set. So \( R \) is a filter.

Let \( A \in S \). Then \( A \subseteq \bigcup S \):
\[
R \supseteq \{ K_0 \cap^3 \ldots \cap^3 K_n \mid K_i \in A \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N} \} = A.
\]
Consequently \( A \supseteq R \).

Let now \( B \in \mathcal{F} \) and \( \forall A \in S; A \supseteq B \). Then \( \forall A \in S; A \subseteq B; B \supseteq \bigcup S \). From this \( B \supseteq T \) for every finite set \( T \subseteq \bigcup S \). Consequently \( B \supseteq \bigcap T \). Thus \( B \supseteq R; B \subseteq R \).

Comparing we get \( \bigcap^3 S = R. \) \( \square \)

**Theorem 4.111.** If \( \mathcal{F} \) is a distributive lattice then for any \( \mathcal{F}_0, \ldots, \mathcal{F}_m \in \mathcal{F} \) \( m \in \mathbb{N} \)
\[
\mathcal{F}_0 \cap^3 \ldots \cap^3 \mathcal{F}_m = \{ K_0 \cap^3 \ldots \cap^3 K_m \mid K_i \in \mathcal{F}_i \text{ where } i = 0, \ldots, n \text{ for } m \in \mathbb{N} \}.
\]

**Proof.** Let’s denote the right part of the equality to be proven as \( R \). First we will prove that \( R \) is a filter. Obviously \( R \) is nonempty.

Let \( A, B \in R \). Then \( A = X_0 \cap^3 \ldots \cap^3 X_m, B = Y_0 \cap^3 \ldots \cap^3 Y_m \) where \( X_i, Y_i \in \mathcal{F}_i \).
\[
A \cap^3 B = (X_0 \cap^3 Y_0) \cap^3 \ldots \cap^3 (X_m \cap^3 Y_m),
\]
consequently \( A \cap^3 B \in R \).

Let \( R \supseteq C \supseteq A \).
\[
C = A \sqcup^3 C = (X_0 \sqcup^3 C) \cap^3 \ldots \cap^3 (X_m \sqcup^3 C) \in R.
\]
So \( R \) is a filter.

Let \( P_i \in \mathcal{F}_i \). Then \( P_i \in R \) because \( P_i = (P_i \sqcup^3 P_0) \cap^3 \ldots \cap^3 (P_i \sqcup^3 P_m) \). So \( \mathcal{F}_i \subseteq R; \mathcal{F}_i \supseteq R \).

Let now \( B \in \mathcal{F} \) and \( \forall i \in \{0, \ldots, m\}; \mathcal{F}_i \supseteq B \). Then \( \forall i \in \{0, \ldots, m\}; \mathcal{F}_i \subseteq B \).

\( L_i \in B \) for every \( L_i \in \mathcal{F}_i \). \( L_0 \cap^3 \ldots \cap^3 L_m \in B \). So \( B \supseteq R; B \subseteq R \).

So \( \mathcal{F}_0 \cap^3 \ldots \cap^3 \mathcal{F}_m = R. \) \( \square \)

### 4.3.10 Separability of Core for Primary Filtrators

**Theorem 4.112.** A primary filtrator with least element, whose core is a distributive lattice, is with separable core.

**Proof.** Let \( A \equiv^3 B \) where \( A, B \in \mathcal{F} \).
\[
A \equiv^3 B = \{ A \equiv^3 B \mid A \in A, B \in B \}.
\]
So
\[
0 \in A \equiv^3 B \iff \exists A \in A, B \in B; A \equiv^3 B = 0 \iff \exists A \in A, B \in B; \uparrow A \equiv^3 \uparrow B = 0^\mathcal{F} \iff \exists A \in A, B \in B; \uparrow A \equiv^3 \uparrow B = 0^\mathcal{F} \iff \exists A \in \text{up } A, B \in \text{up } B; A \equiv^3 B = 0^\mathcal{F}
\]
(used the proposition 4.97). \( \square \)

### 4.3.11 Distributivity of the Lattice of Filters

**Theorem 4.113.** If \( \mathcal{F} \) is a distributive lattice with greatest element, \( S \in \mathcal{P} \mathcal{F} \) and \( A \in \mathcal{F} \) then
\[
A \sqcup^3 \bigcap^3 S = \bigcap^3 (A \sqcup^3 )S.
\]
Proof. Taking into account the previous section, we have:

\[
\mathcal{A} \sqcup^\delta \bigcap S = \\
\mathcal{A} \cap \bigcap^\delta S = \\
\mathcal{A} \cap \{K_0 \sqcap^\delta \ldots \sqcap^\delta K_n \mid K_i \in \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\} = \\
\{K_0 \sqcap^\delta \ldots \sqcap^\delta K_n \mid K_0 \sqcap^\delta \ldots \sqcap^\delta K_n \in \mathcal{A}, K_i \in \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\} = \\
\{K_0 \sqcap^\delta \ldots \sqcap^\delta K_n \mid K_i \in \mathcal{A} \cap \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\} = \\
\{K_0 \sqcap^\delta \ldots \sqcap^\delta K_n \mid K_i \in \bigcup (A \cap S) \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\} = \\
\{K_0 \sqcap^\delta \ldots \sqcap^\delta K_n \mid K_i \in \bigcup \{A \sqcap X \mid X \in S\} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\} = \\
\bigcap \{A \sqcap^\delta X \mid X \in S\} = \\
\bigcap (A \sqcup^\delta) S.
\]

\[\square\]

Corollary 4.114. If \(\mathfrak{A}\) is a distributive lattice with greatest element, then \(\mathfrak{F}\) is also a distributive lattice.

Corollary 4.115. If \(\mathfrak{A}\) is a distributive lattice with greatest element, then \(\mathfrak{F}\) is a co-brouwerian lattice.

4.3.12 Filters over Boolean Lattices

Theorem 4.116. If \(\mathfrak{A}\) is a boolean lattice then \(a \setminus \mathfrak{F} B = a \cap \mathfrak{F} B\) for every \(a \in \mathfrak{A}\), \(B \in \mathfrak{F}\) (where the complement is taken on \(\mathfrak{F}\)).

Proof. \(\mathfrak{A}\) is a distributive lattice by corollary 4.114. Our filtrator is finitely meet-closed by the theorem 4.44 and with join-closed core by the theorem 4.25. It is also up and down aligned. So we can apply the proposition 4.74. \(\square\)

4.3.12.1 Distributivity for an Element of Boolean Core

Lemma 4.117. Let \(\mathfrak{A}\) be the poset of filters over a boolean lattice \(\mathfrak{A}\).

Then \(A \cap \mathfrak{F}\) is a lower adjoint of \(\mathfrak{A} \sqcup \mathfrak{F}\) for every \(A \in \mathfrak{F}\).

Proof. Lemma 4.75. \(\square\)

Theorem 4.118. Let \(\mathfrak{A}\) be the poset of filters over a boolean lattice \(\mathfrak{A}\). Then \(A \cap \mathfrak{F} \sqcup \mathfrak{F} S = \bigcup \mathfrak{F} (A \cap \mathfrak{F}) S\) for every \(A \in \mathfrak{F}\) and every set \(S \in \mathfrak{P}\).

Proof. Direct consequence of the lemma. \(\square\)

4.3.13 Generalized Filter Base

Definition 4.119. Generalized filter base is a filter base on the set \(\mathfrak{A}\).

Definition 4.120. If \(S\) is a generalized filter base and \(A = \bigcap^\delta S\), then we call \(S\) a generalized filter base of a filter \(A\).

Theorem 4.121. If \(\mathfrak{A}\) is a distributive lattice and \(S\) is a generalized filter base of a filter \(\mathcal{F}\) then for any \(K \in \mathfrak{A}\)

\[K \in \mathcal{F} \Leftrightarrow \exists L \in S: K \in L.\]
Proof.
\[ \leftarrow \quad \text{Because } F = \prod S. \]
\[ \Rightarrow \quad \text{Let } K \in F. \text{ Then (taken into account distributivity of } \mathfrak{A} \text{ and that } S \text{ is nonempty) there exist } X_1, \ldots, X_n \subseteq \bigcup S \text{ such that } X_1 \cap \ldots \cap X_n = K \text{ that is } \uparrow X_1 \cap \ldots \cap X_n = \uparrow K. \text{ Consequently (by theorem 4.44) } \uparrow X_1 \cap \ldots \cap \uparrow X_n = \uparrow K. \text{ Replacing every } \uparrow X_i \text{ with such } X_i \in S \text{ that } X_i \in X_i \text{ (this is obviously possible to do), we get a finite set } T_0 \subseteq S \text{ such that } \prod T_0 \subseteq \uparrow K. \text{ From this there exists } \mathcal{C} \subseteq S \text{ such that } \mathcal{C} \subseteq \prod T_0 \subseteq \uparrow K \text{ and so } K \in \mathcal{C}. \]

Corollary 4.122. If \( \mathfrak{F} \) is a distributive lattice with least element and \( S \) is a generalized filter base of a filter \( F \) then \( 0^\mathfrak{F} \in S \Leftrightarrow F = 0^\mathfrak{F} \).

Proof. Substitute \( 0^\mathfrak{F} \) as \( K \).

Theorem 4.123. Let \( \mathfrak{F} \) be a distributive lattice with least element and \( S \) is a nonempty set of filters on \( \mathfrak{F} \) such that \( F_0 \cap \ldots \cap F_n \neq 0^\mathfrak{F} \) for every \( F_0, \ldots, F_n \in S \). Then \( \prod S \neq 0^\mathfrak{F} \).

Proof. Consider the set
\[ S' = \{ F_0 \cap \ldots \cap F_n \mid F_0, \ldots, F_n \in S \}. \]

Obviously \( S' \) is nonempty and finitely meet-closed. So \( S' \) is a generalized filter base. Obviously \( 0^\mathfrak{F} \notin S \). So by properties of generalized filter bases \( \prod S' \neq 0^\mathfrak{F} \). But obviously \( \prod S = \prod S' \). So \( \prod S \neq 0^\mathfrak{F} \).

Corollary 4.124. Let \( \mathfrak{F} \) be a distributive lattice with least element and let \( S \in \mathcal{P} \mathfrak{F} \) such that \( S \neq \emptyset \) and \( A_0 \cap \ldots \cap A_n \neq 0^\mathfrak{F} \) for every \( A_0, \ldots, A_n \in S \). Then \( \prod (\uparrow S) \neq 0^\mathfrak{F} \).

Proof. Because \( (\mathfrak{F}; \mathfrak{F}) \) is finitely meet-closed (by the theorem 4.44).

4.3.14 Stars for filters

Theorem 4.125. Let \( \mathfrak{F} \) be a bounded distributive lattice with greatest element. Then \( \partial a \) is a free star for each \( a \in \mathfrak{F} \).

Proof. \( \mathfrak{F} \) is a distributive lattice by the corollary 4.114. The filtrator \((\mathfrak{F}; \mathfrak{F})\) is finitely join-closed by corollary 4.96. So we can apply the theorem 4.47.

4.3.14.1 Stars of Filters on Boolean Lattices

In this section we will consider the set of filters \( \mathfrak{F} \) on a boolean lattice \( \mathfrak{F} \).

Note that \( \mathcal{P} \mathfrak{F} \) is also a boolean lattice. We will take complements on \( \mathcal{P} \mathfrak{F} \) without specifying that the complement is taken on \( \mathfrak{F} \).

Theorem 4.126. If \( \mathfrak{F} \) is a boolean lattice, \( X \in \text{up } \mathcal{A} \Leftrightarrow \overline{X} \notin \partial \mathcal{A} \) (where complement is taken on the boolean lattice \( \mathcal{P} \mathfrak{F} \)) for every \( X \in \mathcal{P} \mathfrak{F}, \mathcal{A} \in \mathfrak{F}. \)

Proof. \( X \in \text{up } \mathcal{A} \Leftrightarrow X \nabla \mathcal{A} \Leftrightarrow \overline{X} \notin \partial \mathcal{A} \) for any \( X \in \mathcal{P} \mathfrak{F} \) (taking into account theorems 4.44, 4.112, 4.43).

Corollary 4.127. If \( \mathfrak{F} \) is a boolean lattice and \( \mathcal{A} \in \mathfrak{F} \) then
\[ 1. \partial \mathcal{A} = \{ \overline{X} \mid X \in \mathfrak{F} \setminus \text{up } \mathcal{A} \}; \]
\[ 2. \text{up } \mathcal{A} = \{ \overline{X} \mid X \in \mathfrak{F} \setminus \partial \mathcal{A} \} \]
(where complement is taken on the boolean lattice \( \mathfrak{F} \)).

Corollary 4.128. If \( \mathfrak{F} \) is a boolean lattice, \( \partial \) is an injection.

For boolean lattices free stars bijectively correspond to filters:
Theorem 4.129. If $\mathfrak{3}$ is a boolean lattice, then for any set $S \in \mathcal{P}\mathfrak{3}$ there exists a filter $\mathcal{A}$ such that $\partial \mathcal{A} = S$ iff $S$ is a free star.

Proof.

$\Rightarrow$. That $0^\mathfrak{3} \notin S$ is obvious. For every $A, B \in \mathfrak{A}$

\[ A \sqcup^\mathfrak{3} B \in S \iff (A \sqcup^\mathfrak{3} B) \cap \mathfrak{3} A \neq 0^\mathfrak{3} \iff (A \sqcup^\mathfrak{3} B) \cap \mathfrak{3} A \neq 0^\mathfrak{3} \iff (A \sqcup^\mathfrak{3} A) \sqcap (B \sqcap \mathfrak{3} A) \neq 0^\mathfrak{3} \iff A \sqcap \mathfrak{3} A \neq 0^\mathfrak{3} \vee B \sqcap \mathfrak{3} A \neq 0^\mathfrak{3} \iff A \in S \vee B \in S \]

(taken into account corollary 4.114 and theorem 4.25).

$\Leftarrow$. $0^\mathfrak{3} \notin S$ and $\forall A, B \in S: (A \sqcup^\mathfrak{3} B \in S \iff A \in S \vee B \in S)$. Let $T = \{ X | X \in \mathfrak{P} \setminus S \}$. We will prove that $T$ is a filter.

$1^\mathfrak{3} \in T$ because $0^\mathfrak{3} \notin S$; so $T$ is nonempty. To prove that $T$ is a filter it is enough to show $\forall X, Y \in \mathfrak{P}: (X, Y \in T \iff X \sqcap^\mathfrak{3} Y \in T)$. In fact,

\[
X, Y \in T \iff X, Y \notin S \iff \\
\neg (X \in S \vee Y \in S) \iff \\
\neg X \sqcup^\mathfrak{3} Y \notin S \iff \\
\neg X \sqcup^\mathfrak{3} Y \in T \iff \\
X \sqcap^\mathfrak{3} Y \in T.
\]

So $T$ is a filter. To finish the proof we will show that $\partial T = S$. In fact, for every $X \in \mathfrak{P}$

\[
X \in \partial T \iff X \notin \mathfrak{A} \uparrow T \iff X \notin T \iff X \in S.
\]

$\square$

Proposition 4.130. If $\mathfrak{3}$ is a boolean lattice then $A \sqsubseteq B \iff \partial A \subseteq \partial B$ for every $A, B \in \mathfrak{3}$.

Proof.

\[
\partial A \subseteq \partial B \iff \\
\{ X | X \in \mathfrak{P} \setminus \mathcal{A} \} \subseteq \{ X | X \in \mathfrak{P} \setminus \mathcal{B} \} \iff \\
\mathfrak{P} \setminus \mathcal{A} \subseteq \mathfrak{P} \setminus \mathcal{B} \iff \\
\mathfrak{P} \setminus \mathcal{A} \subseteq \mathfrak{P} \setminus \mathcal{B}.
\]

$\square$

Corollary 4.131. $\partial$ is a straight monotone map if $\mathfrak{3}$ is a boolean lattice.

Theorem 4.132. If $\mathfrak{3}$ is a boolean lattice then $\partial \bigcup^\mathfrak{3} S = \bigcup \langle \partial \rangle S$ for every $S \in \mathcal{P}\mathfrak{3}$.

Proof. For boolean lattices $\partial$ is an order embedding from the poset $\mathfrak{3}$ to the complete lattice $\mathcal{P}\mathfrak{P}$.

So accordingly the lemma 4.104 it is enough to prove that there exists $\mathcal{F} \in \mathfrak{3}$ such that $\partial \mathcal{F} = \bigcup \langle \partial \rangle S$.

To prove this it is enough to show that $0^\mathfrak{3} \notin \bigcup \langle \partial \rangle S$ and

\[
\forall A, B \in S: (A \sqcup^\mathfrak{3} B \in \bigcup \langle \partial \rangle S \iff A \in \bigcup \langle \partial \rangle S \vee B \in \bigcup \langle \partial \rangle S). 
\]

$0^\mathfrak{3} \notin \bigcup \langle \partial \rangle S$ is obvious.

Let $A \sqcup^\mathfrak{3} B \in \bigcup \langle \partial \rangle S$. Then there exists $Q \in \langle \partial \rangle S$ such that $A \sqcup^\mathfrak{3} B \in Q$. Then $A \in Q \vee B \in Q$, consequently $A \in \bigcup \langle \partial \rangle S \vee B \in \bigcup \langle \partial \rangle S$. Let now $A \in \bigcup \langle \partial \rangle S$. Then there exists $Q \in \langle \partial \rangle S$ such as $A \in Q$, consequently $A \sqcup^\mathfrak{3} B \in Q$ and $A \sqcup^\mathfrak{3} B \in \bigcup \langle \partial \rangle S$. $\square$
4.3.15 More about the Lattice of Filters

Definition 4.133. Atoms of \( \mathfrak{F} \) (for any poset \( \mathfrak{A} \)) are called ultrafilters.

Definition 4.134. Principal ultrafilters are also called trivial ultrafilters.

Theorem 4.135. If \( \mathfrak{A} \) is a bounded distributive lattice with least element then \( \mathfrak{F} \) is an atomic lattice.

Proof. Let \( F \in \mathfrak{F} \). Let choose (by Kuratowski’s lemma) a maximal chain \( S \) from \( 0^\mathfrak{F} \) to \( F \). Let \( S' = S \setminus \{0^\mathfrak{F}\} \). \( a = \bigcap^\mathfrak{F} S' \neq 0^\mathfrak{F} \) by properties of generalized filter bases (the corollary 4.122 which uses the fact that \( \mathfrak{F} \) is a distributive lattice with least element). If \( a \notin S \) then the chain \( S \) can be extended adding there element \( a \) because \( 0^\mathfrak{F} \subseteq a \bowtie \mathfrak{A} \) for any \( \mathfrak{X} \in S' \) what contradicts to maximality of the chain. So \( a \in S \) and consequently \( a \in S' \). Obviously \( a \) is the minimal element of \( S' \). Consequently (taking into account maximality of the chain) there is no \( \mathfrak{Y} \in \mathfrak{F} \) such that \( 0^\mathfrak{F} \subseteq \mathfrak{Y} \subseteq a \). So \( a \) is an atomic filter. Obviously \( a \subseteq F \). □

Obvious 4.136. If \( \mathfrak{A} \) is a boolean lattice then \( \mathfrak{F} \) is separable.

Theorem 4.137. If \( \mathfrak{A} \) is a boolean lattice then \( \mathfrak{F} \) is an atomistic lattice.

Proof. Because (used the theorem 3.20) \( \mathfrak{F} \) is atomic (theorem 4.135) and separable. □

Corollary 4.138. If \( \mathfrak{A} \) is a boolean lattice then \( \mathfrak{F} \) is atomically separable.

Proof. By the theorem 3.19. □

Theorem 4.139. When \( \mathfrak{A} \) is a boolean lattice, then the filtrator \( (\mathfrak{F}; \mathfrak{P}) \) is central.

Proof. We can conclude that \( \mathfrak{F} \) is atomically separable (the corollary 4.138), with separable core (the theorem 4.112), and with join-closed core (corollary 4.96).

We need to prove \( Z(\mathfrak{F}) = \mathfrak{P} \).

Let \( \mathfrak{X} \in Z(\mathfrak{F}) \). Then there exists \( \mathfrak{Y} \in Z(\mathfrak{F}) \) such that \( \mathfrak{X} \cap^\mathfrak{F} \mathfrak{Y} = 0^\mathfrak{F} \) and \( \mathfrak{X} \cup^\mathfrak{F} \mathfrak{Y} = 1^\mathfrak{F} \). Consequently there is \( X \in \text{up} \mathfrak{X} \) such that \( X \cap^\mathfrak{F} \mathfrak{Y} = 0^\mathfrak{F} \); we also have \( X \cup^\mathfrak{F} \mathfrak{Y} = 1^\mathfrak{F} \). Suppose \( X \subsetneq \mathfrak{X} \). Then there exists \( a \in \text{atoms}^\mathfrak{F} X \) such that \( a \notin \text{atoms}^\mathfrak{F} \mathfrak{X} \). We can conclude also \( a \notin \text{atoms}^\mathfrak{F} (\mathfrak{X} \cup^\mathfrak{F} \mathfrak{Y}) \) (otherwise \( X \cap^\mathfrak{F} \mathfrak{Y} = 0^\mathfrak{F} \)). Thus \( a \notin \text{atoms}^\mathfrak{F} (\mathfrak{X} \cup^\mathfrak{F} \mathfrak{Y}) \) and consequently \( \mathfrak{X} \cup^\mathfrak{F} \mathfrak{Y} \neq 1^\mathfrak{F} \) what is a contradiction. We have \( \mathfrak{X} = X \in \mathfrak{P} \).

Let now \( X \in \mathfrak{P} \). Let \( Y = \overline{X} \). We have \( X \cap^\mathfrak{F} Y = 0^\mathfrak{F} \) and \( X \cup^\mathfrak{P} Y = 1^\mathfrak{F} \). Thus \( X \cap^\mathfrak{F} Y = \bigcap^\mathfrak{P} \{X \cap^\mathfrak{F} Y\} = 0^\mathfrak{F} \); \( X \cup^\mathfrak{P} Y = X \cup^\mathfrak{P} Y = 1^\mathfrak{F} \). We have shown that \( X \in Z(\mathfrak{F}) \). □

4.3.16 Atomic Filters

Proposition 4.140. If \( \mathfrak{A} \) is a meet-semilattice with least element, then \( a \) is an atom of \( \mathfrak{P} \) iff \( a \in \mathfrak{P} \) and \( a \) is an atom of \( \mathfrak{F} \).

Proof. It is semifiltered by the corollary 4.95, finitely meet-closed by proposition 4.97. So we can apply the theorem 4.49. □

Proposition 4.141. If \( \mathfrak{A} \) is a meet-semilattice with least element then, \( a \in \mathfrak{F} \) is an atom of \( \mathfrak{F} \) iff \( \text{up} a = \partial a \).

Proof. It is semifiltered by the corollary 4.95, \( \mathfrak{F} \) is a meet-semilattice by the corollary 4.107. So we can apply theorem 4.50. □

Proposition 4.142. If \( \mathfrak{A} \) is bounded distributive lattice, then atomic elements of the filtrator \( (\mathfrak{F}; \mathfrak{A}) \) are prime.

Proof. \( (\mathfrak{F}; \mathfrak{P}) \) is with finitely join-closed core by the theorem 4.96, \( \mathfrak{F} \) is a distributive lattice by theorem 4.114. So we can apply proposition 4.52. □
The following theorem is essentially borrowed from [18]:

**Theorem 4.143.** Let $\mathfrak{F}$ be a boolean lattice. Let $a$ be a filter. Then the following are equivalent:

1. $a$ is prime.
2. For every $A \in \mathfrak{F}$ exactly one of $\{A, \neg A\}$ is in $a$.
3. $a$ is an atom of $\mathfrak{F}$.

**Proof.**

(1) $\Rightarrow$ (2). Let $a$ be prime. Then $A \cup^{\mathfrak{F}} \neg A = 1^{\mathfrak{F}} \in a$. Therefore $A \in a \lor \neg A \in a$. But since $A \cap^{\mathfrak{F}} \neg A = 0^{\mathfrak{F}} \notin a$ it is impossible $A \in a \land \neg A \in a$.

(2) $\Rightarrow$ (3). Obviously $a \neq 0^{\mathfrak{F}}$. Let a filter $b \sqsubseteq a$. So $b \supset a$. Let $X \in b \setminus a$. Then $X \notin a$ and thus $\neg X \in a$ and consequently $\neg X \in b$. So $0^{\mathfrak{F}} = X \cap^{\mathfrak{F}} \neg X \in b$ and thus $b = 0^{\mathfrak{F}}$. So $a$ is atomic.

(3) $\Rightarrow$ (1). By the previous proposition. □

4.3.17 Some Criteria

**Proposition 4.144.** Let $\mathfrak{F}$ be an atomic complete boolean lattice. Then the following conditions are equivalent for any $F \in \mathfrak{F}$:

1. $F \in \mathfrak{P}$;
2. $\forall S \in \mathfrak{P}: (F \cap^{\mathfrak{F}} S \neq 0 \Rightarrow \exists K \in S: F \cap^{\mathfrak{F}} K \neq 0)$;
3. $\forall S \in \mathfrak{P}: (F \cap^{\mathfrak{F}} \cup S \neq 0 \Rightarrow \exists K \in S: F \cap^{\mathfrak{F}} K \neq 0)$.

**Proof.** The filtrator $(\mathfrak{F}; \mathfrak{P})$ is semfiltered by the corollary 4.95, star separable by 4.128, with finitely meet-closed core by proposition 4.97, with separable core by theorem 4.112. $\mathfrak{P}$ is atomistic because every atomic complete boolean lattice is atomistic. $\mathfrak{F}$ is atomistic by theorem 4.137. So we can apply the theorem 4.53. □

**Theorem 4.145.** If $\mathfrak{F}$ is a complete boolean lattice then for each $F \in \mathfrak{F}$

$$F \in \mathfrak{P} \Leftrightarrow \forall S \in \mathfrak{P}: \left( \bigcup S \subseteq \partial F \Rightarrow S \cap \partial F \neq \emptyset \right)$$

**Proof.**

$$\forall S \in \mathfrak{P}: \left( \bigcup S \subseteq \partial F \Rightarrow S \cap \partial F \neq \emptyset \right)$$

$$\forall S \in \mathfrak{P}: \left( \bigcap S \subseteq \partial F \Rightarrow S \cup \partial F = \emptyset \right)$$

$$\forall S \in \mathfrak{P}: \left( \bigcap S \subseteq \uparrow F \Rightarrow (\neg) S \subseteq F \right)$$

$$\forall S \in \mathfrak{P}: \left( \bigcap S \subseteq \uparrow F \Rightarrow S \subseteq \uparrow F \right),$$

but

$$\forall S \in \mathfrak{P}: \left( \bigcap S \subseteq \uparrow F \Rightarrow S \subseteq \uparrow F \right) \Rightarrow \bigcap S \subseteq \uparrow F \Rightarrow F \in \mathfrak{P}.$$
**Theorem 4.146.** Let $\mathfrak{F}$ be a boolean lattice. For any $S \in \mathfrak{P}$, the condition $\exists F \in \mathfrak{F}. S = \ast F$ is equivalent to conjunction of the following items:

1. $S$ is a free star on $\mathfrak{F}$;
2. $S$ is filter closed.

**Proof.**

$\Rightarrow$.

1. That $0^\mathfrak{F} \notin \ast F$ is obvious. For every $a, b \in \mathfrak{F}$

$$a \cup^\mathfrak{F} b \in \ast F \iff (a \cup^\mathfrak{F} b) \cap^\mathfrak{F} F \neq 0^\mathfrak{F} \iff (a \cap^\mathfrak{F} F) \cup^\mathfrak{F} (b \cap^\mathfrak{F} F) \neq 0^\mathfrak{F} \iff a \cap^\mathfrak{F} F \neq 0^\mathfrak{F} \lor b \cap^\mathfrak{F} F \neq 0^\mathfrak{F} \iff a \in \ast F \lor b \in \ast F$$

(taken into account the corollary 4.114). So $\ast F$ is a free star on $\mathfrak{F}$.

2. We have $T \subseteq S$ and need to prove that $\bigcap^\mathfrak{F} T \cap^\mathfrak{F} F \neq 0^\mathfrak{F}$. Because $\langle F \cap^\mathfrak{F} \rangle T$ is a generalized filter base, $0^\mathfrak{F} \in \langle F \cap^\mathfrak{F} \rangle T \iff \bigcap^\mathfrak{F} (F \cap^\mathfrak{F}) T = 0^\mathfrak{F} \iff \bigcap^\mathfrak{F} T \cap^\mathfrak{F} F \neq 0^\mathfrak{F}$. So it is left to prove $0^\mathfrak{F} \notin \langle F \cap^\mathfrak{F} \rangle T$ what follows from $T \subseteq S$.

$\Leftarrow$. Let $S$ be a free star on $\mathfrak{F}$. Then for every $A, B \in \mathfrak{P}$

$$A, B \in S \cap \mathfrak{P} \iff A, B \in S \iff A \cup^\mathfrak{F} B \in S \iff A \cup^\mathfrak{F} B \in S \iff A \cup^\mathfrak{F} B \in S$$

(taken into account the theorem 4.25). So $S \cap \mathfrak{P}$ is a free star on $\mathfrak{P}$.

Thus there exists $F \in \mathfrak{F}$ such that $\partial F = S \cap \mathfrak{P}$. We have up $X \subseteq S \Rightarrow X \in S$ (because $S$ is filter closed) for every $X \in \mathfrak{F}$; then (taking into account properties of generalized filter bases)

$$X \in S \iff \text{up } X \subseteq S \iff \text{up } X \subseteq \partial F \iff \forall X \in \text{up } X; X \cap^\mathfrak{F} F \neq 0^\mathfrak{F} \iff 0^\mathfrak{F} \notin \langle F \cap^\mathfrak{F} \rangle \text{up } X \iff \bigcap^\mathfrak{F} (F \cap^\mathfrak{F}) \text{up } X \neq 0^\mathfrak{F} \iff \bigcap^\mathfrak{F} F \cap^\mathfrak{F} \text{up } X \neq 0^\mathfrak{F} \iff X \in \ast F.$$

\hfill \Box

### 4.3.18 Filters and a Special Sublattice

**Theorem 4.147.** Let $(\mathfrak{F}; 3)$ be a primary filtrator where $\mathfrak{F}$ is a boolean lattice. Let $A \in \mathfrak{F}$. Then for each $X \in \mathfrak{F}$

$$X \in Z(D A) \iff \exists X \in \mathfrak{P}; X = X \cap^\mathfrak{F} A.$$

$\Leftarrow$. Let $X = X \cap^\mathfrak{F} A$ where $X \in \mathfrak{P}$. Let also $\mathfrak{Y} = \overline{X} \cap^\mathfrak{F} A$. Then $X \cap^\mathfrak{F} \mathfrak{Y} = X \cap^\mathfrak{F} \overline{X} \cap^\mathfrak{F} A = (X \cap^\mathfrak{P} X) \cap^\mathfrak{F} A = 0^\mathfrak{F}$ (used theorem 4.44) and $X \cup^\mathfrak{F} \mathfrak{Y} = (X \cup^\mathfrak{F} X) \cap^\mathfrak{F} A = (X \cup^\mathfrak{P} X) \cap^\mathfrak{F} A = 1^\mathfrak{F} \cap^\mathfrak{F} A = A$ (used the theorems 4.25 and corollary 4.114). So $X \in Z(D A)$. 
Let $X \in Z(DA)$. Then there exists $Y \in Z(DA)$ such that $X \cap \delta Y = 0$ and $X \cup \delta Y = A$. Then (used theorem 4.112) there exists $X \in \text{up} X$ such that $X \cap \delta Y = 0$. We have

$$X = X \cup \delta (X \cap \delta Y) = X \cap \delta (X \cup \delta Y) = X \cap \delta A.$$ 

### 4.3.19 Core Part and Atomic Elements

**Proposition 4.148.** Let $\mathfrak{A}$ be an atomistic lattice. Then for every $a \in \mathfrak{A}$ such that $\text{Core}' a$ exists we have

$$\text{Core}' a = \exists \{x \mid x \text{ is an atom of } \mathfrak{A}, x \subseteq a\}.$$ 

**Proof.** $(\mathfrak{A}; \mathfrak{P})$ is with join-closed core by corollary 4.96. So we can apply theorem 4.67. □

### 4.3.20 Complements and Core Parts

**Proposition 4.149.** Let $\mathfrak{A}$ be a complete boolean lattice. Then $a^* = a^+ = \overline{\text{Core} a}$ for every $a \in \mathfrak{A}$.

**Proof.** The filtrator $(\mathfrak{A}; \mathfrak{P})$ is filtered by the corollary 4.95. $\mathfrak{A}$ is a complete lattice by corollary 4.107. $(\mathfrak{A}; \mathfrak{P})$ is with co-separable core by theorem 4.73. Thus we can apply the theorem 4.60. $(\mathfrak{A}; \mathfrak{P})$ is filtered by corollary 4.95, finitely meet-closed by proposition 4.97, with separable core by theorem 4.112. $\mathfrak{A}$ is a complete lattice by corollary 4.107. So we can apply the theorem 4.62. □

**Proposition 4.150.** Let $\mathfrak{A}$ be a complete lattice. Then $a^* \in \mathfrak{A}$.

**Proof.** $\mathfrak{A}$ is a complete lattice by 4.107. $(\mathfrak{A}; \mathfrak{P})$ is a filtrator with join-closed core by corollary 4.96. $(\mathfrak{A}; \mathfrak{P})$ is a filtrator with separable core by theorem 4.112. So we can apply theorem 4.64. □

**Proposition 4.151.** If $\mathfrak{A}$ is a complete boolean lattice, then $a^+$ is dual pseudocomplement of $a$, that is

$$a^+ = \min \{c \in \mathfrak{A} \mid c \cup \delta a = 1\}$$

for every $a \in \mathfrak{A}$.

**Proof.** $(\mathfrak{A}; \mathfrak{P})$ is filtered by the corollary 4.95. It is with co-separable core by theorem 4.73. $\mathfrak{A}$ is a complete lattice by corollary 4.107. So we can apply the theorem 4.65. □

**Proposition 4.152.** For a primary filtrator over a complete boolean lattice both edge part and dual edge part are always defined.

**Proof.** Core part and dual core part are defined because the core is a complete lattice. Using the theorem 4.116. □

**Proposition 4.153.** If $\mathfrak{A}$ is a complete lattice, then for every $a, b \in \mathfrak{A}$

$$\text{Core}(a \cap \delta b) = \text{Core}(a \cap \delta \text{Core} b).$$

**Proof.** $(\mathfrak{A}; \mathfrak{P})$ is with join-closed core by corollary 4.96. $\mathfrak{A}$ is a meet-semilattice by corollary 4.107. So we can apply theorem 4.68. Then apply proposition 4.100. □

**Proposition 4.154.** If $\mathfrak{A}$ is a complete lattice, then for every $S \in \mathfrak{A}$

$$\text{Core} \bigcap \mathfrak{A} S = \bigcap \mathfrak{P} \text{Core} S.$$ 

**Proof.** By theorem 4.69. □
Proposition 4.163. Let $\mathfrak{F}$ be a complete lattice. Then for every $a, b \in \mathfrak{F}$
\[ \text{Cor}(a \sqcup \mathfrak{F}) b = \text{Cor} a \sqcup \mathfrak{F} \text{ Cor} b. \]

Proof. Let $\mathfrak{F}$ be a complete lattice. Then $(a \sqcup \mathfrak{F}) b = a \sqcup \mathfrak{F} b$ for every $a, b \in \mathfrak{F}$.

Theorem 4.157. Let $\mathfrak{F}$ be a complete boolean lattice. Then $(a \sqcup \mathfrak{F}) b = a \sqcup \mathfrak{F} b$ for every $a, b \in \mathfrak{F}$.

Proof. Let $\mathfrak{F}$ be a complete boolean lattice. Then $(a \sqcup \mathfrak{F}) b = a \sqcup \mathfrak{F} b$ for every $a, b \in \mathfrak{F}$.

4.3.21 Complementive Filters and Factoring by a Filter

Definition 4.159. Let $\mathfrak{A}$ be a meet-semilattice and $\mathcal{A} \in \mathfrak{A}$. The relation $\sim$ on $\mathfrak{A}$ is defined by the formula
\[ \forall X, Y \in \mathfrak{A}: (X \sim Y \Leftrightarrow X \cap \mathfrak{A} A = Y \cap \mathfrak{A} A). \]

Proposition 4.160. The relation $\sim$ is an equivalence relation.

Proof.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Obvious.

Definition 4.161. When $X, Y \in \mathfrak{F}$ and $\mathcal{A} \in \mathfrak{F}$ we define $X \sim Y \Leftrightarrow \uparrow X \sim \uparrow Y$.

Theorem 4.162. Let $\mathfrak{F}$ be a distributive lattice, $\mathcal{A} \in \mathfrak{F}$. Then for every $X, Y \in \mathfrak{F}$
\[ X \sim Y \Leftrightarrow \exists A \in \mathcal{A}: X \cap \mathfrak{F} A = Y \cap \mathfrak{F} A. \]

Proof. $\exists A \in \mathcal{A}: X \cap \mathfrak{F} A = Y \cap \mathfrak{F} A \Rightarrow \exists A \in \mathcal{A}: \uparrow X \cap \mathfrak{F} A = \uparrow Y \cap \mathfrak{F} A \Rightarrow \exists A \in \mathcal{A}: \uparrow X \cap \mathfrak{F} A = \uparrow Y \cap \mathfrak{F} A \Rightarrow \exists A \in \mathcal{A}: X \sim Y \Rightarrow X \sim Y.$

On the other hand, $\uparrow X \cap \mathfrak{F} A = \uparrow Y \cap \mathfrak{F} A \Rightarrow \{X \cap \mathfrak{F} A \mid A \in \mathcal{A}\} = \exists \mathcal{A}, A \in \mathcal{A} \Rightarrow \exists \mathcal{A}, A \in \mathcal{A} \Rightarrow \exists \mathcal{A}, A \in \mathcal{A} \Rightarrow \exists \mathcal{A}, A \in \mathcal{A} \Rightarrow \exists \mathcal{A}, A \in \mathcal{A}.$

Proposition 4.163. The relation $\sim$ is a congruence for each of the following:

1. a meet-semilattice $\mathfrak{A}$;
2. a distributive lattice $\mathfrak{A}$.

Proof. Let $a_0, a_1, b_0, b_1 \in \mathfrak{A}$ and $a_0 \sim a_1$ and $b_0 \sim b_1$.

1. $a_0 \cap b_0 \sim a_1 \cap b_1$ because $(a_0 \cap b_0) \cap A = a_0 \cap (b_0 \cap A) = a_0 \cap (b_1 \cap A) = b_1 \cap (a_1 \cap A) = (a_1 \cap b_1) \cap A$.

2. Taking the above into account, we need to prove only $a_0 \cup b_0 \sim a_1 \cup b_1$. We have

$$(a_0 \cup b_0) \cap A = (a_0 \cap A) \cup (b_0 \cap A) = (a_1 \cap A) \cup (b_1 \cap A) = (a_1 \cup b_1) \cap A.$$ 

\[\square\]

Definition 4.164. We will denote $A / (\sim) = A / (\sim) \cap \mathcal{A}$ for a set $A$ and an equivalence relation $\sim$ on a set $B \supseteq A$. I will call $\sim$ a congruence on $A$ when $(\sim) \cap \mathcal{A} \times \mathcal{A}$ is a congruence on $A$.

Theorem 4.165. Let $\mathfrak{F}$ be the set of filters over a boolean lattice $\mathfrak{B}$ and $\mathcal{A} \in \mathfrak{F}$. Consider the function $\gamma : Z(D \mathcal{A}) \to \mathfrak{F} / \sim$ defined by the formula (for every $p \in Z(D \mathcal{A})$)

$$\gamma p = \{ X \in \mathfrak{F} | \uparrow X \cap \mathcal{A} = p \}.$$ 

Then:

1. $\gamma$ is a lattice isomorphism.
2. $\forall Q \in q : \gamma^{-1} q = \uparrow Q \cap \mathcal{A}$ for every $q \in \mathfrak{F} / \sim$.

Proof. $\forall p \in Z(D \mathcal{A}) : \gamma p \neq \emptyset$ because of theorem 4.147. Thus it is easy to see that $\gamma p \in \mathfrak{F} / \sim$ and that $\gamma$ is an injection.

Let’s prove that $\gamma$ is a lattice homomorphism:

$$\gamma(p_0 \cap \mathcal{A} p_1) = \{ X \in \mathfrak{F} | \uparrow X \cap \mathcal{A} = p_0 \cap \mathcal{A} p_1 \} = \{ X_0 \in \mathfrak{F} | \uparrow X \cap \mathcal{A} = p_0 \cap \mathcal{A} p_1 \} \cap \mathcal{A} \diamond \{ X_1 \in \mathfrak{F} | \uparrow X \cap \mathcal{A} = p_1 \} \cap \mathcal{A} \diamond \{ X_0 \in \mathfrak{F} | \uparrow X \cap \mathcal{A} = p_0 \cap \mathcal{A} p_1 \} \cap \mathcal{A} \diamond \{ X_1 \in \mathfrak{F} | \uparrow X \cap \mathcal{A} = p_1 \} \cap \mathcal{A}$$

Because $\gamma p_0 \cap \mathcal{A} p_1$ and $\gamma(p_0 \cap \mathcal{A} p_1)$ are equivalence classes, thus follows $\gamma(p_0 \cap \mathcal{A} p_1) = \gamma(p_0 \cap \mathcal{A} p_1)$. $\gamma(p_0 \cap \mathcal{A} p_1)$ are equivalence classes, thus follows $\gamma p_0 \cap \mathcal{A} p_1 = \gamma(p_0 \cap \mathcal{A} p_1)$. $\gamma (p_0 \cap \mathcal{A} p_1)$ are equivalence classes, thus follows $\gamma p_0 \cap \mathcal{A} p_1 = \gamma(p_0 \cap \mathcal{A} p_1)$.

To finish the proof it is enough to show that $\forall Q \in q : q = \gamma(\uparrow Q \cap \mathcal{A})$ for every $q \in \mathfrak{F} / \sim$. (From this it follows that $\gamma$ is surjective because $q$ is not empty and thus $\exists Q \in q : q = \gamma(\uparrow Q \cap \mathcal{A})$.) Really,

$$\gamma(\uparrow Q \cap \mathcal{A}) = \{ X \in \mathfrak{F} | \uparrow X \cap \mathcal{A} = \uparrow Q \cap \mathcal{A} \} = [Q] = q.$$ 

This isomorphism is useful in both directions to reveal properties of both lattices $Z(D \mathcal{A})$ and $\mathfrak{F} / \sim$.

Corollary 4.166. If $\mathfrak{B}$ is a boolean lattice then $\mathfrak{F} / \sim$ is a boolean lattice.

4.1. See Wikipedia for a definition of congruence.
Proof. Because $Z(D.A)$ is a boolean lattice (theorem 2.79).

4.3.22 Pseudodifference of filters

Proposition 4.167. For a lattice $\mathfrak{F}$ of filters over a boolean lattice and $a, b \in \mathfrak{F}$ the following expressions are always equal:

1. $a \setminus b = \bigcap \{ z \in \mathfrak{A} \mid a \subseteq b \cup z \}$ (quasidifference of $a$ and $b$);
2. $a \neq b = \bigcup \{ z \in \mathfrak{A} \mid z \subseteq a \land z \cap b = \emptyset \}$ (second quasidifference of $a$ and $b$);
3. $\bigcup \{ \text{atoms } a \setminus \text{atoms } b \}$.

Proof. Theorem 3.43, taking into account corollary 4.115 theorem 4.137.

4.4 Filters on a Set

In this section we will consider filters on the poset $\mathfrak{F} = \mathcal{P} \mathfrak{U}$ (where $\mathfrak{U}$ is some fixed set) with the order $A \subseteq B \iff A \subseteq B$ (for $A, B \in \mathcal{P} \mathfrak{U}$).

In fact, it is a complete boolean lattice with $\bigcap S = \bigcap S, \bigcup S = \bigcup S, \mathfrak{T} = \mathfrak{U} \setminus A$ for every $S \in \mathcal{P} \mathfrak{U}$ and $A \in \mathcal{P} \mathfrak{U}$.

Definition 4.168. I will call a filter on the lattice of all subsets of a given set $\mathfrak{U}$ as a filter on set.

Definition 4.169. I will denote the set on which a filter $F$ is defined as $Base(F)$.

Obvious 4.170. $Base(F) = \bigcup F$.

Definition 4.171. I will call the primary filtrator for $\mathfrak{F} = \mathcal{P} \mathfrak{U}$ (with order on $\mathfrak{F}$ defined as $A \subseteq B \iff A \subseteq B$) for some set $\mathfrak{U}$ as powerset filtrator.

Proposition 4.172. The following are equivalent for a non-empty set $F \in \mathcal{P} \mathfrak{U}$:

1. $F$ is a filter.
2. $\forall X, Y \in F : X \cap Y \in F$ and $F$ is an upper set.
3. $\forall X, Y \in \mathfrak{F} : (X, Y \in F \iff X \cap Y \in F)$.

Proof. By theorem 4.82.

Obvious 4.173. The minimal filter on $\mathcal{P} \mathfrak{U}$ is $\mathcal{P} \mathfrak{U}$.

Obvious 4.174. The maximal filter on $\mathcal{P} \mathfrak{U}$ is $\{ \mathfrak{U} \}$.

I will denote $\uparrow A = \uparrow^{\mathfrak{U}} A = \uparrow^{\mathcal{P} \mathfrak{U}} A$. (The distinction between conflicting notations $\uparrow^{\mathfrak{U}} A$ and $\uparrow^{\mathcal{P} \mathfrak{U}} A$ will be clear from the context.)

Proposition 4.175. The powerset filtrator is both up-aligned and down-aligned.

Proof. By theorem 4.98.

Proposition 4.176. Every powerset filtrator is filtered.

Proof. By corollary 4.95.

Proposition 4.177. Every powerset filtrator is with join-closed core.

Proof. By corollary 4.96.
Proposition 4.178. Every powerset filtrator is with finitely meet-closed core.


Proposition 4.179. Every powerset filtrator is with separable core.

Proof. By theorem 4.112.

Proposition 4.180. Every powerset filtrator is with co-separable core.


Proposition 4.181. \( \text{Cor}^\prime a = \text{Cor} a = \uparrow^{\text{Base}(a)} \cap a \) for every filter \( a \) on a set.

Proof. By proposition 4.100.

Proposition 4.182. \( \text{Cor} a \subseteq a \) for every filter \( a \) on a set.


Proposition 4.183. \( \text{Cor} a = \max \downarrow a \) for every filter \( a \) on a set.

Proof. By proposition 4.102.

Proposition 4.184. For the lattice \( \mathfrak{F} \) of filters on a set \( \mathfrak{U} \), \( A \in \mathfrak{F} \), \( B \in \mathfrak{P} \) we have:

1. \( B \equiv \delta A \Leftrightarrow \overline{B} \not\subseteq A \);
2. \( B \equiv \delta A \Leftrightarrow \overline{B} \subseteq A \).

Proof. By theorem 4.103.

Proposition 4.185. \( \bigcup \overline{\delta} S = \cap S \) for a set \( S \) of filters on a powerset.

Proof. By theorem 4.106.

Corollary 4.186. A set of filters on a powerset is always a complete lattice.

Corollary 4.187. \( A \cup B = A \cap B \) for filters \( A \) and \( B \) on a powerset.

Proposition 4.188. For \( S \in \mathcal{P} \mathfrak{F} \setminus \{\emptyset\} \) where \( \mathfrak{F} \) are filters on a powerset

\[
\bigcap \overline{\delta} S = \{ K_0 \cap ... \cap K_n \mid K_i \in \bigcup S \text{ where } i = 0, ..., n \text{ for } n \in \mathbb{N} \}.
\]

Proof. By theorem 4.110.

Proposition 4.189. For every \( \mathcal{F}_0, ..., \mathcal{F}_m \in \mathfrak{F} \) (\( m \in \mathbb{N} \)) where \( \mathfrak{F} \) are filters on a powerset

\[
\mathcal{F}_0 \cap \overline{\delta} ... \cap \overline{\delta} \mathcal{F}_m = \{ K_0 \cap ... \cap K_m \mid K_i \in \mathcal{F}_i \text{ where } i = 0, ..., n \text{ for } m \in \mathbb{N} \}.
\]

Proof. By theorem 4.111.

Proposition 4.190. If \( A \in \mathfrak{F} \) and \( S \in \mathcal{P} \mathfrak{F} \) where \( \mathfrak{F} \) are filters on a powerset then

\[
A \cup \overline{\delta} \bigcap \overline{\delta} S = \bigcap \overline{\delta} (A \cup \overline{\delta}) S.
\]


Corollary 4.191. The poset of filters on a powerset is a distributive lattice.
**Corollary 4.192.** The poset of filters on a powerset is a co-brouwerian lattice.

**Proposition 4.193.** \( a \sqcap \beta B = a \sqcap \beta B \) for every \( a \in \mathfrak{F}, B \in \mathfrak{P} \) (where \( \mathfrak{F} \) is filters on a powerset and the complement is taken on \( \mathfrak{P} \)).

**Proof.** By theorem 4.116. \( \square \)

**Proposition 4.194.** Let \( \mathfrak{F} \) be the poset of filters on a powerset. \( A \sqcup \beta \bigcup S = \bigcup (A \sqcup \beta S) \) for every \( A \in \mathfrak{P} \) and every set \( S \in \mathcal{P}_\mathfrak{F} \).

**Proof.** By theorem 4.118. \( \square \)

**Proposition 4.195.** If \( S \) is a generalized filter base of a filter \( \mathcal{F} \) on a set \( \Omega \) then for any \( K \in \mathcal{P} \Omega \)

\[ K \in \mathcal{F} \Leftrightarrow \exists L \in S: K \in L. \]

**Proof.** By theorem 4.121. \( \square \)

**Proposition 4.196.** If \( S \) is a generalized filter base of a filter \( \mathcal{F} \) on a set \( U \) then

\[ 0^\beta \in S \Leftrightarrow \mathcal{F} = 0^\beta. \]

**Proof.** By corollary 4.122. \( \square \)

**Proposition 4.197.** Let \( S \) be a nonempty set of filters on a set such that \( \mathcal{F}_0 \sqcap \beta \cdots \sqcap \beta \mathcal{F}_n \neq 0^\beta \) for every \( \mathcal{F}_0, \ldots, \mathcal{F}_n \in S \). Then \( \bigcap^n S \neq 0^\beta \).

**Proof.** By theorem 4.123. \( \square \)

**Proposition 4.198.** Let \( S \in \mathcal{P} \Omega \setminus \{\emptyset\} \) where \( \Omega \) is a set and \( A_0 \cap \ldots \cap A_n \neq 0^\beta \) for every \( A_0, \ldots, A_n \in S \). Then \( \bigcap^n (\lceil S \rceil) S \neq 0^\beta \).

**Proof.** By corollary 4.124. \( \square \)

**Proposition 4.199.** \( \partial a \) is a free star for each filter \( a \) on a set.

**Proof.** By theorem 4.125. \( \square \)

**Proposition 4.200.** For a filter \( \mathcal{A} \) on a set: \( X \in \uparrow \mathcal{A} \Leftrightarrow X \notin \partial \mathcal{A} \) for every \( X \in \mathfrak{P}, A \in \mathfrak{F} \).

**Proof.** By theorem 4.126. \( \square \)

**Proposition 4.201.** For a filter \( \mathcal{A} \) on a set:

1. \( \partial \mathcal{A} = \{ X \mid X \in \mathfrak{P} \setminus \uparrow \mathcal{A} \} \);
2. \( \uparrow \mathcal{A} = \{ X \mid X \in \mathfrak{P} \setminus \partial \mathcal{A} \} \)

(where complement is taken on the boolean lattice \( \mathfrak{P} \)).

**Proof.** By corollary 4.127. \( \square \)

**Proposition 4.202.** \( \partial \) is an injection for filters on sets.

**Proof.** By corollary 4.128. \( \square \)

**Proposition 4.203.** For filters on a set: for any set \( S \in \mathcal{P} \mathfrak{P} \) there exists a filter \( \mathcal{A} \) such that \( \partial \mathcal{A} = S \) iff \( S \) is a free star.

**Proof.** By theorem 4.129. \( \square \)

**Proposition 4.204.** \( A \sqsubseteq B \Leftrightarrow \partial A \sqsubseteq \partial B \) for every filters \( A, B \) on a set.
Proof. By proposition 4.130.

Proposition 4.205. \( \partial \) is a straight monotone map for filters on a set.

Proof. By corollary 4.131.

Proposition 4.206. \( \partial \bigcup S = \bigcup (\partial S) \) for every \( S \in \mathcal{P} \) where \( \mathcal{F} \) are filters on a set.

Proof. By theorem 4.132.

Proposition 4.207. The poset of filters on a set is atomic.


Proposition 4.208. The poset of filters on a set is separable.


Proposition 4.209. The poset of filters on a set is atomistic.

Proof. By theorem 4.137.


Proposition 4.211. The filtrator on a powerset is central.

Proof. By theorem 4.139.

Proposition 4.212. \( a \) is an atom of \( \mathcal{P} \) iff \( a \in \mathcal{P} \) and \( a \) is an atom of \( \mathcal{F} \) for filters on a set.

Proof. By proposition 4.140.

Proposition 4.213. \( a \in \mathcal{F} \) is an atom of \( \mathcal{F} \) iff up \( a = \partial a \) for filters on a set.

Proof. By proposition 4.141.

Theorem 4.214. Let \( a \) be a filter on a set. Then the following are equivalent:

1. \( a \) is prime.
2. For every \( A \in \mathcal{F} \) exactly one of \( \{A, \overline{A}\} \) is in \( a \).
3. \( a \) is an atom of \( \mathcal{P} \).

Proof. By theorem 4.143.

Proposition 4.215. The following conditions are equivalent for every filter \( \mathcal{F} \) on a set:

1. \( \mathcal{F} \in \mathcal{F} \);
2. \( \forall S \in \mathcal{P} \mathcal{F}: (\mathcal{F} \cap S \neq 0 \Rightarrow \exists K \in \mathcal{F}: \mathcal{F} \cap K \neq 0) \);
3. \( \forall S \in \mathcal{P} \mathcal{P}: (\mathcal{F} \cap S \neq 0 \Rightarrow \exists K \in \mathcal{F}: \mathcal{F} \cap K \neq 0) \).

Proof. By proposition 4.144.

Proposition 4.216. For every filter \( \mathcal{F} \) on a set

\[
\mathcal{F} \in \mathcal{P} \Leftrightarrow \forall S \in \mathcal{P} \mathcal{P}: \left( \bigcup S \in \partial \mathcal{F} \Rightarrow S \cap \partial \mathcal{F} \neq 0 \right).
\]

**Theorem 4.217.** For any \( S \in \mathcal{P} \mathcal{F} \), where \( \mathcal{F} \) are filters on a set, the condition \( \exists F \in \mathcal{F}: S = *F \) is equivalent to conjunction of the following items:

1. \( S \) is a free star on \( \mathcal{F}; \)
2. \( S \) is filter closed.

Proof. By theorem 4.146.

**Proposition 4.218.** Let \( \mathcal{F} \) be filters on a set. Let \( A \in \mathcal{F} \). Then for each \( \mathcal{X} \in \mathcal{F} \)

\[ \mathcal{X} \in Z(DA) \Leftrightarrow \exists X \in \Psi: \mathcal{X} = X \cap A. \]

Proof. By theorem 4.147.

**Proposition 4.219.** Cor \( a = \{ p \in U | \{ p \} \subseteq a \} \) and \( \forall a = \{ p \in U | \{ p \} \subseteq a \} \) for every filter \( a \) on a set.


**Proposition 4.220.** For every filter \( a \) on a set \( a^* = a^+ = \overline{\operatorname{Cor} a} = \overline{\operatorname{Cor} a}. \)


**Corollary 4.221.** For every filter \( a \) on a set \( a^* = A^* \in \Psi \).

**Proposition 4.222.** If \( a \) is a filter on a set, then \( a^* \) is dual pseudocomplement of \( a \), that is

\[ a^+ = \min \{ c \in A \mid c \cup A = 1 \}. \]


**Proposition 4.223.** If \( a, b \) are filters on a set, then

1. \( \cap (a \cap b) = \cap a \cap b; \)
2. \( \cap (a \cup b) = \cap a \cup b. \)


**Proposition 4.224.** \( \cap (a \cap b) = \cap (\cap b). \)


**Proposition 4.225.** If \( a, b \) are filters on a set, then

1. \( (a \cap b)^* = a^* \cup b^*; \)
2. \( (a \cup b)^* = a^* \cap b^* \).

Proof. By propositions 4.157 and 4.158.

**Proposition 4.226.** For every \( X, Y \in \mathcal{P} U \) and filter \( F \) on \( U \) we have:

\[ \uparrow X \sim \uparrow Y \Leftrightarrow \exists A \in A: X \cap A = Y \cap A. \]

Proof. By theorem 4.162.

**Proposition 4.227.** Let \( \mathcal{F} \) be the set of filters on a set \( \mathcal{U} \) and \( A \in \mathcal{F} \). Consider the function \( \gamma: Z(DA) \to (\mathcal{P} \mathcal{U})/\sim \) defined by the formula (for every \( p \in Z(DA) \))

\[ \gamma p = \{ X \in \mathcal{F} \mid \uparrow X \cap A = p \}. \]
Then:
1. $\gamma$ is a lattice isomorphism.
2. $\forall Q \in q: \gamma^{-1} q = \uparrow Q \cap \mathcal{A}$ for every $q \in (\mathcal{P}\Omega)/\sim$.

**Proof.** By theorem 4.165. □

**Proposition 4.228.** $(\mathcal{P}\Omega)/\sim$ is a boolean lattice.

**Proof.** By corollary 4.166. □

**Proposition 4.229.** For a lattice $\mathfrak{F}$ of filters on a set and $a, b \in \mathfrak{F}$ the following expressions are always equal:
1. $a \setminus b = \bigcap \{ z \in \mathfrak{A} | a \subseteq b \cup z \}$ (quasidifference of $a$ and $b$);
2. $a \# b = \bigcup \{ z \in \mathfrak{A} | z \subseteq a \land z \cap b = 0 \}$ (second quasidifference of $a$ and $b$);
3. $\bigcup$ (atoms $a \setminus$ atoms $b$).

**Proof.** Theorem 4.167. □

**Conjecture 4.230.** $a \setminus b = a \# b$ for arbitrary filters $a, b$ on powersets is not provable in ZF (without axiom of choice).

### 4.4 Filters on a Set

The consideration below is about filters on a set $\Omega$, but this can be generalized for filters on complete atomic boolean algebras due complete atomic boolean algebras are isomorphic to algebras of sets on some set $\Omega$.

**Definition 4.231.** $\Omega = \{ \Omega \setminus X | X \text{ is a finite subset of } \Omega \}$ is called either Fréchet filter or cofinite filter.

It is trivial that Frechet filter is a filter.

**Proposition 4.232.** $\text{Cor} \Omega = 0^\mathfrak{P}; \bigcap \Omega = \emptyset$.

**Proof.** This can be deduced from the formula $\forall \alpha \in \Omega \exists X \in \Omega: \alpha \notin X$. □

**Theorem 4.233.** $\max \{ \mathcal{X} \in \mathfrak{F} | \text{Cor} \mathcal{X} = 0^\mathfrak{P} \} = \max \{ \mathcal{X} \in \mathfrak{F} | \bigcap \mathcal{X} = \emptyset \} = \Omega$.

**Proof.** Due the last proposition, it is enough to show that $\text{Cor} \mathcal{X} = 0^\mathfrak{P} \Rightarrow \mathcal{X} \subseteq \Omega$ for every filter $\mathcal{X}$. Let $\text{Cor} \mathcal{X} = 0^\mathfrak{P}$ for some filter $\mathcal{X}$. Let $X \in \Omega$. We need to prove that $X \in \mathcal{X}$. Let $X = \Omega \setminus \{ \alpha_0, ..., \alpha_n \}$. $U \setminus \{ \alpha_i \} \in \mathcal{X}$ because otherwise $\alpha_i \in \uparrow^{-1} \text{Cor} \mathcal{X}$. So $X \in \mathcal{X}$. □

**Theorem 4.234.** $\Omega = \bigcup \{ x | x \text{ is a non-trivial ultrafilter} \}$.

**Proof.** It follows from the facts that $\text{Cor} x = 0^\mathfrak{P}$ for every non-trivial ultrafilter $x$, that $\mathfrak{F}$ is an atomic lattice, and the previous theorem. □

**Theorem 4.235.** Cor is the lower adjoint of $\Omega \cup \mathfrak{F} -$.

**Proof.** Because both Cor and $\Omega \cup \mathfrak{F} -$ are monotone, it is enough (theorem 2.98) to prove (for every filters $\mathcal{X}$ and $\mathcal{Y}$)

$\mathcal{X} \subseteq \Omega \cup \mathfrak{F} \text{ Cor } \mathcal{X}$ and $\text{Cor} (\Omega \cup \mathfrak{F} \mathcal{Y}) \subseteq \mathcal{Y}$.

$\text{Cor} (\Omega \cup \mathfrak{F} \mathcal{Y}) = \text{Cor} \Omega \cup \mathfrak{F} \mathcal{Y} = 0^\mathfrak{P} \cup \mathfrak{P} \text{ Cor } \mathcal{Y} = \text{Cor } \mathcal{Y} \subseteq \mathcal{Y}$.

$\Omega \cup \mathfrak{F} \text{ Cor } \mathcal{X} \supseteq \text{Edg } \mathcal{X} \cup \mathfrak{F} \text{ Cor } \mathcal{X} = \mathcal{X}$. □
Corollary 4.236. Cor \( \mathcal{X} = \mathcal{X} \setminus \Omega \) for every filter on a set.

Proof. By the theorem 2.115.

Corollary 4.237. Cor \( \bigsqcup S = \bigsqcup (\text{Cor})S \) for any set \( S \) of filters.

4.4.2 Number of Filters on a Set

Definition 4.238. A collection \( Y \) of sets has finite intersection property iff intersection of any finite subcollection of \( Y \) is non-empty.

The following was borrowed from [7]. Thanks to Andreas Blass for email support about his proof.

Lemma 4.239. (by Hausdorff) For an infinite set \( X \) there is a family \( \mathcal{F} \) of \( 2^{\text{card}X} \) many subsets of \( X \) such that given any disjoint finite subfamilies \( \mathcal{A}, \mathcal{B} \), the intersection of sets in \( \mathcal{A} \) and complements of sets in \( \mathcal{B} \) is nonempty.

Proof. Let

\[
X' = \{(P; Q) \mid P \in \mathcal{P}X \text{ is finite, } Q \in \mathcal{P}P\}.
\]

It’s easy to show that \( \text{card } X' = \text{card } X \). So it is enough to show this for \( X' \) instead of \( X \). Let

\[
\mathcal{F} = \{(P; Q) \in X' \mid Y \cap P \in Q \} \mid Y \in \mathcal{P}X\}.
\]

To finish the proof we show that for every disjoint finite \( Y_+ \in \mathcal{P}PX \) and finite \( Y_- \in \mathcal{P}PX \) that there exist \( (P; Q) \in X' \) such that

\[
\forall Y \in Y_+: (P; Q) \in X' \mid Y \cap P \in Q \quad \text{and} \quad \forall Y \in Y_-: (P; Q) \notin \{(P; Q) \in X' \mid Y \cap P \in Q\}.
\]

what is equivalent to existence \( (P; Q) \in X' \) such that

\[
\forall Y \in Y_+: Y \cap P \in Q \quad \text{and} \quad \forall Y \in Y_-: Y \cap P \notin Q.
\]

For existence of this \( (P; Q) \), it is enough existence of \( P \) such that intersections \( Y \cap P \) are different for different \( Y \in Y_+ \cup Y_- \).

Really, for each pair of distinct \( Y_0, Y_1 \in Y_+ \cup Y_- \) choose a point which lies in one of the sets \( Y_0, Y_1 \) and not in another, and call the set of such points \( P \). Then \( Y \cap P \) are different for different \( Y \in Y_+ \cup Y_- \).

Corollary 4.240. For an infinite set \( X \) there is a family \( \mathcal{F} \) of \( 2^{\text{card}X} \) many subsets of \( X \) such that for arbitrary disjoint subfamilies \( \mathcal{A}, \mathcal{B} \) the set \( \mathcal{A} \cup \{X \setminus A \mid A \in \mathcal{B}\} \) has finite intersection property.

Theorem 4.241. Let \( X \) be a set. The number of ultrafilters on \( X \) is \( 2^{2^{\text{card}X}} \) if \( X \) is infinite and card \( X \) if \( X \) is finite.

Proof. The finite case follows from the fact that every ultrafilter on a finite set is trivial. Let \( X \) be infinite. From the lemma, there exists a family of \( 2^{\text{card}X} \) many subsets of \( X \) such that for every \( \mathcal{G} \in \mathcal{P} \mathcal{F} \) we have \( \Phi(\mathcal{F}; \mathcal{G}) = \bigsqcup \{ \mathcal{G}_n \} \bigsqcup \{ \{X \setminus A \mid A \in \mathcal{F} \setminus \mathcal{G}\} \neq 0^{\text{card}(X)} \}.
\)

This filter contains all sets from \( \mathcal{G} \) and does not contain any sets from \( \mathcal{F} \setminus \mathcal{G} \). So for every suitable pairs \( (\mathcal{F}_0; \mathcal{G}_0) \) and \( (\mathcal{F}_1; \mathcal{G}_1) \) there are \( A \in \Phi(\mathcal{F}_0; \mathcal{G}_0) \) such that \( \mathcal{A} \in \Phi(\mathcal{F}_1; \mathcal{G}_1) \). Consequently all filters \( \Phi(\mathcal{F}; \mathcal{G}) \) are disjoint. So for every pair \( (\mathcal{F}; \mathcal{G}) \) where \( \mathcal{G} \in \mathcal{P} \mathcal{F} \) there exist a distinct ultrafilter under \( \Phi(\mathcal{F}; \mathcal{G}) \), but the number of such pairs \( (\mathcal{F}; \mathcal{G}) \) is \( 2^{2^{\text{card}X}} \). Obviously the number of all filters is not above \( 2^{2^{\text{card}X}} \).

Corollary 4.242. The number of filters on \( \Omega \) is \( 2^{2^{\text{card}\Omega}} \) if \( \Omega \) is infinite and \( 2^{\text{card}\Omega} \) if \( \Omega \) is finite.

Proof. The finite case is obvious. The infinite case follows from the theorem and the fact that filters are collections of sets and there cannot be more than \( 2^{2^{\text{card}\Omega}} \) collections of sets on \( \Omega \).
4.5 Some Counter-Examples

**Example 4.243.** There exist a bounded distributive lattice which is not lattice with separable center.

**Proof.** The lattice with the following Hasse diagram is bounded and distributive because it does not contain "diamond lattice" nor "pentagon lattice" as a sublattice [40].

![Hasse diagram](image)

Figure 4.1.

It’s center is \(\{0, 1\}\). \(x \cap y = 0\) despite up \(x = \{x, a, 1\}\) but \(y \cap 1 \neq 0\) consequently the lattice is not with separable center.

For further examples we will use the filter \(\Delta\) defined by the formula

\[
\Delta = \{ \uparrow(-\varepsilon; \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \}
\]

and more general

\[
\Delta + a = \{ \uparrow(a - \varepsilon; a + \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \}.
\]

**Example 4.244.** There exists \(A \in \mathcal{P}U\) such that \(\bigcap \mathfrak{S} \uparrow \not\uparrow \Delta A\).

**Proof.** \(\uparrow \cap \{(-a; a) \mid a \in \mathbb{R}, a > 0\} = \uparrow \{0\} \not= \Delta\).

**Example 4.245.** There exists a set \(U\) and a filter \(a\) and a set \(S\) of filters on the set \(U\) such that \(a \uparrow \cap \mathfrak{S} S \not= \bigcup \mathfrak{S} \{a \cap \mathfrak{S} S\}\).

**Proof.** Let \(a = \Delta\) and \(S = \{ \uparrow(\varepsilon; +\infty) \mid \varepsilon > 0 \}\). Then \(a \cap \mathfrak{S} \bigcup \mathfrak{S} S = \Delta \cap \mathfrak{S} \{0; +\infty\} \not= 0\) while \(\bigcup \mathfrak{S} \{a \cap \mathfrak{S} S\} \not= \mathfrak{S} \{0\}\).

**Example 4.246.** There are tornings which are not weak partitions.

**Proof.** \(\{\Delta \cap \mathfrak{S} \mid a \in \mathbb{R}\}\) is a torning but not weak partition of the real line.

**Lemma 4.247.** Let \(\mathfrak{S}\) be the set of filters on a set \(U\). Then \(\uparrow X \cap \mathfrak{S} \Omega \subseteq \uparrow Y \cap \mathfrak{S} \Omega\) iff \(X \setminus Y\) is a finite set, for every sets \(X, Y \in \mathcal{P}U\).

**Proof.** \(\uparrow X \cap \mathfrak{S} \Omega \subseteq \uparrow Y \cap \mathfrak{S} \Omega\) \(\iff\) \(X \setminus K_X \subseteq \{ Y \cap K_Y \mid K_Y \in \Omega \}\) \(\iff\) \(\forall K_Y \in \Omega \exists K_X \in \Omega : Y \setminus K_Y = X \setminus K_X \Rightarrow \forall L_Y \in M \exists L_X \subseteq M : Y \setminus L_Y = X \setminus L_X \Rightarrow \forall L_Y \in M : X \setminus (Y \setminus L_Y) \subseteq M \Rightarrow X \setminus Y \subseteq M\), where \(M\) is the set of finite subsets of \(U\).

**Example 4.248.** There exists a filter \(A\) on a set \(U\) such that \((\mathcal{P}U) / \sim \) and \(Z(D A)\) are not complete lattices.

**Proof.** Due to the isomorphism it is enough to prove for \((\mathcal{P}U) / \sim\).

---

4.2. See Wikipedia for a definition of Hasse diagrams.
Let take $U = N$ and $A = \Omega$ be the Fréchet filter on $N$.

Partition $N$ into infinitely many infinite sets $A_0, A_1, \ldots$. To withhold our example we will prove that the set $\mathcal{E} = \{[A_0], [A_1], \ldots \}$ has no supremum in $(\mathcal{P}(U))/\sim$.

Let $[X]$ be an upper bound of $\{A_0, A_1, \ldots \}$ that is $\forall i \in N: \uparrow X \cap \exists \Omega \supseteq \uparrow A_i \cap \exists \Omega$ that is $A_i \setminus X$ is finite. Consequently $X$ is infinite. So $X \cap A_i \neq \emptyset$.

Choose for every $i \in N$ some $z_i \in X \cap A_i$. Then $\{z_0, z_1, \ldots \}$ is an infinite subset of $X$ (take into account that $z_i \neq z_j$ for $i \neq j$). Let $Y = X \setminus \{z_0, z_1, \ldots \}$. Then $\uparrow Y \cap \exists \Omega \supseteq \uparrow A_i \cap \exists \Omega$ because $A_i \setminus Y = A_i \setminus (X \setminus \{z_i\}) = (A_i \setminus X) \cup \{z_i\}$ which is finite because $A_i \setminus X$ is finite. Thus $[Y]$ is an upper bound for $\{[A_0], [A_1], \ldots \}$.

Suppose $\uparrow Y \cap \exists \Omega \supseteq \uparrow X \cap \exists \Omega$. Then $Y \setminus X$ is finite what is not true. So $\uparrow Y \cap \exists \Omega \subset \uparrow X \cap \exists \Omega$ that is $[Y]$ is below $[X]$. □

### 4.5.1 Weak and Strong Partition

**Definition 4.249.** A family $S$ of subsets of a countable set is independent iff the intersection of any finitely many members of $S$ and the complements of any other finitely many members of $S$ is infinite.

**Lemma 4.250.** The “infinite” at the end of the definition could be equivalently replaced with “non-empty” if we assume that $S$ is infinite.

**Proof.** Suppose that some sets from the above definition has a finite intersection $J$ of cardinality $n$. Then (thanks $S$ is infinite) get one more set $X \in S$ and we have $J \cap X \neq \emptyset$ and $J \cap (N \setminus X) \neq \emptyset$. So $\text{card}(J \cap X) < n$. Repeating this, we prove that for some finite family of sets we have empty intersection what is a contradiction. □

**Lemma 4.251.** There exists an independent family on $N$ of cardinality $\aleph$.  

**Proof.** Let $C$ be the set of finite subsets of $\mathbb{Q}$. Since $\text{card} C = \text{card} N$, it suffices to find $\aleph$ independent subsets of $C$. For each $r \in \mathbb{R}$ let

$$E_r = \{ F \in C \mid \text{card}(F \cap (-\infty; r)) \text{ is even} \}.$$  

All $E_{r_1}$ and $E_{r_2}$ are distinct for distinct $r_1, r_2 \in \mathbb{R}$ since we may consider $F = \{r' \in C$ where a rational number $r'$ is between $r_1$ and $r_2$ and thus $F$ is a member of exactly one of the sets $E_{r_1}$ and $E_{r_2}$. Thus $\text{card}(E_r \mid r \in \mathbb{R}) = \aleph$. 

We will show that $\{E_r \mid r \in \mathbb{R}\}$ is independent. Let $r_1, \ldots, r_k, s_1, \ldots, s_k$ be distinct reals. It is enough to show that these have a nonempty intersection, that is existence of some $F$ such that $F$ belongs to all the $E_r$, and none of $E_s$.

But this can be easily accomplished taking $F$ having zero or one element in each of intervals to which $r_1, \ldots, r_k, s_1, \ldots, s_k$ split the real line. □

**Example 4.252.** There exists a weak partition of a filter on a set which is not a strong partition.

**Proof.** (suggested by Andreas Blass) Let $\{X_r \mid r \in \mathbb{R}\}$ be an independent family of subsets of $N$.

Let $\mathcal{F}_a$ be a filter generated by $X_a$ and the complements $N \setminus X_b$ for all $b \in R, b \neq a$. Independence implies that $\mathcal{F}_a \neq 0^\exists$ (by properties of filter bases).

Let $S = \{F_r \mid r \in \mathbb{R}\}$. We will prove that $S$ is a weak partition but not a strong partition.

Let $a \in \mathbb{R}$. Then $X_a \in \mathcal{F}_a$ while $\forall b \in \mathbb{R} \setminus \{a\}: N \setminus X_a \in \mathcal{F}_b$ and therefore $N \setminus X_a \in \bigcup^\exists \mathcal{F}_b \setminus R \ni a \neq a$. Therefore $\mathcal{F}_a \cap \exists \mathcal{F}_b \ni \exists \mathcal{F}_b \setminus R \ni a \neq a \equiv 0^\exists$. Thus $S$ is a weak partition. Suppose $S$ is a strong partition. Then for each set $Z \in \mathcal{P} \mathbb{R}$

$$\exists \mathcal{F}_b \setminus b \in Z \cap \exists \mathcal{F}_b \setminus b \in \mathbb{R} \setminus Z = 0^\exists$$

what is equivalent to existence of $M(Z) \in \mathcal{P} \mathbb{N}$ such that

$$M(Z) \in \exists \mathcal{F}_b \setminus b \in Z \quad \text{and} \quad N \setminus M(Z) \in \exists \mathcal{F}_b \setminus b \in \mathbb{R} \setminus Z$$
that is
\[ \forall b \in Z: M(Z) \in \mathcal{F}_b \quad \text{and} \quad \forall b \in \mathcal{R} \setminus Z: N \setminus M(Z) \in \mathcal{F}_b. \]
Suppose \( Z \neq Z' \in \mathcal{P} \mathcal{N}. \) Without loss of generality we may assume that some \( b \in Z \) but \( b \notin Z' \). Then \( M(Z) \in \mathcal{F}_b \) and \( N \setminus M(Z') \in \mathcal{F}_b. \) If \( M(Z) = M(Z') \) then \( \mathcal{F}_b = 0 \) what contradicts to the above.

So \( M \) is an injective function from \( \mathcal{P} \mathcal{R} \) to \( \mathcal{P} \mathcal{N} \) what is impossible due cardinality issues.

**Lemma 4.253.** (by Niels Diepeveen, with help of Karl Kronenfeld) Let \( K \) be a collection of free ultrafilters. We have \( \bigcup K = \Omega \) if \( \exists \mathcal{G} \in K: A \in \mathcal{G} \) for every infinite set \( A \).

**Proof.**
\[ \Rightarrow. \] Suppose \( \bigcup K = \Omega \) and let \( A \) be a set such that \( \exists \mathcal{G} \in K: A \in \mathcal{G} \). Let’s prove \( A \) is finite.

Really, \( \forall \mathcal{G} \in K: \mathcal{G} \in \mathcal{G} \); \( \mathcal{G} \setminus A \in \Omega \); \( A \) is finite.

\[ \Leftarrow. \] Let \( \exists \mathcal{G} \in K: A \in \mathcal{G} \). Suppose \( A \) is a set in \( \bigcup K \).

To finish the proof it’s enough to show that \( \mathcal{G} \setminus A \) is finite.

Suppose \( \mathcal{G} \setminus A \) is infinite. Then \( \exists \mathcal{G} \in K: \mathcal{G} \setminus A \in \mathcal{G} \); \( \exists \mathcal{G} \in K: A \notin \mathcal{G} \); \( A \notin \bigcup K \), contradiction.

**Lemma 4.254.** (by Niels Diepeveen) If \( K \) is a non-empty set of ultrafilters such that \( \bigcup K = \Omega \), then for every \( \mathcal{G} \in K \) we have \( \bigcup (K \setminus \{ \mathcal{G} \}) = \Omega \).

**Proof.** \( \exists \mathcal{F} \in K: A \in \mathcal{F} \) for every infinite set \( A \).

The set \( A \) can be partitioned into two infinite sets \( A_1, A_2 \).

Take \( \mathcal{F}_1, \mathcal{F}_2 \in K \) such that \( A_1 \in \mathcal{F}_1 \), \( A_2 \in \mathcal{F}_2 \).

\( \mathcal{F}_1 \neq \mathcal{F}_2 \) because otherwise \( A_1 \) and \( A_2 \) are not disjoint.

Obviously \( A \in \mathcal{F}_1 \) and \( A \in \mathcal{F}_2 \).

So there exist two different \( \mathcal{F} \in K \) such that \( A \in \mathcal{F} \). Consequently \( \exists \mathcal{F} \in K \setminus \{ \mathcal{G} \} : A \in \mathcal{F} \) that is \( \bigcup (K \setminus \{ \mathcal{G} \}) = \Omega \). \[ \Box \]

**Example 4.255.** There exists a filter on a set which cannot be weakly partitioned into ultrafilters.

**Proof.** Consider cofinite filter \( \Omega \) on any infinite set.

Suppose \( K \) is its weak partition into ultrafilters. Then \( \exists x \in K \setminus \{ x \} \) for some ultrafilter \( x \in K \).

We have \( \bigcup (K \setminus \{ x \}) \subset \bigcup K \) (otherwise \( x \subset \bigcup (K \setminus \{ x \}) \)) what is impossible due the last lemma. \[ \Box \]

**Corollary 4.256.** There exists a filter on a set which cannot be strongly partitioned into ultrafilters.

### 4.6 Open problems about filters

In this section, I will formulate some conjectures about lattices of filters on a set. If a conjecture comes true, it may be generalized for more general lattices (such as, for example, lattices of filters on arbitrary lattices). I deem that the main challenge is to prove the special case about lattices of filters on a set, and generalizing the conjectures is expected to be an easy task.

#### 4.6.1 Partitioning

Consider the complete lattice \( [S] \) generated by the set \( S \) where \( S \) is a strong partition of some element \( a \).

**Conjecture 4.257.** \( [S] = \{ \bigcup X \mid X \in \mathcal{P} \cdot S \} \), where \( [S] \) is the complete lattice generated by a strong partition \( S \) of filter on a set.

**Proposition 4.258.** Provided that the last conjecture is true, we have that \( [S] \) is a complete atomic boolean lattice with the set of its atoms being \( S \).
**Remark 4.259.** Consequently $[S]$ is atomistic, completely distributive and isomorphic to a power set algebra (see [39]).

**Proof.** Completeness of $[S]$ is obvious. Let $A \in [S]$. Then there exists $X \in \mathcal{P}S$ such that $A = \bigsqcup^\delta X$. Let $B = \bigsqcup^\delta (S \setminus X)$. Then $B \in [S]$ and $A \cap^\delta B = 0^\delta$. $A \cup^\delta B = \bigsqcup^\delta S$ is the greatest element of $[S]$. So we have proved that $[S]$ is a boolean lattice.

Now let prove that $[S]$ is atomic with the set of atoms being $S$. Let $z \in S$ and $A \in [S]$. If $A \neq z$ then either $A = 0^\delta$ or $x \in X$ where $A = \bigsqcup^\delta X$, $X \in \mathcal{P}S$ and $x \neq z$. Because $S$ is a partition, $\bigsqcup^\delta (X \setminus \{z\}) \cap^\delta z = 0^\delta$ and $\bigsqcup^\delta (X \setminus \{z\}) \neq 0^\delta$. So $A = \bigsqcup^\delta X = \bigsqcup^\delta (X \setminus \{z\}) \cup^\delta z \neq z$.

Finally we will prove that elements of $[S] \setminus S$ are not atoms. Let $A \in [S] \setminus S$ and $A \neq 0$. Then $A \supseteq x \cup^\delta y$ where $x, y \in S$ and $x \neq y$. If $A$ is an atom then $A = x = y$ what is impossible. \[\Box\]

**Proposition 4.260.** The conjecture about the value of $[S]$ is equivalent to closedness of $\{\bigsqcup^\delta X \mid X \in \mathcal{P}S\}$ under arbitrary meets and joins.

**Proof.** If $\{\bigsqcup^\delta X \mid X \in \mathcal{P}S\} = [S]$ then trivially $\{\bigsqcup^\delta X \mid X \in \mathcal{P}S\}$ is closed under arbitrary meets and joins.

If $\{\bigsqcup^\delta X \mid X \in \mathcal{P}S\}$ is closed under arbitrary meets and joins, then it is the complete lattice generated by the set $S$ because it cannot be smaller than the set of all suprema of subsets of $S$. \[\Box\]

That $\{\bigsqcup^\delta X \mid X \in \mathcal{P}S\}$ is closed under arbitrary joins is trivial. I have not succeeded to prove that it is closed under arbitrary meets, but have proved a weaker statement that is is closed under finite meets:

**Proposition 4.261.** $\{\bigsqcup^\delta X \mid X \in \mathcal{P}S\}$ is closed under finite meets.

**Proof.** Let $R = \{\bigsqcup^\delta X \mid X \in \mathcal{P}S\}$. Then for every $X, Y \in \mathcal{P}S$

- $\bigsqcup^\delta X \cap^\delta \bigsqcup^\delta Y = \bigsqcup^\delta ((X \cap Y) \cup^\delta (X \setminus Y)) \cap^\delta \bigsqcup^\delta Y = (\bigsqcup^\delta (X \cap Y) \cup^\delta (X \setminus Y)) \cap^\delta \bigsqcup^\delta Y = (\bigsqcup^\delta (X \cap Y) \cap^\delta \bigsqcup^\delta Y) \cup^\delta (\bigsqcup^\delta (X \setminus Y) \cap^\delta \bigsqcup^\delta Y) = (\bigsqcup^\delta (X \cap Y) \cap^\delta \bigsqcup^\delta Y) \cup^\delta \emptyset^\delta = \bigsqcup^\delta (X \cap Y) \cap^\delta \bigsqcup^\delta Y$.

Applying the formula $\bigsqcup^\delta X \cap^\delta \bigsqcup^\delta Y = \bigsqcup^\delta (X \cap Y) \cap^\delta \bigsqcup^\delta Y$ twice we get

- $\bigsqcup^\delta X \cap^\delta \bigsqcup^\delta Y = \bigsqcup^\delta (X \cap Y) \cap^\delta \bigsqcup^\delta Y = \bigsqcup^\delta (X \cap Y)$.

But for any $A, B \in R$ there exist $X, Y \in \mathcal{P}S$ such that $A = \bigsqcup^\delta X$, $B = \bigsqcup^\delta Y$. So $A \cap^\delta B = \bigsqcup^\delta X \cap^\delta \bigsqcup^\delta Y = \bigsqcup^\delta (X \cap Y) \in R$. \[\Box\]
4.6.2 Quasidifference

**Conjecture 4.26.** \( a \upharpoonright b = \bigsqcup \{ a \cap \uparrow (U \setminus B) \mid B \in b \} \) for all \( a, b \in \mathcal{F} \) for each lattice \( \mathcal{F} \) of filters on a set \( U \).

4.6.3 Non-Formal Problems

Should we research the lattice of free stars?

Find a common generalization of two theorems:

1. If \( \mathfrak{A} \) is a meet-semilattice with greatest element then for any \( \mathcal{A}, \mathcal{B} \in \mathcal{F} \)
   \[ \mathcal{A} \sqcup \mathcal{B} = \mathcal{A} \cap \mathcal{B}. \]

2. If \( \mathfrak{A} \) is a join-semilattice then \( \mathcal{F} \) is a join-semilattice then and for any \( \mathcal{A}, \mathcal{B} \in \mathcal{F} \)
   \[ \mathcal{A} \sqcup \mathcal{B} = \mathcal{A} \cap \mathcal{B}. \]

Under which conditions \( a \upharpoonright b \) and \( a \# b \) are complementive to \( a \)?

Generalize straight maps for arbitrary posets.
Chapter 5
Common knowledge, part 2 (topology)

In this chapter I describe basics of the theory known as general topology. Starting with the next chapter after this one I will describe generalizations of customary objects of general topology described in this chapter.

The reason why I’ve written this chapter is to show to the reader kinds of objects which I generalize below in this book. For example, funcoids and a generalization of proximity spaces, and funcoids are a generalization of pretopologies. To understand the intuitive meaning of funcoids one needs first know what are proximities and what are pretopologies.

Having said that, customary topology is not used in my definitions and proofs below. It is just to feed your intuition.

5.1 Metric spaces

The theory of topological spaces started immediately with the definition would be completely non-intuitive for the reader. It is the reason why I first describe metric spaces and show that metric spaces give rise for a topology (see below). Topological spaces are understandable as a generalization of topologies induced by metric spaces.

Metric spaces is a formal way to express the notion of distance. For example, there are distance $|x - y|$ between real numbers $x$ and $y$, distance between points of a plane, etc.

**Definition 5.1.** A metric space is a set $U$ together with a function $d: U \times U \to \mathbb{R}$ (distance) such that for every $x, y, z \in U$:

1. $d(x, y) \geq 0$;
2. $d(x, y) = 0 \iff x = y$;
3. $d(x, y) = d(y, x)$ (symmetry);
4. $d(x, z) \leq d(x, y) + d(y, z)$ (triangle inequality).

**Exercise 5.1.** Show that the Euclid space $\mathbb{R}^n$ (with the standard distance) is a metric space for every $n \in \mathbb{N}$.

**Definition 5.2.** Open ball of radius $r > 0$ centered at point $a \in U$ is the set $B_r(a) = \{x \in U \mid d(r, x) < r\}$.

**Definition 5.3.** Closed ball of radius $r > 0$ centered at point $a \in U$ is the set $B_r[a] = \{x \in U \mid d(r, x) \leq r\}$.

5.1.1 Open and closed sets

**Definition 5.4.** A set $A$ in a metric space is called open when $\forall a \in A \exists r > 0: B_r(a) \subseteq A$.

**Definition 5.5.** A set $A$ in a metric space is closed when its complement $U \setminus A$ is open.

**Definition 5.6.** Closure $\text{cl}(A)$ of a set $A$ in a metric space is the set of points $y$ such that $\forall \varepsilon > 0 \exists a \in A: d(y, a) < \varepsilon$. 

89
Proposition 5.7. $\text{cl}(A) \supseteq A$.

Proof. It follows from $d(a, a) = 0 < \varepsilon$. \hfill $\square$

**Exercise 5.2.** Prove $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$ for every subsets $A$ and $B$ of a metric space.

### 5.1.2 Continuity

**Definition 5.8.** A function $f$ from a metric space $\mathfrak{A}$ to a metric space $\mathfrak{B}$ is called continuous at point $a \in \mathfrak{A}$ when

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in \mathfrak{A}: (d(a, x) < \delta \Rightarrow d(f(a), f(x)) < \varepsilon).$$

**Definition 5.9.** A function $f$ is called continuous when it is continuous at every point of its domain.

### 5.2 Pretopological spaces

**Pretopological space** can be defined in two equivalent ways: a neighborhood system or a preclosure operator. To be more clear I will call pretopological space only the first (neighborhood system) and the second call a preclosure space.

**Definition 5.10.** Pretopological space is a set $U$ together with a filter $\Delta(x)$ on $U$ for every $x \in U$, such that $\uparrow^U \{x\} \subseteq \Delta(x)$. $\Delta(x)$ is called a pretopology on $U$.

**Definition 5.11.** Preclosure on a set $U$ is an unary operation $\text{cl}$ on $\mathcal{P}U$ such that for every $A, B \in \mathcal{P}U$:

1. $\text{cl}(\emptyset) = \emptyset$;
2. $\text{cl}(A) \supseteq A$;
3. $\text{cl}(A \cup B) = \text{cl}(A) \cup \text{cl}(B)$.

I call a preclosure together with a set $U$ as preclosure space.

**Theorem 5.12.** Small pretopological spaces and small preclosure spaces bijectively correspond to each other by the formulas:

$$\text{cl}(A) = \{x \in U \mid A \in \partial \Delta(x)\};$$

$$\Delta(x) = \{A \in \mathcal{P}U \mid x \notin \text{cl}(U \setminus A)\}. \quad (5.1) \quad \text{(5.2)}$$

Proof. First let's prove that $\text{cl}$ defined by formula (5.1) is really a preclosure.

$\text{cl}(\emptyset) = \emptyset$ is obvious. If $x \in A$ then $A \in \partial \Delta(x)$ and so $\text{cl}(A) \supseteq A$. $\text{cl}(A \cup B) = \{x \in U \mid A \cup B \in \partial \Delta(x)\} = \{x \in U \mid A \in \partial \Delta(x) \vee B \in \partial \Delta(x)\} = \text{cl}(A) \cup \text{cl}(B)$. So, it is really a preclosure.

Next let's prove that $\Delta$ defined by formula (5.2) is a pretopology. That $\Delta(x)$ is an upper set is obvious. Let $A, B \in \Delta(x)$. Then $x \notin \text{cl}(U \setminus A) \land x \notin \text{cl}(U \setminus B)$; $x \notin \text{cl}(U \setminus A) \cup \text{cl}(U \setminus B) = \text{cl}(U \setminus (A \cap B)); A \cap B \in \Delta(x)$, we have proved that $\Delta(x)$ is a filter.

Let's prove $\uparrow^U \{x\} \subseteq \Delta(x)$. If $A \in \Delta(x)$ then $x \notin \text{cl}(U \setminus A)$ and consequently $x \notin U \setminus A; x \in A; A \in \uparrow^U \{x\}$. So $\uparrow^U \{x\} \subseteq \Delta(x)$ and thus $\Delta$ is a pretopology.

It is left to prove that the functions defined by the above formulas are mutually inverse.

Let $\text{cl}_0$ be a preclosure, let $\Delta$ is the pretopology induced by $\text{cl}_0$ by the formula (5.2), let $\text{cl}_1$ is the preclosure induced by $\Delta$ by the formula (5.1). Let's prove $\text{cl}_1 = \text{cl}_0$. Really, $x \in \text{cl}_1(A) \Leftrightarrow A \in \Delta(x) \Leftrightarrow \uparrow^U \{x\} \subseteq \Delta(x) \Leftrightarrow \forall X \in \mathcal{P}U: (x \notin \text{cl}_0(X) \Rightarrow A \setminus X \neq \emptyset) \Leftrightarrow \forall X \in \mathcal{P}U: (x \notin \text{cl}_0(X) \Rightarrow \exists X' \in \mathcal{P}U: (A \setminus X' = \emptyset \Rightarrow x \notin \text{cl}_0(X'))) \Leftrightarrow x \in \text{cl}_0(A).$ So $\text{cl}_1(A) = \text{cl}_0(A)$.

Let now $\Delta_0$ be a pretopology, let $\text{cl}$ is the closure induced by $\Delta_0$ by the formula (5.1), let $\Delta_1$ is the pretopology induced by $\text{cl}$ by the formula (5.2). Really $A \in \Delta_1(x) \Leftrightarrow x \notin \text{cl}(U \setminus A) \Leftrightarrow \text{cl}_0(x) = \uparrow^U (U \setminus A) \Leftrightarrow \uparrow^U A \supseteq \Delta_0(x) \Leftrightarrow A \in \Delta_0(x)$ (used proposition 4.184). So $\Delta_1(x) = \Delta_0(x)$. 

That these functions are mutually inverse, is now proved.

5.2.1 Pretopology induced by a metric
Every metric space induces a pretopology by the formula:

\[ \Delta(x) = \bigcap \{ \bigcup B_r(x) \mid r \in \mathbb{R}, r > 0 \}. \]

**Exercise 5.3.** Show that it is a pretopology.

**Proposition 5.13.** The preclosure corresponding to this pretopology is the same as the preclosure of the metric space.

**Proof.** I denote the preclosure of the metric space as \( \text{cl}_M \) and the preclosure corresponding to our pretopology as \( \text{cl}_P \). We need to show \( \text{cl}_P = \text{cl}_M \).

Really: \( \text{cl}_P(A) = \{ x \in U \mid A \notin \partial \Delta(x) \} = \{ x \in U \mid \forall \varepsilon > 0: B_\varepsilon(x) \notin A \} = \{ y \in U \mid \forall \varepsilon > 0 \exists a \in A: d(y, a) < \varepsilon \} = \text{cl}_M(A) \) for every set \( A \in \mathcal{P}U \).

5.3 Topological spaces

**Proposition 5.14.** For the set of open sets of a metric space \((U; d)\) it holds:
1. Union of any (possibly infinite) number of open sets is an open set.
2. Intersection of a finite number of open sets is an open set.
3. \( U \) is an open set.

**Proof.** Let \( S \) be a set of open sets. Let \( a \in \bigcup S \). Then there exists \( A \in S \) such that \( a \in A \). Because \( A \) is open we have \( B_r(a) \subseteq A \) for some \( r > 0 \). Consequently \( B_r(a) \subseteq \bigcup S \) that is \( \bigcup S \) is open.

Let \( A_0, ..., A_n \) be open sets. Let \( a \in A_0 \cap ... \cap A_n \) for some \( n \in \mathbb{N} \) where \( A_i \) are open sets for every \( i = 0, ..., n \). Then there exist \( r_i \) such that \( B_{r_i}(a) \subseteq A_i \). So \( B_{r}(a) \subseteq A_0 \cap ... \cap A_n \) for \( r = \min \{ r_0, ..., r_n \} \) that is \( A_0 \cap ... \cap A_n \) is open.

That \( U \) is an open set is obvious.

**Definition 5.15.** A **topology** on a set \( U \) is a collection \( \mathcal{O} \) (called the set of **open sets**) of subsets of \( U \) such that:
1. Union of any (possibly infinite) number of open sets is an open set.
2. Intersection of a finite number of open sets is an open set.
3. \( U \) is an open set.

The pair \((U; \mathcal{O})\) is called a **topological space**.

**Remark 5.16.** From the above it is clear that every metric induces a topology.

**Proposition 5.17.** Empty set is always open.

**Proof.** Empty set is union of an empty set.

**Definition 5.18.** A **closed set** is a complement of an open set.

Topology can be equivalently expresses in terms of closed sets:
A **topology** on a set \( U \) is a collection (called the set of **closed sets**) of subsets of \( U \) such that:
1. Intersection of any (possibly infinite) number of closed sets is a closed set.
2. Union of a finite number of closed sets is a closed set.
3. \( \emptyset \) is a closed set.

**Exercise 5.4.** Show that the definitions using open and closed sets are equivalent.
5.3.1 Relationships between pretopologies and topologies

5.3.1.1 Topological space induced by preclosure space

Having a preclosure space \((U; \text{cl})\) we define a topological space whose closed sets are such sets \(A \in \mathcal{P}U\) that \(\text{cl}(A) = A\).

**Proposition 5.19.** This really defines a topology.

**Proof.** Let \(S\) be a set of closed sets. First, we need to prove that \(\bigcap S\) is a closed set. We have \(\text{cl}(\bigcap S) \subseteq A\) for every \(A \in S\). Thus \(\text{cl}(\bigcap S) \subseteq \bigcap S\) and consequently \(\text{cl}(\bigcap S) = \bigcap S\). So \(\bigcap S\) is a closed set.

Let now \(A_0, ..., A_n\) be closed sets then

\[
\text{cl}(A_0 \cup ... \cup A_n) = \text{cl}(A_0) \cup ... \cup \text{cl}(A_n) = A_0 \cup ... \cup A_n
\]

that is \(A_0 \cup ... \cup A_n\) is a closed set.

That \(\emptyset\) is a closed set is obvious. \(\Box\)

Having a pretopological space \((U; \Delta)\) we define a topological space whose open sets are

\[
\{X \in \mathcal{P}U \mid \forall x \in X: X \in \Delta(x)\}.
\]

**Proposition 5.20.** This really defines a topology.

**Proof.** Let set \(S \subseteq \{X \in \mathcal{P}U \mid \forall x \in X: X \in \Delta(x)\}\). Then \(\forall X \in S \forall x \in X: X \in \Delta(x)\). Thus

\[
\forall x \in \bigcup S \exists x \in S: X \in \Delta(x)
\]

and so \(\forall x \in \bigcup S: \bigcup S \in \Delta(x)\). So \(\bigcup S\) is an open set.

Let now \(A_0, ..., A_n \in \{X \in \mathcal{P}U \mid \forall x \in X: X \in \Delta(x)\}\) for \(n \in \mathbb{N}\). Then \(\forall x \in A_i: A_i \in \Delta(x)\) and so

\[
\forall x \in A_0 \cap ... \cap A_n: A_i \in \Delta(x);
\]

thus \(\forall x \in A_0 \cap ... \cap A_n: A_0 \cap ... \cap A_n \in \Delta(x)\). So \(A_0 \cap ... \cap A_n \in \{X \in \mathcal{P}U \mid \forall x \in X: X \in \Delta(x)\}\).

That \(U\) is an open set is obvious. \(\Box\)

**Proposition 5.21.** Topology \(\tau\) defined by a pretopology and topology \(\rho\) defined by the corresponding preclosure, are the same.

**Proof.** Let \(A \in \mathcal{P}U\).

A is \(\rho\)-closed \(\iff \text{cl}(A) = A \iff cl(A) \subseteq A \iff \forall x \in U: (A \in \partial \Delta(x) \Rightarrow x \in A)\);

A is \(\tau\)-open \(\iff \forall x \in A: A \in \Delta(x) \iff \forall x \in U: (x \in A \Rightarrow A \in \Delta(x)) \iff \forall x \in U: (x \notin U \setminus A \Rightarrow U \setminus A \notin \partial \Delta(x))\).

So \(\rho\)-closed and \(\tau\)-open are negations of each other. It follows \(\rho = \tau\). \(\Box\)

5.3.1.2 Preclosure space induced by topological space

We define a preclosure and a pretopology induced by a topology and then show these two are equivalent.

Having a topological space we define a preclosure space by the formula

\[
\text{cl}(A) = \bigcap \{X \in \mathcal{P}U \mid X \text{ is a closed set, } X \supseteq A\}.
\]

**Proposition 5.22.** It is really a preclosure.

**Proof.** \(\text{cl}(\emptyset) = \emptyset\) because \(\emptyset\) is a closed set. \(\text{cl}(A) \supseteq A\) is obvious. \(\text{cl}(A \cup B) = \bigcap \{X \in \mathcal{P}U \mid X \text{ is a closed set, } X \supseteq A \cup B\}\) = \(\bigcap \{X_1 \cup X_2 \mid X_1, X_2 \in \mathcal{P}U \text{ are closed sets, } X_1 \supseteq A, X_2 \supseteq B\}\) = \(\bigcap \{X_1 \in \mathcal{P}U \mid X_1 \text{ is closed a set, } X_1 \supseteq A\} \cup \bigcap \{X_2 \in \mathcal{P}U \mid X_2 \text{ is closed a set, } X_2 \supseteq B\}\) = \(\text{cl}(A) \cup \text{cl}(B)\). Thus \(\text{cl}\) is a preclosure.

Or: \(\Delta(x) = \bigcap \{\downarrow U X \mid X \in \mathcal{O}, x \in X\}\).
It is trivially a pretopology (used the fact that $U \in \mathcal{O}$).

**Proposition 5.23.** The preclosure and the pretopology defined in this section above correspond to each other (by the formulas from theorem 5.12).

**Proof.** We need to prove $\text{cl}(A) = \{x \in U \mid \Delta(x) \neq \uparrow U A\}$, that is

$$\bigcap \{X \in \mathcal{P}U \mid X \text{ is a closed set, } X \supseteq A\} = \{x \in U \mid \bigcap \{\uparrow U X \mid X \in \mathcal{O}, x \in X\} \neq \uparrow U A\}.$$  

Equivalently transforming it, we get:

$$\bigcap \{X \in \mathcal{P}U \mid X \text{ is a closed set, } X \supseteq A\} = \{x \in U \mid \forall X \in \mathcal{O} : (x \in X \Rightarrow \uparrow U X \neq \uparrow U A)\};$$

$$\bigcap \{X \in \mathcal{P}U \mid X \text{ is a closed set, } X \supseteq A\} = \{x \in U \mid \forall X \in \mathcal{O} : (x \in X \Rightarrow X \neq A)\}.$$  

It is trivially a pretopology (used the fact that $0 \in \mathcal{O}$).

**Proposition 5.24.** If $\tau$ is the topology induced by pretopology $\pi$, in turn induced by topology $\rho$, then $\tau = \rho$.

**Proof.** The set of closed sets of $\tau$ is $\{A \in \mathcal{P}U \mid \text{cl}_\tau(A) = A\} = \{A \in \mathcal{P}U \mid \bigcap \{X \in \mathcal{P}U \mid X \text{ is a closed set in } \rho, X \supseteq A\} = A\} = \{A \in \mathcal{P}U \mid A \text{ is a closed set in } \rho\}$ (taken into account that intersecting closed sets is a closed set).

**Definition 5.25.** Idempotent closures are called Kuratowski closures.

**Theorem 5.26.** The above defined correspondences between topologies and pretopologies, restricted to Kuratowski closures, is a bijection.

**Proof.** Taking into account the above proposition, it’s enough to prove that:

If $\tau$ is the pretopology induced by topology $\pi$, in turn induced by a Kuratowski closure $\rho$, then $\tau = \rho$.

$$\text{cl}_\tau(A) = \bigcap \{X \in \mathcal{P}U \mid X \text{ is a closed set in } \tau, X \supseteq A\} = \bigcap \{X \in \mathcal{P}U \mid \text{cl}_\tau(X) = X, X \supseteq A\} = \bigcap \{\text{cl}_\rho(X) \mid X \in \mathcal{P}U, \text{cl}_\rho(X) = X, X \supseteq \text{cl}_\rho(A)\} = \bigcap \{\text{cl}_\rho(\text{cl}_\rho(X)) \mid X = A\} = \text{cl}_\rho(\text{cl}_\rho(A)) = \text{cl}_\rho(A).$$

### 5.3.1.3 Topology induced by a metric

**Definition 5.27.** Every metric space induces a topology in this way: A set $X$ is open iff

$$\forall x \in X \exists \varepsilon > 0. B_x(\varepsilon) \subseteq X.$$  

**Exercise 5.5.** Prove it is really a topology and this topology is the same as the topology, induced by the pretopology, in turn induced by our metric space.

### 5.4 Proximity spaces

Let $(U; d)$ be metric space. We will define distance between sets $A, B \in \mathcal{P}U$ by the formula

$$d(A, B) = \inf \{d(a, b) \mid a \in A, b \in B\}.$$  

(Here “inf” denotes infimum on the real line.)

**Definition 5.28.** Sets $A, B \in \mathcal{P}U$ are near (denoted $A \delta B$) iff $d(A, B) = 0$.

$\delta$ defined in this way (for a metric space) is an example of proximity as defined below.

**Definition 5.29.** A proximity space is a set $(U; \delta)$ conforming to the following axioms (for every $A, B, C \in \mathcal{P}U$):

1. $A \cap B \neq \emptyset \Rightarrow A \delta B$;
2. if \( A \delta B \) then \( A \neq \emptyset \) and \( B \neq \emptyset \);
3. \( A \delta B \Rightarrow B \delta A \) (symmetry);
4. \( (A \cup B) \delta C \Rightarrow A \delta C \vee B \delta C \);
5. \( C \delta (A \cup B) \Rightarrow C \delta A \vee C \delta B \);
6. \( A \delta B \) implies existence of \( P, Q \in \mathcal{P}U \) with \( A \delta P \), \( B \delta Q \) and \( P \cup Q = U \).

**Exercise 5.6.** Show that proximity generated by a metric space is really a proximity (conforms to the above axioms).

**Definition 5.30.** *Quasi-proximity* is defined as the above but without the symmetry axiom.

**Definition 5.31.** Closure is generated by a proximity by the following formula:

\[
\text{cl}(A) = \{ a \in U \mid \{a\} \delta A \}.
\]

**Proposition 5.32.** Every closure generated by a proximity is a Kuratowski closure.

**Proof.** First prove it is a preclusion. \( \text{cl}(\emptyset) = \emptyset \) is obvious. \( \text{cl}(A) \supseteq A \) is obvious. \( \text{cl}(A \cup B) = \{ a \in U \mid \{a\} \delta A \cup \{a\} \delta B \} = \{ a \in U \mid \{a\} \delta A \} \cup \{ a \in U \mid \{a\} \delta B \} = \text{cl}(A) \cup \text{cl}(B) \).

It is remained to prove that \( \text{cl} \) is idempotent, that is \( \text{cl}(\text{cl}(A)) = \text{cl}(A) \). It is enough to show \( \text{cl}(\text{cl}(A)) \subseteq \text{cl}(A) \), that is if \( x \notin \text{cl}(A) \) then \( x \notin \text{cl}(\text{cl}(A)) \).

If \( x \notin \text{cl}(A) \) then \( \{x\} \delta A \). So there are \( P, Q \in \mathcal{P}U \) such that \( \{x\} \delta P \), \( A \delta Q \), \( P \cup Q = U \). Then \( U \setminus Q \subseteq P \), so \( \{x\} \delta U \setminus Q \) and hence \( x \in Q \). Hence \( U \setminus \text{cl}(A) \subseteq Q \), and so \( \text{cl}(A) \subseteq U \setminus Q \subseteq P \). Consequently \( \{x\} \delta \text{cl}(A) \) and hence \( x \notin \text{cl}(\text{cl}(A)) \). \[\square\]
Chapter 6

Funcoids

In this chapter (and several following chapters) the word filter will refer to a filter on a set (rather than a filter on an arbitrary poset).

6.1 Informal introduction into funcoids

Funcoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also funcoids are a generalization of binary relations.

That funcoids are a common generalization of “spaces” (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement “$f$ is a continuous function from a space $\mu$ to a space $\nu$” can be described in terms of funcoids as the formula $f \circ \mu \subseteq \nu \circ f$ (see below for details).

Most naturally funcoids appear as a generalization of proximity spaces.

Let $\delta$ be a proximity that is a certain binary relation so that $AB$ is defined for every sets $A$ and $B$.

We will extend it from sets to filters by the formula:

$$A \delta^0 B, \forall A \in A, B \in B: A \delta B.$$ 

Then (as it will be proved below) there exist two functions $\alpha, \beta \in \mathfrak{F}$ such that

$$A \delta^0 B \iff B \cap \alpha A \neq 0^\delta \iff A \cap \beta B \neq 0^\delta.$$ 

The pair $(\alpha; \beta)$ is called funcoid when $B \cap \alpha A \neq 0^\delta \iff A \cap \beta B \neq 0^\delta$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first $\alpha$ and the second $\beta$. The first component of a funcoid $f$ is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of principal funcoids (see below) these coincide.)

One of the most important properties of a funcoid is that it is uniquely determined by just one of its components. That is a funcoid $f$ is uniquely determined by the function $\langle f \rangle$. Moreover a funcoid $f$ is uniquely determined by values of $\langle f \rangle$ on principal filters.

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let $\alpha$ be a pretopological space that is a map $\alpha \in \mathfrak{S}$ for some set $\mathfrak{S}$. Then we define $\alpha' X = \bigcup \{ \alpha x \mid x \in X \}$ for every set $X \in \mathfrak{P}$. We will prove that there exists a unique funcoid $f$ such that $\alpha' = \langle f \rangle |_{\mathfrak{P}} \circ \uparrow$ where $\mathfrak{P}$ is the set of principal filters on $\mathfrak{S}$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator $p$ on a set $\mathfrak{S}$ it exists a unique funcoid $f$ such that $\langle f \rangle |_{\mathfrak{P}} \circ \uparrow = \uparrow \circ p$.

For every binary relation $p$ on a set $\mathfrak{S}$ there exists unique funcoid $f$ such that

$$\forall X \in \mathfrak{P}: \langle f \rangle |_{\mathfrak{P}} \uparrow X = \uparrow(p) X$$

(where $\langle p \rangle$ is defined in the introduction), recall that a funcoid is uniquely determined by the values of its first component on sets. I will call such funcoids principal. So funcoids are a generalization of binary relations.
Composition of binary relations (i.e. of principal funcoids) complies with the formulas:

\[(g \circ f) = (g) \circ (f) \quad \text{and} \quad ((g \circ f)^{-1}) = (f^{-1}) \circ (g^{-1}).\]

By the same formulas we can define composition of every two funcoids. Funcoids with this composition form a category (the category of funcoids).

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula \((\alpha; \beta)^{-1} = (\beta; \alpha)\). In the particular case if \(\mu\) is a proximity we have \(\mu^{-1} = \mu\) because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filters instead of acting on sets. Below these will be defined domain and image of a funcoid (the domain and the image of a funcoid are filters).

### 6.2 Basic definitions

**Definition 6.1.** Let us call a **funcoid** from a set \(A\) to a set \(B\) a quadruple \((A; B; \alpha; \beta)\) where \(\alpha \in \mathcal{F}(B)^{\mathcal{F}(A)}, \beta \in \mathcal{F}(A)^{\mathcal{F}(B)}\) such that

\[\forall \mathcal{X} \in \mathcal{F}(A), \mathcal{Y} \in \mathcal{F}(B): (\mathcal{Y} \not= \mathcal{X} \iff \mathcal{X} \not= \mathcal{Y}).\]

Further we will assume that all funcoids in consideration are small without mentioning it explicitly.

**Definition 6.2.** **Source** and **destination** of every funcoid \((A; B; \alpha; \beta)\) are defined as:

\[\text{Src}(A; B; \alpha; \beta) = A \quad \text{and} \quad \text{Dst}(A; B; \alpha; \beta) = B.\]

I will denote FCD\((A; B)\) the set of funcoids from \(A\) to \(B\).

I will denote FCD the set of all funcoids (for small sets).

**Definition 6.3.** \((A; B; \alpha; \beta)\) def for a funcoid \((A; B; \alpha; \beta)\).

**Definition 6.4.** The **reverse** funcoid \((A; B; \alpha; \beta)^{-1} = (B; A; \beta; \alpha)\) for a funcoid \((A; B; \alpha; \beta)\).

**Note 6.5.** The reverse funcoid is **not** an inverse in the sense of group theory or category theory.

**Proposition 6.6.** If \(f\) is a funcoid then \(f^{-1}\) is also a funcoid.

**Proof.** It follows from symmetry in the definition of funcoid. \(\square\)

**Obvious 6.7.** \((f^{-1})^{-1} = f\) for a funcoid \(f\).

**Definition 6.8.** The relation \([f] \in \mathcal{P}(\mathcal{F}(\text{Src } f) \times \mathcal{F}(\text{Dst } f))\) is defined (for every funcoid \(f\) and \(\mathcal{X} \in \mathcal{F}(\text{Src } f), \mathcal{Y} \in \mathcal{F}(\text{Dst } f)\)) by the formula \(\mathcal{X} [f] \mathcal{Y} \iff \mathcal{Y} \not= (f^{-1}) \mathcal{X}\).

**Obvious 6.9.** \(\mathcal{X} [f] \mathcal{Y} \iff \mathcal{Y} \not= (f^{-1}) \mathcal{X}\) for every funcoid \(f\) and \(\mathcal{X} \in \mathcal{F}(\text{Src } f), \mathcal{Y} \in \mathcal{F}(\text{Dst } f)\).

**Obvious 6.10.** \([f^{-1}] = [f]^{-1}\) for a funcoid \(f\).

**Theorem 6.11.** Let \(A, B\) be small sets.

1. For given value of \((f)\) there exists no more than one funcoid \(f \in \text{FCD}(A; B)\).

2. For given value of \([f]\) there exists no more than one funcoid \(f \in \text{FCD}(A; B)\).

**Proof.** Let \(f, g \in \text{FCD}(A; B)\).

Obviously, \((f) = (g) \Rightarrow [f] = [g]\) and \((f^{-1}) = (g^{-1}) \Rightarrow [f] = [g]\). So it’s enough to prove that \([f] = [g] \Rightarrow (f) = (g)\).
Provided that $[f]=[g]$ we have $\mathcal{Y} \neq (f)\mathcal{X} \Leftrightarrow \mathcal{X} [f] \Rightarrow \mathcal{Y} [g] \Rightarrow \mathcal{Y} \neq (g)\mathcal{X}$ and consequently $(f)\mathcal{X} = (g)\mathcal{X}$ for every $\mathcal{X} \in \mathcal{F}(A), \mathcal{Y} \in \mathcal{F}(B)$ because a set of filters is separable, thus $(f) = (g)$. □

**Proposition 6.12.** $(f)0^{\exists}([\text{Src } f]) = 0^{\exists}([\text{Dst } f])$ for every funcoid $f$.

**Proof.** $\mathcal{Y} \neq (f)0^{\exists}([\text{Src } f]) \Leftrightarrow 0^{\exists}([\text{Src } f]) \neq (f)^{-1}\mathcal{Y} \Leftrightarrow 0 \Leftrightarrow \mathcal{Y} \neq 0^{\exists}([\text{Dst } f])$. Thus $(f)0^{\exists}([\text{Src } f]) = 0^{\exists}([\text{Dst } f])$ by separability of filters. □

**Proposition 6.13.** $(f)(\mathcal{I} \sqcup \mathcal{J}) = (f)(\mathcal{I} \sqcup (f)\mathcal{J})$ for every funcoid $f$ and $\mathcal{I}, \mathcal{J} \in \mathcal{F}(\text{Src } f)$.

**Proof.**

$$\begin{align*}
\ast((f)(\mathcal{I} \sqcup \mathcal{J})) &= \{\mathcal{Y} \in \mathcal{F} | \mathcal{Y} \neq (f)(\mathcal{I} \sqcup \mathcal{J})\} \\
&= \{\mathcal{Y} \in \mathcal{F} | \mathcal{I} \sqcup \mathcal{J} \neq (f)^{-1}\mathcal{Y}\} \\
&= \{\mathcal{Y} \in \mathcal{F} | \mathcal{I} \neq (f)^{-1}\mathcal{Y} \lor \mathcal{J} \neq (f)^{-1}\mathcal{Y}\} \\
&= \{\mathcal{Y} \in \mathcal{F} | \mathcal{Y} \neq (f)^{-1}\mathcal{I} \lor \mathcal{Y} \neq (f)^{-1}\mathcal{J}\} \\
&= \{\mathcal{Y} \in \mathcal{F} | \mathcal{Y} \neq (f)(\mathcal{I} \sqcup (f)\mathcal{J})\} \\
&= \ast((f)(\mathcal{I} \sqcup (f)\mathcal{J})).
\end{align*}$$

Thus $(f)(\mathcal{I} \sqcup \mathcal{J}) = (f)(\mathcal{I} \sqcup (f)\mathcal{J})$ because $\mathcal{F}([\text{Dst } f])$ is separable. □

**Proposition 6.14.** For every $f \in \text{FCD}(A; B)$ for every sets $A$ and $B$ we have:

1. $\mathcal{K} [f] \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K} [f] \mathcal{I} \lor \mathcal{K} [f] \mathcal{J}$ for every $\mathcal{I}, \mathcal{J} \in \mathcal{F}(B), \mathcal{K} \in \mathcal{F}(A)$.
2. $\mathcal{I} \sqcup \mathcal{J} [f] \mathcal{K} \Leftrightarrow \mathcal{I} [f] \mathcal{K} \lor \mathcal{J} [f] \mathcal{K}$ for every $\mathcal{I}, \mathcal{J} \in \mathcal{F}(A), \mathcal{K} \in \mathcal{F}(B)$.

**Proof.**

1. $\mathcal{K} [f] \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow (\mathcal{I} \sqcup \mathcal{J}) \cap (f) \mathcal{K} \neq 0^{\exists}(B) \Leftrightarrow \mathcal{I} \sqcup (f) \mathcal{K} \neq 0^{\exists}(B) \lor \mathcal{J} \sqcup (f) \mathcal{K} \neq 0^{\exists}(B) \Leftrightarrow \mathcal{K} [f] \mathcal{I} \lor \mathcal{K} [f] \mathcal{J}.$

2. Similar. □

### 6.2.1 Composition of funcoids

**Definition 6.15.** Funcoids $f$ and $g$ are **composable** when $\text{Dst } f = \text{Src } g$.

**Definition 6.16.** Composition of composable funcoids is defined by the formula

$$(B; C; \alpha_2; \beta_2) \circ (A; B; \alpha_1; \beta_1) = (A; C; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

**Proposition 6.17.** If $f$, $g$ are composable funcoids then $g \circ f$ is a funcoid.

**Proof.** Let $f = (A; B; \alpha_1; \beta_1), f = (B; C; \alpha_2; \beta_2)$. For every $\mathcal{X} \in \mathcal{F}(A), \mathcal{Y} \in \mathcal{F}(B)$ we have $\mathcal{Y} \neq (\alpha_2 \circ \alpha_1)\mathcal{X} \Leftrightarrow \mathcal{Y} \neq \alpha_2 \alpha_1 \mathcal{X} \Leftrightarrow \alpha_1 \mathcal{X} \neq \beta_2 \mathcal{Y} \Leftrightarrow \mathcal{X} \neq \beta_1 \beta_2 \mathcal{Y} \Leftrightarrow \mathcal{X} \neq (\beta_1 \circ \beta_2) \mathcal{Y}$.

So $(A; C; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)$ is a funcoid. □

**Obvious 6.18.** $(g \circ f) = (g) \circ (f)$ for every composable funcoids $f$ and $g$.

**Proposition 6.19.** $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable funcoids $f$, $g$, $h$.

**Proof.** $(h \circ g) \circ f = (h \circ g) \circ (f) = (h \circ (g) \circ (f)) = (h \circ (g \circ (f))) = (h \circ (g \circ f))$ = $(h \circ (g \circ f))$. □

**Theorem 6.20.** $(g \circ f)^{-1} = (f^{-1} \circ g^{-1})$ for every composable funcoids $f$ and $g$.

**Proof.** $((g \circ f)^{-1}) = (f^{-1} \circ g^{-1}) = (f^{-1} \circ g^{-1})$. □
6.3 Funcoid as continuation

Let $f$ be a funcoid.

**Definition 6.21.** $(f)^*$ is the function $\mathcal{P}(\text{Src} f) \to \mathfrak{F}(\text{Dst} f)$ defined by the formula

$$(f)^* X = (f)_{\uparrow \text{Src} f} X.$$

**Definition 6.22.** $[f]^*$ is the relation between $\mathcal{P}(\text{Src} f)$ and $\mathcal{P}(\text{Dst} f)$ defined by the formula

$$X [f]^* Y \iff (f)_{\uparrow \text{Src} f} X [f]_{\uparrow \text{Dst} f} Y.$$

**Obvious 6.23.**

1. $(f)^* = (f) \circ (\uparrow \text{Src} f)$;
2. $[f]^* = (\uparrow \text{Dst})^{-1} \circ [f] \circ (\uparrow \text{Src} f)$.

**Obvious 6.24.** $(g \circ f)^* X = (g \circ f)^* X$ for every $X \in \mathcal{P}(\text{Src} f)$.

**Theorem 6.25.** For every funcoid $f$ and $X \in \mathcal{F}(\text{Src} f)$, $Y \in \mathcal{F}(\text{Dst} f)$

1. $(f) X = \bigcap (\langle (f)^* \rangle X$;
2. $X [f]^* Y \iff \forall X \in X, Y \in Y : X [f]^* Y$.

**Proof.**

2. $X [f]^* Y \iff \forall X \in X, Y \in Y : (f)_{\uparrow \text{Src} f} X [f]_{\uparrow \text{Dst} f} Y$. Analogously $X [f]^* Y \iff \forall X \in X : (f)_{\uparrow \text{Src} f} X [f] Y$. Combining these two equivalences we get

$$X [f]^* Y \iff \forall X \in X, Y \in Y : (f)_{\uparrow \text{Src} f} X [f]_{\uparrow \text{Dst} f} Y.$$

1. $\forall X \in X, Y \in Y : (f)_{\uparrow \text{Src} f} X [f]_{\uparrow \text{Dst} f} Y \iff \forall X \in X : (f)_{\uparrow \text{Src} f} X [f]_{\uparrow \text{Dst} f} Y$.

Let's denote $W = \{X \cap (f)^* X \mid X \in X \}$. We will prove that $W$ is a generalized filter base. To prove this it is enough to show that $V = \{X \mid X \in X \}$ is a generalized filter base.

Let $P, Q \in V$. Then $P \cap (f)^* A, Q \cap (f)^* B$ where $A, B \in X$; $A \cap B \in X$ and $R \subseteq P \cap Q$ for $R = (f)^* (A \cap B) \in V$. So $V$ is a generalized filter base and thus $W$ is a generalized filter base.

$0 \notin \mathcal{F}(f) \notin W \iff \bigcap W = 0 \notin \mathcal{F}(f) \notin W$ by properties of generalized filter bases. That is

$$\forall X \in X : (f) \cap (f)^* X \neq 0 \notin \mathcal{F}(f) \iff \forall X \in X : (f) \cap (f)^* X \neq 0 \notin \mathcal{F}(f).$$

Comparing with the above, $\forall X \in X, Y \in Y : (f)_{\uparrow \text{Src} f} X [f]_{\uparrow \text{Dst} f} Y \iff \forall X \in X : (f)_{\uparrow \text{Src} f} X [f]_{\uparrow \text{Dst} f} Y$ because the lattice of filters is separable.

**Corollary 6.26.** Let $f$ be a funcoid.

1. The value of $f$ can be restored knowing $(f)^*$.
2. The value of $f$ can be restored knowing $[f]^*$.

**Proposition 6.27.** For every $f \in FCD(A; B)$ we have (for every $I, J \in \mathcal{P} A$

$$(f)^* 0 = 0 \notin B, \quad (f)^* (I \cup J) = (f)^* I \cup (f)^* J$$

and

$$\neg (f) I \cup (f) J, \quad I \cup J \neg K \iff (f)^* K \lor J \neq K \quad \text{for every } I, J \in \mathcal{P} A, K \in \mathcal{P} B,$$

$$\neg (f) I \cup (f) J, \quad (f)^* I \lor J \lor (f)^* J \quad \text{for every } I, J \in \mathcal{P} A, K \in \mathcal{P} B.$$

**Proof.** $(f)^* 0 = (f)^* I \cup (f) J = (f)^* I \cup (f) J = (f)^* I \cup (f) J = (f)^* I \cup (f) J$.

$I [f]^* 0 \iff (f)^* I \cup (f) J \iff 0; I \cup J \neg K \iff (f)^* I \lor J \lor (f)^* J \lor (f)^* J \lor (f)^* J \lor (f)^* J \lor (f)^* J \lor (f)^* J \lor (f)^* J \lor (f)^* J \lor (f)^* J \lor (f)^* J$.

The rest follows from symmetry.
Theorem 6.28. Fix sets $A$ and $B$. Let $L_F = \lambda f \in \mathcal{FCD}(A; B): (f)^*$ and $L_R = \lambda f \in \mathcal{FCD}(A; B): [f]^*$.

1. $L_F$ is a bijection from the set $\mathcal{FCD}(A; B)$ to the set of functions $\alpha \in \mathcal{G}(B)^{\mathcal{P}A}$ that obey the conditions (for every $I, J \in \mathcal{P}A$)

$$\alpha \emptyset = 0^{\delta(B)}, \quad \alpha(I \cup J) = \alpha I \cup \alpha J.$$ (6.1)

For such $\alpha$ it holds (for every $X \in \mathcal{G}(A)$)

$$\{L_{F^{-1}} \alpha\} X = \prod_{\alpha} \{\alpha\} X.$$ (6.2)

2. $L_R$ is a bijection from the set $\mathcal{FCD}(A; B)$ to the set of binary relations $\delta \in \mathcal{P}(\mathcal{P}A \times \mathcal{P}B)$ that obey the conditions

$$\neg(I \delta \emptyset), \quad I \cup J \delta K \Leftrightarrow I \delta K \lor J \delta K \quad \text{(for every } I, J \in \mathcal{P}A, K \in \mathcal{P}B),$$

$$\neg(\emptyset \delta I), \quad K \delta I \cup J \Leftrightarrow K \delta I \lor K \delta J \quad \text{(for every } I, J \in \mathcal{P}B, K \in \mathcal{P}A).$$ (6.3)

For such $\delta$ it holds (for every $X \in \mathcal{G}(A)$, $Y \in \mathcal{G}(B)$)

$$X \setminus [L_{R^{-1}} \delta] Y \Leftrightarrow \forall X \in X, Y \in Y: X \delta Y.$$ (6.4)

**Proof.** Injectivity of $L_F$ and $L_R$, formulas (6.2) (for $\alpha \in \mathcal{L}m L_{F^{-1}}$ and (6.4) (for $\delta \in \mathcal{L}m L_{R}$), formulas (6.1) and (6.3) follow from two previous theorems. The only thing remained to prove is that for every $\alpha$ and $\delta$ that obey the above conditions a corresponding functor $f$ exists.

2. Let define $\alpha \in \mathcal{G}(B)^{\mathcal{P}A}$ by the formula $\alpha(X) = \{Y \in \mathcal{P}B \mid X \delta_{\operatorname{Y}} Y\}$ for every $X \in \mathcal{P}A$. (It is obvious that $\{Y \in \mathcal{P}B \mid X \delta_{\operatorname{Y}} Y\}$ is a free star.) Analogously it can be defined $\beta \in \mathcal{G}(A)^{\mathcal{P}B}$ by the formula $\beta(X) = \{Y \in \mathcal{P}A \mid X \delta_{\operatorname{Y}} Y\}$. Let’s continue $\alpha$ and $\beta$ to $\alpha' \in \mathcal{G}(B)^{\mathcal{P}A}$ and $\beta' \in \mathcal{G}(A)^{\mathcal{P}B}$ by the formulas

$$\alpha(X) \supseteq \prod_{\alpha} \{\alpha\} X' \quad \text{and} \quad \beta(Y) \supseteq \prod_{\beta} \{\beta\} Y'$$

and $\delta$ to $\delta' \in \mathcal{P}(\mathcal{G}(A) \pi \mathcal{G}(B))$ by the formula

$$X \delta' Y \Leftrightarrow \forall X \in X, Y \in Y': X \delta Y.$$ (6.4)

$\mathcal{G}(A)$ is a generalized filter base: To prove it is enough to show that $\{\alpha\} X$ is a generalized filter base. If $A, B \in \mathcal{L}(\alpha)X$ then exist $X_1, X_2 \in \mathcal{L}X$ such that $A = \alpha X_1, B = \alpha X_2$.

Then $\alpha(X_1 \pi X_2) \in \{\alpha\} X$. So $\{\alpha\} X$ is a generalized filter base and thus $W$ is a generalized filter base.

By properties of generalized filter bases, $\prod_{\{\mathcal{G}(A)\} X \neq 0^{\delta(B)}}$ is equivalent to

$$\forall X \in X: Y \setminus \alpha X \neq 0^{\delta(B)},$$

what is equivalent to $\forall X \in X, Y \in Y: Y \setminus \alpha X \neq 0^{\delta(B)} \Leftrightarrow \forall X \in X, Y \in Y: Y \setminus \alpha X \neq 0^{\delta(B)}$. Combining the equivalencies we get $Y \setminus \alpha X \neq 0^{\delta(B)} \Leftrightarrow X \delta' Y$. Analogously $Y \setminus \alpha' X \neq 0^{\delta(A)} \Leftrightarrow X \delta' Y$. So $Y \setminus \alpha' X \neq 0^{\delta(A)} \Leftrightarrow X \setminus \alpha Y \neq 0^{\delta(A)}$. That is $(A; B, \alpha'; \beta')$ is a functor. From the formula $Y \setminus \alpha' X \neq 0^{\delta(A)} \Leftrightarrow X \delta' Y$ it follows that

$$X ([A; B; \alpha'; \beta'])^* X \Leftrightarrow Y \setminus \alpha' X \neq 0^{\delta(B)} \Leftrightarrow \alpha' X \delta' \setminus Y \Leftrightarrow X \delta Y.$$ (6.4)

1. Let define the relation $\delta \in \mathcal{P}(\mathcal{P}A \pi \mathcal{P}B)$ by the formula $X \delta Y \Leftrightarrow Y \setminus \alpha X \neq 0^{\delta(B)}$.

That $\neg(I \delta \emptyset)$ and $\neg(\emptyset \delta I)$ is obvious. We have $I \cup J \delta K \Leftrightarrow I \delta K \lor J \delta K \lor I \delta J \lor J \delta K$ and $K \delta I \cup J \Leftrightarrow I \delta K \lor J \delta K \lor J \delta K \lor I \delta J$. That is the formulas (6.3) are true. Accordingly to the above there exists a functor $f$ such that

$$X \setminus \{f\} Y \Leftrightarrow \forall X \in X, Y \in Y: X \delta Y.$$
\[ \forall X \in \mathcal{P}A, Y \in \mathcal{P}B : (\uparrow^B Y \cap f) \uparrow^A X \neq 0^\mathcal{B}(B) \iff \uparrow^A X [f] \uparrow^B Y \iff X \not\in (f) \not\in \alpha X \neq 0^\mathcal{B}(B) \), consequently \( \forall X \in \mathcal{P}A : \alpha X = (f) \uparrow^A X = (f)^* X \). \] 

Note that by the last theorem to every proximity \( \delta \) corresponds a unique funcoid. So funcoids are a generalization of (quasi-)proximity structures. Reverse funcoids can be considered as a generalization of conjugate quasi-proximity.

**Definition 6.29.** Any (multivalued) function \( F : A \rightarrow B \) corresponds to a funcoid \( \uparrow^\text{FCD}(A; B) F \in \text{FCD}(A; B) \), where by definition \( \uparrow^\text{FCD}(A; B) F \{X\} = \{ \uparrow^B f \} \{X\} \) for every \( X \in \mathfrak{F}(A) \).

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff. (Take \( \alpha = \uparrow^B \circ (f) \).

**Definition 6.30.** \( \uparrow^\text{FCD} f \stackrel{\text{def}}{=} (\mathrm{Src} f ; \mathrm{Dst} f ; \uparrow^\text{FCD}(\mathrm{Src} f ; \mathrm{Dst} f) \mathfrak{R} f) \) for every Rel-morphism \( f \).

**Definition 6.31.** Funcoids corresponding to a binary relation (= multivalued function) are called principal funcoids.

We may equate principal funcoids with corresponding binary relations by the method of appendix B in \[29\]. This is useful for describing relationships of funcoids and binary relations, such as for the formulas of continuous functions and continuous funcoids (see below).

**Theorem 6.32.** If \( S \) is a generalized filter base on \( \mathrm{Src} f \) then \( \langle f \rangle \cap S = \cap \{ \langle f \rangle P \mid P \in S \} \) for every funcoid \( f \).

**Proof.** \( \langle f \rangle \cap S \subseteq \langle f \rangle X \) for every \( X \in S \) and thus \( \langle f \rangle \cap S \subseteq \cap \{ \langle f \rangle S \} \).

By properties of generalized filter bases:

\( \langle f \rangle \cap S = \cap \{ \langle f \rangle \langle P \rangle \} \cap S = \cap \{ \langle f \rangle P \mid P \in S \} \cap \{ \langle f \rangle \langle P \rangle \} = \cap \{ \langle f \rangle P \mid P \in S \} \subseteq \cap \{ \langle f \rangle S \} \). \]

\[ \bbox[5pt]{6.4 \text{ Lattices of funcoids}} \]

**Definition 6.33.** \( f \subseteq g \stackrel{\text{def}}{=} [f] \subseteq [g] \) for \( f, g \in \text{FCD} \).

Thus every \( \text{FCD}(A; B) \) is a poset. (It’s taken into account that \( [f] \neq [g] \) when \( f \neq g \).)

We will consider filtrators (filtrators of funcoids) whose base is \( \text{FCD}(A; B) \) and whose core are principal funcoids from \( A \) to \( B \).

**Lemma 6.34.** \( \langle f \rangle^* X = \cap \{ \{ F \}^* X \mid F \in \uparrow^f \} \) for every funcoid \( f \) and set \( X \in \mathcal{P}(\mathrm{Src} f) \).

**Proof.** Obviously \( \langle f \rangle^* X \subseteq \cap \{ \{ F \}^* X \mid F \in \uparrow^f \} \).

Let \( B \in \langle f \rangle^* X \). Let \( F_B = X \subseteq B \cap (\text{Src} f) \) \( \times \mathrm{Dst} f \).

\( \langle f \rangle P = \uparrow^\text{FCD}(\text{Src} f ; \text{Dst} f) f \). By properties of generalized filter bases:

\( \forall f \in \langle f \rangle^* X \cap \{ \{ F \}^* X \mid F \in \uparrow^f \} \) because \( B \in \langle f \rangle^* X \).

Thus \( \forall B \in \langle f \rangle^* X : B \in \cap \{ \{ F \}^* X \mid F \in \uparrow^f \} \).

**Theorem 6.35.** \( \{ X \} \cap \{ \{ F \}^* X \mid F \in \uparrow^f \} = \{ X \} \cap \{ \{ F \}^* X \mid F \in \uparrow^f \} \) for every funcoid \( f \) and \( X \in \mathfrak{R}(\mathrm{Src} f) \).

**Proof.** \( \{ X \} \cap \{ \{ F \}^* X \mid F \in \uparrow^f \} = \{ X \} \cap \{ \{ F \}^* X \mid F \in \uparrow^f \} = \{ X \} \cap \{ \{ F \}^* X \mid F \in \uparrow^f \} \) (the lemma used). 

**Conjecture 6.36.** Every filtrator of funcoids is:

1. with separable core;
2. with co-separable core.

Below it is shown that $FCD(A; B)$ are complete lattices for every sets $A$ and $B$. We will apply lattice operations to subsets of such sets without explicitly mentioning $FCD(A; B)$.

**Theorem 6.37.** $FCD(A; B)$ is a complete lattice (for every sets $A$ and $B$). For every $R \in \mathcal{P}FCD(A; B)$ and $X \in \mathcal{P}A$, $Y \in \mathcal{P}B$

1. $X \bigsqcup R^* Y \Leftrightarrow \exists f \in R: X \in (f)^* Y$;
2. $(\bigsqcup R)^* X = \bigsqcup \{(f)^* X \mid f \in R\}$.

**Proof.** Accordingly [26] to prove that it is a complete lattice it’s enough to prove existence of all joins.

1. $\alpha X \overset{\text{def}}{=} \bigsqcup \{(f)^* X \mid f \in R\}$. We have $\alpha \emptyset = 0^{\text{Dst}(f)}$;
   
   \[
   \alpha(I \sqcup J) = \bigsqcup \{(f)^*(I \sqcup J) \mid f \in R\} = \bigsqcup \{(f)^*I \sqcup (f)^*J \mid f \in R\} = \bigsqcup \{(f)^*I \mid f \in R\} \sqcup \bigsqcup \{(f)^*J \mid f \in R\} = \alpha I \sqcup \alpha J.
   \]

So $\langle h \rangle^* = \alpha$ for some funcoid $h$. Obviously

\[
\forall f \in R: h \sqsupseteq f. \tag{6.5}
\]

And $h$ is the least funcoid for which holds the condition (6.5). So $h = \bigsqcup R$.

1. $X \bigsqcup R^* Y \Leftrightarrow \bigcup \{\chi \mid \exists f \in R: X \in (f)^* Y\}$.

In the next theorem, compared to the previous one, the class of infinite joins is replaced with lesser class of finite joins and simultaneously class of sets is changed to more wide class of filters.

**Theorem 6.38.** For every $f, g \in FCD(A; B)$ and $X \in \mathfrak{F}(A)$ (for every sets $A, B$)

1. $(f \cup g)X = \langle f \rangle X \sqcup \langle g \rangle X$;
2. $[f \cup g] = [f] \sqcup [g]$.

**Proof.** 1. Let $\alpha X \overset{\text{def}}{=} \langle f \rangle X \sqcup \langle g \rangle X$; $\beta Y \overset{\text{def}}{=} \langle f^{-1} \rangle Y \sqcup \langle g^{-1} \rangle Y$ for every $X \in \mathfrak{F}(A)$, $Y \in \mathfrak{F}(B)$. Then

\[
\forall \chi \in \mathfrak{F}(A)\exists \chi \neq 0^{\mathfrak{F}(B)} \forall Y \in \mathfrak{F}(B) \Rightarrow \chi \sqcup \langle f \rangle X \neq 0^{\mathfrak{F}(B)} \lor Y \sqcup \langle g \rangle X \neq 0^{\mathfrak{F}(B)}
\]

So $h = (A; B; \alpha; \beta)$ is a funcoid. Obviously $h \sqsupseteq f$ and $h \sqsupseteq g$. If $p \sqsubseteq f$ and $p \sqsubseteq g$ for some funcoid $p$ then $(p)\chi \sqsubseteq (f)\chi \sqcup (g)\chi = \langle h \rangle \chi$. That is $p \sqsubseteq h$. So $f \sqcup g = h$.

2. $X \in (f \cup g)Y \Leftrightarrow \forall \chi \in \mathfrak{F}(A)\exists \chi \neq 0^{\mathfrak{F}(B)} \forall Y \in \mathfrak{F}(B) \Rightarrow \chi \sqcup (f)X \neq 0^{\mathfrak{F}(B)} \lor \chi \sqcup (g)X \neq 0^{\mathfrak{F}(B)}$.

**Definition 6.39.** $\text{GR} f \overset{\text{def}}{=} \{F \in \mathcal{P}(\text{Src } f \times \text{Dst } f) \mid F \sqsubseteq f\}$.

**Definition 6.40.** $\text{xyGR} f \overset{\text{def}}{=} \{(\text{Src } f ; \text{Dst } f ; F) \mid F \in \text{GR } f\}$.

**Remark 6.41.** $\text{xyGR } f$ is a set of Rel-morphisms.

### 6.5 More on composition of funcoids

**Proposition 6.42.** $[g \circ f] = [g] \circ (f) = \langle g^{-1} \rangle^{-1} \circ [f]$ for every composable funcoids $f$ and $g$. 


Proof. \( X \cdot [g \circ f] \cdot Y \Leftrightarrow Y \cap (g \circ f) \cdot X \neq 0 \cdot (\text{Dist} \cdot g) \Leftrightarrow Y \cap (g) \cdot (f) \cdot X \neq 0 \cdot (\text{Dist} \cdot g) \Leftrightarrow (f) \cdot X \cdot [g] \cdot Y \Leftrightarrow X \cdot ([g \circ f]) \cdot Y \) for every \( X \in \mathfrak{F}(\text{Src} \cdot f) \), \( Y \in \mathfrak{F}(\text{Dist} \cdot g) \). \( [g \circ f] = ([f^{-1} \circ g^{-1}]^{-1} \circ [f^{-1} \circ g^{-1}])^{-1} = ([f^{-1} \circ g^{-1}])^{-1} = (g^{-1})^{-1} \circ [f] \). □

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that \( x \cdot (y \circ f) \cdot z \Leftrightarrow \exists y : (x \cdot f \cdot y \cdot g \cdot z) \) for every \( x \) and \( z \) and every binary relations \( f \) and \( g \).

**Theorem 6.43.** For every sets \( A, B, C \) and \( f \in \text{FCD}(A; B) \), \( g \in \text{FCD}(B; C) \) and \( X \in \mathfrak{F}(A) \), \( Z \in \mathfrak{F}(C) \)

\[ X \cdot [g \circ f] \cdot Z \Leftrightarrow \exists y \in \text{atoms}^B (X \cdot [f] \cdot y \cdot [g] \cdot Z). \]

**Proof.**

\[ \exists y \in \text{atoms}^B (X \cdot [f] \cdot y \cdot [g] \cdot Z) \Rightarrow \exists y \in \text{atoms}^B (Z \cap (g) \cdot y \neq 0 \cdot (C) \land y \cap (f) \cdot X \neq 0 \cdot (B)) \Rightarrow \exists y \in \text{atoms}^B (Z \cap (g) \cdot (f) \cdot X \neq 0 \cdot (C)) \Rightarrow Z \cap (g) \cdot (f) \cdot X \neq 0 \cdot (C) \Rightarrow X \cdot [g \circ f] \cdot Z. \]

Reversely, if \( X \cdot [g \circ f] \cdot Z \) then \( (f) \cdot X \cdot [g] \cdot Z \), consequently there exists \( y \in \text{atoms} (f) \cdot X \) such that \( y \cdot [g] \cdot Z \); we have \( X \cdot [f] \cdot y \).

**Theorem 6.44.** For every sets \( A, B, C \)

1. \( f \cdot (g \cup h) = f \cdot g \cup f \cdot h \) for \( g, h \in \text{FCD}(A; B) \) and \( f \in \text{FCD}(B; C) \);
2. \( (g \cup h) \cdot f = g \cdot f \cup h \cdot f \) for \( g, h \in \text{FCD}(B; C) \) and \( f \in \text{FCD}(A; B) \).

**Proof.** I will prove only the first equality because the other is analogous.

For every \( X \in \mathfrak{F}(A) \), \( Z \in \mathfrak{F}(C) \)

\[ X \cdot [f \circ (g \cup h)] \cdot Z \Leftrightarrow \exists y \in \text{atoms}^B (X \cdot [g \cup h] \cdot y \land [f] \cdot Z) \Rightarrow \exists y \in \text{atoms}^B ((X \cdot [g] \cdot y \lor X \cdot [h] \cdot y \land [f] \cdot Z)) \Rightarrow \exists y \in \text{atoms}^B ((X \cdot [g] \cdot y \lor X \cdot [h] \cdot y \land [f] \cdot Z)) \lor \exists y \in \text{atoms}^B (X \cdot [h] \cdot y \land [f] \cdot Z) \Rightarrow X \cdot [f \circ g] \cdot Z \lor X \cdot [f \circ h] \cdot Z \Rightarrow X \cdot [f \circ g \cup f \circ h] \cdot Z. \]

\[ \Box \]

### 6.6 Domain and range of a funcoid

**Definition 6.45.** Let \( A \) be a set. The **identity funcoid** \( \text{id}^{\text{FCD}}(A) = (A; A; \text{id}_A; \text{id}_A) \).

**Obvious 6.46.** The identity funcoid is a funcoid.

**Definition 6.47.** Let \( A \) be a set, \( A \in \mathfrak{F}(A) \). The **restricted identity funcoid** \( \text{id}_{\mathfrak{F}}^{\text{CD}} = (A; A \cap; A \cap) \).

**Proposition 6.48.** The restricted identity funcoid is a funcoid.

**Proof.** We need to prove that \( (A \cap \mathcal{A}) \cap \mathcal{Y} \neq 0 \cdot (A) \Leftrightarrow (A \cap \mathcal{Y}) \cap \mathcal{X} \neq 0 \cdot (A) \) what is obvious. □

**Obvious 6.49.**

1. \( (\text{id}^{\text{FCD}}(A))^{-1} = \text{id}^{\text{FCD}}(A) \);
2. \( (\text{id}_{\mathfrak{F}}^{\text{CD}})^{-1} = \text{id}_{\mathfrak{F}}^{\text{CD}} \).
Obvious 6.50. For every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}(A)$
1. $\mathcal{X} \uplus [\text{id}_{\text{FCD}(A)}] \mathcal{Y} \iff \mathcal{X} \cap \mathcal{Y} = \emptyset$.
2. $\mathcal{X} \uplus [\text{id}_{\text{FCD}}] \mathcal{Y} \iff \mathcal{A} \cap \mathcal{X} \cap \mathcal{Y} = \emptyset$.

Definition 6.51. I will define restricting of a funcoid $f$ to a filter $\mathcal{A} \in \mathfrak{F}(\text{Src } f)$ by the formula
\[ f|_{\mathcal{A}} = f \circ \text{id}_{\text{FCD}}. \]

Definition 6.52. Image of a funcoid $f$ will be defined by the formula $\text{im } f = \langle f \rangle 1\mathfrak{F}(\text{Src } f)$.

Domain of a funcoid $f$ is defined by the formula $\text{dom } f = \text{im } f^{-1}$.

Obvious 6.53. For every binary relation $f$ between sets $A$ and $B$
1. $\text{im } f \uplus [\text{FCD}(A; B)] f = \uparrow B \text{im } f$;
2. $\text{dom } f \uplus [\text{FCD}(A; B)] f = \uparrow A \text{dom } f$.

Proposition 6.54. $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \cap \text{dom } f)$ for every $f \in \text{FCD}, \mathcal{X} \in \mathfrak{F}(\text{Src } f)$.

Proof. For every $\mathcal{Y} \in \mathfrak{F}(\text{Dst } f)$ we have $\mathcal{Y} \cap \langle f \rangle (\mathcal{X} \cap \text{dom } f) = \emptyset$ and $\mathcal{Y} \cap \langle f \rangle (\mathcal{X} \cap \text{dom } f^{-1}) = \emptyset$. Thus $\langle f \rangle (\mathcal{X} \cap \text{dom } f) = \langle f \rangle \mathcal{X}$ because the lattice of filters is separable.

Proposition 6.55. $\mathcal{X} \cap \text{dom } f \neq \emptyset \Rightarrow \mathcal{Y} \cap \langle f \rangle \mathcal{X} \neq \emptyset \Rightarrow (f) \mathcal{X} \neq \emptyset$ for every $f \in \text{FCD}, \mathcal{X} \in \mathfrak{F}(\text{Src } f)$.

Proof. $\mathcal{X} \cap \text{dom } f \neq \emptyset$ implies $\mathcal{X} \cap \langle f \rangle (\mathcal{Y} \cap (f)^{-1}) = \emptyset \Rightarrow \mathcal{Y} \cap \langle f \rangle (\mathcal{X} \cap (f)^{-1}) \neq \emptyset \Rightarrow \langle f \rangle (\mathcal{X} \cap (f)^{-1}) \neq \emptyset$.

Corollary 6.56. $\text{dom } f = \bigcap \{ a \in \text{atoms } \mathfrak{F}(\text{Src } f) \mid \langle f \rangle a \neq \emptyset \}$.

Proof. This follows from the fact that $\mathfrak{F}(\text{Src } f)$ is an atomistic lattice.

Proposition 6.57. $\text{dom } (f|_{\mathcal{A}}) = \mathcal{A} \cap \text{dom } f$ for every funcoid $f$ and $\mathcal{A} \in \mathfrak{F}(\text{Src } f)$.

Proof. $\text{dom } f|_{\mathcal{A}} = \text{im } (\text{id}_{\mathcal{A}} \circ f^{-1}) = \langle \text{id}_{\mathcal{A}} \rangle \langle f \rangle (\mathcal{Y} \cap \text{dom } f^{-1}) = \mathcal{A} \cap \langle f \rangle \text{dom } f$.

Theorem 6.58. $\text{im } f = \bigcap \langle \text{im } \rangle \text{up } f$ and $\text{dom } f = \bigcap \langle \text{dom } \rangle \text{up } f$ for every funcoid $f$.

Proof. $\text{im } f = \langle f \rangle 1\mathfrak{F}(\text{Src } f) = \bigcap \{ \langle f \rangle | F \in \text{up } f \} = \bigcap \{ \text{im } f \mid F \in \text{up } f \} = \bigcap \langle \text{im } \rangle \text{up } f$.

The second formula follows from symmetry.

Proposition 6.59. For every composable funcoids $f, g$:
1. If $\text{im } f \supseteq \text{dom } g$ then $\text{im } (g \circ f) = \text{im } g$.
2. If $\text{im } f \subseteq \text{dom } g$ then $\text{dom } (g \circ f) = \text{dom } f$.

Proof. $\text{im } (g \circ f) = \langle g \circ f \rangle 1\mathfrak{F}(\text{Src } f) = \langle g \rangle \langle f \rangle 1\mathfrak{F}(\text{Src } f) = \langle g \rangle \text{im } f = \langle g \rangle \text{dom } \mathcal{A} = \langle g \rangle \mathcal{A} = \text{im } g$.

2. $\text{dom } (g \circ f) = \langle f \rangle \mathcal{A} \circ \mathcal{A}^{-1}$ what by proved above is equal to $\text{im } f^{-1}$.

Lemma 6.60. $\lambda \mathcal{B} \in \mathfrak{F}(B): 1\mathfrak{A} \times \text{FCD } \mathcal{B}$ is an upper adjoint of $\lambda \mathcal{B} \in \mathfrak{F}(A; B): \text{im } f$ (for every sets $A, B$).

Proof. We need to prove $\text{im } f \subseteq \mathcal{B} \iff f \subseteq 1\mathfrak{A} \times \text{FCD } \mathcal{B}$ what is obvious.

6.7 Categories of funcoids

I will define two categories, the category of funcoids and the category of funcoid triples.

The category of funcoids is defined as follows:

- Objects are small sets.
- The set of morphisms from a set \( A \) to a set \( B \) is \( \text{FCD}(A; B) \).
- The composition is the composition of funcoids.
- Identity morphism for a set is the identity funcoid for that set.

To show it is really a category is trivial.

The category of funcoid triples is defined as follows:

- Objects are filters on small sets.
- The morphisms from a filter \( \mathcal{A} \) to a filter \( \mathcal{B} \) are triples \( (\mathcal{A}; \mathcal{B}; f) \) where \( f \in \text{FCD}(	ext{Base}(\mathcal{A}); \text{Base}(\mathcal{B})) \) and \( \text{dom} f \subseteq A \land \text{im} f \subseteq B \).
- The composition is defined by the formula \( (\mathcal{A}; \mathcal{B}; g) \circ (\mathcal{A}; \mathcal{B}; f) = (\mathcal{A}; \mathcal{C}; g \circ f) \).
- Identity morphism for a filter \( \mathcal{A} \) is \( \text{id}_{\text{FCD}}^{\mathcal{A}} \).

To prove that it is really a category is trivial.

6.8 Specifying funcoids by functions or relations on atomic filters

**Theorem 6.62.** For every funcoid \( f \) and \( X \in \mathfrak{F}^{\text{Src} f} \), \( Y \in \mathfrak{F}^{\text{Dst} f} \)

1. \( \langle f \rangle X = \bigsqcup \langle \langle f \rangle \rangle \text{atoms} X \)
2. \( X [f] Y \iff \exists x \in \text{atoms} X, y \in \text{atoms} Y; x [f] y. \)

**Proof.**

1. \( Y \cap \langle f \rangle X \neq 0^{\text{Dst} f} \iff X \cap \langle f^{-1} \rangle Y \neq 0^{\text{Src} f} \)
   \( \iff \exists x \in \text{atoms} X; x \cap \langle f^{-1} \rangle Y \neq 0^{\text{Src} f} \)
   \( \iff \exists x \in \text{atoms} X; Y \cap \langle f \rangle x \neq 0^{\text{Dst} f} \).

\( \partial \langle f \rangle X = \bigsqcup \langle \partial \rangle \langle \langle f \rangle \rangle \text{atoms} X = \partial \bigsqcup \langle \langle f \rangle \rangle \text{atoms} X. \) So \( \langle f \rangle X = \bigsqcup \langle \langle f \rangle \rangle \text{atoms} X \) by proposition 4.202.

2. If \( X [f] Y \), then \( Y \cap \langle f \rangle X \neq 0^{\text{Dst} f} \), consequently there exists \( y \in \text{atoms} Y \) such that \( y \cap \langle f \rangle X \neq 0^{\text{Dst} f} \), \( X [f] y \). Repeating this second time we get that there exists \( x \in \text{atoms} X \) such that \( x [f] y \). From this it follows

\( \exists x \in \text{atoms} X, y \in \text{atoms} Y; x [f] y. \)

The reverse is obvious.

**Theorem 6.63.** Let \( A \) and \( B \) be sets.

1. A function \( \alpha \in \mathfrak{F}(B)^{\text{atoms}^{\mathcal{A}}} \) such that (for every \( a \in \text{atoms}^{\mathcal{A}} \))

\[
\alpha a = \bigsqcup \langle \circ \alpha \circ \text{atoms} \circ \uparrow \rangle a \quad (6.6)
\]

can be continued to the function \( \langle f \rangle \) for a unique \( f \in \text{FCD}(A; B) \);

\[
\langle f \rangle X = \bigsqcup \langle \alpha \rangle \text{atoms} X \quad (6.7)
\]
for every $X \in \mathfrak{G}(A)$.

2. A relation $\delta \in \mathfrak{P}(\text{atoms}^{\mathfrak{G}(A)} \times \text{atoms}^{\mathfrak{G}(B)})$ such that (for every $a \in \text{atoms}^{\mathfrak{G}(A)}$, $b \in \text{atoms}^{\mathfrak{G}(B)}$)

$$\forall X \in a, Y \in b \exists x \in \text{atoms} \uparrow^A X, y \in \text{atoms} \uparrow^B Y: x \delta y \Rightarrow a \delta b$$

(6.8)

can be continued to the relation $[f]$ for a unique $f \in \text{FCD}(A; B)$;

$$\mathcal{X} \uparrow f \mathcal{Y} \Leftrightarrow \exists x \in \text{atoms} \mathcal{X}, y \in \text{atoms} \mathcal{Y}: x \delta y$$

(6.9)

for every $X \in \mathfrak{G}(A), \mathcal{Y} \in \mathfrak{G}(B)$.

**Proof.** Existence of no more than one such funcoids and formulas (6.7) and (6.9) follow from the previous theorem.

1. Consider the function $\alpha' \in \mathfrak{G}(B)^{\mathfrak{P}A}$ defined by the formula (for every $X \in \mathfrak{P}A$)

$$\alpha'X = \bigcup \langle \alpha \rangle \text{atoms} \uparrow^A X.$$ 

Obviously $\alpha' \emptyset = \emptyset^{\mathfrak{G}(B)}$. For every $I, J \in \mathfrak{P}A$

$$\alpha'(I \cup J) = \bigcup \langle \alpha \rangle \text{atoms} \uparrow^A (I \cup J)$$

$$= \bigcup \langle \alpha \rangle \text{atoms} \uparrow^A I \cup \text{atoms} \uparrow^A J$$

$$= \bigcup \langle \alpha \rangle \text{atoms} \uparrow^A I \cup \langle \alpha \rangle \text{atoms} \uparrow^A J$$

$$= \bigcup \langle \alpha \rangle \text{atoms} \uparrow^A I \cup \bigcup \langle \alpha \rangle \text{atoms} \uparrow^A J$$

$$= \alpha' I \cup \alpha' J.$$ 

Let continue $\alpha'$ till a funcoid $f$ (by the theorem 6.28): $(f) \mathcal{X} = \bigcap \langle \alpha' \rangle \mathcal{X}$.

Let’s prove the reverse of (6.6):

$$\prod \langle \bigcup \langle \alpha \rangle \circ \text{atoms} \circ \uparrow^A \rangle a = \prod \langle \bigcup \langle \alpha \rangle \circ \text{atoms} \rangle \langle \uparrow^A \rangle a$$

$$\subseteq \prod \langle \bigcup \langle \alpha \rangle \circ \text{atoms} \rangle \{\{a\}\}$$

$$= \prod \{\bigcup \langle \alpha \rangle \{a\}\}$$

$$= \prod \{\bigcup \{\alpha a\}\} = \prod \{\alpha a\} = \alpha a.$$

Finally,

$$\alpha a = \prod \langle \bigcup \langle \alpha \rangle \circ \text{atoms} \circ \uparrow^A \rangle \{a\} = \prod \langle \alpha' \rangle a = \langle f \rangle a,$$

so $(f)$ is a continuation of $\alpha$.

2. Consider the relation $\delta' \in \mathfrak{P}(\mathfrak{P}A \times \mathfrak{P}B)$ defined by the formula (for every $X \in \mathfrak{P}A, Y \in \mathfrak{P}B$)

$$X \delta' Y \Leftrightarrow \exists x \in \text{atoms} \uparrow^A X, y \in \text{atoms} \uparrow^B Y: x \delta y.$$

Obviously $\neg(X \delta' \emptyset)$ and $\neg(\emptyset \delta' Y)$.

For suitable $I$ and $J$ we have:

$$(I \cup J) \delta' Y \Leftrightarrow \exists x \in \text{atoms} \uparrow^A (I \cup J), y \in \text{atoms} \uparrow^B Y: x \delta y$$

$$\Leftrightarrow \exists x \in \text{atoms} \uparrow^A I \cup \text{atoms} \uparrow^A J, y \in \text{atoms} \uparrow^B Y: x \delta y$$

$$\Leftrightarrow \exists x \in \text{atoms} \uparrow^A I, y \in \text{atoms} \uparrow^B Y: x \delta y \vee \exists x \in \text{atoms} \uparrow^A J, y \in \text{atoms} \uparrow^B Y: x \delta y$$

$$\Leftrightarrow I \delta' Y \lor J \delta' Y;$$

similarly $X \delta'(I \cup J) \Leftrightarrow X \delta' I \lor X \delta' J$ for suitable $I$ and $J$. Let’s continue $\delta'$ till a funcoid $f$ (by the theorem 6.28):

$$\mathcal{X} \uparrow f \mathcal{Y} \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y}: X \delta' Y.$$

The reverse of (6.8) implication is trivial, so

$$\forall X \in a, Y \in b \exists x \in \text{atoms} \uparrow^A X, y \in \text{atoms} \uparrow^B Y: x \delta y \Leftrightarrow a \delta b.$$

$$\forall X \in a, Y \in b \exists x \in \text{atoms} \uparrow^A X, y \in \text{atoms} \uparrow^B Y: x \delta y \Leftrightarrow \forall X \in a, Y \in b: X \delta' Y \Leftrightarrow a \{f\} b.$$
So \( a \delta b \Leftrightarrow a[f] b \), that is \([f]\) is a continuation of \( \delta \). \( \square \)

One of uses of the previous theorem is the proof of the following theorem:

**Theorem 6.64.** If \( A \) and \( B \) are sets, \( R \in \mathcal{P}FCD(A; B) \), \( x \in \text{atoms}^\mathfrak{3}(A) \), \( y \in \text{atoms}^\mathfrak{3}(B) \), then

1. \( \langle \bigcap R \rangle x = \bigcap \{ \langle f \rangle x \mid f \in R \} \);
2. \( x \bigcap R \ y \Leftrightarrow \forall f \in R; x[f] y \).

**Proof.** 2. Let denote \( x \delta y \Leftrightarrow \forall f \in R; x[f] y \). For every \( a \in \text{atoms}^\mathfrak{3}(A) \), \( b \in \text{atoms}^\mathfrak{3}(B) \)

\[
\forall X \in a, Y b \sqcap x \in \text{atoms} \uparrow X, y \in \text{atoms} \uparrow Y; x \delta y \Rightarrow
\forall f \in R, X \in a, Y b \sqcap x \in \text{atoms} \uparrow X, y \in \text{atoms} \uparrow Y; x[f] y \Rightarrow
\forall f \in R; a[f] b \Leftrightarrow
\]

So by theorem 6.63, \( \delta \) can be continued till \([p]\) for some funcoid \( p \in \text{FCD}(A; B) \).

For every funcoid \( q \in \text{FCD}(A; B) \) such that \( \forall f \in R; q \sqsubseteq f \) we have \( x[q] y \Rightarrow \forall f \in R; x[f] y \Leftrightarrow x \delta y \Leftrightarrow x[p] y \), so \( q \sqsubseteq p \). Consequently \( p = \bigcap R \).

From this \( x \bigcap R \ y \Leftrightarrow \forall f \in R; x[f] y \).

1. From the former \( y \in \text{atoms} \bigcap R \rangle x \Leftrightarrow y \sqcap \bigcap R \rangle x \neq 0^\mathfrak{3}(B) \Leftrightarrow \forall f \in R; y \sqcap \langle f \rangle x \neq 0^\mathfrak{3}(B) \Leftrightarrow y \in \bigcap \{ \text{atoms} \langle f \rangle x \mid f \in R \} \Rightarrow y \in \text{atoms} \bigcap \{ \langle f \rangle x \mid f \in R \} \) for every \( y \in \text{atoms}^\mathfrak{3}(A) \). From this it follows \( \bigcap R \rangle x = \bigcap \{ \langle f \rangle x \mid f \in R \} \). \( \square \)

**Theorem 6.65.** \( g \circ f = \bigcap \{ G \circ F \mid F \in \text{up} f, G \in \text{up} g \} \) for every composable funcoids \( f \) and \( g \).

**Proof.** Let \( x \in \text{atoms}^\mathfrak{3}(\text{Sec} f) \). Then

\[
(g \circ f)x =
\]

\[
\bigcap \{ G \langle f \rangle x \mid G \in \text{up} g \} = \text{theorem 6.35}
\]

\[
\bigcap \{ \langle G \rangle \langle f \rangle x \mid \langle G \rangle \in \text{up} f \} \mid G \in \text{up} g \} = \text{theorem 6.32}
\]

\[
\bigcap \{ G \langle F \rangle x \mid F \in \text{up} f, G \in \text{up} g \} = \text{theorem 6.64}
\]

\[
\bigcap \{ G \circ F x \mid F \in \text{up} f, G \in \text{up} g \} = \text{theorem 6.64}
\]

Thus \( g \circ f = \bigcap \{ G \circ F \mid F \in \text{up} f, G \in \text{up} g \} \). \( \square \)

**Theorem 6.66.** Let \( A, B, C \) be sets, \( f \in \text{FCD}(A; B) \), \( g \in \text{FCD}(B; C) \), \( h \in \text{FCD}(A; C) \). Then

\( g \circ f \neq h \Leftrightarrow \exists g \neq h \circ f^{-1} \).

**Proof.**

\[
g \circ f \neq h \Leftrightarrow \exists a \in \text{atoms}^\mathfrak{3}(A), c \in \text{atoms}^\mathfrak{3}(C) \; a \sqcap (g \circ f) \sqcap c \Leftrightarrow
\]

\[
\exists a \in \text{atoms}^\mathfrak{3}(A), c \in \text{atoms}^\mathfrak{3}(C) \; (a \sqcap g \circ f) \sqcap a \sqcap c \Leftrightarrow
\]

\[
\exists b \in \text{atoms}^\mathfrak{3}(B), c \in \text{atoms}^\mathfrak{3}(C) \; (g \circ f) \sqcap b \sqcap c \Leftrightarrow
\]

\[
\exists b \in \text{atoms}^\mathfrak{3}(B), c \in \text{atoms}^\mathfrak{3}(C) \; (g \circ f) \sqcap c \sqcap b \Leftrightarrow
\]

\[
g \neq h \circ f^{-1}.
\]

\( \square \)
6.9 Direct product of filters

A generalization of Cartesian product of two sets is funcoidal product of two filters:

**Definition 6.67.** Funcoidal product of filters $A$ and $B$ is such a funcoid $A \times_{\text{FCD}} B \in \text{FCD} (\text{Base}(A); \text{Base}(B))$ that for every $\mathcal{X} \in \mathcal{F} (\text{Base}(A)), \mathcal{Y} \in \mathcal{F} (\text{Base}(B))$

\[ \mathcal{X} \times_{\text{FCD}} B \Rightarrow \mathcal{X} \neq A \land \mathcal{Y} \neq B. \]

**Proposition 6.68.** $A \times_{\text{FCD}} B$ is really a funcoid and

\[ (A \times_{\text{FCD}} B), \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \neq A \\ \text{for some } \mathcal{B} \in \mathcal{F} (\text{Base}(B)) & \text{if } \mathcal{X} \supset A. \end{cases} \]

**Proof.** Obvious. □

**Obvious 6.69.** $\uparrow^{\text{FCD}(U;V)} (A \times B) = \uparrow^{U} A \times_{\text{FCD}} \uparrow^{V} B$ for sets $A \subseteq U$ and $B \subseteq V$.

**Proposition 6.70.** $f \subseteq A \times_{\text{FCD}} B \Rightarrow \text{dom } f \subseteq A \land \text{im } f \subseteq B$ for every $f \in \text{FCD}(A; B)$ and $A \in \mathcal{F}(A), B \in \mathcal{F}(B)$.

**Proof.** If $f \subseteq A \times_{\text{FCD}} B$ then $\text{dom } f \subseteq \text{dom}(A \times_{\text{FCD}} B) \subseteq A$, $\text{im } f \subseteq \text{im}(A \times_{\text{FCD}} B) \subseteq B$. If $f \subseteq A \land \text{im } f \subseteq B$ then

\[ \forall \mathcal{X} \in \mathcal{F}(A), \mathcal{Y} \in \mathcal{F}(B); (\mathcal{X} \uparrow f) \Rightarrow \mathcal{X} \cap A \neq 0 \land \mathcal{Y} \cap B \neq 0 \Rightarrow \mathcal{X} \cap B \neq 0 ; \]

consequently $f \subseteq A \times_{\text{FCD}} B$. □

The following theorem gives a formula for calculating an important particular case of a meet on the lattice of funcoids:

**Theorem 6.71.** $f \cap (A \times_{\text{FCD}} B) = \text{id}^\text{FCD}_B \circ f \circ \text{id}^\text{FCD}_A$ for every funcoid $f$ and $A \in \mathcal{F}(\text{Src } f), B \in \mathcal{F}(\text{Dst } f)$.

**Proof.** $h \equiv \text{id}^\text{FCD}_B \circ f \circ \text{id}^\text{FCD}_A$. For every $\mathcal{X} \in (\text{Src } f)$

\[ \langle h \rangle \mathcal{X} = \langle \text{id}^\text{FCD}_B \rangle \langle f \rangle \langle \text{id}^\text{FCD}_A \rangle \mathcal{X} = \mathcal{B} \cap \langle f \rangle (A \cap \mathcal{X}). \]

From this, as easy to show, $h \subseteq f$ and $h \subseteq A \times_{\text{FCD}} B$. If $g \subseteq f \land g \subseteq A \times_{\text{FCD}} B$ for a $g \in \text{FCD}(\text{Src } f; \text{Dst } f)$ then $\text{dom } g \subseteq A$, $\text{im } g \subseteq B,$

\[ \langle g \rangle \mathcal{X} = \mathcal{B} \cap \langle g \rangle (A \cap \mathcal{X}) \subseteq B \cap \langle f \rangle (A \cap \mathcal{X}) = \langle \text{id}^\text{FCD}_B \rangle \langle f \rangle \langle \text{id}^\text{FCD}_A \rangle \mathcal{X} = \langle h \rangle \mathcal{X}, \]

$g \subseteq h$. So $h = f \cap (A \times_{\text{FCD}} B)$. □

**Corollary 6.72.** $f \mid_A = f \cap (A \times_{\text{FCD}} 1_{\mathcal{F}(\text{Dst } f)})$ for every $f \in \text{FCD}$ and $A \in \mathcal{F}(\text{Src } f)$.

**Proof.** $f \cap (A \times_{\text{FCD}} 1_{\mathcal{F}(\text{Dst } f)}) = \text{id}^\text{FCD}_1 \circ f \circ \text{id}^\text{FCD}_A = f \circ \text{id}^\text{FCD}_A = f \mid_A$. □

**Corollary 6.73.** $f \neq A \times_{\text{FCD}} B \Leftrightarrow A \not\subseteq f \circ \text{id}^\text{FCD}_A$ for every $f \in \text{FCD}$, $A \in \mathcal{F}(\text{Src } f), B \in \mathcal{F}(\text{Dst } f)$.

**Proof.** The set of funcoidal products of principal filters is a separation subset of the lattice of funcoids. □
Theorem 6.75. Let $A$, $B$ be sets. If $S \in \mathcal{P}(\mathfrak{A}(A) \times \mathfrak{A}(B))$ then
$$
\bigcap \{A \times \mathcal{FCD} B \mid (A; B) \in S\} = \bigcap \text{dom } S \times \mathcal{FCD} \bigcap \text{im } S.
$$

Proof. If $x \in \text{atoms } \mathfrak{A}(A)$ then by the theorem 6.64
$$
\langle \bigcap \{A \times \mathcal{FCD} B \mid (A; B) \in S\} \rangle x = \bigcap \{\langle A \times \mathcal{FCD} B \rangle x \mid (A; B) \in S\}.
$$
If $x \not\in \bigcap \text{dom } S$ then
$$
\forall (A; B) \in S: (x \cap A \neq 0^{\mathfrak{A}(A)} \land \langle A \times \mathcal{FCD} B \rangle x = B);
$$
$$
\langle \langle A \times \mathcal{FCD} B \rangle x \mid (A; B) \in S \rangle = \text{im } S;
$$
if $x \in \bigcap \text{dom } S$ then
$$
\exists (A; B) \in S: (x \cap A = 0^{\mathfrak{A}(A)} \land \langle A \times \mathcal{FCD} B \rangle x = 0^{\mathfrak{A}(B)});
$$
$$
\langle \langle A \times \mathcal{FCD} B \rangle x \mid (A; B) \in S \rangle \ni 0^{\mathfrak{A}(B)}.
$$
So
$$
\langle \bigcap \{A \times \mathcal{FCD} B \mid (A; B) \in S\} \rangle x = \begin{cases}
\bigcap \text{im } S & \text{if } x \not\in \bigcap \text{dom } S \\
0^{\mathfrak{A}(B)} & \text{if } x \in \bigcap \text{dom } S.
\end{cases}
$$
From this the statement of the theorem follows. 

Corollary 6.76. For every $A_0, A_1 \in \mathfrak{A}(A), B_0, B_1 \in \mathfrak{A}(B)$ (for every sets $A, B$
$$(A_0 \times \mathcal{FCD} B_0) \cap (A_1 \times \mathcal{FCD} B_1) = (A_0 \cap A_1 \times \mathcal{FCD} (B_0 \cap B_1)).$$

Proof. $(A_0 \times \mathcal{FCD} B_0) \cap (A_1 \times \mathcal{FCD} B_1) = \bigcap \{A_0 \times \mathcal{FCD} B_0, A_1 \times \mathcal{FCD} B_1\}$ what is by the last theorem equal to $(A_0 \cap A_1) \times \mathcal{FCD} (B_0 \cap B_1)$. 

Theorem 6.77. If $A, B$ are sets and $A \in \mathfrak{A}(A)$ then $A \times \mathcal{FCD}$ is a complete homomorphism from the lattice $\mathfrak{A}(B)$ to the lattice $\mathcal{FCD}(A; B)$, if also $A \neq 0^{\mathfrak{A}(A)}$ then it is an order embedding.

Proof. Let $S \in \mathcal{P}(\mathfrak{A}(B), X \in \mathcal{P}A, x \in \text{atoms } \mathfrak{A}(A)$.
$$
\bigcup (A \times \mathcal{FCD})S X = \bigcup \{\langle A \times \mathcal{FCD} B \rangle x \mid B \in S\}
$$
$$
= \begin{cases}
\bigcup S & \text{if } X \in \partial A \\
0^{\mathfrak{A}(B)} & \text{if } X \notin \partial A
\end{cases} = \langle A \times \mathcal{FCD} \bigcup S \rangle x;
$$
$$
\bigcap (A \times \mathcal{FCD})S x = \bigcap \{\langle A \times \mathcal{FCD} B \rangle x \mid B \in S\}
$$
$$
= \begin{cases}
\bigcap S & \text{if } x \notin A \\
0^{\mathfrak{A}(B)} & \text{if } x \in A
\end{cases} = \langle A \times \mathcal{FCD} \bigcap S \rangle x.
$$
Thus $\bigcup (A \times \mathcal{FCD})S = A \times \mathcal{FCD} \bigcup S$ and $\bigcap (A \times \mathcal{FCD})S = A \times \mathcal{FCD} \bigcap S$.
If $A \neq 0^{\mathfrak{A}(A)}$ then obviously the function $A \times \mathcal{FCD}$ is injective. 

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a funcoidal product of filters) funcoid (of atomic width).

Proposition 6.78. If $f \in \mathcal{FCD}$ and $a$ is an atomic filter on $\text{Src } f$ then
$$
\langle f \rangle a = a \times \mathcal{FCD} \langle f \rangle a.
$$

Proof. Let $X \in \mathfrak{A}(\text{Src } f)$.
$$
X \neq a \Rightarrow \langle f \rangle a X = \langle f \rangle a, \quad X \not\sim a \Rightarrow \langle f \rangle a X = 0^{\mathfrak{A}(\text{Det } f)}.
$$

\[\square\]
6.10 Atomic funcoids

Theorem 6.79. An $f \in \text{FCD}(A; B)$ is an atom of the lattice $\text{FCD}(A; B)$ (for some sets $A$, $B$) iff it is a funcoidal product of two ultrafilters.

Proof. Let $f \in \text{FCD}(A; B)$ be an atom of the lattice $\text{FCD}(A; B)$. Let's get elements $a \in \text{atoms} \, \text{dom} \, f$ and $b \in \text{atoms} \, \text{ran} \, f$. Then for every $X \in \mathcal{F}(A)$
\[ X \cap a = \langle a \times \text{FCD} \, b \rangle \, X = 0^{\mathcal{F}(B)} \subseteq \langle f \rangle \, X, \quad X \neq a \Rightarrow \langle a \times \text{FCD} \, b \rangle \, X = b \subseteq \langle f \rangle \, X. \]
So $a \times \text{FCD} \, b \subseteq f$; because $f$ is atomic we have $f = a \times \text{FCD} \, b$.
\[
\Rightarrow \quad f \subseteq \langle (f) \, a \rangle \text{ (for every sets } A, B) \quad \Rightarrow \quad a \in \text{atoms} \, \mathcal{F}(A), \quad b \in \text{atoms} \, \mathcal{F}(B), \quad f \subseteq \langle f \rangle \, a \times \text{FCD} \, b \subseteq f. \]

Theorem 6.80. The lattice $\text{FCD}(A; B)$ is atomic (for every sets $A$, $B$).

Proof. Let $f$ be a non-empty funcoid from $A$ to $B$. Then dom $f \neq 0^{\mathcal{F}(A)}$, thus by the theorem 4.207 there exists $a \in \text{atoms} \, \text{dom} \, f$. So $\langle f \rangle \, a \neq 0^{\mathcal{F}(B)}$ thus it exists $b \in \text{atoms} \, \langle f \rangle \, a$. Finally the atomic funcoid $a \times \text{FCD} \, b \subseteq f$.

Theorem 6.81. The lattice $\text{FCD}(A; B)$ is separable (for every sets $A$, $B$).

Proof. Let $f, g \in \text{FCD}(A; B)$, $f \sqsubseteq g$. Then there exists $a \in \text{atoms} \, \mathcal{F}(A)$ such that $\langle f \rangle \, a \sqsubseteq \langle g \rangle \, a$. So because the lattice $\mathcal{F}(B)$ is atomically separable, there exists $b \in \text{atoms} \, \mathcal{F}(B)$ such that $\langle f \rangle \, a \cap b = 0^{\mathcal{F}(B)}$ and $b \subseteq \langle g \rangle \, a$. For every $x \in \text{atoms} \, \mathcal{F}(A)$
\[ \langle f \rangle \, x \cap \langle a \times \text{FCD} \, b \rangle \, x = 0^{\mathcal{F}(B)} \quad \text{and consequently } f \neq a \times \text{FCD} \, b. \]
Thus $\langle f \rangle \, x \cap \langle a \times \text{FCD} \, b \rangle \, x = 0^{\mathcal{F}(B)}$ and consequently $f \neq a \times \text{FCD} \, b$.
\[ \langle a \times \text{FCD} \, b \rangle \, a = b \subseteq \langle g \rangle \, a, \quad x \neq a \Rightarrow \langle a \times \text{FCD} \, b \rangle \, x = 0^{\mathcal{F}(B)} \subseteq \langle g \rangle \, x. \]
Thus $\langle a \times \text{FCD} \, b \rangle \, x \subseteq \langle g \rangle \, x$ and consequently $a \times \text{FCD} \, b \subseteq g$.
So the lattice $\text{FCD}(A; B)$ is separable by the theorem 3.14.

Corollary 6.82. The lattice $\text{FCD}(A; B)$ is:
1. separable;
2. atomically separable;
3. conforming to Wallman’s disjunction property.


Remark 6.83. For more ways to characterize (atomic) separability of the lattice of funcoids see subsections “Separation subsets and full stars” and “Atomically separable lattices”.

Corollary 6.84. The lattice $\text{FCD}(A; B)$ is an atomistic lattice.

Proof. Let $f \in \text{FCD}(A; B)$. Suppose contrary to the statement to be proved that $\bigcup \text{ atoms } f \neq f$. Then there exist $a \in \text{ atoms } f$ such that $a \cap \bigcup \text{ atoms } f = 0^{\text{FCD}(A; B)}$ what is impossible.

Proposition 6.85. $\text{atoms}(f \cup g) = \text{atoms} \, f \cup \text{atoms} \, g$ for every funcoids $f, g \in \text{FCD}(A; B)$ (for every sets $A$, $B$).
Proof. $a \times^{\text{FCD}} b \neq f \cup g \Leftrightarrow a [f \cup g] b \Leftrightarrow a [f] b \cup a [g] b \Leftrightarrow a \times^{\text{FCD}} b \neq f \cup a \times^{\text{FCD}} b \neq g$ for every atomic filters $a$ and $b$. \hfill \Box

**Theorem 6.86.** For every $f, g, h \in \text{FCD}(A; B)$, $R \in \mathcal{P}\text{FCD}(A; B)$ (for every sets $A$ and $B$)

1. $f \cap (g \cup h) = (f \cap g) \cup (f \cap h)$;
2. $f \cup \bigcap R = \bigcap (\{f \cup \cdot\} R)$.

**Proof.** We will take into account that the lattice of funcoids is an atomistic lattice.

1. atoms$(f \cap (g \cup h)) = \text{atoms } f \cap \text{atoms } (g \cup h) = \text{atoms } f \cap \{\text{atoms } g \cup \text{atoms } h\} = \{\text{atoms } f \cap \text{atoms } g\} \cup \{\text{atoms } f \cap \text{atoms } h\} = \{\text{atoms } f\} \cap \{\text{atoms } g\} \cup \{\text{atoms } f\} \cap \{\text{atoms } h\}$.
2. atoms$(f \cup \bigcap R) = \text{atoms } f \cup \{\text{atoms } f\} \cap \{\text{atoms } R\} = \text{atoms } f \cup \{\text{atoms } f\} \cap \{\text{atoms } R\} = \bigcap \{\text{atoms } f\} \cap \{\text{atoms } R\} = \bigcap \{\text{atoms } f\} \cap \{\text{atoms } R\}$ (it was used the following equality.)

Note that distributivity of the lattice of funcoids is proved through using atoms of this lattice.

I have never seen such method of proving distributivity.

**Corollary 6.87.** The lattice $\text{FCD}(A; B)$ is co-brouwerian (for every sets $A$, $B$).

**Conjecture 6.88.** Distributivity of the lattice $\text{FCD}(A; B)$ of funcoids (for arbitrary sets $A$ and $B$) is not provable in ZF (without axiom of choice).

The next proposition is one more (among the theorem 6.43) generalization for funcoids of composition of relations.

**Proposition 6.89.** For every composable funcoids $f$, $g$

\[
\text{atoms}(g \circ f) = \left\{ \left[ x \times^{\text{FCD}} z \right] | x \in \text{atoms}^{\text{FCD}}(\text{Src } f), z \in \text{atoms}^{\text{FCD}}(\text{Dst } g), \exists y \in \text{atoms}^{\text{FCD}}(\text{Dst } f); (x \times^{\text{FCD}} y \in \text{atoms } f \land y \times^{\text{FCD}} z \in \text{atoms } g) \right\}.
\]

**Proof.** $x \times^{\text{FCD}} z \neq g \circ f \Leftrightarrow x [g \circ f] z \Leftrightarrow \exists y \in \text{atoms}^{\text{FCD}}(\text{Dst } f); (x \times^{\text{FCD}} y \neq f \land y \times^{\text{FCD}} z \neq g)$ (it was used the theorem 6.43). \hfill \Box

**Corollary 6.90.** $g \circ f = \bigsqcup \{G \circ F | F \in \text{atoms } f, G \in \text{atoms } g\}$ for every composable funcoids $f$, $g$.

**Theorem 6.91.** Let $f$ be a funcoid.

1. $\mathcal{X} \langle f \rangle \mathcal{Y} \Leftrightarrow \exists F \in \text{atoms } f; \mathcal{X} \langle F \rangle \mathcal{Y}$ for every $\mathcal{X} \in \mathcal{F}(\text{Src } f)$, $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$;
2. $(\langle f \rangle) \mathcal{X} = \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathcal{F}(\text{Src } f)$.

**Proof.** 1. $\exists F \in \text{atoms } f; \mathcal{X} \langle F \rangle \mathcal{Y} \Leftrightarrow \exists a \in \mathcal{F}(\text{Src } f), b \in \mathcal{F}(\text{Dst } f); (a \times^{\text{FCD}} b \neq f \land a \times^{\text{FCD}} b \neq \mathcal{X} \times^{\text{FCD}} \mathcal{Y}) \Leftrightarrow \exists F \in \text{atoms } f; (F \neq f \land F \neq a \times^{\text{FCD}} b \neq a \times^{\text{FCD}} b \neq \mathcal{X} \times^{\text{FCD}} \mathcal{Y}) \Leftrightarrow \mathcal{X} \langle F \rangle \mathcal{Y}$.
2. Let $\mathcal{Y} \in \mathcal{F}(\text{Dst } f)$. Suppose $\mathcal{Y} \neq \langle f \rangle \mathcal{X}$. Then $\mathcal{X} \langle f \rangle \mathcal{Y} \Leftrightarrow \exists F \in \text{atoms } f; \mathcal{X} \langle F \rangle \mathcal{Y} \Leftrightarrow \exists F \in \text{atoms } f; \mathcal{Y} \neq \langle F \rangle \mathcal{X}$. So $(\langle f \rangle) \mathcal{X} \subseteq \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$. The contrary $(\langle f \rangle) \mathcal{X} \supseteq \bigsqcup_{F \in \text{atoms } f} \langle F \rangle \mathcal{X}$ is obvious. \hfill \Box
Problem 6.92. Let $A$ and $B$ be infinite sets. Characterize the set of all coatoms of the lattice $\text{FCD}(A; B)$ of funcoids from $A$ to $B$. Particularly, is this set empty? Is $\text{FCD}(A; B)$ a coatomic lattice? coatomistic lattice?

### 6.11 Complete funcoids

**Definition 6.93.** I will call **co-complete** such a funcoid $f$ that $(f)^*X$ is a principal filter for every $X \in \mathcal{P}(\text{Src } f)$.

**Obvious 6.94.** Funcoid $f$ is co-complete iff $(f)X \in \mathfrak{Q}$ for every $X \in \mathfrak{Q}$.

**Remark 6.95.** I will call **generalized closure** such a function $\alpha \in \mathcal{P}B^{\mathcal{P}A}$ (for some sets $A$, $B$) that

1. $\alpha \emptyset = \emptyset$;
2. $\forall I, J \in \mathcal{P}A: \alpha(I \cup J) = \alpha I \cup \alpha J$.

**Obvious 6.96.** A funcoid $f$ is co-complete iff $(f)^* = \uparrow^{\text{Dst } f} \circ \alpha$ for a generalized closure $\alpha$.

**Remark 6.97.** Thus funcoids can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of funcoids.

**Definition 6.98.** I will call a **complete funcoid** a funcoid whose reverse is co-complete.

**Theorem 6.99.** The following conditions are equivalent for every funcoid $f$:

1. funcoid $f$ is complete;
2. $\forall S \in \mathcal{P}\mathfrak{S}(\text{Src } f), J \in \mathcal{P}(\text{Dst } f): (\bigcup S \uparrow f) \uparrow^{\text{Dst } f} J \Leftrightarrow \exists I \in S: I \uparrow f \cap (f)^{-1}\uparrow^{\text{Src } f} J \neq \emptyset$;
3. $\forall S \in \mathcal{P}\mathfrak{P}(\text{Src } f), J \in \mathcal{P}(\text{Dst } f): (\bigcup S \uparrow f) \uparrow f \cap (f)^{-1}J \neq \emptyset$;
4. $\forall S \in \mathcal{P}\mathfrak{P}(\text{Src } f): \langle f \rangle \bigcup S = \bigcup \langle (f)^* \rangle S$;
5. $\forall S \in \mathcal{P}\mathfrak{P}(\text{Src } f): \langle f \rangle \downarrow S = \bigcup \langle (f)^* \rangle S$;
6. $\forall A \in \mathcal{P}(\text{Src } f): (f)^* A = \bigcup \langle (f)^* \rangle a \mid a \in A$.

**Proof.**

(3) $\Rightarrow$ (1). For every $S \in \mathcal{P}\mathfrak{P}(\text{Src } f), J \in \mathcal{P}(\text{Dst } f)$

$$\uparrow^{\text{Src } f} \bigcup S \cap (f)^{-1}\uparrow^{\text{Src } f} J \neq \emptyset \Leftrightarrow \exists I \in S: \uparrow^{\text{Src } f} I \cap (f)^{-1}\uparrow^{\text{Src } f} J \neq \emptyset,$$

consequently by proposition 4.215 we have that $(f)^{-1}\uparrow^{\text{Src } f} J$ is a principal filter.

(1) $\Rightarrow$ (2). For every $S \in \mathcal{P}\mathfrak{S}(\text{Src } f), J \in \mathcal{P}(\text{Dst } f)$ we have $(f)^{-1}\uparrow^{\text{Src } f} J$ is a principal filter, consequently

$$\bigcup S \cap (f)^{-1}\uparrow^{\text{Src } f} J \neq \emptyset \Leftrightarrow \exists I \in S: I \cap (f)^{-1}\uparrow^{\text{Src } f} J \neq \emptyset.$$  

From this follows (2).

(6) $\Rightarrow$ (5). $(f)^* \bigcup S = \bigcup \{ (f)^* a \mid a \in A \} \mid A \in S = \bigcup \{ \{ (f)^* a \mid a \in A \} \mid A \in S \} = \bigcup \{ (f)^* A \mid A \in S \} = \bigcup \langle (f)^* \rangle S$.

(2) $\Rightarrow$ (4). $\uparrow^{\text{Dst } f} J \neq \langle f \rangle \bigcup S \Rightarrow \bigcup S \uparrow f \cap (f)^{\downarrow^{\text{Dst } f} J} \Leftrightarrow \exists I \in S: I \uparrow f \cap (f)^{\downarrow^{\text{Dst } f} J} \Leftrightarrow \exists I \in S: \uparrow^{\text{Dst } f} J \neq \langle f \rangle I \Leftrightarrow \uparrow^{\text{Dst } f} J \neq \bigcup \langle (f)^* \rangle S$ (used theorem 4.215).

(2) $\Rightarrow$ (3), (4) $\Rightarrow$ (5), (5) $\Rightarrow$ (3), (5) $\Rightarrow$ (6). Obvious.

The following proposition shows that complete funcoids are a direct generalization of pretopological spaces.
Proposition 6.100. To specify a complete funcoid $f$ it is enough to specify $\langle f \rangle^*$ on one-element sets, values of $\langle f \rangle^*$ on one element sets can be specified arbitrarily.

Proof. From the above theorem is clear that knowing $\langle f \rangle^*$ on one-element sets $\langle f \rangle^*$ can be found on every set and then the value of $\langle f \rangle$ can be inferred for every filter.

Choosing arbitrarily the values of $\langle f \rangle^*$ on one-element sets we can define a complete funcoid the following way: $\langle f \rangle^* X = \bigcup \{ \langle f \rangle^* \{ \alpha \} : \alpha \in X \}$ for every $X \in \mathcal{P}(\text{Src } f)$. Obviously it is really a complete funcoid. \hfill \Box

Theorem 6.101. A funcoid is principal iff it is both complete and co-complete.

Proof.

$\Rightarrow$. Obvious.

$\Leftarrow$. Let $f$ be both a complete and co-complete funcoid. Consider the relation $g$ defined by that $\exists^\text{Def} f \{ \alpha \} = \langle f \rangle^* \{ \alpha \}$ ($g$ is correctly defined because $f$ corresponds to a generalized closure). Because $f$ is a complete funcoid $f$ is the funcoid corresponding to $g$. \hfill \Box

Theorem 6.102. If $R \in \mathcal{P}(\text{FCD}(A; B))$ is a set of co-complete funcoids then $\bigcup R$ is a co-complete funcoid (for every sets $A$ and $B$).

Proof. It is enough to prove only for co-complete funcoids. Let $R \in \mathcal{P}(\text{FCD}(A; B))$ be a set of co-complete funcoids. Then for every $X \in \mathcal{P}(\text{Src } f)$

$$\langle \bigcup R \rangle^* X = \bigcup \{ \langle f \rangle^* X \mid f \in R \}$$

is a principal filter (used theorem 6.37). \hfill \Box

Corollary 6.103. If $R$ is a set of binary relations between sets $A$ and $B$ then $\bigcup \langle \exists^\text{FCD}(A; B) \rangle R = \exists^\text{FCD}(A; B) \bigcup R$.

Proof. From two last theorems. \hfill \Box

Theorem 6.104. Filtrators of funcoids are filtered.

Proof. It’s enough to prove that every funcoid is representable as an (infinite) meet (on the lattice $\text{FCD}(A; B)$) of some set of principal funcoids.

Let $f \in \text{FCD}(A; B)$, $X \in \mathcal{P}A$, $Y \in \langle f \rangle X$, $g(X; Y) \equiv \exists^\text{FCD} \uparrow A X \times \uparrow B Y \cup \uparrow A X \times \exists^\text{FCD} \uparrow B (Y)$. For every $K \in \mathcal{P}A$

$$\langle g(X; Y) \rangle^* K = (\exists^\text{FCD} \uparrow B Y)^* K \cup (\exists^\text{FCD} \uparrow A X)^* K = \begin{cases} 0^\text{B}(Y) & \text{if } K = \emptyset \\
\uparrow B Y & \text{if } 0 \neq K \subseteq X \\
1^\text{B}(B) & \text{if } K \not\subseteq X \end{cases} \downarrow \langle f \rangle^* K;$$

so $g(X; Y) \supseteq f$. For every $X \in \mathcal{P}A$ 

$$\prod \{ \langle g(X; Y) \rangle^* X \mid Y \in \langle f \rangle^* X \} = \prod \{ \uparrow B Y \mid Y \in \langle f \rangle^* X \} = \langle f \rangle^* X;$$

consequently

$$\langle \prod \{ g(X; Y) \mid X \in \mathcal{P}A, Y \in \langle f \rangle^* X \} \rangle^* X \subseteq \langle f \rangle^* X$$

that is

$$\prod \{ g(X; Y) \mid X \in \mathcal{P}A, Y \in \langle f \rangle^* X \} \subseteq f$$

and finally

$$f = \prod \{ g(X; Y) \mid X \in \mathcal{P}A, Y \in \langle f \rangle^* X \}. \hfill \Box$$

Theorem 6.105.

1. $g$ is metacomplete if $g$ is a complete funcoid.
2. \( g \) is co-metacomplete if \( g \) is a co-complete funcoid.

**Proof.**

1. \( (g \circ \bigcup R)^\ast X = (g)(\bigcup R)^\ast X = (g)\bigcup \{(f)^\ast X \ | \ f \in R\} = \bigcup \{(g \circ f)^\ast X \ | \ f \in R\} = (g \circ \bigcup R)^\ast X \) for every set \( X \subseteq A \). So \( g \circ (\bigcup R) = (g \circ \bigcup R). \)

2. By duality. \( \square \)

**Conjecture 6.106.** \( g \) is complete if \( g \) is a metacomplete funcoid.

I will denote \( \text{ComplFCD} \) and \( \text{CoComplFCD} \) the sets of small complete and co-complete funcoids correspondingly. \( \text{ComplFCD}(A; B) \) are complete funcoids from \( A \) to \( B \) and likewise with \( \text{CoComplFCD}(A; B) \).

**Obvious 6.107.** \( \text{ComplFCD} \) and \( \text{CoComplFCD} \) are closed regarding composition of funcoids.

**Proposition 6.108.** \( \text{ComplFCD} \) and \( \text{CoComplFCD} \) (with induced order) are complete lattices.

**Proof.** It follows from the theorem 6.102. \( \square \)

**Theorem 6.109.** Atoms of the lattice \( \text{ComplFCD}(A; B) \) are exactly funcoidal products of the form \( \uparrow^A \{\alpha\} \times \text{FCD} b \) where \( \alpha \in A \) and \( b \) is an ultrafilter on \( B \).

**Proof.** First, it’s easy to see that \( \uparrow^A \{\alpha\} \times \text{FCD} b \) are elements of \( \text{ComplFCD}(A; B) \). Also \( \text{FCD}(A; B) \) is an element of \( \text{ComplFCD}(A; B) \).

\( \uparrow^A \{\alpha\} \times \text{FCD} b \) are atoms of \( \text{ComplFCD}(A; B) \) because these are atoms of \( \text{FCD}(A; B) \).

It remains to prove that if \( f \) is an atom of \( \text{ComplFCD}(A; B) \) then \( f = \uparrow^A \{\alpha\} \times \text{FCD} b \) for some \( \alpha \in A \) and an ultrafilter \( b \) on \( B \).

Suppose \( f \in \text{FCD}(A; B) \) is a non-empty complete funcoid. Then there exists \( \alpha \in A \) such that \( (f\ast \{\alpha\}) \neq 0(\beta) \). Thus \( \uparrow^A \{\alpha\} \times \text{FCD} b \subseteq f \) for some ultrafilter \( b \) on \( B \). If \( f \) is an atom then \( f = \uparrow^A \{\alpha\} \times \text{FCD} b \). \( \square \)

**Theorem 6.110.**

1. A funcoid \( f \) is complete iff there exists a function \( G: \text{Src } f \to X(\text{Dst } f) \) such that

\[
\begin{align*}
f &= \bigsqcup \{a^{\text{Src } f} \{\alpha\} \times \text{FCD} G(\alpha) \ | \ \alpha \in \text{Src } f\}. \\
\end{align*}
\]

(6.10)

2. A funcoid \( f \) is co-complete iff there exists a function \( G: \text{Dst } f \to X(\text{Src } f) \) such that

\[
\begin{align*}
f &= \bigsqcup \{G(\alpha) \times \text{FCD} \uparrow^\text{Dst } f \{\alpha\} \ | \ \alpha \in \text{Dst } f\}.
\end{align*}
\]

**Proof.** We will prove only the first as the second is symmetric.

\( \Rightarrow \). Let \( f \) be complete. Then take

\[
G(\alpha) = \bigsqcup \{b \in X(\text{Dst } f) \ | \ \uparrow^\text{Src } f \{\alpha\} \times \text{FCD} b \subseteq f\}
\]

and we have (6.10) obviously.

\( \Leftarrow \). Let (6.10) hold. Then \( G(\alpha) = \bigsqcup \text{atoms } G(\alpha) \) and thus

\[
f = \bigsqcup \{\uparrow^\text{Src } f \{\alpha\} \times \text{FCD} b \ | \ \alpha \in \text{Src } f, b \in \text{atoms } G(\alpha)\}
\]

and so \( f \) is complete. \( \square \)

**Theorem 6.111.**

1. For a complete funcoid \( f \) there exists exactly one function \( F \in X(\text{Dst } f^{\text{Src } f} \) such that

\[
f = \bigsqcup \{\uparrow^\text{Src } f \{\alpha\} \times \text{FCD } F(\alpha) \ | \ \alpha \in \text{Src } f\}.
\]
2. For a co-complete funcoid \( f \) there exists exactly one function \( F \in \mathcal{F}(\text{Src} f)^{\text{Dst} f} \) such that
\[
f = \bigsqcup \{ F(\alpha) \times ^{\text{FCD}}_{\text{Dst} f} \{ \alpha \} \mid \alpha \in \text{Dst} f \}.
\]

**Proof.** We will prove only the first as the second is similar. Let
\[
f = \bigsqcup \{ ^{\text{Src} f} \{ \alpha \} \times ^{\text{FCD}} F(\alpha) \mid \alpha \in \text{Src} f \} = \bigsqcup \{ ^{\text{Src} f} \{ \alpha \} \times ^{\text{FCD}} G(\alpha) \mid \alpha \in \text{Src} f \}
\]
for some \( F, G \in \mathcal{F}(\text{Dst} f)^{\text{Src} f} \). We need to prove \( F = G \). Let \( \beta \in \text{Src} f \).
\[
\langle f \rangle^* \{ \beta \} = \bigsqcup \{ ^{\text{Src} f} \{ \alpha \} \times ^{\text{FCD}} F(\alpha))^* \{ \beta \} \mid \alpha \in \text{Src} f \} = F(\beta).
\]
Similarly \( \langle f \rangle^* \{ \beta \} = G(\beta) \). So \( F(\beta) = G(\beta) \).

\[\square\]

### 6.12 Funcoids corresponding to pretopologies

Let \( \Delta \) be a pretopology on a set \( U \) and \( \text{cl} \) the preclosure corresponding to it (see theorem 5.12).

Both induce a funcoid, I will show that these two funcoids are reverse of each other:

**Theorem 6.112.** Let \( f \) be a complete funcoid defined by the formula \( \langle f \rangle^* \{ x \} = \Delta(x) \) for every \( x \in U \), let \( g \) be a co-complete funcoid defined by the formula \( \langle g \rangle^* X = \uparrow^U \text{cl}(X) \) for every \( X \in \mathcal{P}U \). Then \( g = f^{-1} \).

**Remark 6.113.** It is obvious that funcoids \( f \) and \( g \) exist.

**Proof.** \( X \downarrow^y X \iff \uparrow^U \downarrow^U \iff Y \neq \text{cl}(X) \iff \exists y \in Y : \Delta(y) \neq \uparrow^U \iff \exists y \in Y : \langle f \rangle^* \{ y \} = \uparrow^U \iff \langle f \rangle^* \{ y \} = \uparrow^U \iff Y \downarrow^y \iff Y \downarrow^y X \).

So \( g = f^{-1} \).

\[\square\]

### 6.13 Completion of funcoids

**Theorem 6.114.** \( \text{Cor} f = \text{Cor}' f \) for an element \( f \) of a filtrator of funcoids.

**Proof.** By theorems 4.34 and 6.104.

**Definition 6.115.** Completion of a funcoid \( f \in \text{FCD}(A; B) \) is the complete funcoid \( \text{Compl} f \in \text{FCD}(A; B) \) defined by the formula \( (\text{Compl} f)^* \{ \alpha \} = \langle f \rangle^* \{ \alpha \} \) for \( \alpha \in \text{Src} f \).

**Definition 6.116.** Co-completion of a funcoid \( f \) is defined by the formula
\[
\text{CoCompl} f = (\text{Compl} f^{-1})^{-1}.
\]

**Obvious 6.117.** \( \text{Compl} f \sqsubseteq f \) and \( \text{CoCompl} f \sqsubseteq f \).

**Proposition 6.118.** The filtrator \( \text{FCD}(A; B); \text{ComplFCD}(A; B) \) is filtered.

**Proof.** Because the filtrator of funcoids is filtered.

**Theorem 6.119.** \( \text{Compl} f = \text{Cor} \text{FCD}(A; B); \text{ComplFCD}(A; B) \) \( f = \text{Cor} \text{FCD}(A; B); \text{ComplFCD}(A; B) \) \( f \) for every funcoid \( f \in \text{FCD}(A; B) \).

**Proof.** \( \text{Cor} \text{FCD}(A; B); \text{ComplFCD}(A; B) \) \( f = \text{Cor} \text{FCD}(A; B); \text{ComplFCD}(A; B) \) \( f \) using theorem 4.34 since the filtrator \( \text{FCD}(A; B); \text{ComplFCD}(A; B) \) is filtered.

Let \( g \in \text{up} \text{FCD}(A; B); \text{ComplFCD}(A; B) \) \( f \). Then \( g \in \text{ComplFCD}(A; B) \) and \( g \sqsubseteq f \). Thus \( g = \text{Compl} g \sqsubseteq \text{Compl} f \).
Thus \( \forall g \in \text{up}(\text{ComplFCD}(A;B) ; \text{FCD}(A;B)) \) \( f; g \geq \text{Compl} \).

Let \( \forall g \in \text{up}(\text{ComplFCD}(A;B) ; \text{FCD}(A;B)) \) \( f; h \geq g \) for some \( h \in \text{ComplFCD}(A;B) \).

Then \( h \sqsubseteq \bigcup \text{up}(\text{ComplFCD}(A;B) ; \text{FCD}(A;B)) \) \( f = f \) and consequently \( h = \text{Compl} h \sqsubseteq \text{Compl} f \).

Thus

\[
\text{Compl} f = \bigcap \text{up}(\text{ComplFCD}(A;B) ; \text{FCD}(A;B)) f = \text{Cor}(\text{ComplFCD}(A;B) ; \text{FCD}(A;B)) f.
\]

**Theorem 6.120.** \( (\text{CoCompl} f)^* X = \text{Cor} (f)^* X \) for every funcoid \( f \) and set \( X \in \mathcal{P}(\text{Src} f) \).

**Proof.** CoCompl \( f \sqsubseteq f \) thus \( (\text{CoCompl} f)^* X \sqsubseteq (f)^* X \) but \( (\text{CoCompl} f)^* X \) is a principal filter thus \( (\text{CoCompl} f)^* X \sqsubseteq \text{Cor} (f)^* X \).

Let \( X = \text{Cor} (f)^* X \). Then \( \alpha \emptyset = 0 \text{(_{\text{Dest} f})} \) and

\[
\alpha (X \cup Y) = \text{Cor} (f)^* (X \cup Y) = \text{Cor} ((f)^* X \cup (f)^* Y) = \text{Cor} (f)^* X \cup \text{Cor} (f)^* Y = \alpha X \cup \alpha Y
\]

(used the theorem 4.223). Thus \( \alpha \) can be continued till \( \langle g \rangle \) for some funcoid \( g \). This funcoid is co-complete.

Evidently \( g \) is the greatest co-complete element of \( \text{FCD}(\text{Src} f; \text{Dest} f) \) which is lower than \( f \).

Thus \( g = \text{CoCompl} f \) and \( \text{Cor} (f)^* X = \alpha X = \langle g \rangle^* X = (\text{CoCompl} f)^* X \).

**Theorem 6.121.** \( \text{ComplFCD}(A;B) \) is an atomistic lattice.

**Proof.** Let \( f \in \text{ComplFCD}(A;B) \). \( (f)^* X = \bigcup \{ (f)^* \{x\} \mid x \in X \} = \bigcup \{ (f)^* \{x\} \mid x \in X \} = \bigcup \{ (f)^* \{x\} \mid x \in X \} \) is a join of atoms of \( \text{ComplFCD}(A;B) \).

**Theorem 6.122.** A funcoid \( f \) is complete iff it is a join (on the lattice \( \text{FCD}(A;B) \)) of atomic complete funcoids.

**Proof.** It follows from the theorem 6.102 and the previous theorem.

**Corollary 6.123.** \( \text{ComplFCD}(A;B) \) is join-closed.

**Theorem 6.124.** \( \text{Compl} \bigcup R = \bigcup \{ \text{Compl} R \} \) for every \( R \in \mathcal{P}(\text{FCD}(A;B)) \) (for every sets \( A, B \)).

**Proof.** \( (\text{Compl} \bigcup R)^* X = \bigcup \{ (\bigcup R)^* \{ \alpha \} \mid \alpha \in X \} = \bigcup \{ (f)^* \{ \alpha \} \mid f \in R \} \mid \alpha \in X \} = \bigcup \{ (f)^* \{ \alpha \} \mid \alpha \in X \} \mid f \in R \} = \bigcup \{ (\text{Compl} f)^* X \mid f \in R \} = \bigcup (\text{Compl} R)^* X \) for every set \( X \).

**Corollary 6.125.** \( \text{Compl} \) is a lower adjoint.

**Conjecture 6.126.** \( \text{Compl} \) is not an upper adjoint (in general).

**Proposition 6.127.** \( \text{Compl} f = \bigcup \{ f|_{\text{Src} f \{ \alpha \}} \mid \alpha \in \text{Src} f \} \) for every funcoid \( f \).

**Proof.** Let denote \( R \) the right part of the equality to prove.

\[
(R)^* \{ \beta \} = \bigcup \{ (f)^* \{ \alpha \} \mid \alpha \in \text{Src} f \} = (f)^* \{ \beta \}
\]

for every \( \beta \in \text{Src} f \) and \( R \) is complete as a join of complete funcoids.

Thus \( R \) is the completion of \( f \).

**Conjecture 6.128.** \( \text{Compl} f = f \downarrow \{ \Omega \times \text{FCD} \uparrow \} \) for every funcoid \( f \).

This conjecture may be proved by considerations similar to these in the section “Fréchet filter”.

**Lemma 6.129.** Co-completion of a complete funcoid is complete.

**Proof.** Let \( f \) be a complete funcoid.
\( (\text{CoCompl } f)^* X = \text{Cor } (f)^* X = \text{Cor } \bigsqcup \{(f)^*\{x\} | x \in X\} = \bigsqcup \{\text{Cor } (f)^*\{x\} | x \in X\} = \bigsqcup \{\text{CoCompl } f)^*\{x\} | x \in X\} \) for every set \( X \). Thus \( \text{CoCompl } f \) is complete. \( \square \)

**Theorem 6.130.** \( \text{Compl } \text{CoCompl } f = \text{CoCompl } \text{Compl } f = \text{Cor } f \) for every funcoid \( f \).

**Proof.** \( \text{Compl } \text{CoCompl } f \) is co-complete since (used the lemma) \( \text{CoCompl } f \) is co-complete. Thus \( \text{Compl } \text{CoCompl } f \) is a principal funcoid. \( \text{CoCompl } f \) is the greatest co-complete funcoid under \( f \) and \( \text{Compl } \text{CoCompl } f \) is the greatest complete funcoid under \( \text{CoCompl } f \). So \( \text{Compl } \text{CoCompl } f \) is greater than any principal funcoid under \( \text{CoCompl } f \) which is greater than any principal funcoid under \( f \). Thus \( \text{Compl } \text{CoCompl } f \) is the greatest principal funcoid under \( f \). Thus \( \text{Compl } \text{CoCompl } f = \text{Cor } f \). Similarly \( \text{CoCompl } \text{Compl } f = \text{Cor } f \). \( \square \)

**Question 6.131.** Is \( \text{Compl } \text{FCD}(A; B) \) a co-brouwerian lattice for every sets \( A, B \)?

### 6.13.1 More on completion of funcoids

**Proposition 6.132.** For every composable funcoids \( f \) and \( g \)

1. \( \text{Compl}(g \circ f) \supseteq \text{Compl } g \circ \text{Compl } f \);
2. \( \text{CoCompl}(g \circ f) \supseteq \text{CoCompl } g \circ \text{CoCompl } f \).

**Proof.**

1. \( \text{Compl } g \circ \text{Compl } f = \text{Compl } (\text{Compl } g \circ \text{Compl } f) \subseteq \text{Compl } (g \circ f) \).
2. By duality. \( \square \)

**Proposition 6.133.** For every composable funcoids \( f \) and \( g \)

1. \( \text{Compl}(g \circ f) = (\text{Compl } g) \circ f \) if \( f \) is a co-complete funcoid.
2. \( \text{CoCompl}(f \circ g) = f \circ \text{CoCompl } g \) if \( f \) is a complete funcoid.

**Proof.**

1. \( (\text{Compl } (g \circ f))^* X = \text{Cor } (g \circ f)^* X = \text{Cor } (g)(f)^* X = (\text{Compl } g)(f)^* X = ((\text{Compl } g) \circ f)^* X \) for every \( X \).
2. \( (\text{Compl } (g \circ f))^{-1} = f^{-1} \circ (\text{Compl } g)^{-1} \); \( \text{CoCompl } (g \circ f)^{-1} = f^{-1} \circ \text{CoCompl } g^{-1} \);
   \( \text{CoCompl } f^{-1} \circ g^{-1} = f^{-1} \circ \text{CoCompl } g^{-1} \). After variable replacement we get \( \text{CoCompl } (f \circ g) = f \circ \text{CoCompl } g \) (after the replacement \( f \) is a complete funcoid). \( \square \)

**Corollary 6.134.** \( \text{CoCompl } ((\text{Compl } g) \circ f) = \text{Compl } (\text{Compl } g \circ (\text{CoCompl } f)) = (\text{Compl } g) \circ (\text{CoCompl } f) \).

**Proof.** By the theorem:

\( \text{Compl } (g \circ (\text{CoCompl } f)) = (\text{Compl } g) \circ (\text{CoCompl } f) \);
\( \text{CoCompl } ((\text{Compl } f) \circ g) = (\text{CoCompl } f) \circ (\text{Compl } g) \).
After variable replacement \( \text{CoCompl } ((\text{Compl } g) \circ f) = (\text{Compl } g) \circ (\text{CoCompl } f) \). \( \square \)

**Proposition 6.135.** For every composable funcoids \( f \) and \( g \)

1. \( \text{Compl } (g \circ (\text{Compl } f)) = \text{Compl } (g \circ f) \);
2. \( \text{CoCompl } ((\text{CoCompl } g) \circ f) = \text{CoCompl } (g \circ f) \).

**Proof.**

1. \( (g \circ (\text{Compl } f))^* \{x\} = (g)(\text{Compl } f)^* \{x\} = (g)(f)^* \{x\} = (g \circ f)^* \{x\} \).
   Thus \( \text{Compl } (g \circ (\text{Compl } f)) = \text{Compl } (g \circ f) \).
2. \( (\text{Compl } (g \circ (\text{Compl } f)))^{-1} = (\text{Compl } (g \circ f))^{-1} \); \( \text{CoCompl } (g \circ (\text{Compl } f))^{-1} = \text{CoCompl } (g \circ f)^{-1} \);
   \( \text{CoCompl } ((\text{Compl } f)^{-1} \circ g^{-1}) = \text{CoCompl } (f^{-1} \circ g^{-1}) \); \( \text{CoCompl } ((\text{CoCompl } f^{-1}) \circ g^{-1}) = \text{CoCompl } (f^{-1} \circ g^{-1}) \). After variable replacement \( \text{CoCompl } ((\text{CoCompl } g) \circ f) = \text{CoCompl } (g \circ f) \). \( \square \)
### 6.13.1.1 Open maps

**Definition 6.136.** An open map from a topological space to a topological space is a function which maps open sets into open sets.

An obvious generalization of this is an open map \( f \) from an endofuncoid \( \mu \) to an endofuncoid \( \nu \), which is by definition a function (or rather a principal, entirely defined, monovalued funcoid) from \( \text{Ob} \mu \) to \( \text{Ob} \nu \) such that

\[
\forall x \in \text{Ob} \mu, V \in \langle \mu \rangle^* \{x\}: \langle f \rangle^* V \supseteq \langle \nu \rangle \langle f \rangle^* \{x\}.
\]

This formula is equivalent (exercise!) to

\[
\forall x \in \text{Ob} \mu: \langle f \rangle \langle \mu \rangle^* \{x\} \supseteq \langle \nu \rangle \langle f \rangle^* \{x\}.
\]

It can be abstracted/simplified further (now for an arbitrary funcoid \( f \) from \( \text{Ob} \mu \) to \( \text{Ob} \nu \)):

\[
\text{Compl}(f) \supseteq \text{Compl}(\nu) \circ f.
\]

**Definition 6.137.** An open funcoid from an endofuncoid \( \mu \) to an endofuncoid \( \nu \) is a funcoid \( f \) from \( \text{Ob} \mu \) to \( \text{Ob} \nu \) such that

\[
\text{Compl}(f) \supseteq \text{Compl}(\nu) \circ f.
\]

**Theorem 6.138.** Let \( \mu, \nu, \pi \) be endofuncoids. Let \( f \) be a co-complete open funcoid from \( \text{Ob} \mu \) to \( \text{Ob} \nu \) and \( g \) is an open funcoid from \( \text{Ob} \nu \) to \( \text{Ob} \pi \). Then \( g \circ f \) is an open funcoid from \( \text{Ob} \mu \) to \( \text{Ob} \pi \).

**Proof.** Let \( \text{Compl}(f) \supseteq \text{Compl}(\nu) \circ f \) and \( \text{Compl}(g \circ \nu) \supseteq \text{Compl}(\pi \circ g) \).

\[
\text{Compl}(g \circ f \circ \mu) \supseteq \text{Compl}(g \circ \text{Compl}(f \circ \mu)) \supseteq \text{Compl}(g \circ \text{Compl}(\nu) \circ f) = \text{Compl}(g \circ \text{Compl}(\nu)) \circ f = \text{Compl}(g \circ \nu) \circ f \supseteq \text{Compl}(\pi \circ g) \circ f = \text{Compl}(\pi \circ g \circ f).
\]

**Obvious 6.139.** A funcoid \( f \) is open iff \( f \circ \mu \supseteq \text{Compl}(\nu) \circ f \).

**Corollary 6.140.** A co-complete funcoid \( f \) is open iff \( f \circ \mu \supseteq (\text{Compl}(\nu) \circ f) \). Thus \( f \) is open iff it is a continuous morphism from \( \mu \) to \( \text{Compl} \nu \) with the reverse order of funcoids. (See a definition of a continuous morphism below.)

### 6.14 Monovalued and injective funcoids

Following the idea of definition of monovalued morphism let’s call monovalued such a funcoid \( f \) that \( f \circ f^{-1} \subseteq \text{id}_{\text{dom} f}^{\text{FCD}} \).

Similarly, I will call a funcoid injective when \( f^{-1} \circ f \subseteq \text{id}_{\text{dom} f}^{\text{FCD}} \).

**Obvious 6.141.** A funcoid \( f \) is:

- monovalued iff \( f \circ f^{-1} \subseteq \text{id}^{\text{FCD}}(\text{Dom} f) \);
- injective iff \( f^{-1} \circ f \subseteq \text{id}^{\text{FCD}}(\text{Src} f) \).

In other words, a funcoid is monovalued (injective) when it is a monovalued (injective) morphism of the category of funcoids. Monovaluedness is dual of injectivity.

**Obvious 6.142.**

1. A morphism \( (A; B; f) \) of the category of funcoid triples is monovalued iff the funcoid \( f \) is monovalued.
2. A morphism \( (A; B; f) \) of the category of funcoid triples is injective iff the funcoid \( f \) is injective.

**Theorem 6.143.** The following statements are equivalent for a funcoid \( f \):

1. \( f \) is monovalued.
2. \( \forall a \in \text{atoms}^3(\text{Src } f); \langle f \rangle a = \text{atoms}^3(\text{Dst } f) \cup \{0^3(\text{Dst } f)\} \).
3. \( \forall I, J \in \mathcal{G}(\text{Dst } f); \langle f^{-1} \rangle (I \cap J) = (f^{-1})I \cap (f^{-1})J \).
4. \( \forall I, J \in \mathcal{P}(\text{Dst } f); \langle f^{-1} \rangle^* (I \cap J) = (f^{-1})^* I \cap (f^{-1})^* J \).

**Proof.**

(2) \( \Rightarrow (3) \). Let \( a \in \text{atoms}^3(\text{Src } f), \langle f \rangle a = b \). Then because \( b \in \text{atoms}^3(\text{Dst } f) \cup \{0^3(\text{Dst } f)\} \)

\[
(I \cap J) \cap b \neq 0^3(\text{Dst } f) \Leftrightarrow I \cap b \neq 0^3(\text{Dst } f) \land J \cap b \neq 0^3(\text{Dst } f);
\]

so

\[
a \cup \langle f^{-1} \rangle (I \cap J) \neq 0^3(\text{Src } f) \Leftrightarrow a \cup \langle f^{-1} \rangle I \neq 0^3(\text{Src } f) \land a \cup \langle f^{-1} \rangle J \neq 0^3(\text{Src } f);
\]

or

\[
\langle f^{-1} \rangle (I \cap J) = (f^{-1})I \cap (f^{-1})J.
\]

(3) \( \Rightarrow (1) \). \( \langle f^{-1} \rangle a \cap \langle f^{-1} \rangle b = \langle f^{-1} \rangle (a \cap b) = \langle f^{-1} \rangle (0^3(\text{Dst } f) = 0^3(\text{Src } f) \) for every two distinct ultrafilters \( a \) and \( b \) on \( \text{Dst } f \). This is equivalent to \( \neg((f^{-1})a \cup f^{-1}b) \Leftrightarrow (f^{-1})a \cup f^{-1}b = (f^{-1})a \land (f^{-1})b = (f^{-1})a \land (f^{-1})b \). So \( f \circ (f^{-1})b \Rightarrow a = b \) for every ultrafilters \( a \) and \( b \). This is possible only then \( f \circ f^{-1} \subseteq \text{id}_{\text{func}(\text{Dst } f)} \).

(4) \( \Rightarrow (3) \). Obvious.

\( \neg(2) \Rightarrow \neg(1) \). Suppose \( \langle f \rangle a \notin \text{atoms}^3(\text{Dst } f) \cup \{0^3(\text{Dst } f)\} \) for some \( a \in \text{atoms}^3(\text{Src } f) \). Then there exist two atomic filters \( p \) and \( q \) on \( \text{Dst } f \) such that \( p \neq q \) and \( \langle f \rangle a = p \cup \langle f \rangle a = q \). Consequently \( p = (f^{-1})p \cup (f^{-1})p = (f^{-1})p \cup (f^{-1})p = (f^{-1})p \cup (f^{-1})p = (f^{-1})p \cup (f^{-1})p \neq 0^3(\text{Dst } f) \). So it cannot be \( f \circ f^{-1} \subseteq \text{id}_{\text{func}(\text{Dst } f)} \).

**Corollary 6.144.** A binary relation corresponds to a monovalued funcoid if and only if it is a function.

**Proof.** Because \( \forall I, J \in \mathcal{P}(\text{im } f); \langle f^{-1} \rangle^* (I \cap J) = (f^{-1})^* I \cap (f^{-1})^* J \) is true for a funcoid \( f \) corresponding to a binary relation if and only if it is a function.

**Remark 6.145.** This corollary can be reformulated as follows: For binary relations (principal funcoids) the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a funcoid are the same.

**Proposition 6.146.** Every monovalued funcoid is metamovalued.

**Proof.** \( \langle (\bigcap G) \circ f \rangle x = (\bigcap G) \langle f \rangle x = \bigcap_{g \in G} \langle g \rangle \langle f \rangle x = \bigcap_{g \in G} \langle g \circ f \rangle x = \bigcap_{g \in G} \langle g \circ f \rangle x \) for every ultrafilter \( x \in \text{atoms}^3(\text{Src } f) \). Thus \( \langle (\bigcap G) \circ f \rangle = \bigcap_{g \in G} \langle g \circ f \rangle \).

**Corollary 6.147.** Every injective funcoid is metainjective.

**Conjecture 6.148.** Every metamovalued funcoid is monovalued.

### 6.15 \( T_0, T_1, T_2, \) and \( T_3 \)-separable funcoids

For funcoids it can be generalized \( T_0, T_1, T_2, \) and \( T_3 \)-separability. Whilethout prove \( T_0 \) and \( T_2 \) separability is defined through \( T_1 \) separability.

**Definition 6.149.** Let call \( T_1 \)-separable such endofuncoid \( f \) that for every \( \alpha, \beta \in \text{Ob } f \) is true

\[
\alpha \neq \beta \Rightarrow \neg(\{\alpha\} [f^* \{\beta\}]).
\]
Proposition 6.150. An endofuncoid \( f \) is \( T_1 \)-separable iff \( \text{Cor} f \subseteq \text{id}^{\text{FCD}(\text{Ob} f)} \).

Proof. \( \text{Cor} f \subseteq \text{id}^{\text{FCD}(\text{Ob} f)} \iff \forall x, y \in \text{Ob} f: \{(x, y) \mid (\text{Cor} f)^*(y) \Rightarrow x = y\} \iff \text{Cor} f \subseteq \text{id}^{\text{FCD}(\text{Ob} f)} \). \( \square \)

Definition 6.151. Let call \( T_0 \)-separable such funcoid \( f \in \text{FCD}(A; A) \) that \( f \cap f^{-1} \) is \( T_1 \)-separable.

Definition 6.152. Let call \( T_2 \)-separable such funcoid \( f \) that \( f^{-1} \circ f \) is \( T_1 \)-separable.

For symmetric transitive funcoids \( T_1 \)- and \( T_2 \)-separability are the same (see theorem 3.51).

Obvious 6.153. A funcoid \( f \) is \( T_2 \)-separable iff \( \forall \alpha \neq \beta \Rightarrow (f)^*(\{\alpha\}) \neq (f)^*(\{\beta\}) \) for every \( \alpha, \beta \in \text{Src} f \).

Definition 6.154. Regular funcoid is an endofuncoid \( f \) such that \( (f)(f^{-1}) C \succeq (f)^*(p) \iff p \notin C \) for every \( p \in \text{Ob} f \) and \( C \in \mathcal{P} \text{Ob} f \).

Obvious 6.155. Funcoid \( f \) is regular iff:
1. \( (f \circ f^{-1})^* C \succeq (f)^*(p) \iff p \notin C \);
2. \( (f^{-1} \circ f \circ f^{-1})^* C \supseteq \text{Ob} f(p) \iff p \notin C \);
3. \( (f^{-1} \circ f \circ f^{-1}) C \subseteq \text{Ob} f(C) \);
4. \( f^{-1} \circ f \circ f^{-1} \subseteq \text{id}^{\text{FCD}(\text{Ob} f)} \).

Definition 6.156. An endofuncoid is \( T_3 \) iff it is both \( T_2 \) and regular.

6.16 Filters closed regarding a funcoid

Definition 6.157. Let’s call closed regarding a funcoid \( f \in \text{FCD}(A; A) \) such filter \( A \in \mathfrak{F}(\text{Src} f) \) that \( \langle f \rangle A \subseteq A \).

This is a generalization of closedness of a set regarding an unary operation.

Proposition 6.158. If \( \mathcal{I} \) and \( \mathcal{J} \) are closed (regarding some funcoid \( f \)), \( S \) is a set of closed filters on \( \text{Src} f \), then
1. \( \mathcal{I} \cup \mathcal{J} \) is a closed filter;
2. \( \bigcap S \) is a closed filter.

Proof. Let denote the given funcoid as \( f \). \( \langle f \rangle(\mathcal{I} \cup \mathcal{J}) = \langle f \rangle \mathcal{I} \cup \langle f \rangle \mathcal{J} \subseteq \mathcal{I} \cup \mathcal{J} \). \( \langle f \rangle \bigcap S \subseteq \bigcap \langle (f) \rangle S \subseteq \bigcap S \). Consequently the filters \( \mathcal{I} \cup \mathcal{J} \) and \( \bigcap S \) are closed. \( \square \)

Proposition 6.159. If \( S \) is a set of filters closed regarding a complete funcoid, then the filter \( \bigcup S \) is also closed regarding our funcoid.

Proof. \( \langle f \rangle \bigcup S = \bigcup \langle (f) \rangle S \subseteq \bigcup S \) where \( f \) is the given funcoid. \( \square \)
Chapter 7
Reloids

7.1 Basic definitions

Definition 7.1. I call a reloid from a set $A$ to a set $B$ a triple $(A; B; F)$ where $F \in \mathfrak{F}(A \times B)$.

Definition 7.2. Source and destination of every reloid $(A; B; F)$ are defined as

$$\text{Src}(A; B; F) = A \quad \text{and} \quad \text{Dst}(A; B; F) = B.$$ 

I will denote $\text{RLD}(A; B)$ the set of reloids from $A$ to $B$. I will denote $\text{RLD}$ the set of all reloids (for small sets).

Definition 7.3. $\text{GR}(A; B; F) \overset{\text{def}}{=} \{ (A; B; K) \mid K \in F \}$ for every reloid $(A; B; F)$. Note that $xy\text{GR}(A; B; F)$ is a set of morphisms of the category Rel.

Definition 7.4.
- $\uparrow_{\text{RLD}}(A; B; f) \overset{\text{def}}{=} (A; B; \uparrow A \times B f)$ for every relation $f \in \mathcal{P}(A \times B)$.
- $\uparrow_{\text{RLD}} f = (\text{Src} f; \text{Dst} f; \uparrow \text{Src} f \times \text{Dst} f \text{GR} f)$ for every Rel-morphism $f$.

Definition 7.5. I call members of a set $\uparrow_{\text{RLD}} \text{Rel}(A; B)$ as principal reloids.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations.

Definition 7.6. The reverse reloid of a reloid $f$ is defined by the formula

$$(A; B; F)^{-1} = (B; A; \{ K^{-1} \mid K \in F \}).$$

Note 7.7. The reverse reloid is not an inverse in the sense of group theory or category theory.

Reverse reloid is a generalization of conjugate quasi-uniformity.

Definition 7.8. Every set $\text{RLD}(A; B)$ is a poset by the formula $f \subseteq g \Leftrightarrow \text{GR} f \subseteq \text{GR} g$. We will apply lattice operations to subsets of $\text{RLD}(A; B)$ without explicitly mentioning $\text{RLD}(A; B)$.

Obvious 7.9. The poset $\text{RLD}(A; B)$ is isomorphic to the poset $\mathfrak{F}(A \times B)$ for every sets $A$, $B$.

7.2 Composition of reloids

Definition 7.10. Reloids $f$ and $g$ are composable when $\text{Dst} f = \text{Src} g$.

Definition 7.11. Composition of (composable) reloids is defined by the formula

$g \circ f = \bigsqcup \{ \uparrow_{\text{RLD}} (G \circ F) \mid F \in \text{xyGR} f, G \in \text{xyGR} g \}.$

Obvious 7.12. Composition of reloids is a reloid.
**Theorem 7.13.** $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable reloids $f$, $g$, $h$.

**Proof.** For two nonempty collections $A$ and $B$ of sets $I$ will denote

$$A \sim B \iff \forall K \in A \exists L \in B : L \subseteq K \land \forall K' \in B \exists L \in A : L \subseteq K.$$ 

It is easy to see that $\sim$ is a transitive relation.

Let first prove that for every nonempty collection of relations $A$, $B$, $C$

$$A \sim B \iff A \circ C \sim B \circ C.$$ 

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B : K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for every $P \in A \circ C$ there exists $P' \in B \circ C$ such that $P' \subseteq P$; the vice versa is analogous. So $A \circ C \sim B \circ C$.

GR((h $\circ$ g) $\circ$ f) $\sim$ GR(h $\circ$ g) $\circ$ GR f, GR(h $\circ$ g) $\sim$ (GR h) $\circ$ (GR g). By proven above GR((h $\circ$ g) $\circ$ f) $\sim$ (GR h) $\circ$ (GR g) $\circ$ (GR f).

Analogously GR(h $\circ$ (g $\circ$ f)) $\sim$ (GR h) $\circ$ (GR g) $\circ$ (GR f).

So GR(h $\circ$ (g $\circ$ f)) $\sim$ GR((h $\circ$ g) $\circ$ f) what is possible only if GR(h $\circ$ (g $\circ$ f)) = GR((h $\circ$ g) $\circ$ f).

Thus $(h \circ g) \circ f = h \circ (g \circ f)$.

**Theorem 7.14.** For every reloid $f$:

1. $f \circ f = \{\{F \circ (F \circ F) \mid F \in \text{xyGR} f\} \mid \text{Src} f = \text{Dst} f\}$
2. $f^{-1} \circ f = \{\{F \circ (F^{-1} \circ F) \mid F \in \text{xyGR} f\} \mid \text{Src} f = \text{Dst} f\}$
3. $f \circ f^{-1} = \{\{F \circ (F \circ F^{-1}) \mid F \in \text{xyGR} f\} \mid \text{Src} f = \text{Dst} f\}$

**Proof.** I will prove only (1) and (2) because (3) is analogous to (2).

1. It’s enough to show that $\forall F, G \in \text{xyGR} f \exists H \in \text{xyGR} f : H \circ H \subseteq G \circ F$. To prove it take $H = F \cap G$.

2. It’s enough to show that $\forall F, G \in \text{xyGR} f \exists H \in \text{xyGR} f : H^{-1} \circ H \subseteq F \cap G \circ F$. To prove it take $H = F \cap G$. Then $H^{-1} \circ H = (F \cap G)^{-1} \circ (F \cap G) \subseteq F \cap G$.

**Theorem 7.15.** For every sets $A$, $B$, $C$ if $g, h \in \text{RLD}(A; B)$ then

1. $f \circ (g \circ h) = f \circ g \cup f \circ h$ for every $f \in \text{RLD}(B; C)$;
2. $(g \circ h) \circ f = g \circ f \circ h \circ f$ for every $f \in \text{RLD}(C; A)$.

**Proof.** We’ll prove only the first as the second is dual.

By the infinite distributivity law for filters we have

$$f \circ g \cup f \circ h = \bigcap \{\{F \circ (F \circ G) \mid F \in \text{xyGR} f, G \in \text{xyGR} g\} \cup \{F \circ H \mid F \in \text{xyGR} f, H \in \text{xyGR} h\} \mid F \in \text{xyGR} f, G \in \text{xyGR} g, H \in \text{xyGR} h\}$$

Obviously

$$\bigcap \{\{F \circ (F \circ G) \mid F \in \text{xyGR} f, G \in \text{xyGR} g, H \in \text{xyGR} h\} \cup \{F \circ H \mid F \in \text{xyGR} f, H \in \text{xyGR} h\} \mid F \in \text{xyGR} f, G \in \text{xyGR} g, H \in \text{xyGR} h\} = \bigcap \{F \circ (G \cup H) \mid F \in \text{xyGR} f, G \in \text{xyGR} g, H \in \text{xyGR} h\}.$$

Because $G \in \text{xyGR} g \land H \in \text{xyGR} h \Rightarrow G \cup H \in \text{xyGR} (g \cup h)$ we have

$$\bigcap \{F \circ (G \cup H) \mid F \in \text{xyGR} f, G \in \text{xyGR} g, H \in \text{xyGR} h\}$$

Because $G \in \text{xyGR} g \land H \in \text{xyGR} h \Rightarrow G \cup H \in \text{xyGR} (g \cup h)$ we have

$$\bigcap \{F \circ (G \cup H) \mid F \in \text{xyGR} f, G \in \text{xyGR} g, H \in \text{xyGR} h\} \cup \{F \circ K \mid F \in \text{xyGR} f, K \in \text{xyGR} (g \cup h)\} = f \circ (g \cup h).$$
Thus we have proved \( f \circ g \cup f \circ h \supseteq f \circ (g \cup h) \). But obviously \( f \circ (g \cup h) \supseteq f \circ g \) and \( f \circ (g \cup h) \supseteq f \circ h \) and so \( f \circ (g \cup h) \supseteq f \circ g \cup f \circ h \). Thus \( f \circ (g \cup h) = f \circ g \cup f \circ h \). \( \square \)

**Theorem 7.16.** Let \( A, B, C \) be sets, \( f \in RLD(A; B) \), \( g \in RLD(B; C) \), \( h \in RLD(A; C) \). Then

\[
g \circ f \neq h \Leftrightarrow g \neq h \circ f^{-1}.
\]

**Proof.** \( g \circ f \neq h \Leftrightarrow \bigcap f^{RLD}(G \circ f) \subseteq x \in RLD(\text{Src } f; \text{Dst } g) \). Thus we have proved

\[
7.3 \text{ Direct product of filters}
\]

**Theorem 7.17.** For every composable reloids \( f \) and \( g \)

1. \( g \circ f = \bigsqcup \{ g \circ f \mid F \in \text{atoms } f \} \).
2. \( g \circ f = \bigsqcup \{ G \circ f \mid G \in \text{atoms } g \} \).

**Proof.** We will prove only the first as the second is dual.

\[
\bigsqcup \{ g \circ f \mid F \in \text{atoms } f \} = g \circ f \Leftrightarrow \forall x \in RLD(\text{Src } f; \text{Dst } g) \): (\( x \neq g \circ f \Leftrightarrow x \neq \bigcup \{ g \circ f \mid F \in \text{atoms } f \} \)) \Leftrightarrow \forall x \in RLD(\text{Src } f; \text{Dst } g) \): (\( g^{-1} \circ x \neq f \Leftrightarrow \exists F \in \text{atoms } f \): (\( g^{-1} \circ x \neq F \)) what is obviously true.

**Corollary 7.18.** If \( f \) and \( g \) are composable reloids, then

\[
g \circ f = \bigsqcup \{ G \circ f \mid F \in \text{atoms } f, G \in \text{atoms } g \}.
\]

**Proof.** \( g \circ f = \bigsqcup \{ g \circ f \mid F \in \text{atoms } f \} = \bigsqcup \{ G \circ f \mid G \in \text{atoms } g \} \mid F \in \text{atoms } f \) = \( \bigsqcup \{ G \circ f \mid F \in \text{atoms } f, G \in \text{atoms } g \} \). \( \square \)

### 7.3 Direct product of filters

**Definition 7.19.** *Reloidal product* of filters \( A \) and \( B \) is defined by the formula

\[
\text{Def. } A \times \text{RLD } B = \bigcap \{ f^{RLD}(A \times B) \mid A \in \mathcal{A}, B \in \mathcal{B} \}.
\]

**Obvious 7.20.** \( \uparrow A \times \text{RLD } U \uparrow B = \downarrow \text{RLD } U \downarrow V \) for every sets \( A \subseteq U, B \subseteq V \).

**Theorem 7.21.** \( A \times \text{RLD } B = \bigsqcup \{ a \times \text{RLD } b \mid a \in \text{atoms } A, b \in \text{atoms } B \} \) for every filters \( A, B \).

**Proof.** Obviously

\[
A \times \text{RLD } B \supseteq \bigsqcup \{ a \times \text{RLD } b \mid a \in \text{atoms } A, b \in \text{atoms } B \}.
\]

Reversely, let

\[
K \in \text{GR} \bigcup \{ a \times \text{RLD } b \mid a \in \text{atoms } A, b \in \text{atoms } B \}.
\]

Then \( K \in \text{GR}(a \times \text{RLD } b) \) for every \( a \in \text{atoms } A, b \in \text{atoms } B \); \( K \supseteq X_a \times Y_b \) for some \( X_a \in A, Y_b \in B \);\( K \supseteq \bigcup \{ X_a \times Y_b \mid a \in \text{atoms } A, b \in \text{atoms } B \} = \bigcup \{ X_a \mid a \in \text{atoms } A \} \times \bigcup \{ Y_b \mid b \in \text{atoms } B \} \supseteq A \times B \)

where \( A \in \mathcal{A}, B \in \mathcal{B} \); \( K \in \text{GR}(A \times \text{RLD } B) \). \( \square \)

**Theorem 7.22.** If \( A_0, A_1 \in \mathcal{F}(A), B_0, B_1 \in \mathcal{F}(B) \) for some sets \( A, B \) then

\[
(A_0 \times \text{RLD } B_0) \cap (A_1 \times \text{RLD } B_1) = (A_0 \cap A_1) \times \text{RLD } (B_0 \cap B_1).
\]
Proposition 7.30. Restricting reloid to a filter. Domain and image

Corollary 7.24. If \( S \in \mathcal{P}(\mathfrak{F}(A) \times \mathfrak{F}(B)) \) for some sets \( A, B \) then
\[
\bigcap \{ A \times \mathsf{RLD} \mathcal{B} \mid (A; \mathcal{B}) \in S \} = \bigcap \mathsf{dom} \mathcal{S} \times \mathsf{RLD} \mathcal{P} \bigcap \mathsf{im} \mathcal{S}.
\]

Theorem 7.23. Let \( \mathcal{P} = \bigcap \mathsf{dom} \mathcal{S}, \mathcal{Q} = \bigcap \mathsf{im} \mathcal{S} \), \( l = \bigcap \{ A \times \mathsf{RLD} \mathcal{B} \mid (A; \mathcal{B}) \in S \} \).

Proof. Let \( \mathcal{P} = \bigcap \mathsf{dom} \mathcal{S}, \mathcal{Q} = \bigcap \mathsf{im} \mathcal{S} \), \( l = \bigcap \{ A \times \mathsf{RLD} \mathcal{B} \mid (A; \mathcal{B}) \in S \} \).

Let \( \mathcal{P} \times \mathsf{RLD} \mathcal{Q} \subseteq l \) is obvious.

Corollary 7.24. If \( A \) is a filter and \( T \) is a set of filters with common base.

Proof. Take \( S = \{ A \} \times T \) where \( T \) is a set of filters.

Then \( \bigcap \{ A \times \mathsf{RLD} \mathcal{B} \mid \mathcal{B} \in T \} = A \times \mathsf{RLD} \mathcal{P} \bigcap T \) if \( A \) is a filter.

\[ \square \]

Definition 7.25. I will call a reloid convex if it is a join of direct products.

7.4 Restricting reloid to a filter. Domain and image

Definition 7.26. Identity reloid for a set \( A \) is defined by the formula \( \mathsf{id}^{\mathsf{RLD}(A)} = \uparrow^{\mathsf{RLD}(A;A)} \mathsf{id}_A^A \).

Obvious 7.27. \( (\mathsf{id}^{\mathsf{RLD}(A)})^{-1} = \mathsf{id}^{\mathsf{RLD}(A)} \).

Definition 7.28. I call restricting a reloid \( f \) to a filter \( A \) as \( f \mid_A = f \cap (A \times \mathsf{RLD} \mathfrak{B}(\mathsf{Src} f)) \).

Definition 7.29. Domain and image of a reloid \( f \) are defined as follows:
\[
\mathsf{dom} f = \bigcap (\langle \mathsf{Src} f \rangle \mathcal{G} f); \quad \mathsf{im} f = \bigcap (\langle \mathsf{Dst} f \rangle \mathcal{G} f).
\]

Proposition 7.30. \( f \subseteq A \times \mathsf{RLD} \mathcal{B} \iff \mathsf{dom} f \subseteq A \land \mathsf{im} f \subseteq B \) for every reloid \( f \) and filters \( A \in \mathfrak{B}(\mathsf{Src} f), B \in \mathfrak{B}(\mathsf{Dst} f) \).

Proof. \( \Rightarrow \). It follows from \( \mathsf{dom}(A \times \mathsf{RLD} \mathcal{B}) \subseteq A \land \mathsf{im}(A \times \mathsf{RLD} \mathcal{B}) \subseteq B \).

\( \Leftarrow \). \( \mathsf{dom} f \subseteq A \Rightarrow \forall A \in A \exists F \in \mathcal{G} f; \mathsf{dom} F \subseteq A \). Analogously \( \mathsf{im} f \subseteq B \Rightarrow \forall B \in \mathcal{G} f; \mathsf{im} F \subseteq B \).

Let \( \mathsf{dom} f \subseteq A \land \mathsf{im} f \subseteq B \), \( A \in A, B \in B \). Then there exist \( F, G \in \mathcal{G} f \) such that \( \mathsf{dom} F \subseteq A \land \mathsf{im} G \subseteq B \). Consequently \( \mathsf{F} \cap \mathsf{G} \in \mathcal{G} f \), \( \mathsf{dom}(\mathsf{F} \cap \mathsf{G}) \subseteq A, \mathsf{im}(\mathsf{F} \cap \mathsf{G}) \subseteq B \) that is \( \mathsf{F} \cap \mathsf{G} \subseteq A \times B \). So there exists \( H \in \mathcal{G} f \) such that \( H \subseteq A \times B \) for every \( A \in A, B \in B \). So \( f \subseteq A \times \mathsf{RLD} \mathcal{B} \).

\[ \square \]

Definition 7.31. I call restricted identity reloid for a filter \( A \) the reloid
\[ \mathsf{id}^{\mathsf{RLD}(A)} = (\mathsf{id}^{\mathsf{RLD}(\mathsf{Base}(A))}) \mid_A. \]
Theorem 7.35. \( \text{id}^{\text{RLD}}_A = \bigcap \{ \text{id}^{\text{RLD}}(\text{Base}(A); \text{Base}(A)) \text{id}_A | A \in A \} \) for every filter \( A \).

Proof. Let \( K \in \text{GR} \bigcap \{ \text{id}^{\text{RLD}}(\text{Base}(A); \text{Base}(A)) : \text{id}_A | A \in A \} \), then there exists \( A \in A \) such that \( K \supseteq \text{id}_A \). Then

\[
\text{id}^{\text{RLD}}_A \subseteq \text{id}^{\text{RLD}}(\text{Base}(A); \text{Base}(A)) \cdot \text{id}_A | A \in A \}
\]

Thus \( K \in \text{GR} \text{id}^{\text{RLD}}_A \).

Reversely let \( K \in \text{GR} \text{id}^{\text{RLD}}_A = \text{GR}(\text{id}^{\text{RLD}}(\text{Base}(A); \text{Base}(A)) | (\text{id}^{\text{RLD}}(\text{Base}(A); \text{Base}(A))(A \times \text{Base}(A)) = \text{id}^{\text{RLD}}(\text{Base}(A); \text{Base}(A)) \text{id}_A \supseteq \text{id}^{\text{RLD}}(\text{Base}(A); \text{Base}(A)) \text{id}_A \}
\]

Thus \( K \in \text{GR} \text{id}^{\text{RLD}}_A \).

Corollary 7.33. \( (\text{id}^{\text{RLD}}_A)^{-1} = \text{id}^{\text{RLD}}_A \).

Theorem 7.34. \( \text{id}^{\text{RLD}}(\text{Base}(A); \text{Base}(A)) \text{id}_A = \text{id}^{\text{RLD}}_A \) for every reloid \( f \) and \( A \in \mathfrak{F}(\text{Src}(f)) \).

Proof. We need to prove that \( f \cap (A \times \text{RLD} \mathfrak{F}(\text{Src}(f))) = f \cap \{ \text{id}^{\text{RLD}}(\text{Src}(f) ; \text{Src}(f)) \text{id}_A | A \in A \} \).

We have \( f \circ \{ \text{id}^{\text{RLD}}(\text{Src}(f) ; \text{Src}(f)) \text{id}_A | A \in A \} = \bigcap \{ \text{id}^{\text{RLD}}(\text{Src}(f) \times \text{RLD} f)(f \cap \text{id}_A) | F \in \text{GR} f, A \in A \} \bigcap \{ \text{id}^{\text{RLD}}(\text{Src}(f) \times \text{RLD} f)(A \times \text{Dust}(f)) | F \in \text{GR} f, A \in A \} = \bigcap \{ \text{id}^{\text{RLD}}(\text{Src}(f) \times \text{RLD} f)(f \cap (A \times \text{Dust}(f))) | F \in \text{GR} f, A \in A \} = f \cap (A \times \text{RLD} \mathfrak{F}(\text{Src}(f))). \]

Theorem 7.35. \( (g \circ f)|_A = g \circ (f|_A) \) for every composable reloids \( f \) and \( g \) and \( A \in \mathfrak{F}(\text{Src}(f)) \).

Proof. \( (g \circ f)|_A = (g \circ f) \text{id}^{\text{RLD}}_A = g \circ (f \circ \text{id}^{\text{RLD}}_A) = g \circ (f|_A). \)

Theorem 7.36. \( \cap (A \times \text{RLD} B) = \text{id}^{\text{RLD}}_A \circ f \circ \text{id}^{\text{RLD}}_A \) for every reloid \( f \) and \( A \in \mathfrak{F}(\text{Src}(f)), B \in \mathfrak{F}(\text{Dst}(f)). \)

Proof. \( f \cap (A \times \text{RLD} B) = f \cap (A \times \text{RLD} \mathfrak{F}(\text{Src}(f))) \cap (\mathfrak{F}(\text{Src}(f) \times \text{RLD} B) = f|_A \cap (\mathfrak{F}(\text{Src}(f) \times \text{RLD} B) = (f \circ \text{id}^{\text{RLD}}_A) \cap (\mathfrak{F}(\text{Src}(f) \times \text{RLD} B)) = (f \circ \text{id}^{\text{RLD}}_A)^{-1} \cap (\mathfrak{F}(\text{Src}(f) \times \text{RLD} B)^{-1} = \text{id}^{\text{RLD}}_A \circ f \circ \text{id}^{\text{RLD}}_A. \)

Theorem 7.37. \( f|_{\text{Src}(\alpha)} = \text{RLD} \text{im}(f|_{\text{Src}(\alpha)}) \) for every reloid \( f \) and \( \alpha \in \text{Src}(f). \)

Proof. First,

\[
\text{im}(f|_{\text{Src}(\alpha)}) = \bigcap \{ \text{GR} f|_{\text{Src}(\alpha)} \} \bigcap \{ \text{GR} f \cap (\{ \alpha \} \times \text{Dust}(f)) | F \in \text{GR} f \} = \bigcap \{ \text{GR} f \cap (\{ \alpha \} \times \text{Dust}(f)) | F \in \text{GR} f \}.
\]

Taking this into account we have:

\[
\text{RLD} \text{im}(f|_{\text{Src}(\alpha)}) = \bigcap \{ \text{RLD}(\text{Src}(f \times \text{Dust}) f)((\{ \alpha \} \times K) | K \in \text{im}(f|_{\text{Src}(\alpha)}) \} = \bigcap \{ \text{RLD}(\text{Src}(f \times \text{Dust}) f)((\{ \alpha \} \times \text{im}(f|_{\alpha})) | F \in \text{GR} f \} = \bigcap \{ \text{RLD}(\text{Src}(f \times \text{Dust}) f)((\{ \alpha \} \times \text{im}(f|_{\alpha})) | F \in \text{GR} f \} = \bigcap \{ \text{RLD}(\text{Src}(f \times \text{Dust}) f) | F \in \text{GR} f \} \cap \text{RLD}(\text{Src}(f \times \text{Dust}) f)((\{ \alpha \} \times \text{Dust} f) = f|_{\text{Src}(\alpha)}. \]
Lemma 7.38. \( \lambda \mathcal{B} \in \mathfrak{B}(B): 1^3 \times^{\text{ECF}} \mathcal{B} \) is an upper adjoint of \( \lambda f \in \text{RLD}(A; B): \text{im } f \) (for every sets \( A, B \)).

Proof. We need to prove \( \text{im } f \subseteq B \Rightarrow f \subseteq 1^3 \times^{\text{RLD}} \mathcal{B} \) what is obvious. \( \square \)


Proof. By properties of Galois connections and duality. \( \square \)

### 7.5 Categories of reloids

I will define two categories, the category of reloids and the category of reloid triples. The category of reloids is defined as follows:

- Objects are small sets.
- The set of morphisms from a set \( A \) to a set \( B \) is \( \text{RLD}(A; B) \).
- The composition is the composition of reloids.
- Identity morphism for a set is the identity reloid for that set.

To show it is really a category is trivial.

The category of reloid triples is defined as follows:

- Objects are filters on small sets.
- The morphisms from a filter \( A \) to a filter \( B \) are triples \( (A; B; f) \) where \( f \in \text{RLD}(\text{Base}(A); \text{Base}(B)) \) and \( \text{dom } f \subseteq A, \text{im } f \subseteq B \).
- The composition is defined by the formula \( (B; C; g) \circ (A; B; f) = (A; C; g \circ f) \).
- Identity morphism for a filter \( A \) is \( \text{id}_{A}^{\text{RLD}} \).

To prove that it is really a category is trivial.

### 7.6 Monovalued and injective reloids

Following the idea of definition of monovalued morphism let’s call monovalued such a reloid \( f \) that \( f \circ f^{-1} \subseteq \text{id}_{\text{im } f}^{\text{RLD}} \). Similarly, I will call a reloid injective when \( f^{-1} \circ f \subseteq \text{id}_{\text{dom } f}^{\text{RLD}} \).

Obvious 7.40. A reloid \( f \) is

- monovalued iff \( f \circ f^{-1} \subseteq \text{id}_{\text{im } f}^{\text{RLD}} \);  
- injective iff \( f^{-1} \circ f \subseteq \text{id}_{\text{dom } f}^{\text{RLD}} \).

In other words, a reloid is monovalued (injective) when it is a monovalued (injective) morphism of the category of reloids.

Monovaluedness is dual of injectivity.

Obvious 7.41.

1. A morphism \( (A; B; f) \) of the category of reloid triples is monovalued iff the reloid \( f \) is monovalued.
2. A morphism \( (A; B; f) \) of the category of reloid triples is injective iff the reloid \( f \) is injective.

Theorem 7.42.

1. A reloid \( f \) is a monovalued iff there exists a function (monovalued binary relation) \( F \in \text{GR } f \).
2. A reloid \( f \) is a injective iff there exists an injective binary relation \( F \in \text{GR } f \).
3. A reloid \( f \) is a both monovalued and injective iff there exists an injection (a monovalued and injective binary relation = injective function) \( F \in \text{GR} f \).

**Proof.** The reverse implications are obvious. Let’s prove the direct implications:

1. Let \( f \) be a monovalued reloid. Then \( f \circ f^{-1} \subseteq \text{id}_{\text{RLD}(\text{Dst} f)} \). So there exists \( h \in \text{GR}((f \circ f)^{-1}) = \text{GR} \bigcap \{ f_{\text{RLD}(\text{Dst} f): \text{GR}(f)}(F \circ f^{-1}) \mid F \in \text{GR} f \} \) such that \( (f \circ f)^{-1}_{\text{RLD}(\text{Dst} f)}h \subseteq \text{id}_{\text{RLD}(\text{Dst} f)} \). It’s simple to show that \( \{ F \circ f^{-1} \mid F \in \text{GR} f \} \) is a filter base. Consequently there exists \( F \in \text{GR} f \) such that \( F \circ f^{-1} \subseteq \text{id}_{\text{Dst} f} \) that is \( F \) is a function.

2. Similar.

3. Let \( f \) be a both monovalued and injective reloid. Then by proved above there exist \( F, G \in \text{GR} f \) such that \( F \) is monovalued and \( G \) is injective. Thus \( F \cap G \in \text{GR} f \) is both monovalued and injective.

**Conjecture 7.43.** A reloid \( f \) is monovalued iff

\[
\forall g \in \text{RLD}(\text{Src} f; \text{Dst} f): (g \subseteq f \Rightarrow \exists A \in \mathfrak{F}(\text{Src} f): g = f|_A).
\]

## 7.7 Complete reloids and completion of reloids

**Definition 7.44.** A complete reloid is a reloid representable as a join of reloidal products \( \uparrow A \{ \alpha \} \times_{\text{RLD}} b \) where \( \alpha \in A \) and \( b \) is an ultrafilter on \( B \) for some sets \( A \) and \( B \).

**Definition 7.45.** A co-complete reloid is a reloid representable as a join of reloidal products \( a \times_{\text{RLD}} \uparrow B \{ \beta \} \) where \( \beta \in B \) and \( a \) is an ultrafilter on \( A \) for some sets \( A \) and \( B \).

I will denote the sets of complete and co-complete reloids correspondingly as \( \text{ComplRLD} \) and \( \text{CoComplRLD} \).

**Obvious 7.46.** Complete and co-complete are dual.

**Theorem 7.47.**

1. A reloid \( f \) is complete iff there exists a function \( G: \text{Src} f \to \mathfrak{F}(\text{Dst} f) \) such that

\[
\overline{f} = \bigsqcup \{ \uparrow \text{Src} f \{ \alpha \} \times_{\text{RLD}} G(\alpha) \mid \alpha \in \text{Src} f \}.
\]

2. A reloid \( f \) is co-complete iff there exists a function \( G: \text{Dst} f \to \mathfrak{F}(\text{Src} f) \) such that

\[
\overline{f} = \bigsqcup \{ G(\alpha) \times_{\text{RLD}} \uparrow \text{Dst} f \{ \alpha \} \mid \alpha \in \text{Dst} f \}.
\]

**Proof.** We will prove only the first as the second is symmetric.

\( \Rightarrow. \) Let \( f \) be complete. Then take

\[
G(\alpha) = \bigsqcup \{ b \in \text{atoms} \mathfrak{F}(\text{Dst} f) \mid \uparrow \text{Src} f \{ \alpha \} \times_{\text{RLD}} b \subseteq f \}
\]

and we have (7.1) obviously.

\( \Leftarrow. \) Let (7.1) hold. Then \( G(\alpha) = \bigsqcup \text{atoms} G(\alpha) \) and thus

\[
f = \bigsqcup \{ \uparrow \text{Src} f \{ \alpha \} \times_{\text{RLD}} b \mid \alpha \in \text{Src} f, b \in \text{atoms} G(\alpha) \}
\]

and so \( f \) is complete.

**Obvious 7.48.** Complete and co-complete reloids are convex.
**Obvious 7.49.** Principal reloids are complete and co-complete.

**Obvious 7.50.** Join (on the lattice of reloids) of complete reloids is complete.

**Corollary 7.51.** ComplRLD (with the induced order) is a complete lattice.

**Theorem 7.52.** A reloid which is both complete and co-complete is principal.

**Proof.** Let $f$ be a complete and co-complete reloid. We have

$$f = \bigsqcup \{ \uparrow_{\text{Src}} f \{ \alpha \} \times \text{RLD} G(\alpha) \mid \alpha \in \text{Src} f \}$$

and

$$f = \bigsqcup \{ H(\beta) \times \text{RLD} \uparrow_{\text{Dst}} f \{ \beta \} \mid \beta \in \text{Dst} f \}$$

for some functions $G: \text{Src} f \to \mathcal{F}(\text{Dst} f)$ and $H: \text{Dst} f \to \mathcal{F}(\text{Src} f)$. For every $\alpha \in \text{Src} f$ we have

$$G(\alpha) = \im f \uparrow_{\text{Src} f} = \im \bigg( f \cap \left( \uparrow_{\text{Src}} f \{ \alpha \} \times \text{RLD} 1_{\mathcal{F}(\text{Dst} f)} \right) \bigg) = (*)$$

$$\im \bigg( (H(\beta) \times \text{RLD} \uparrow_{\text{Dst}} f \{ \beta \}) \cap \left( \uparrow_{\text{Src}} f \{ \alpha \} \times \text{RLD} 1_{\mathcal{F}(\text{Dst} f)} \right) \bigg) = \beta \in \text{Dst} f$$

Thus $G(\alpha)$ is a principal filter that is $G(\alpha) = \uparrow_{\text{Dst}} f g(\alpha)$ for some $g: \text{Src} f \to \text{Dst} f$; $\uparrow_{\text{Src}} f \{ \alpha \} \times \text{RLD} G(\alpha) = \uparrow_{\text{RLD} \{ \text{Src} f ; \text{Dst} f \}}(\{ \alpha \} \times g(\alpha))$; $f$ is principal as a join of principal reloids. □

**Conjecture 7.53.** Composition of complete reloids is complete.

**Theorem 7.54.**

1. For a complete reloid $f$ there exists exactly one function $F \in \mathcal{F}(\text{Dst} f)_{\text{Src} f}$ such that

$$f = \bigsqcup \{ \uparrow_{\text{Src}} f \{ \alpha \} \times \text{RLD} F(\alpha) \mid \alpha \in \text{Src} f \}.$$  

2. For a co-complete reloid $f$ there exists exactly one function $F \in \mathcal{F}(\text{Src} f)_{\text{Dst} f}$ such that

$$f = \bigsqcup \{ F(\alpha) \times \text{RLD} \uparrow_{\text{Dst}} f \{ \alpha \} \mid \alpha \in \text{Dst} f \}.$$  

**Proof.** We will prove only the first as the second is similar. Let

$$f = \bigsqcup \{ \uparrow_{\text{Src}} f \{ \alpha \} \times \text{RLD} F(\alpha) \mid \alpha \in \text{Src} f \} = \bigsqcup \{ \uparrow_{\text{Src}} f \{ \alpha \} \times \text{RLD} G(\alpha) \mid \alpha \in \text{Src} f \}$$

for some $F, G \in \mathcal{F}(\text{Dst} f)_{\text{Src} f}$. We need to prove $F = G$. Let $\beta \in \text{Src} f$.

$$f \cap \left( \uparrow_{\text{Src}} f \{ \beta \} \times \text{RLD} 1_{\mathcal{F}(\text{Dst} f)} \right) = (\text{proposition 4.194})$$

$$\uparrow_{\text{Src}} f \{ \beta \} \times \text{RLD} \left( F(\alpha) \cap \left( \uparrow_{\text{Src}} f \{ \beta \} \times \text{RLD} 1_{\mathcal{F}(\text{Dst} f)} \right) \mid \alpha \in \text{Src} f \right) = \uparrow_{\text{Src}} f \{ \beta \} \times \text{RLD} F(\beta).$$

Similarly $f \cap \left( \uparrow_{\text{Src}} f \{ \beta \} \times \text{RLD} 1_{\mathcal{F}(\text{Dst} f)} \right) = \uparrow_{\text{Src}} f \{ \beta \} \times \text{RLD} G(\beta)$. Thus $\uparrow_{\text{Src}} f \{ \beta \} \times \text{RLD} F(\beta) = \uparrow_{\text{Src}} f \{ \beta \} \times \text{RLD} G(\beta)$ and so $F(\beta) = G(\beta)$. □

**Definition 7.55.** Completion and co-completion of a reloid $f \in \text{RLD}(A; B)$ are defined by the formulas:

$$\text{Compl } f = \text{Cor}^{\text{(RLD}(A; B); \text{ComplRLD}((A; B))} f; \quad \text{CoCompl } f = \text{Cor}^{\text{(RLD}(A; B); \text{CoComplRLD}((A; B))} f.$$
7.7 Complete reloids and completion of reloids

**Theorem 7.56.** Atoms of the lattice $\text{ComplRLD}(A; B)$ are exactly reoidal products of the form $\uparrow^A \{ \alpha \} \times_{\text{RLD}} b$ where $\alpha \in A$ and $b$ is an ultrafilter on $B$.

**Proof.** First, it’s easy to see that $\uparrow^A \{ \alpha \} \times_{\text{RLD}} b$ are elements of $\text{ComplRLD}(A; B)$. Also $\emptyset_{\text{RLD}(A; B)}$ is an element of $\text{ComplRLD}(A; B)$.

$\uparrow^A \{ \alpha \} \times_{\text{RLD}} b$ are atoms of $\text{ComplRLD}(A; B)$ because these are atoms of $\text{RLD}(A; B)$.

It remains to prove that if $f$ is an atom of $\text{ComplRLD}(A; B)$ then $f = \uparrow^A \{ \alpha \} \times_{\text{RLD}} b$ for some $\alpha \in A$ and an ultrafilter $b$ on $B$.

Suppose $f$ is a non-empty complete reloid. Then $\uparrow^A \{ \alpha \} \times_{\text{RLD}} b \subseteq f$ for some $\alpha \in A$ and an ultrafilter $b$ on $B$. If $f$ is an atom then $f = \uparrow^A \{ \alpha \} \times_{\text{RLD}} b$. $\square$

**Obvious 7.57.** $\text{ComplRLD}(A; B)$ is an atomistic lattice.

**Proposition 7.58.** $\text{Compl} f = \bigsqcup \{ f \mid_{\uparrow^A \{ \alpha \}} \mid \alpha \in \text{Src } f \}$ for every reloid $f$.

**Proof.** Let’s denote $R$ the right part of the equality to be proven.

That $R$ is a complete reloid follows from the equality $$f \mid_{\uparrow^A \{ \alpha \}} = \uparrow^A \{ \alpha \} \times_{\text{RLD}} \text{im}(f \mid_{\uparrow^A \{ \alpha \}}).$$

The only thing left to prove is that $g \subseteq R$ for every complete reloid $g$ such that $g \subseteq f$.

Really let $g$ is a complete reloid such that $g \subseteq f$. Then

$$g = \bigsqcup \{ f \mid_{\uparrow^A \{ \alpha \}} \mid \alpha \in \text{Src } f \}$$

for some function $G: \text{Src } f \to \mathfrak{F}(\text{Dst } f)$.

We have $\uparrow^A \{ \alpha \} \times_{\text{RLD}} G(\alpha) = g \mid_{\uparrow^A \{ \alpha \}} \subseteq f \mid_{\uparrow^A \{ \alpha \}}$. Thus $g \subseteq R$. $\square$

**Conjecture 7.59.** $\text{Compl } f \cap \text{Compl } g = \text{Compl } (f \cap g)$ for every $f, g \in \text{RLD}(A; B)$.

**Theorem 7.60.** $\text{Compl} \bigsqcup R = \bigsqcup \langle \text{Compl} \rangle R$ for every set $R \in \mathcal{P} \text{RLD}(A; B)$ for every sets $A, B$.

**Proof.**

$$\text{Compl} \bigsqcup R = \bigsqcup \{ \bigsqcup \{ f \mid_{\uparrow^A \{ \alpha \}} \mid \alpha \in A \} \mid f \in R \} = (\text{proposition 4.194})$$

$$\bigsqcup \langle \text{Compl} \rangle R. \square$$

**Lemma 7.61.** Completion of a co-complete reloid is principal.

**Proof.** Let $f$ be a co-complete reloid. Then there is a function $F: \text{Dst } f \to \mathfrak{F}(\text{Src } f)$ such that

$$f = \bigsqcup \{ F(\alpha) \times_{\text{RLD}} \uparrow^A \{ \alpha \} \mid \alpha \in \text{Dst } f \}.$$

So

$$\text{Compl } f = \bigsqcup \{ \bigsqcup \{ F(\alpha) \times_{\text{RLD}} \uparrow^A \{ \alpha \} \mid \alpha \in \text{Dst } f \} \mid_{\uparrow^A \{ \beta \}} \mid \beta \in \text{Src } f \} = \bigsqcup \{ (F(\alpha) \times_{\text{RLD}} \uparrow^A \{ \alpha \}) \cap (\uparrow^A \{ \beta \} \times_{\text{RLD}} \mathfrak{F}(\text{Dst } f)) \mid \beta \in \text{Src } f \} = (*)$$

$$\bigsqcup \{ \bigsqcup \{ (F(\alpha) \times_{\text{RLD}} \uparrow^A \{ \alpha \}) \cap (\uparrow^A \{ \beta \} \times_{\text{RLD}} \mathfrak{F}(\text{Dst } f)) \mid \alpha \in \text{Dst } f \} \mid \beta \in \text{Src } f, \uparrow^A \{ \beta \} \subseteq F(\alpha) \}.$$  

* Proposition 4.194.

Thus $\text{Compl } f$ is principal. $\square$

**Theorem 7.62.** $\text{Compl } \text{CoCompl } f = \text{CoCompl } \text{Compl } f = \text{Cor } f$ for every reloid $f$. 
**Proof.** We will prove only $\text{Compl } \text{CoCompl } f = \text{Cor } f$. The rest follows from symmetry.

From the lemma $\text{Compl } \text{CoCompl } f$ is principal. It is obvious $\text{Compl } \text{CoCompl } f \subseteq f$. So to finish the proof we need to show only that for every principal reloid $F \subseteq f$ we have $F \subseteq \text{Compl } \text{CoCompl } f$.

Really, obviously $F \subseteq \text{CoCompl } f$ and thus $F = \text{Compl } F \subseteq \text{Compl } \text{CoCompl } f$. □

**Question 7.63.** Is $\text{ComplRLD}(A; B)$ a distributive lattice? Is $\text{ComplRLD}(A; B)$ a co-brouwerian lattice?

**Conjecture 7.64.** If $f$ is a complete reloid, then it is metacomplete.

**Conjecture 7.65.** If $f$ is a metacomplete reloid, then it is complete.

**Conjecture 7.66.** $\text{Compl } f = f \setminus \left( \Omega^{\text{Src } f} \times_{\text{RLD}} 1^{\text{Dst } f} \right)$ for every reloid $f$.

By analogy with similar properties of funcoids described above:

**Proposition 7.67.** For composable reloids $f$ and $g$ it holds

1. $\text{Compl}(g \circ f) \supseteq (\text{Compl } g) \circ (\text{Compl } f)$;
2. $\text{CoCompl}(g \circ f) \supseteq (\text{CoCompl } g) \circ (\text{CoCompl } f)$.

**Proof.**

1. $\text{Compl } g \circ \text{Compl } f \subseteq \text{Compl } ((\text{Compl } g) \circ (\text{Compl } f)) \subseteq \text{Compl } (g \circ f)$.
2. By duality. □

**Conjecture 7.68.** For composable reloids $f$ and $g$ it holds

1. $\text{Compl}(g \circ f) = (\text{Compl } g) \circ f$ if $f$ is a co-complete reloid;
2. $\text{CoCompl}(f \circ g) = f \circ \text{CoCompl } g$ if $f$ is a complete reloid;
3. $\text{CoCompl}((\text{Compl } g) \circ f) = \text{Compl}(g \circ (\text{CoCompl } f)) = (\text{Compl } g) \circ (\text{CoCompl } f)$;
4. $\text{Compl}(g \circ (\text{Compl } f)) = \text{Compl}(g \circ f)$;
5. $\text{CoCompl}((\text{CoCompl } g) \circ f) = \text{CoCompl}(g \circ f)$. 


Chapter 8
Relationships between funcoids and reloids

8.1 Funcoid induced by a reloid

Every reloid \( f \) induces a funcoid \((FCD)f \in FCD(Src \ f; Dst \ f)\) by the following formulas (for every \( \mathcal{X} \in \mathfrak{F}(Src \ f), \mathcal{Y} \in \mathfrak{F}(Dst \ f)\)):

\[
\mathcal{X}[(FCD)f] \mathcal{Y} \Leftrightarrow \forall F \in xyGR \ f: \mathcal{X}[\lceil FCD \rceil F] \mathcal{Y};
\]

\[
((FCD)f)\mathcal{X} = \bigcap \{ \lceil FCD \rceil F \mathcal{X} | F \in xyGR \ f \}.
\]

We should prove that \((FCD)f\) is really a funcoid.

**Proof.** We need to prove that

\[
\mathcal{X}[(FCD)f] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap ((FCD)f)\mathcal{X} \neq 0 \mathfrak{F}(Dst \ f) \Leftrightarrow \mathcal{X} \cap ((FCD)f)\mathcal{Y} \neq 0 \mathfrak{F}(Src \ f).
\]

The above formula is equivalent to:

\[
\forall F \in xyGR \ f: \mathcal{X}[\lceil FCD \rceil F] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap \bigcap \{ \lceil FCD \rceil F \mathcal{X} | F \in xyGR \ f \} \neq 0 \mathfrak{F}(Src \ f) \Leftrightarrow \mathcal{X} \cap ((FCD)f)\mathcal{Y} \neq 0 \mathfrak{F}(Src \ f).
\]

We have \( \forall F \in xyGR \ f: \mathcal{X}[\lceil FCD \rceil F] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap \bigcap \{ \lceil FCD \rceil F \mathcal{X} | F \in xyGR \ f \} \neq 0 \mathfrak{F}(Src \ f) \).

Let’s denote \( W = \{ \mathcal{Y} \cap \bigcap \{ \lceil FCD \rceil F \mathcal{X} | F \in xyGR \ f \} | F \in xyGR \ f \} \).

\( W \) is a generalized filter base.

\( \forall F \in xyGR \ f: \mathcal{X}[\lceil FCD \rceil F] \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap \bigcap \{ \lceil FCD \rceil F \mathcal{X} | F \in xyGR \ f \} \neq 0 \mathfrak{F}(Src \ f) \).

We need to prove only that \( 0 \mathfrak{F}(Src \ f) \notin W \Rightarrow \mathcal{Y} \cap \bigcap \{ \lceil FCD \rceil F \mathcal{X} | F \in xyGR \ f \} \neq 0 \mathfrak{F}(Src \ f) \). (The rest follows from symmetry.)

This follows from the fact that \( W \) is a generalized filter base.

Let’s prove that \( W \) is a generalized filter base. For this it’s enough to prove that \( V = \{ \mathcal{Y} \cap \bigcap \{ \lceil FCD \rceil F \mathcal{X} | F \in xyGR \ f \} | F \in xyGR \ f \} \) is a generalized filter base. Let \( \mathcal{A}, \mathcal{B} \in V \) that is \( \mathcal{A} = \lceil FCD \rceil P \mathcal{X}, \mathcal{B} = \lceil FCD \rceil Q \mathcal{X} \) where \( P, Q \in xyGR \ f \). Then for \( \mathcal{C} = \lceil FCD \rceil (P \cap Q) \mathcal{X}\) is true both \( \mathcal{C} \in V \) and \( \mathcal{C} \subseteq \mathcal{A}, \mathcal{B} \). So \( V \) is a generalized filter base and thus \( W \) is a generalized filter base.

**Proposition 8.1.** \((FCD)^{RLD}f = \lceil FCD \rceil f\) for every Rel-morphism \( f \).

**Proof.** \( \mathcal{X}[(FCD)^{RLD}f] \mathcal{Y} \Leftrightarrow \forall F \in xyGR \ (FCD)^{RLD}f: \mathcal{X}[\lceil FCD \rceil F] \mathcal{Y} \Leftrightarrow \mathcal{X}[\lceil FCD \rceil f] \mathcal{Y} \) (for every \( \mathcal{X} \in \mathfrak{F}(Src \ f), \mathcal{Y} \in \mathfrak{F}(Dst \ f) \)).

**Theorem 8.2.** \( \mathcal{X}[(FCD)f] \mathcal{Y} \Leftrightarrow \mathcal{X} \times RL \mathcal{Y} \neq f \) for every reloid \( f \) and \( \mathcal{X} \in \mathfrak{F}(Src \ f), \mathcal{Y} \in \mathfrak{F}(Dst \ f) \).

**Proof.**

\[
\mathcal{X} \times RL \mathcal{Y} \neq f \Leftrightarrow \forall F \in GR \ f, P \in \mathcal{X} \times RL \mathcal{Y}: P \neq F
\]

\[
\Leftrightarrow \forall F \in GR \ f, X \in \mathcal{X}, Y \in \mathcal{Y}: X \times Y \neq F
\]

\[
\Leftrightarrow \forall F \in GR \ f, X \in \mathcal{X}, Y \in \mathcal{Y}: \lceil SRC/X [\lceil FCD(Src \ f; Dst \ f) F] \rceil Dst \fY
\]

\[
\Leftrightarrow \forall F \in GR \ f: \mathcal{X}[\lceil FCD(Src \ f; Dst \ f) F] \rceil \mathcal{Y}
\]

\[
\Leftrightarrow \mathcal{X}[(FCD)f] \mathcal{Y}.
\]

**Theorem 8.3.** \((FCD)f = \bigcap \{ \lceil FCD \rceil xyGR \ f \) for every reloid \( f \).
Proof. Let \(a\) be an ultrafilter on \(\text{Src} f\).
\[
\langle (\text{FCD}) f \rangle a = \bigcap \{ \langle \text{FCD} F \rangle a \mid F \in \text{xyGR} f \}
\]
by the definition of \((\text{FCD})\).
\[
\bigcap \{ \langle \text{FCD} \text{xyGR} f \rangle a \mid a \}
\]
by theorem 6.64.
So \(\langle (\text{FCD}) f \rangle a = \bigcap \{ \langle \text{FCD} \text{xyGR} f \rangle a \mid a\}
\) for every \(a\).
\[\square\]

Lemma 8.4. For every two filter bases \(S\) and \(T\) of morphisms \(\text{Rel}(U; V)\) and every set \(A \subseteq U\)
\[
\bigcap \{ \langle \text{RLD} \rangle S \} = \bigcap \{ \langle \text{RLD} \rangle T \} = \bigcap \{ \langle \text{V} \rangle F A \mid F \in S \} = \bigcap \{ \langle \text{V} \rangle G A \mid G \in T \}.
\]
Proof. Let \(\bigcap \{ \langle \text{RLD} \rangle S \} = \bigcap \{ \langle \text{RLD} \rangle T \}.
\]
First let prove that \(\{ \langle F \rangle A \mid F \in S \} \) is a filter base. Let \(X, Y \in \{ \langle F \rangle A \mid F \in S \} \). Then \(X = \langle F_X \rangle A\) and \(Y = \langle F_Y \rangle A\) for some \(F_X, F_Y \in S\). Because \(S\) is a filter base, we have \(S \ni F_Z \subseteq F_X \cap F_Y\). So \(\langle F_Z \rangle A \subseteq X \cap Y \) and \(\langle F_Z \rangle A \in \{ \langle F \rangle A \mid F \in S \} \). So \(\{ \langle F \rangle A \mid F \in S \} \) is a filter base.

Suppose \(X \in \bigcap \{ \langle \text{V} \rangle F A \mid F \in S \} \). Then there exists \(X' \in \{ \langle F \rangle A \mid F \in S \} \) where \(X \supseteq X'\) because \(\{ \langle F \rangle A \mid F \in S \} \) is a filter base. That is \(X' = \langle F \rangle A\) for some \(F \in S\). Then \(X \supseteq X'\) because \(\langle F \rangle \subseteq \langle F \rangle\). So \(\{ \langle F \rangle A \mid F \in S \} \) is a filter base. Let \(Y' = \langle G \rangle A\). We have \(Y' \subseteq X' \cap X\) and \(\langle G \rangle A \in \{ \langle F \rangle \mid F \in T \}\). \(Y' \in \bigcap \{ \langle \text{V} \rangle G A \mid G \in T \}\). \(X \in \bigcap \{ \langle \text{V} \rangle G A \mid G \in T \}\). The reverse is symmetric. \(\square\)

Lemma 8.5. \(\{ \langle G \circ F \rangle \mid F \in \text{GR} f, G \in \text{GR} g \} \) is a filter base for every reloid \(f\) and \(g\).
Proof. Let denote \(D = \{ \langle G \circ F \rangle \mid F \in \text{GR} f, G \in \text{GR} g \} \). Let \(A \subseteq D \cap B \subseteq D\). Then \(A = \langle G_A \circ F_A \rangle \cap B = \langle G_B \circ F_B \rangle\) for some \(F_A, F_B \in \text{GR} f, G_A, G_B \in \text{GR} g\). So \(A \subseteq D \ni (G_A \cap G_B) \circ (F_A \cap F_B) \in D\) because \(F_A \cap F_B \in \text{GR} f, G_A \cap G_B \in \text{GR} g\). \(\square\)

Theorem 8.6. \((\text{FCD})(g \circ f) = (\text{FCD})g \circ (\text{FCD})f\) for every composable reloids \(f\) and \(g\).
Proof.
\[
\langle (\text{FCD})(g \circ f) \rangle X = \bigcap \{ \langle \text{Det} g \rangle \langle H \rangle X \mid H \in \text{GR}(g \circ f) \} = \bigcap \{ \langle \text{Det} g \rangle \langle H \rangle X \mid H \in \text{GR} \} \bigcap \{ \langle \text{RLD} \rangle (G \circ F) \mid F \in \text{xyGR} f, G \in \text{xyGR} g \}.
\]
Obviously
\[
\bigcap \{ \langle \text{RLD} \rangle (G \circ F) \mid F \in \text{xyGR} f, G \in \text{xyGR} g \} = \bigcap \{ \langle \text{Det} g \rangle \langle H \rangle X \mid H \in \text{GR} \} \bigcap \{ \langle \text{V} \rangle G A \mid F \in \text{xyGR} f, G \in \text{xyGR} g \};
\]
from this by the lemma 8.4 (taking into account that
\[
\{ \langle G \circ F \rangle \mid F \in \text{GR} f, G \in \text{GR} g \}
\]
and
\[
\text{GR} \bigcap \{ \langle \text{RLD} \rangle (G \circ F) \mid F \in \text{xyGR} f, G \in \text{xyGR} g \}
\]
are filter bases)
\[
\bigcap \{ \langle \text{Det} g \rangle \langle H \rangle X \mid H \in \text{GR} \} \bigcap \{ \langle \text{RLD} \rangle (G \circ F) \mid F \in \text{xyGR} f, G \in \text{xyGR} g \} = \bigcup \{ \langle \text{Det} g \rangle \langle G \circ F \rangle X \mid F \in \text{GR} f, G \in \text{GR} g \}.
\]
On the other side
\[
\langle (\text{FCD})g \circ (\text{FCD})f \rangle X = \langle (\text{FCD})g \rangle \langle (\text{FCD})f \rangle X = \langle (\text{FCD})g \rangle \bigcap \{ \langle \text{Det} g \rangle \langle F \rangle X \mid F \in \text{xyGR} f \} = \bigcup \{ \langle \text{Det} g \rangle \langle G \circ F \rangle X \mid F \in \text{GR} f, G \in \text{GR} g \}.
\]
Let’s prove that \(\{ \langle F \rangle X \mid F \in xyGR f \} \) is a filter base. If \(A, B \in \{ \langle F \rangle X \mid F \in xyGR f \} \) then \(A = \langle F_1 \rangle X, B = \langle F_2 \rangle X\) where \(F_1, F_2 \in xyGR f\). \(A \cap B \ni \langle F \cap F \rangle X \in \{ \langle F \rangle X \mid F \in xyGR f \} \). So \(\{ \langle F \rangle X \mid F \in xyGR f \} \) is really a filter base.

By theorem 6.32 we have
\[
\langle \text{FCD} \rangle \bigcap \{ \langle \text{Det} g \rangle \langle F \rangle X \mid F \in \text{xyGR} f \} = \bigcap \{ \langle \text{Det} g \rangle \langle G \rangle \langle F \rangle X \mid F \in \text{xyGR} f \}.
\]
So continuing the above equalities,

\[ (((\text{FCD})g) \circ (\text{FCD})f) \circ X = \prod \left\{ \prod \{ \{ (\text{FCD})g \circ (\text{FCD})f \} \mid F \in \text{GR} f \} \mid G \in \text{GR} g \} \right\} \]

Combining these equalities we get \(((\text{FCD})g \circ (\text{FCD})f) \circ X = (((\text{FCD})g) \circ (\text{FCD})f) \circ X\) for every set \(X \in \mathcal{P}((\text{Src} f))\).

Corollary 8.7.
1. \((\text{FCD})f\) is a monovalued funcoid if \(f\) is a monovalued reloid.
2. \((\text{FCD})f\) is an injective funcoid if \(f\) is an injective reloid.

Proof. We will prove only the first as the second is dual. Let \(f\) be a monovalued reloid. Then \(f \circ f^{-1} \subseteq \text{id}_{\text{RLD}(\text{Dst} f)}\); \((\text{FCD})f \circ f^{-1} \subseteq \text{id}_{\text{FCD}(\text{Dst} f)}\); \((\text{FCD})f \circ ((\text{FCD})f)^{-1} \subseteq \text{id}_{\text{FCD}(\text{Dst} f)}\) that is \((\text{FCD})f\) is a monovalued funcoid.

Proposition 8.8. \((\text{FCD})\text{id}_{\text{RLD}} = \text{id}_{\text{FCD}}\) for every filter \(A\).

Proof. Recall that \(\text{id}_{\text{RLD}} = \prod \{ \{ \text{Base}(A) \mid A \in A \} \}. For every \(X, Y \in \mathcal{G}(\text{Base}(A))\) we have:

\[
X [\text{FCD}(\text{id}_{\text{RLD}})] Y \Leftrightarrow X \times \text{RLD} Y \neq \text{id}_{\text{RLD}}[X \times \text{RLD} Y \neq \text{id}_{\text{RLD}}[\text{Base}(A) \times \text{Base}(B)] \text{id}_{\text{RLD}}] Y \Leftrightarrow \forall A \in A, X \times \text{RLD} Y \neq \text{id}_{\text{RLD}}[\text{Base}(A) \times \text{Base}(B)] \text{id}_{\text{RLD}} Y (\text{used properties of generalized filter bases}).\]

Proposition 8.9. \((\text{FCD})(A \times \text{RLD} B) = A \times \text{FCD} B\) for every filters \(A, B\).

Proof. \(X [\text{FCD}((A \times \text{RLD} B)] Y \Leftrightarrow \forall F \in \text{GR}(A \times \text{RLD} B) \colon X [\text{FCD} F] Y \) (for every \(X \in \mathcal{G}(\text{Base}(A)), Y \in \mathcal{G}(\text{Base}(B))\)).

Evidently

\[
\forall F \in \text{GR}(A \times \text{RLD} B) \colon X [\text{FCD} F] Y \Leftrightarrow \forall A \in A, B \in B : X [\text{FCD}([\text{Base}(A) \times \text{Base}(B)] \text{F} \) \)] Y.
\]

Let \(\forall A \in A, B \in B : X [\text{FCD}([\text{Base}(A) \times \text{Base}(B)] \text{F} \) \)] Y. Then if \(F \in \text{GR}(A \times \text{RLD} B)\), there are \(A \in A, B \in B\) such that \(F \geq A \times B\). So \(X [\text{FCD}([\text{Base}(A) \times \text{Base}(B)] \text{F} \) \)] Y.

Thus \(X [\text{FCD}((A \times \text{RLD} B)] Y \Leftrightarrow X [A \times \text{FCD} B] Y \).

Proposition 8.10. \text{dom}(\text{FCD})f = \text{dom} f and \text{im}(\text{FCD})f = \text{im} f for every reloid \(f\).

Proof. \(\text{im}(\text{FCD})f = ((\text{FCD})f) \circ \text{Src} f = \prod \{ \{ \text{Src} f \mid F \in \text{GR} f \} \mid F \in \text{GR} f \} = \prod \{ \{ \text{Src} f \mid \text{GR} f \} \mid F \in \text{GR} f \} = \prod \{ \{ \text{Src} f \mid \text{GR} f \} \mid F \in \text{GR} f \} = \text{dom}(\text{FCD})f = \text{dom} f \) is similar.

Proposition 8.11. \((\text{FCD})(f \cap (A \times \text{RLD} B)) = (\text{FCD})f \cap (A \times \text{FCD} B)\) for every reloid \(f\) and \(A \in \mathcal{G}(\text{Src} f)\) and \(B \in \mathcal{G}(\text{Dst} f)\).

Proof. \((\text{FCD})(f \cap (A \times \text{RLD} B)) = (\text{FCD})(\text{id}_{\text{RLD}} \circ f \circ \text{id}_{\text{RLD}}) = (\text{FCD})(\text{id}_{\text{RLD}} \circ (\text{FCD})f \circ \text{id}_{\text{RLD}}) = \text{id}_{\text{FCD}} \circ (\text{FCD})f \circ \text{id}_{\text{FCD}} = (\text{FCD})(f \cap (A \times \text{FCD} B))\).
Proof. $\text{(FCD)} f = \bigcup \{ x \times \text{FCD} \ y \mid x \in \text{atoms}^{3}(\text{Src} f), y \in \text{atoms}^{3}(\text{Src} f), x \times \text{FCD} \ y \neq (\text{FCD}) f \}$, but $x \times \text{FCD} \ y \neq (\text{FCD}) f \Leftrightarrow x [(\text{FCD}) f] y \Leftrightarrow x \times \text{RLD} \ y \neq f$, thus follows the theorem.

### 8.2 Reloids induced by a funcoid

Every funcoid $f \in \text{FCD}(A; B)$ induces a reloid from $A$ to $B$ in two ways, intersection of outward relations and union of inward relations and union of filters:

- $\text{(RLD)}_{\text{out}} f = \bigcap \{(\text{RLD}) x y f \}$
- $\text{(RLD)}_{\text{in}} f = \bigcup \{ A \times \text{RLD} \ B \mid A \in \mathcal{G}(A), B \in \mathcal{G}(B), A \times \text{FCD} \ B \subseteq f \}$

**Theorem 8.15.** $\text{(RLD)}_{\text{in}} f = \bigcup \{ a \times \text{RLD} \ b \mid a \in \text{atoms}^{3}(A), b \in \text{atoms}^{3}(B), a \times \text{FCD} \ b \subseteq f \}$

**Proof.** It follows from theorem 7.21.

**Remark 8.16.** It seems that $\text{(RLD)}_{\text{in}}$ has smoother properties and is more important than $\text{(RLD)}_{\text{out}}$. (However see also the exercise below for $\text{(RLD)}_{\text{in}}$ not preserving identities.)

**Proposition 8.17.** $\text{GR}^{\text{RLD}} f = \text{GR}^{\text{FCD}} f$ for every Rel-morphism $f$.

**Proof.** $X \in \text{GR}^{\text{RLD}} f \Leftrightarrow X \supseteq f \Leftrightarrow X \in \text{GR}^{\text{FCD}} f$.

**Proposition 8.18.** $\text{(RLD)}_{\text{out}} \text{GR}^{\text{FCD}} f = \text{GR}^{\text{RLD}} f$ for every Rel-morphism $f$.

**Proof.** $\text{(RLD)}_{\text{out}} \text{GR}^{\text{FCD}} f = \bigcap \{(\text{RLD}) x y f \}$ taking into account the previous proposition.

Surprisingly, a funcoid is greater inward than outward:

**Theorem 8.19.** $\text{(RLD)}_{\text{out}} f \subseteq \text{(RLD)}_{\text{in}} f$ for every funcoid $f$.

**Proof.** We need to prove

- $\text{(RLD)}_{\text{out}} f \subseteq \bigcup \{ A \times \text{RLD} \ B \mid A \in \mathcal{G}(\text{Src} f), B \in \mathcal{G}(\text{Dst} f), A \times \text{FCD} \ B \subseteq f \}$

Let

- $K \subseteq \bigcup \{ A \times \text{RLD} \ B \mid A \in \mathcal{G}(\text{Src} f), B \in \mathcal{G}(\text{Dst} f), A \times \text{FCD} \ B \subseteq f \}$

Then

- $K \subseteq (\text{RLD})_{\text{out}} \text{GR}^{\text{FCD}} f \subseteq \bigcup \{ X_{A} \times B | A \in \mathcal{G}(\text{Src} f), B \in \mathcal{G}(\text{Dst} f), A \times \text{FCD} \ B \subseteq f \}$

- $= (\text{RLD})_{\text{out}} \text{GR}^{\text{FCD}} f (X_{A} \times Y) = (\text{RLD})_{\text{out}} \text{GR}^{\text{FCD}} f (X \times Y) \\
- \subseteq (\text{RLD})_{\text{out}} \text{GR}^{\text{FCD}} f (X \times Y) \subseteq \text{GR}^{\text{RLD}} f (X \times Y) \subseteq \text{GR}^{\text{FCD}} f (X \times Y) \subseteq \text{GR}^{\text{RLD}} f (X \times Y)

Theorem 8.20. $(\text{FCD})(\text{RLD})_{\text{in}} f = f$ for every funcoid $f$.

**Proof.** For every sets $X \in \mathcal{P}(\text{Src} f), Y \in \mathcal{P}(\text{Dst} f)$

$$X [(\text{FCD})(\text{RLD})_{\text{in}} f]^{*} Y \Leftrightarrow \uparrow^{\text{RLD}} f (X \times \text{RLD} \ Y) \neq (\text{RLD})_{\text{in}} f \Leftrightarrow \uparrow^{\text{RLD}} (X \times Y) \neq \bigcup \{ a \times \text{RLD} \ b | a \in \text{atoms}^{3}(A), b \in \text{atoms}^{3}(B), a \times \text{FCD} \ b \subseteq f \} \Leftrightarrow (*)

\exists a \in \text{atoms}^{3}(A), b \in \text{atoms}^{3}(B): (a \times \text{FCD} \ b \subseteq f \land a \subseteq (\text{RLD})_{\text{in}} f \uparrow^{\text{Src}} X \land b \subseteq \uparrow^{\text{Dst}} Y) \Leftrightarrow X f^{*} Y.$$
**Remark 8.21.** The above theorem allows to represent funcoids as reloids.

**Obvious 8.22.** \((\text{RLD})_{\text{in}}(A \times_{\text{FCD}} B) = A \times_{\text{RLD}} B\) for every filters \(A, B\).

**Conjecture 8.23.** \((\text{RLD})_{\text{out}}\text{id}_A^{\text{FCD}} = \text{id}_A^{\text{RLD}}\) for every filter \(A\).

**Exercise 8.1.** Prove that generally \((\text{RLD})_{\text{in}}\text{id}_A^{\text{FCD}} \neq \text{id}_A^{\text{RLD}}\).

**Conjecture 8.24.** \(\text{dom}(\text{RLD})_{\text{in}} f = \text{dom} f\) and \(\text{im}(\text{RLD})_{\text{in}} f = \text{im} f\) for every funcoid \(f\).

**Proposition 8.25.** \(\text{dom}(f | A) = A \cap \text{dom} f\) for every reloid \(f\) and filter \(A \in \mathcal{F}(\text{Src} f)\).

**Proof.** \(\text{dom}(f | A) = \text{dom}(\text{FCD})(f | A) = \text{dom}(\text{FCD})f | A = A \cap \text{dom}(\text{FCD})f = A \cap \text{dom} f\).

**Theorem 8.26.** For every composable reloids \(f, g\):

1. If \(f \sqsupseteq \text{dom} g\) then \(\text{im}(g \circ f) = \text{im} g\).
2. If \(f \sqsubseteq \text{dom} g\) then \(\text{dom}(g \circ f) = \text{dom} g\).

**Proof.**

1. \(\text{im}(g \circ f) = \text{im}(\text{FCD})(g \circ f) = \text{im}(\text{FCD} g \circ (\text{FCD}) f) = \text{im}(\text{FCD}) g = \text{im} g\).
2. Similar.

**Conjecture 8.27.** \((\text{RLD})_{\text{in}} (g \circ f) = (\text{RLD})_{\text{in}} g \circ (\text{RLD})_{\text{in}} f\) for every composable funcoids \(f, g\).

**Theorem 8.28.** \(a \times_{\text{RLD}} b \subseteq (\text{RLD})_{\text{in}} f \Leftrightarrow a \times_{\text{FCD}} b \subseteq f\) for every funcoid \(f\) and \(a \in \text{atoms}^{\mathcal{F}(\text{Src} f)}\), \(b \in \text{atoms}^{\mathcal{F}(\text{Dst} f)}\).

**Proof.** \(a \times_{\text{FCD}} b \subseteq f \Rightarrow a \times_{\text{RLD}} b \subseteq (\text{RLD})_{\text{in}} f\) is obvious.

\[ a \times_{\text{RLD}} b \subseteq (\text{RLD})_{\text{in}} f \Rightarrow a \times_{\text{RLD}} b \neq (\text{RLD})_{\text{in}} f \Rightarrow a [\text{FCD}(\text{RLD})_{\text{in}} b] \Rightarrow a [f] b \Rightarrow a \times_{\text{FCD}} b \subseteq f.\]

**Conjecture 8.29.** If \(A \times_{\text{RLD}} B \subseteq (\text{RLD})_{\text{in}} f\) then \(A \times_{\text{FCD}} B \subseteq f\) for every funcoid \(f\) and \(A \in \mathcal{F}(\text{Src} f), B \in \mathcal{F}(\text{Dst} f)\).

**Theorem 8.30.** \(\text{GR}(\text{FCD})g \supseteq \text{GR} g\) for every reloid \(g\).

**Proof.** Let \(K \in \text{GR} g\). Then for every filters \(X \in \mathcal{P}\text{Src} g, Y \in \mathcal{P}\text{Dst} g\), \(X [K] Y \Rightarrow X [\text{FCD} K] Y \Rightarrow X [\text{FCD}) g] Y \Rightarrow X [\text{FCD} g] Y\)

Thus \(\text{FCD} K \supseteq (\text{FCD} g)\) that is \(K \in \text{GR} (\text{FCD} g)\).

**Theorem 8.31.** \(g \circ (A \times_{\text{RLD}} B) \circ f = (\text{FCD})(f^{-1}) A \times_{\text{RLD}} (\text{FCD} g) B\) for every reloids \(f, g\) and filters \(A \in \mathcal{F}(\text{Src} f), B \in \mathcal{F}(\text{Dst} g)\).

**Proof.** \(g \circ (A \times_{\text{RLD}} B) \circ f = \bigcap \{\text{RLD}(\text{Src} f; \text{Dst} g)(G \circ (A \times B) \circ f) \mid F \in \text{GR} f, G \in \text{GR} g, A \in A, B \in B\} = \bigcap \{\text{RLD}(\text{Src} f; \text{Dst} g)(\text{FCD})(g) B \mid F \in \text{GR} f, G \in \text{GR} g, A \in A, B \in B\} = \bigcap \{\text{RLD}(\text{Src} f; \text{Dst} g)(\text{FCD})(g) B \mid F \in \text{GR} f, G \in \text{GR} g, A \in A, B \in B\}\).

**Corollary 8.32.**

1. \((A \times_{\text{RLD}} B) \circ f = (\text{FCD})(f^{-1}) A \times_{\text{RLD}} B\).
2. \( g \circ (A \times \text{RLD} B) = A \times \text{RLD} ((\text{FCD})g)B. \)

### 8.3 Galois connections between funcoids and reloids

**Theorem 8.33.** \((\text{FCD})\): \(\text{RLD}(A; B) \rightarrow \text{FCD}(A; B)\) is the lower adjoint of \((\text{RLD})_\text{in}; \text{FCD}(A; B) \rightarrow \text{RLD}(A; B)\) for every sets \(A, B\).

**Proof.** Because \((\text{FCD})\) and \((\text{RLD})_\text{in}\) are trivially monotone, it’s enough to prove (for every \(f \in \text{RLD}(A; B), g \in \text{FCD}(A; B)\))

\[
f \subseteq (\text{RLD})_\text{in}(\text{FCD})f \quad \text{and} \quad \text{FCD}(\text{RLD})_\text{in}g \subseteq g.
\]

The second formula follows from the fact that \((\text{FCD})(\text{RLD})_\text{in}g = g\).

**Corollary 8.34.**

1. \((\text{FCD})\bigcup S = \bigcup ((\text{FCD}))S\) if \(S \in \mathcal{P}\text{RLD}(A; B)\).
2. \((\text{RLD})_\text{in}\bigcap S = \bigcap ((\text{RLD})_\text{in})S\) if \(S \in \mathcal{P}\text{FCD}(A; B)\).

**Proposition 8.35.** \((\text{RLD})_\text{in}(f \cap (A \times \text{FCD} B)) = ((\text{RLD})_\text{in}f) \cap (A \times \text{RLD} B)\) for every funcoid \(f\) and \(A \in \mathfrak{F}(\text{Src} f), B \in \mathfrak{F}(\text{Dst} f)\).

**Proof.** \((\text{RLD})_\text{in}(f \cap (A \times \text{FCD} B)) = \text{RLD}(\text{in}f \cap \text{RLD}(A \times \text{FCD} B)) = ((\text{RLD})_\text{in}f) \cap (A \times \text{RLD} B). \)

**Corollary 8.36.** \((\text{RLD})_\text{in}(f \downarrow A) = ((\text{RLD})_\text{in}f) \downarrow A\).

**Conjecture 8.37.** \((\text{RLD})_\text{in}\) is not a lower adjoint (in general).

**Conjecture 8.38.** \((\text{RLD})_\text{out}\) is neither a lower adjoint nor an upper adjoint (in general).

**Exercise 8.2.** Prove that \(\text{cardFCD}(A; B) = 2^{2^{\text{card}(A; B)}}\) if \(A\) or \(B\) is an infinite set.

**Lemma 8.39.** \(\uparrow^{\text{FCD}}\{(x; y)\} \subseteq (\text{FCD})g \iff \uparrow^{\text{RLD}}\{(x; y)\} \subseteq g\) for every reloid \(g\).

**Proof.** \(\uparrow^{\text{FCD}}\{(x; y)\} \subseteq (\text{FCD})g \iff \uparrow^{\text{RLD}}\{(x; y)\} \not\subseteq g \iff (\text{FCD})g \not\subseteq \{x\} \not\subseteq \{y\} \iff \uparrow^{\text{RLD}}\{(x; y)\} \subseteq g. \)

**Theorem 8.40.** Cor (\text{FCD})g = (\text{FCD})\text{Cor} g for every reloid \(g\).

**Proof.** Cor (\text{FCD})g = \(\bigcup \{\uparrow^{\text{FCD}}\{(x; y)\} \subseteq (\text{FCD})g\} = \bigcup \{\uparrow^{\text{RLD}}\{(x; y)\} \subseteq g\} = (\text{FCD})\bigcup \{\uparrow^{\text{RLD}}\{(x; y)\} \subseteq g\} = (\text{FCD})\text{Cor} g. \)

**Conjecture 8.41.** For every funcoid \(g\)

1. \(\text{Cor} (\text{RLD})_\text{in}g = (\text{RLD})_\text{in}\text{Cor} g;\)
2. \(\text{Cor} (\text{RLD})_\text{out}g = (\text{RLD})_\text{out}\text{Cor} g.\)
8.4 Funcoidal reloids

**Definition 8.42.** I call *funcoidal* such a reloid $\nu$ that
\[ X \times_{RLD} Y \neq \nu \Rightarrow \exists X' \in \mathfrak{F}(\text{Base}(X)) \setminus \{0\}, Y' \in \mathfrak{F}(\text{Base}(Y)) \setminus \{0\}; (X' \cap X \land Y' \subseteq Y \land X' \times_{RLD} Y' \subseteq \nu) \]
for every $X \in \mathfrak{F}(\text{Src} \nu), Y \in \mathfrak{F}(\text{Dst} \nu)$.

**Proposition 8.43.** A reloid $\nu$ is funcoidal iff $x \times_{RLD} y \neq \nu \Rightarrow x \times_{RLD} y \subseteq \nu$ for every ultrafilters $x$ and $y$ on respective sets.

**Proof.**
\[ \Rightarrow. \ x \times_{RLD} y \neq \nu \Rightarrow \exists X' \in \text{atoms } x, Y' \in \text{atoms } y; X' \times_{RLD} Y' \subseteq \nu \Rightarrow x \times_{RLD} y \subseteq \nu. \]
\[ \Leftarrow. \ X \times_{RLD} Y \neq \nu \Rightarrow \exists x \in \text{atoms } X, y \in \text{atoms } Y; x \times_{RLD} y \neq \nu \Rightarrow \exists x \in \text{atoms } X, y \in \text{atoms } Y; x \times_{RLD} y \subseteq \nu \Rightarrow \exists X' \in \mathfrak{F}(\text{Base}(X)) \setminus \{0\}, Y' \in \mathfrak{F}(\text{Base}(Y)) \setminus \{0\}; (X' \cap X \land Y' \subseteq Y \land X' \times_{RLD} Y' \subseteq \nu). \]

**Proposition 8.44.** $(\text{RLD})_{in}(\text{FCD}) f = \bigcup \{ x \times_{RLD} y \mid x \in \text{atoms}^3(\text{Src} \nu), y \in \text{atoms}^3(\text{Dst} \nu), x \times_{RLD} y \neq f \}$. 

**Proof.** $(\text{RLD})_{in}(\text{FCD}) f = \bigcup \{ a \times_{RLD} b \mid a \in \text{atoms}^3(\text{Src} f), b \in \text{atoms}^3(\text{Dst} f), a \times_{FCD} b \subseteq (\text{FCD}) f \} = \bigcup \{ a \times_{RLD} b \mid a \in \text{atoms}^3(\text{Src} f), b \in \text{atoms}^3(\text{Dst} f), a \in (\text{FCD}) f \} = \bigcup \{ a \times_{RLD} b \mid a \in \text{atoms}^3(\text{Src} f), b \in \text{atoms}^3(\text{Src} f), a \times_{RLD} b \neq f \}$. 

**Definition 8.45.** I call $(\text{RLD})_{in}(\text{FCD}) f$ *funcoidization* of a reloid $f$.

**Lemma 8.46.** $(\text{RLD})_{in}(\text{FCD}) f$ is funcoidal for every reloid $f$.

**Proof.** $x \times_{RLD} y \neq (\text{RLD})_{in}(\text{FCD}) f \Rightarrow x \times_{RLD} y \subseteq (\text{RLD})_{in}(\text{FCD}) f$. 

**Theorem 8.47.** $(\text{RLD})_{in}$ is a bijection from FCD($A; B$) to the set of funcoidal reloids from $A$ to $B$.

**Proof.** Let $f \in$ FCD($A; B$). Prove that $(\text{RLD})_{in} f$ is funcoidal.

Really $(\text{RLD})_{in} f = (\text{RLD})_{in}(\text{FCD})(\text{RLD})_{in} f$ and thus we can use the lemma stating that it is funcoidal.

It remains to prove $(\text{RLD})_{in}(\text{FCD}) f = f$ for a funcoidal reloid $f$. ($(\text{FCD})(\text{RLD})_{in} g = g$ for every funcoid $g$ is already proved above.)

$(\text{RLD})_{in}(\text{FCD}) f = \bigcup \{ x \times_{RLD} y \mid x \in \text{atoms}^3(\text{Src} \nu), y \in \text{atoms}^3(\text{Dst} \nu), x \times_{RLD} y \neq f \} = \bigcup \{ p \in \text{atoms}(x \times_{RLD} y) \mid x \in \text{atoms}^3(\text{Src} \nu), y \in \text{atoms}^3(\text{Dst} \nu), x \times_{RLD} y \neq f \} = \bigcup \{ p \in \text{atoms}(x \times_{RLD} y) \mid x \in \text{atoms}^3(\text{Src} \nu), y \in \text{atoms}^3(\text{Dst} \nu), x \times_{RLD} y \subseteq f \} = \bigcup \text{atoms} f = f$. 

**Corollary 8.48.** Funcoidal reloids are convex.

**Proof.** Every $(\text{RLD})_{in} f$ is obviously convex.
Chapter 9

On distributivity of composition with a principal reloid

9.1 Decomposition of composition of binary relations

Remark 9.1. Sorry for unfortunate choice of terminology: “composition” and “decomposition” are unrelated.

The idea of the proof below is that composition of binary relations can be decomposed into two operations: \( \circ \) and \( \text{dom} \):

\[
g \circ f = \{ (x; z); y \mid x f y \land y g z \}\.
\]

Composition of binary relations can be decomposed: \( g \circ f = \text{dom}(g \circ f) \).

It can be decomposed even further: \( g \circ f = \Theta_0 f \cap \Theta_1 g \) where

\[
\Theta_0 f = \{ (x; z); y \mid x f y, z \in \mathfrak{U} \} \quad \text{and} \quad \Theta_1 f = \{ (x; z); y \mid y f z, x \in \mathfrak{U} \}.
\]

(Here \( \mathfrak{U} \) is the Grothendieck universe.)

Now we will do a similar trick with reloids.

9.2 Decomposition of composition of reloids

A similar thing for reloids:

\[
g \circ f = \bigcap \{ \mathcal{RLD}(\text{Src } f \times \text{Dst } g)(G \circ F) \mid F \in \text{GR } f, G \in \text{GR } g \} = \bigcap \{ \mathcal{RLD}(\text{Src } f \times \text{Dst } g) \text{dom}(G \circ F) \mid F \in \text{GR } f, G \in \text{GR } g \}.
\]

Lemma 9.2. \( \{ G \circ F \mid F \in \text{GR } f, G \in \text{GR } g \} \) is a filter base.

Proof. Let \( P, Q \in \{ G \circ F \mid F \in \text{GR } f, G \in \text{GR } g \} \). Then \( P = G_0 \circ F_0 \), \( Q = G_1 \circ F_1 \) for some \( F_0, F_1 \in f, G_0, G_1 \in g \). Then \( F_0 \cap F_1 \in \text{GR } f, G_0 \cap G_1 \in \text{GR } g \) and thus

\[
P \cap Q \supseteq (F_0 \cap F_1) \cap (G_0 \cap G_1) \in \{ G \circ F \mid F \in \text{GR } f, G \in \text{GR } g \}.
\]

Corollary 9.3. \( \{ \mathcal{RLD}(\text{Src } f \times \text{Dst } g; \mathfrak{U})(G \circ F) \mid F \in \text{GR } f, G \in \text{GR } g \} \) is a generalized filter base.

Proposition 9.4. \( g \circ f = \text{dom} \bigcap \{ \mathcal{RLD}(\text{Src } f \times \text{Dst } g; \mathfrak{U})(G \circ F) \mid F \in \text{GR } f, G \in \text{GR } g \} \).

Proof. \( \mathcal{RLD}(\text{Src } f \times \text{Dst } g) \text{dom}(G \circ F) \supseteq \text{dom} \bigcap \{ \mathcal{RLD}(\text{Src } f \times \text{Dst } g; \mathfrak{U})(G \circ F) \mid F \in \text{GR } f, G \in \text{GR } g \} \).

Thus

\[
g \circ f \supseteq \text{dom} \bigcap \{ \mathcal{RLD}(\text{Src } f \times \text{Dst } g; \mathfrak{U})(G \circ F) \mid F \in \text{GR } f, G \in \text{GR } g \}.
\]

Let \( X \in \text{dom} \bigcap \{ \mathcal{RLD}(\text{Src } f \times \text{Dst } g; \mathfrak{U})(G \circ F) \mid F \in \text{GR } f, G \in \text{GR } g \} \). Then there exist \( Y \) such that

\[
X \times Y \in \text{GR} \bigcap \{ \mathcal{RLD}(\text{Src } f \times \text{Dst } g; \mathfrak{U})(G \circ F) \mid F \in \text{GR } f, G \in \text{GR } g \}.
\]

So because it is a generalized filter base \( X \times Y \supseteq G \circ F \) for some \( F \in \text{GR } f, G \in \text{GR } g \). Thus \( X \in \text{dom}(G \circ F), X \in \text{GR}(g \circ f) \).  

139
We can define $g \otimes f$ for reloids $f, g$:

$$g \otimes f = \{ G \otimes F \mid F \in GR f, G \in GR g \}.$$ 

Then

$$g \circ f = \bigcap \{^{RLD}(\text{Src} f; \text{Dst} g)\}(\text{dom})(g \otimes f) = \text{dom} \bigcap \{^{RLD}(\text{Src} f \times \text{Dst} g; U)\}(g \otimes f).$$

### 9.3 Lemmas for the main result

**Lemma 9.5.** $(g \otimes f) \cap (h \otimes f) = (g \cap h) \otimes f$ for binary relations $f, g, h$.

**Proof.** $(g \cap h) \otimes f = \Theta_0 f \cap \Theta_1 (g \cap h) = \Theta_0 f \cap (\Theta_1 g \cap \Theta_1 h) = (\Theta_0 f \cap \Theta_1 g) \cap (\Theta_0 f \cap \Theta_1 h) = (g \otimes f) \cap (h \otimes f).$ \hfill $\square$

**Lemma 9.6.** Let $F = \uparrow^{RLD}(\text{Src} F; \text{Dst} F)$ be a principal reloid, $T$ is a set of reloids from $\text{Dst} F$ to set $V$.

$$\bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in GR \bigcup T\} = \bigcup \{ \bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in T\} \}$$

**Proof.** $\bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in GR \bigcup T\} \supseteq \bigcup \{ \bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in T\} \}$ is obvious.

Let $K \in \bigcup \{ \bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in T\} \}$. Then for each $G \in T$

$$K \in \bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \}.$$ 

Then $K \in \{\Gamma \cap f \mid \Gamma \in GR G\}$ by properties of generalized filter bases.

$K \in \{\Gamma \cap f \mid \Gamma \in GR G\} = \bigcup_{n \in N, \Gamma_n \in G} \Gamma_n \cap f$.

$\forall G \in T: K \geq \bigcap_{n \in N, \Gamma_n \in G} (\Gamma_n \cap f)$.

$K \geq (\Gamma_0 \cap f) \cap ... \cap (\Gamma_n \cap f)$. So $K \in \{\Gamma \cap f \mid \Gamma \in \bigcup_{n \in N, \Gamma_n \in G} \Gamma_n \} = \{\Gamma \cap f \mid \Gamma \in \bigcup_{n \in N, \Gamma_n \in G} \Gamma_n \}$.

$K \in \{\Gamma \cap f \mid \Gamma \in \bigcup_{n \in N, \Gamma_n \in G} \Gamma_n \}$.

Thus $K \in \{\Gamma \cap f \mid \Gamma \in \bigcup_{n \in N, \Gamma_n \in G} \Gamma_n \}$. 

$\square$

### 9.4 Proof of the main result

**Theorem 9.7.** $(\bigcup T) \circ F = \bigcup \{ G \circ f \mid G \in T \}$ for every principal reloid $F = \uparrow^{RLD}(\text{Src} F; \text{Dst} g)$ and a set $T$ of reloids from $\text{Dst} F$ to some set $V$. (In other words principal reloids are co-metacomplete and thus also metacomplete by duality.)

**Proof.**

$$(\bigcup T) \circ F = \bigcap \{^{RLD}(\text{Src} f; V)(\text{dom})(\bigcup T) \otimes F\}$$

$$= \text{dom} \bigcap \{^{RLD}(\text{Src} f \times V; U)(\bigcup T) \otimes F\}$$

$$= \bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in \bigcup T\};$$

$$\bigcup \{ G \circ f \mid G \in T \} = \bigcap \{ \text{dom} \bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in T\} \}$$

$$= \text{dom} \bigcup \{ \bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in T\} \}$$

It’s enough to prove

$$\bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in GR \bigcup T\} = \bigcup \{ \bigcap \{^{RLD}(\text{Src} f \times V; U)(G \otimes f) \mid G \in T\} \}$$
but this is the statement of the lemma.

9.5 Embedding reloids into funcoids

Definition 9.8. Let \( f \) be a reloid. The funcoid \( \rho f \in \text{FCD}(\text{Src } f; \text{Dst } f) \) is defined by the formulas:
\[
\rho f x = f \circ x \quad \text{and} \quad \rho f^{-1} y = f^{-1} \circ y
\]
where \( x \) are endoreloids on \( \text{Src } f \) and \( y \) are endoreloids on \( \text{Dst } f \).

Proposition 9.9. It is really a funcoid (if we equate reloids \( x \) and \( y \) with corresponding filters on Cartesian products of sets).

Proof. \( y \not\in (\rho f)x \Leftrightarrow y \not\in f \circ x \Leftrightarrow f^{-1} \circ y \not\in x \Leftrightarrow (\rho f^{-1})y \neq x \).

Corollary 9.10. \( (\rho f)^{-1} = \rho f^{-1} \).

Definition 9.11. It can be continued to arbitrary funcoids \( x \) having source \( \text{Src } f \) by the formula
\[
\rho f x = \rho f \circ x = (\rho f) x = ((\rho g) \circ (\rho f)) x.
\]

Proposition 9.12. \( \rho f \) is an injection.

Proof. Consider \( x = \text{id}_{\text{Src } f} \).

Proposition 9.13. \( \rho (g \circ f) = (\rho g) \circ (\rho f) \).

Proof. \( (\rho (g \circ f)) x = g \circ f \circ x = (\rho g) (\rho f) x = ((\rho g) \circ (\rho f)) x \). Thus \( (\rho (g \circ f)) = (\rho g) \circ (\rho f) = ((\rho g) \circ (\rho f)) \) and so \( (\rho (g \circ f)) = (\rho g) \circ (\rho f) \).

Theorem 9.14. \( \rho \bigsqcup F = \bigsqcup (\rho f) \) for a set \( F \) of reloids.

Proof. It’s enough to prove \( (\rho \bigsqcup F)^* X = (\bigsqcup (\rho f))^* X \) for a set \( X \).

Really, \( (\rho \bigsqcup F)^* X = (\rho \bigsqcup F)^\uparrow X = (\bigsqcup (\rho f))^\uparrow X = \bigsqcup \{ f \circ \uparrow X \mid f \in F \} = \bigsqcup \{ (\rho f)^\uparrow X \mid f \in F \} = \bigsqcup \{ f \circ \uparrow X, f \in F \} = \bigsqcup (\rho f)^* X \).

Conjecture 9.15. \( \rho \bigsqcup F = \bigsqcup (\rho f) \) for a set \( F \) of reloids.

Proposition 9.16. \( \rho \text{id}_{\text{RLD}(A)} = \text{id}_{\text{FCD}(A)} \).

Proof. \( (\rho \text{id}_{\text{RLD}(A)}) x = \text{id}_{\text{RLD}(A)} \circ x = x \).

We can try to develop further theory by applying embedding of reloids into funcoids for researching of properties of reloids.

Theorem 9.17. Reloid \( f \) is monovalued iff funcoid \( \rho f \) is monovalued.

Proof. \( \rho f \) is monovalued \( \Leftrightarrow (\rho f) \circ (\rho f)^{-1} \subseteq \text{id}_{\text{FCD}(A)} \Leftrightarrow \rho f \circ f^{-1} \subseteq \text{id}_{\text{FCD}(A)} \Leftrightarrow \rho (f \circ f^{-1}) \subseteq \text{id}_{\text{FCD}(A)} \Leftrightarrow f \) is monovalued.
Chapter 10

Continuous morphisms

This chapter uses the apparatus from the section “Partially ordered dagger categories”.

10.1 Traditional definitions of continuity

In this section we will show that having a funcoid or reloid $\uparrow f$ corresponding to a function $f$ we can express continuity of it by the formula $\uparrow f \circ \mu \subseteq \nu \circ \uparrow f$ (or similar formulas) where $\mu$ and $\nu$ are some spaces.

10.1.1 Pretopology

Let $(A; \text{cl}_A)$ and $(B; \text{cl}_B)$ be preclosure spaces. Then by definition a function $f: A \to B$ is continuous iff $f \text{cl}_A(X) \subseteq \text{cl}_B(f(X))$ for every $X \in \mathcal{P}A$. Let now $\mu$ and $\nu$ be endofuncoids corresponding correspondingly to $\text{cl}_A$ and $\text{cl}_B$. Then the condition for continuity can be rewritten as

$$f \circ \text{cl}_\mu \subseteq \text{cl}_\nu \circ f $$

10.1.2 Proximity spaces

Let $\mu$ and $\nu$ be proximity spaces (which I consider a special case of endofuncoids). By definition a function $f$ is a proximity-continuous map (also called equicontinuous) from $\mu$ to $\nu$ iff

$$\forall X \in \mathcal{P}(\text{Src} \mu), Y \in \mathcal{P}(\text{Dst} \mu); (X [\mu]^* Y \Rightarrow (f)X [\nu]^* (f)Y).$$

Equivalently transforming this formula we get (writing $\uparrow$ instead of $\uparrow_{\text{FCD}(\text{Ob} \mu; \text{Ob} \nu)}$ for brevity):

$$\forall X \in \mathcal{P}(\text{Src} \mu), Y \in \mathcal{P}(\text{Dst} \mu); (X [\mu]^* Y \Rightarrow \uparrow \text{Ob} \nu X \cap (\uparrow f)^{-1} (\uparrow \text{Ob} \mu))$$

So a function $f$ is proximity continuous iff $\mu \subseteq (\uparrow_{\text{FCD}(\text{Ob} \mu; \text{Ob} \nu)} f \circ \nu \circ \uparrow_{\text{FCD}(\text{Ob} \mu; \text{Ob} \nu)} f$.

10.1.3 Uniform spaces

Uniform spaces are a special case of endoreloids.

Let $\mu$ and $\nu$ be uniform spaces. By definition a function $f$ is a uniformly continuous map from $\mu$ to $\nu$ iff

$$\forall \varepsilon \in \text{GR} \nu \exists \delta \in \text{GR} \nu \forall (x; y) \in \delta; (fx; fy) \in \varepsilon.$$
Proposition 10.3.

Our three definitions of continuity

Continuous morphisms

Equivalently transforming this formula we get:

\[ \forall \epsilon \in \text{GR} \nu \exists \delta \in \text{GR} \nu \forall (x; y) \in \delta: \{(fx; fy)\} \subseteq \epsilon; \]

\[ \forall \epsilon \in \text{GR} \nu \exists \delta \in \text{GR} \nu \forall (x; y) \in \delta: f \circ \{(x; y)\} \circ f^{-1} \subseteq \epsilon; \]

\[ \forall \epsilon \in \text{GR} \nu \exists \delta \in \text{GR} \nu; f \circ \delta \circ f^{-1} \subseteq \epsilon; \]

\[ \forall \epsilon \in \text{GR} \nu; f \circ \text{RLD}(\text{Ob} \mu; \text{Ob} \nu) f \circ \mu \circ (f \circ \text{RLD}(\text{Ob} \mu; \text{Ob} \nu) f)^{-1} \subseteq \text{RLD}(\text{Ob} \nu; \text{Ob} \nu) \epsilon; \]

\[ f \circ \text{RLD}(\text{Ob} \mu; \text{Ob} \nu) f \circ \mu \circ (f \circ \text{RLD}(\text{Ob} \mu; \text{Ob} \nu) f)^{-1} \subseteq \nu. \]

So a function \( f \) is uniformly continuous if \( f \circ \text{RLD}(\text{Dest} \mu; \text{Dest} \nu) f \circ \mu \circ (f \circ \text{RLD}(\text{Dest} \mu; \text{Dest} \nu) f)^{-1} \subseteq \nu. \)

10.2 Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let’s summarize these three algebraic formulas:

Let \( \mu \) and \( \nu \) be endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms \( f \) of this precategory which conform to the following formula:

\[ f \in \text{C}(\mu; \nu) \Leftrightarrow f \in \text{Mor} \text{(Ob} \mu; \text{Ob} \nu) \land f \circ \mu \subseteq \nu \circ f. \]

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

\[ f \in \text{C}'(\mu; \nu) \Leftrightarrow f \in \text{Mor} \text{(Ob} \mu; \text{Ob} \nu) \land \mu \subseteq f^1 \circ \nu \circ f; \]

\[ f \in \text{C}''(\mu; \nu) \Leftrightarrow f \in \text{Mor} \text{(Ob} \mu; \text{Ob} \nu) \land f \circ \mu \circ f^1 \subseteq \nu. \]

Remark 10.1. In the examples (above) about funcoids and reloids the “dagger functor” is the reverse of a funcoid or reloid, that is \( f^\dagger = f^{-1}. \)

Proposition 10.2. Every of these three definitions of continuity forms a wide sub-precategory (wide subcategory if the original precategory is a category).

Proof.

C. Let \( f \in \text{C}(\mu; \nu), g \in \text{C}(\nu; \pi). \) Then \( f \circ \mu \subseteq \nu \circ f, g \circ \nu \subseteq \pi \circ g; g \circ f \circ \mu \subseteq g \circ \nu \circ f \subseteq \pi \circ g \circ f. \)

So \( g \circ f \in \text{C}(\mu; \pi). \) \( 1_{\text{Ob} \mu} \in \text{C}(\mu; \mu) \) is obvious.

C'. Let \( f \in \text{C}'(\mu; \nu), g \in \text{C}'(\nu; \pi). \) Then \( \mu \subseteq f^1 \circ \nu \circ f, \nu \subseteq g^1 \circ \pi \circ g; \)

\[ \mu \subseteq f^1 \circ g^1 \circ \pi \circ g \circ f; \quad \mu \subseteq (g \circ f)^1 \circ \pi \circ (g \circ f). \]

So \( g \circ f \in \text{C}'(\mu; \pi). \) \( 1_{\text{Ob} \mu} \in \text{C}'(\mu; \mu) \) is obvious.

C''. Let \( f \in \text{C}''(\mu; \nu), g \in \text{C}''(\nu; \pi). \) Then \( f \circ \mu \circ f^1 \subseteq \nu, g \circ \nu \circ g^1 \subseteq \pi; \)

\[ g \circ f \circ \mu \circ f^1 \circ g^1 \subseteq \pi; \quad (g \circ f)^1 \circ \mu \circ (g \circ f)^1 \subseteq \pi. \]

So \( g \circ f \in \text{C}''(\mu; \pi). \) \( 1_{\text{Ob} \mu} \in \text{C}''(\mu; \mu) \) is obvious. \( \Box \)

Proposition 10.3. For a monovalued morphism \( f \) of a partially ordered dagger category and its endomorphisms \( \mu \) and \( \nu \)

\[ f \in \text{C}'(\mu; \nu) \Rightarrow f \in \text{C}(\mu; \nu) \Rightarrow f \in \text{C}''(\mu; \nu). \]

Proof. Let \( f \in \text{C}'(\mu; \nu). \) Then \( \mu \subseteq f^1 \circ \nu \circ f, f \circ \mu \circ f^1 \circ \nu \circ f \subseteq 1_{\text{Dest} f} \circ \nu \circ f = \nu \circ f; f \in \text{C}(\mu; \nu). \)

Let \( f \in \text{C}(\mu; \nu). \) Then \( f \circ \mu \subseteq \nu \circ f, f \circ \mu \circ f^1 \subseteq \nu \circ f \circ f^1 \subseteq \nu \circ 1_{\text{Dest} f} = \nu; f \in \text{C}''(\mu; \nu). \) \( \Box \)

Proposition 10.4. For an entirely defined morphism \( f \) of a partially ordered dagger category and its endomorphisms \( \mu \) and \( \nu \)

\[ f \in \text{C}''(\mu; \nu) \Rightarrow f \in \text{C}(\mu; \nu) \Rightarrow f \in \text{C}'(\mu; \nu). \]
Proof. Let \( f \in C''(\mu; \nu) \). Then \( f \circ \mu \circ f^! \subseteq \nu; \ f \circ \mu \circ f^! \subseteq f \circ \nu; \ f \circ \mu \circ 1_{\text{Src}} f \subseteq \nu \circ f; \ f \circ \mu \subseteq \nu \circ f; \ f \in C(\mu; \nu) \).

Let \( f \in C(\mu; \nu) \). Then \( f \circ \mu \subseteq \nu; \ f^! \circ f \subseteq f \circ \nu; \ f \circ \mu \subseteq 1_{\text{Src}} f \subseteq f \circ \nu; \ f \subseteq f \circ \nu; \ f \in C''(\mu; \nu) \).

For entirely defined monovalued morphisms our three definitions of continuity coincide:

**Theorem 10.5.** If \( f \) is a monovalued and entirely defined morphism of a partially ordered dagger precategory then

\[ f \in C'(\mu; \nu) \iff f \in C(\mu; \nu) \iff f \in C''(\mu; \nu). \]

**Proof.** From two previous propositions.

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is proximity-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

**Theorem 10.6.** If an entirely defined morphism of the category of reloids \( f \in C''(\mu; \nu) \) for some endomorphisms \( \mu \) and \( \nu \) of the category of reloids, then \( (\text{FCD}) f \in C'(\text{FCD}) \mu; (\text{FCD}) \nu) \).

**Exercise 10.1.** I leave a simple exercise for the reader to prove the last theorem.

### 10.3 Continuity of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids on some set regarding the composition.) Consider also some lattice (10.3 Continuity of a restricted morphism Theorem 10.6.

**Proof.**

Let \( f \in C''(\mu; \nu) \). Then \( f \circ \mu \subseteq \nu; \ f^! \circ f \subseteq f \circ \nu; \ f \circ \mu \subseteq 1_{\text{Src}} f \subseteq f \circ \nu; \ f \subseteq f \circ \nu; \ f \in C''(\mu; \nu) \).

For entirely defined monovalued morphisms our three definitions of continuity coincide:

**Theorem 10.5.** If \( f \) is a monovalued and entirely defined morphism of a partially ordered dagger precategory then

\[ f \in C'(\mu; \nu) \iff f \in C(\mu; \nu) \iff f \in C''(\mu; \nu). \]

**Proof.** From two previous propositions.

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is proximity-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

**Theorem 10.6.** If an entirely defined morphism of the category of reloids \( f \in C''(\mu; \nu) \) for some endomorphisms \( \mu \) and \( \nu \) of the category of reloids, then \( (\text{FCD}) f \in C'(\text{FCD}) \mu; (\text{FCD}) \nu) \).

**Exercise 10.1.** I leave a simple exercise for the reader to prove the last theorem.

### 10.3 Continuity of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids on some set regarding the composition.) Consider also some lattice (10.3 Continuity of a restricted morphism Theorem 10.6.

We will map every object \( A \) to so called **restricted identity element** \( I_A \) of the semigroup (for example restricted identity funcoid or restricted identity reloid). For identity elements we will require

1. \( I_A \circ I_B = I_A \cap I_B \);
2. \( f \circ I_A \subseteq f; \ I_A \circ f \subseteq f \).

In the case when our semigroup is “dagger” (that is is a dagger precategory) we will require also

\( (I_A) = I_A \).

We can define restricting an element \( f \) of our semigroup to an object \( A \) by the formula \( f \mid_A = f \circ I_A \).

We can define **rectangular restricting** an element \( m \) of our semigroup to objects \( A \) and \( B \) as \( I_B \circ f \circ I_A \). Optionally we can define direct product \( A \times B \) of two objects by the formula (true for funcoids and for reloids):

\[ \mu \cap (A \times B) = I_B \circ f \circ I_A. \]

**Square restricting** of an element \( \mu \) to an object \( A \) is a special case of rectangular restricting and is defined by the formula \( I_A \circ f \circ I_A \) (or by the formula \( \mu \cap (A \times A) \)).

**Theorem 10.7.** For every elements \( f, \mu, \nu \) our semigroup and an object \( A \)

1. \( f \in C(\mu; \nu) \Rightarrow f \mid_A \in C(I_A \circ \mu \circ I_A; \nu); \)
2. \( f \in C'(\mu; \nu) \Rightarrow f \mid_A \in C'(I_A \circ \mu \circ I_A; \nu); \)
3. \( f \in C''(\mu; \nu) \Rightarrow f \mid_A \in C''(I_A \circ \mu \circ I_A; \nu); \)

(Two last items are true for the case when our semigroup is dagger.)

**Proof.**

1. \( f \mid_A \in C(I_A \circ \mu \circ I_A; \nu) \iff f \mid_A \circ I_A \circ \mu \circ I_A \subseteq \nu \iff f \mid_A \circ I_A \circ \mu \circ I_A \subseteq \nu \iff f \mid_A \circ \mu \circ I_A \subseteq f \circ \nu \iff f \mid_A \circ \mu \circ I_A \subseteq f \circ \nu \iff f \in C(\mu; \nu) \).
2. \( f \mid_A \in C'(I_A \circ \mu \circ I_A; \nu) \iff I_A \circ \mu \circ I_A \subseteq (f \mid_A) \circ \nu \circ f \iff I_A \circ \mu \circ I_A \subseteq (f \mid_A) \circ \nu \circ f \iff f \in C'(\mu; \nu) \).
3. \( f \mid_A \in C''(I_A \circ \mu \circ I_A; \nu) \iff f \mid_A \circ I_A \circ \mu \circ I_A \circ (f \mid_A) \circ \nu \circ f \iff f \mid_A \circ I_A \circ \mu \circ I_A \circ (f \mid_A) \circ \nu \circ f \iff f \in C''(\mu; \nu) \).
Chapter 11

Connectedness regarding funcoids and reloids

Definition 11.1. I will call endoreloids and endofuncoids reloids and funcoids with the same source and destination.

11.1 Some lemmas

Lemma 11.2. If \( \neg(A \cdot [f]^* B) \land A \cup B \in \text{dom } f \cup \text{im } f \) then \( f \) is closed on \( \uparrow^U A \) for a funcoid \( f \in \text{FCD}(U; U) \) for every sets \( U \) and \( A, B \in \mathcal{P}U \).

Proof. Let \( A \cup B \in \text{dom } f \cup \text{im } f \). \( \neg(A \cdot [f]^* B) \Leftrightarrow \uparrow^U B \cap \langle f \rangle \uparrow^U A = \emptyset \Rightarrow (\text{dom } f \cup \text{im } f) \cap \uparrow^U B \cap \langle f \rangle^* A = \emptyset \Rightarrow (\text{dom } f \cup \text{im } f) \setminus \uparrow^U A \cap \langle f \rangle^* A = \emptyset \Rightarrow \langle f \rangle^* A \subseteq \uparrow^U A \). \( \square \)

Corollary 11.3. If \( \neg(A \cdot [f]^* B) \land A \cup B \in \text{dom } f \cup \text{im } f \) then \( f \) is closed on \( \uparrow^U (A \setminus B) \) for a funcoid \( f \in \text{FCD}(U; U) \) for every sets \( U \) and \( A, B \in \mathcal{P}U \).

Proof. Let \( \neg(A \cdot [f]^* B) \land A \cup B \in \text{dom } f \cup \text{im } f \). Then \( \neg((A \setminus B) \cdot [f]^* B) \land (A \setminus B) \cup B \in \text{dom } f \cup \text{im } f \). \( \square \)

Lemma 11.4. If \( \neg(A \cdot [f]^* B) \land A \cup B \in \text{dom } f \cup \text{im } f \) then \( \neg(A \cdot [f^n]^* B) \) for every whole positive \( n \).

Proof. Let \( \neg(A \cdot [f]^* B) \land A \cup B \in \text{dom } f \cup \text{im } f \). From the above lemma \( \langle f \rangle^* A \subseteq \uparrow^U A \). \( \uparrow^U B \cap \langle f \rangle \uparrow^U A = \emptyset \), consequently \( \langle f \rangle^* A \subseteq \uparrow^U (A \setminus B) \). Because (by the above corollary) \( f \) is closed on \( \uparrow^U (A \setminus B) \), then \( \langle f \rangle \langle f \rangle \uparrow^U A \subseteq \uparrow^U (A \setminus B) \), \( \langle f \rangle \langle f \rangle \langle f \rangle \uparrow^U A \subseteq \uparrow^U (A \setminus B) \), etc. So \( (f^n) \uparrow^U A \subseteq \uparrow^U (A \setminus B) \), \( \uparrow^U B \cap (f^n) \uparrow^U A = \emptyset \Rightarrow \langle f \rangle \uparrow^U A \). \( \square \)

11.2 Endomorphism series

Definition 11.5. \( S_1(\mu) = \mu \sqcup \mu^2 \sqcup \mu^3 \sqcup \ldots \) for an endomorphism \( \mu \) of a precategory with countable join of morphisms.

Definition 11.6. \( S(\mu) = \mu^0 \sqcup S_1(\mu) = \mu^0 \sqcup \mu \sqcup \mu^2 \sqcup \mu^3 \sqcup \ldots \) where \( \mu^0 = 1_{\text{Ob } \mu} \) (identity morphism for the object \( \text{Ob } \mu \)) where \( \text{Ob } \mu \) is the object of endomorphism \( \mu \) for an endomorphism \( \mu \) of a category with countable join of morphisms.

I call \( S_1 \) and \( S \) endomorphism series.

We will consider the collection of all binary relations (on a set \( \mathcal{U} \)), as well as the collection of all funcoids and the collection of all reloids on a fixed set, as categories with single object \( \mathcal{U} \) and the identity morphisms \( \text{id}_\mathcal{U} \), \( \text{id}^{\text{FCD}(\mathcal{U})} \), \( \text{id}^{\text{RLD}(\mathcal{U})} \).

Proposition 11.7. The relation \( S(\mu) \) is transitive for the category of binary relations.
11.3 Connectedness regarding binary relations

Before going to research connectedness for funcoids and reloids we will excursion into the basic special case of connectedness regarding binary relations on a set $\Omega$.

**Definition 11.8.** A set $A$ is called (strongly) connected regarding a binary relation $\mu$ when

$$\forall X \in \mathcal{P}(\text{dom } \mu) \setminus \{\emptyset\}, Y \in \mathcal{P}(\text{im } \mu) \setminus \{\emptyset\}: (X \cup Y = A \Rightarrow X [\mu] Y).$$

Let $\Omega$ be a set.

**Definition 11.9.** Path between two elements $a, b \in \Omega$ in a set $A \subseteq \Omega$ through binary relation $\mu$ is the finite sequence $x_0 \ldots x_n$ where $x_0 = a, x_n = b$ for $n \in \mathbb{N}$ and $x_i(\mu \cap A \times A) x_{i+1}$ for every $i = 0, \ldots, n - 1$. $n$ is called path length.

**Proposition 11.10.** There exists path between every element $a \in \Omega$ and that element itself.

**Proof.** It is the path consisting of one vertex (of length 0).

**Proposition 11.11.** There is a path from element $a$ to element $b$ in a set $A$ through a binary relation $\mu$ iff $a (\mu \cap A \times A) b$ (that is $(a, b) \in \mu A A$).

**Proof.**

$\Rightarrow$. If a path from $a$ to $b$ exists, then $\{b\} \subseteq (\mu \cap A \times A)^n \{a\}$ where $n$ is the path length. Consequently $\{b\} \subseteq (\mu \cap A \times A)^n \{a\}$; $a (\mu \cap A \times A) b$.

$\Leftarrow$. If $a (\mu \cap A \times A) b$ then there exists $n \in \mathbb{N}$ such that $(\mu \cap A \times A)^n b$. By definition of composition of binary relations this means that there exist finite sequence $x_0 \ldots x_n$ where $x_0 = a, x_n = b$ for $n \in \mathbb{N}$ and $x_i(\mu \cap A \times A) x_{i+1}$ for every $i = 0, \ldots, n - 1$. That is there is a path from $a$ to $b$.

**Theorem 11.12.** The following statements are equivalent for a binary relation $\mu$ and a set $A$:

1. For every $a, b \in A$ there is a path between $a$ and $b$ in $A$ through $\mu$.
2. $\mu A A \supseteq A A$.
3. $\mu A A = A A$.
4. $A$ is connected regarding $\mu$.

**Proof.**

(1)$\Rightarrow$(2). Let for every $a, b \in A$ there is a path between $a$ and $b$ in $A$ through $\mu$. Then $a (\mu \cap A \times A) b$ for every $a, b \in A$. It is possible only when $\mu A A \supseteq A A$.

(3)$\Rightarrow$(1). For every two vertices $a$ and $b$ we have $a (\mu \cap A \times A) b$. So (by the previous theorem) for every two vertices $a$ and $b$ there exists a path from $a$ to $b$.

(3)$\Rightarrow$(4). Suppose $\neg (X [\mu \cap A \times A] Y)$ for some $X, Y \in \mathcal{P} \Omega \setminus \{\emptyset\}$ such that $X \cup Y = A$. Then by a lemma $\neg (X [\mu \cap A \times A]^n Y)$ for every $n \in \mathbb{N}$. Consequently $\neg (X [\mu \cap A \times A] Y)$. So $\mu A A \neq A A$.
11.4 Connectedness regarding funcoids and reloids

(4) ⇒ (3). If \( (S(\mu \cap (A \times A))) \{v\} = A \) for every vertex \( v \) then \( S(\mu \cap (A \times A)) = A \times A \). Consider the remaining case when \( V^{\text{def}} = \{S(\mu \cap (A \times A)) \} \{v\} < A \) for some vertex \( v \). Let \( W = A \setminus V \). If \( \operatorname{card} A = 1 \) then \( S(\mu \cap (A \times A)) \supseteq \operatorname{id}_A = A \times A \); otherwise \( W \neq \emptyset \). Then \( V \cup W = A \) and so \( V [\mu] W \) what is equivalent to \( V [\mu \cap (A \times A)] W \) that is \( \langle \mu \cap (A \times A) \rangle V \cap W \neq \emptyset \). This is impossible because \( \langle \mu \cap (A \times A) \rangle V = \langle \mu \cap (A \times A) \rangle S(\mu \cap (A \times A)) \rangle V = \langle S(\mu \cap (A \times A)) \rangle V \subseteq \langle S(\mu \cap (A \times A)) \rangle V = V \).

(2) ⇒ (3). Because \( S(\mu \cap (A \times A)) \subseteq A \times A \). \[
\]

**Corollary 11.13.** A set \( A \) is connected regarding a binary relation \( \mu \) iff it is connected regarding \( \mu \cap (A \times A) \).

**Definition 11.14.** A connected component of a set \( A \) regarding a binary relation \( F \) is a maximal connected subset of \( A \).

**Theorem 11.15.** The set \( A \) is partitioned into connected components (regarding every binary relation \( F \)).

**Proof.** Consider the binary relation \( a \sim b \Leftrightarrow a (S(F)) b \wedge b (S(F)) a \). \( \sim \) is a symmetric, reflexive, and transitive relation. So all points of \( A \) are partitioned into a collection of sets \( Q \). Obviously each component is (strongly) connected. If a set \( R \subseteq A \) is greater than one of that connected components \( A \) then it contains a point \( b \in B \) where \( B \) is some other connected component. Consequently \( R \) is disconnected.

**Proposition 11.16.** A set is connected (regarding a binary relation) iff it has one connected component.

**Proof.** Direct implication is obvious. Reverse is proved by contradiction.

---

**11.4 Connectedness regarding funcoids and reloids**

**Definition 11.17.** \( S^*_1(\mu) = \prod \{ \{ \mu \} S_1(M) \mid M \in \text{xyGR} \mu \} \) for an endoreloid \( \mu \).

**Definition 11.18.** Connectivity reloid \( S^*(\mu) \) for an endoreloid \( \mu \) is defined as follows:

\[
S^*(\mu) = \prod \{ \{ \mu \} S(M) \mid M \in \text{xyGR} \mu \}.
\]

**Remark 11.19.** Do not mess the word connectivity with the word connectedness which means being connected.\(^{11.1}\)

**Proposition 11.20.** \( S^*(\mu) = \text{id}^{\text{RLD}(\text{Ob} \mu)} \sqcup S^*_1(\mu) \) for every endoreloid \( \mu \).

**Proof.** By the proposition 4.190.

**Proposition 11.21.** \( S^*(\mu) = S(\mu) \) if \( \mu \) is a principal reloid.

**Proof.** \( S^*(\mu) = \prod \{ S(\mu) \} = S(\mu) \).

**Definition 11.22.** A filter \( \mathcal{A} \in \mathfrak{F}(\text{Ob} \mu) \) is called connected regarding an endoreloid \( \mu \) when \( S^*(\mu \cap (A \times \text{RLD} \mathcal{A})) \supseteq \mathcal{A} \times \text{RLD} \mathcal{A} \).

**Obvious 11.23.** A filter \( \mathcal{A} \in \mathfrak{F}(\text{Ob} \mu) \) is connected regarding an endoreloid \( \mu \) iff \( S^*(\mu \cap (A \times \text{RLD} \mathcal{A})) = \mathcal{A} \times \text{RLD} \mathcal{A} \).

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\(^{11.1}\) In some math literature these two words are used interchangeably.
Definition 11.24. A filter $\mathcal{A} \in \mathfrak{F}(\text{Ob} \mu)$ is called connected regarding an endofuncoid $\mu$ when
\[ \forall X, Y \in \mathfrak{F}(\text{Ob} \mu) \setminus \{0^{\text{Ob} \mu}\}: (X \cup Y = A \Rightarrow X [\mu] Y). \]

Proposition 11.25. Let $A$ be a set. The filter $\uparrow^{\text{Ob} \mu} A$ is connected regarding an endofuncoid $\mu$ iff
\[ \forall X, Y \in \mathfrak{P}(\text{Ob} \mu) \setminus \{0\}: (X \cup Y = A \Rightarrow X [\mu]^* Y). \]

Proof.
\[ \Rightarrow. \text{ Obvious.} \]
\[ \Leftarrow. \text{ It follows from co-separability of filters.} \]

Theorem 11.26. The following are equivalent for every set $A$ and binary relation $\mu$ on a set $U$:
1. $A$ is connected regarding binary relation $\mu$.
2. $\uparrow^U A$ is connected regarding $\uparrow^{\text{RLD}(U; U)} \mu$.
3. $\uparrow^U A$ is connected regarding $\uparrow^{\text{FCD}(U; U)} \mu$.

Proof.
\[ (1) \Leftrightarrow (2). \quad S^* (\uparrow^{\text{RLD}(U; U)} \mu \cap (\uparrow^U A \times \text{RLD} \uparrow^U A)) = S^* (\uparrow^{\text{RLD}(U; U)} (\mu \cap (A \times A))) = \uparrow^{\text{RLD}(U; U)} S (\mu \cap (A \times A)). \] So $S^* (\uparrow^{\text{RLD}(U; U)} (\mu \cap (A \times A))) \supseteq \uparrow^U A \times \text{RLD} \uparrow^U A \leftrightarrow \uparrow^{\text{RLD}(U; U)} S (\mu \cap (A \times A)) \supseteq \uparrow^{\text{RLD}(U; U)} (A \times A) = \uparrow^U A \times \text{RLD} \uparrow^U A.$

\[ (1) \Leftrightarrow (3). \text{ It follows from the previous proposition.} \]

Next is conjectured a statement more strong than the above theorem:

Conjecture 11.27. Let $\mathcal{A}$ be a filter on a set $U$ and $F$ be a binary relation on $U$.
$\mathcal{A}$ is connected regarding $\uparrow^{\text{FCD}(U; U)} F$ iff $\mathcal{A}$ is connected regarding $\uparrow^{\text{RLD}(U; U)} F$.

Obvious 11.28. A filter $\mathcal{A}$ is connected regarding a reloid $\mu$ iff it is connected regarding the reloid $\mu \cap (A \times \text{RLD} A)$.

Obvious 11.29. A filter $\mathcal{A}$ is connected regarding a funcoid $\mu$ iff it is connected regarding the funcoid $\mu \cap (A \times \text{FCD} A)$.

Theorem 11.30. A filter $\mathcal{A}$ is connected regarding a reloid $f$ iff $\mathcal{A}$ is connected regarding every $F \in (\uparrow^{\text{RLD}}) xyGR f$.

Proof.
\[ \Rightarrow. \text{ Obvious.} \]
\[ \Leftarrow. \text{ $\mathcal{A}$ is connected regarding } \uparrow^{\text{RLD}} f \text{ iff } S(f) = F^0 \cup F^1 \cup F^2 \cup \ldots \in A \times \text{RLD} A. \]
\[ S^*(f) = \bigcap \{ \uparrow^{\text{RLD}} S(F) \mid F \in xyGR f \} \supseteq \bigcap \{ A \times \text{RLD} A \mid F \in xyGR f \} = A \times \text{RLD} A. \]

Conjecture 11.31. A filter $\mathcal{A}$ is connected regarding a funcoid $f$ iff $\mathcal{A}$ is connected regarding every $F \in (\uparrow^{\text{FCD}}) \text{xyGR } f$.

The above conjecture is open even for the case when $\mathcal{A}$ is a principal filter.

Conjecture 11.32. A filter $\mathcal{A}$ is connected regarding a reloid $f$ iff it is connected regarding the funcoid $(\text{FCD}) f$.

The above conjecture is true in the special case of principal filters:

Proposition 11.33. A filter $\uparrow^{\text{Ob} \mu} A$ (for a set $A$) is connected regarding an endoreloid $f$ iff it is connected regarding the endofuncoid $(\text{FCD}) f$.
**Proof.** \( \uparrow^{\text{Ob}} A \) is connected regarding a reloid \( f \) iff \( A \) is connected regarding every \( F \in \text{xyGR} f \) that is when \( \forall F \in \text{xyGR} f \forall X, Y \in \mathfrak{O}(\text{Ob} f) \setminus \{0^{\text{Ob} f}\} : (X \sqcup Y = \uparrow^{\text{Ob}} f A \Rightarrow X \uparrow^{\text{FCD} F} Y) \leftrightarrow \forall X, Y \in \mathfrak{O}(\text{Ob} f) \setminus \{0^{\text{Ob} f}\} : (X \sqcup Y = \uparrow^{\text{Ob}} f A \Rightarrow \forall F \in \text{xyGR} f : X \uparrow^{\text{FCD} F} Y) \leftrightarrow \forall X, Y \in \mathfrak{O}(\text{Ob} f) \setminus \{0^{\text{Ob} f}\} : (X \sqcup Y = \uparrow^{\text{Ob}} f A \Rightarrow \forall F \in \text{xyGR} f : X \uparrow^{\text{FCD} F} Y) \leftrightarrow \forall X, Y \in \mathfrak{O}(\text{Ob} f) \setminus \{0^{\text{Ob} f}\} : (X \sqcup Y = \uparrow^{\text{Ob}} f A \Rightarrow X \uparrow^{\text{FCD} F} Y) \leftrightarrow \) that is when the set \( \uparrow^{\text{Ob}} f A \) is connected regarding the funcoid \( (\text{FCD} f) \). \( \square \)

**Conjecture 11.34.** A set \( A \) is connected regarding an endofuncoid \( \mu \) iff for every \( a, b \in A \) there exists a totally ordered set \( P \subseteq A \) such that \( \min P = a, \max P = b \) and \( \forall q \in P \setminus \{b\} : \{x \in P \mid x < q\} [\mu]_{\uparrow} \{x \in P \mid x > q\} \).

Weaker condition:
\[
\forall q \in P \setminus \{b\} : \{x \in P \mid x < q\} [\mu] \{x \in P \mid x > q\} \forall q \in P \setminus \{a\} : \{x \in P \mid x < q\} [\mu] \{x \in P \mid x > q\}.
\]

### 11.5 Algebraic properties of \( S \) and \( S^* \)

**Theorem 11.35.** \( S^*(S^*(f)) = S^*(f) \) for every endoreloid \( f \).

**Proof.** \( S^*(S^*(f)) = \bigcap \{\uparrow^{\text{RLD}} R | R \in \text{xyGR} S^*(f)\} \subseteq \bigcap \{\uparrow^{\text{RLD}} S(R) | R \in \{S(F) | F \in \text{xyGR} f\}\} = \bigcap \{\uparrow^{\text{RLD}} S(R) | R \in \text{xyGR} f\} = \bigcap \{\uparrow^{\text{RLD}} S(R) | R \in \text{xyGR} f\} = S^*(f). \)

So \( S^*(S^*(f)) \subseteq S^*(f) \). That \( S^*(S^*(f)) \supseteq S^*(f) \) is obvious. \( \square \)

**Corollary 11.36.** \( S^*(S(f)) = S(S^*(f)) = S^*(f) \) for every endoreloid \( f \).

**Proof.** Obviously \( S^*(S(f)) \supseteq S^*(f) \) and \( S(S^*(f)) \supseteq S^*(f) \).

But \( S^*(S(f)) \supseteq S^*(S(f)) = S^*(f) \) and \( S(S^*(f)) \supseteq S^*(S^*(f)) = S^*(f) \). \( \square \)

**Conjecture 11.37.** \( S(S(f)) = S(f) \) for

1. every endoreloid \( f \);
2. every endofuncoid \( f \).

**Conjecture 11.38.** For every endoreloid \( f \)

1. \( S(f) \circ S(f) = S(f) \);
2. \( S^*(f) \circ S^*(f) = S^*(f) \);
3. \( S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f) \).

**Conjecture 11.39.** \( S(f) \circ S(f) = S(f) \) for every endofuncoid \( f \).
Chapter 12
Total boundness of reloids

12.1 Thick binary relations

**Definition 12.1.** I will call \( \alpha \)-thick and denote \( \text{thick}_\alpha(E) \) a Rel-endomorphism \( E \) when there exists a finite cover \( S \) of \( \text{Ob} \) \( E \) such that \( \forall A \in S: A \times A \subseteq \text{GR} E \).

**Definition 12.2.** \( \text{CS}(S) = \bigcup \{ A \times A \mid A \in S \} \) for a collection \( S \) of sets.

**Remark 12.3.** \( \text{CS} \) means “Cartesian squares”.

**Obvious 12.4.** A Rel-endomorphism is \( \alpha \)-thick iff there exists a finite cover \( S \) of \( \text{Ob} E \) such that \( \text{CS}(S) \subseteq \text{GR} E \).

**Definition 12.5.** I will call \( \beta \)-thick and denote \( \text{thick}_\beta(E) \) a Rel-endomorphism \( E \) when there exists a finite set \( B \) such that \( (E)B = \text{Ob} E \).

**Proposition 12.6.** \( \text{thick}_\alpha(E) \Rightarrow \text{thick}_\beta(E) \).

**Proof.** Let \( \text{thick}_\alpha(E) \). Then there exists a finite cover \( S \) of the set \( \text{Ob} E \) such that \( \forall A \in S: A \times A \subseteq \text{GR} E \). Without loss of generality assume \( A \neq \emptyset \) for every \( A \in S \). So \( A \subseteq \langle E \rangle \{ x_A \} \) for some \( x_A \) for every \( A \in S \). So \( \langle E \rangle \{ x_A \mid A \in S \} = \bigcup \{ \langle E \rangle \{ x_A \} \mid A \in S \} = \text{Ob} E \) and thus \( E \) is \( \beta \)-thick. \( \square \)

**Obvious 12.7.** Let \( X \) be a set, \( A \) and \( B \) are Rel-endomorphisms on \( X \) and \( B \supseteq A \). Then:
- \( \text{thick}_\alpha(A) \Rightarrow \text{thick}_\alpha(B) \);
- \( \text{thick}_\beta(A) \Rightarrow \text{thick}_\beta(B) \).

**Example 12.8.** There is a \( \beta \)-thick Rel-morphism which is not \( \alpha \)-thick.

**Proof.** Consider the Rel-morphism on \([0; 1]\) with the below graph:

\[
\Gamma = \{(x; x) \mid x \in [0; 1]\} \cup \{(x; 0) \mid x \in [0; 1]\} \cup \{(0; x) \mid x \in [0; 1]\}.
\]

\( \Gamma \) is \( \beta \)-chick because \( \langle \Gamma \rangle \{0\} = [0; 1] \).

To prove that \( \Gamma \) is not \( \alpha \)-thick it’s enough to prove that every set \( A \) such that \( A \times A \subseteq \Gamma \) is finite.
Suppose for the contrary that $A$ is infinite. Then $A$ contains more than one non-zero points $y, z$ ($y \neq z$). Without loss of generality $y < z$. So we have that $(y, z)$ is not of the form $(y; y)$ nor $(0; y)$ nor $(y, 0)$. Therefore $A \times A$ isn’t a subset of $\Gamma$.

\section{Totally bounded endoreloids}

The below is a straightforward generalization of the customary definition of totally bounded sets on uniform spaces (it’s proved below that for uniform spaces the below definitions are equivalent).

\begin{definition}
An endoreloid $f$ is $\alpha$-totally bounded ($\text{totBound}_{\alpha}(f)$) if every $E \in \text{xyGR} f$ is $\alpha$-thick.
\end{definition}

\begin{definition}
An endoreloid $f$ is $\beta$-totally bounded ($\text{totBound}_{\beta}(f)$) if every $E \in \text{xyGR} f$ is $\beta$-thick.
\end{definition}

\begin{remark}
We could rewrite the above definitions in a more algebraic way like $x \in \text{GR} f$ (with thick, would be defined as a set rather than as a predicate), but we don’t really need this simplification.
\end{remark}

\begin{proposition}
If an endoreloid is $\alpha$-totally bounded then it is $\beta$-totally bounded.
\end{proposition}

\begin{proof}
Because thick$\alpha(E) \Rightarrow$ thick$\beta(E)$.
\end{proof}

\begin{proposition}
If an endoreloid $f$ is reflexive and Ob $f$ is finite then $f$ is both $\alpha$-totally bounded and $\beta$-totally bounded.
\end{proposition}

\begin{proof}
It enough to prove that $f$ is $\alpha$-totally bounded. Really, every $E \in \text{xyGR} f$ is reflexive. Thus $\{x\} \times \{x\} \subseteq E$ for $x \in \text{Ob f}$ and thus $\{\{x\} \mid x \in \text{Ob f}\}$ is a sought for finite cover of Ob $f$.
\end{proof}

\begin{obvious}
\begin{itemize}
\item A principal endoreloid induced by a Rel-morphism $E$ is $\alpha$-totally bounded iff $E$ is $\alpha$-thick.
\item A principal endoreloid induced by a Rel-morphism $E$ is $\beta$-totally bounded iff $E$ is $\beta$-thick.
\end{itemize}
\end{obvious}

\begin{example}
There is a $\beta$-totally bounded endoreloid which is not $\alpha$-totally bounded.
\end{example}

\begin{proof}
It follows from the example above and properties of principal endoreloids.
\end{proof}

\section{Special case of uniform spaces}

\begin{definition}
Uniform space is essentially the same as symmetric, reflexive and transitive endoreloid.
\end{definition}

\begin{exercise}
Prove that it is essentially the same as the standard definition of a uniform space (see Wikipedia or PlanetMath).
\end{exercise}

\begin{theorem}
Let $f$ be such an endoreloid that $f \circ f^{-1} \subseteq f$. Then $f$ is $\alpha$-totally bounded iff it is $\beta$-totally bounded.
\end{theorem}

\begin{proof}
\begin{itemize}
\item\Rightarrow. Proved above.
\item\Leftarrow. For every $\varepsilon \in \text{GR} f$ we have that $\langle \varepsilon \rangle \{c_0\}, \ldots, \langle \varepsilon \rangle \{c_n\}$ covers the space. $\langle \varepsilon \rangle \{c_1\} \times \langle \varepsilon \rangle \{c_1\} \subseteq \varepsilon \circ \varepsilon^{-1}$ because for $x \in \langle \varepsilon \rangle \{c_1\}$ (the same as $c_1 \in \{\varepsilon^{-1}\} \{x\}$) we have $\langle \varepsilon \rangle \{c_1\} \times \langle \varepsilon \rangle \{c_1\} \{x\} = \langle \varepsilon \rangle \{c_1\} \subseteq \langle \varepsilon \rangle \{\varepsilon^{-1}\} \{x\} = \langle \varepsilon \circ \varepsilon^{-1} \rangle \{x\}$. For every $\varepsilon' \in \text{GR} f$ exists $\varepsilon \in \text{GR} f$ such that $\varepsilon \circ \varepsilon^{-1} \subseteq \varepsilon'$ because $f \circ f^{-1} \subseteq f$. Thus for every $\varepsilon'$ we have $\langle \varepsilon \rangle \{c_1\} \times \langle \varepsilon \rangle \{c_1\} \subseteq \varepsilon'$ and so $\langle \varepsilon \rangle \{c_0\}, \ldots, \langle \varepsilon \rangle \{c_n\}$.
\end{itemize}
\end{proof}
Corollary 12.18. A uniform space is $\alpha$-totally bounded iff it is $\beta$-totally bounded.

Proof. From the theorem and the definition of uniform spaces.

12.4 Relationships with other properties

Theorem 12.19. Let $\mu$ and $\nu$ be endoreloids. Let $f$ be a principal $C'(\mu; \nu)$ continuous, monovalued, surjective reloid. Then if $\mu$ is $\beta$-totally bounded then $\nu$ is also $\beta$-totally bounded.

Proof. Let $\varphi$ be the monovalued, surjective function, which induces the reloid $f$.

We have $\mu \subseteq f^{-1} \circ \nu \circ f$.

Let $F \in \text{GR} \nu$. Then there exists $E \in \text{GR} \mu$ such that $E \subseteq \varphi^{-1} \circ F \circ \varphi$.

Since $\mu$ is $\beta$-totally bounded, there exists a finite subset $A$ of $\text{Ob} \mu$ such that $\langle E \rangle A = \text{Ob} \mu$.

We claim $\langle F \rangle \langle \varphi \rangle A = \text{Ob} \nu$.

Indeed let $y \in \text{Ob} \nu$ be an arbitrary point. Since $\varphi$ is surjective, there exists $x \in \text{Ob} \mu$ such that $\varphi x = y$. Since $\langle E \rangle A = \text{Ob} \mu$ there exists $a \in A$ such that $a E x$ and thus $a (\varphi^{-1} \circ F \circ \varphi) x$. So $\langle \varphi a; y \rangle = \langle \varphi a; \varphi x \rangle \in F$. Therefore $y \in \langle F \rangle \langle \varphi \rangle A$.

Theorem 12.20. Let $\mu$ and $\nu$ be endoreloids. Let $f$ be a principal $C''(\mu; \nu)$ continuous, surjective reloid. Then if $\mu$ is $\alpha$-totally bounded then $\nu$ is also $\alpha$-totally bounded.

Proof. Let $\varphi$ be the surjective binary relation which induces the reloid $f$.

We have $f \circ \varphi \circ f^{-1} \subseteq \nu$.

Let $F \in \text{GR} \nu$. Then there exists $E \in \text{GR} \mu$ such that $\varphi \circ E \circ \varphi^{-1} \subseteq F$.

There exists a finite cover $S$ of $\text{Ob} \mu$ such that

$$\bigcup \{ A \times A \mid A \in S \} \subseteq E.$$

Thus $\varphi \circ (\bigcup \{ A \times A \mid A \in S \}) \circ \varphi^{-1} \subseteq F$ that is $\bigcup \{ \langle \varphi \rangle A \times \langle \varphi \rangle A \mid A \in S \} \subseteq F$.

It remains to prove that $\{ \langle \varphi \rangle A \mid A \in S \}$ is a cover of $\text{Ob} \nu$. It is true because $\varphi$ is a surjection and $S$ is a cover of $\text{Ob} \mu$.

A stronger statement (principality requirement removed):

Conjecture 12.21. The image of a uniformly continuous entirely defined monovalued surjective reloid from a ($\alpha$, $\beta$)-totally bounded endoreloid is also ($\alpha$, $\beta$)-totally bounded.

Can we remove the requirement to be entirely defined from the above conjecture?

Question 12.22. Under which conditions it’s true that join of ($\alpha$, $\beta$) totally bounded reloids is also totally bounded?

12.5 Additional predicates

We may consider also the following predicates expressing different kinds of what is intuitively is understood as boundness. Their usefulness is unclear, but I present them for completeness.

- $\forall E \in \text{GR} f \exists n \in \mathbb{N}: \text{thick}_\alpha(E^n)$
- $\forall E \in \text{GR} f \exists n \in \mathbb{N}: \text{thick}_\beta(E^n)$
- $\forall E \in \text{GR} f \exists n \in \mathbb{N}: \text{thick}_\alpha(E^0 \cup \ldots \cup E^n)$
- $\forall E \in \text{GR} f \exists n \in \mathbb{N}: \text{thick}_\beta(E^0 \cup \ldots \cup E^n)$
- $\exists n \in \mathbb{N}: \text{totBound}_\alpha(f^n)$
\begin{itemize}
  \item $\exists n \in \mathbb{N}: \text{totBound}_\alpha(f^n)$
  \item $\exists n \in \mathbb{N}: \text{totBound}_\alpha(f^0 \sqcup \ldots \sqcup f^n)$
  \item $\forall n \in \mathbb{N}: \text{totBound}_\beta(f^0 \sqcup \ldots \sqcup f^n)$
  \item totBound$_\alpha(S(f))$
  \item totBound$_\beta(S(f))$
\end{itemize}

Some of the above defined predicates are equivalent:

**Proposition 12.23.**

- $\forall E \in \text{GR} f \exists n \in \mathbb{N}: \text{thick}_\alpha(E^n) \Leftrightarrow \exists n \in \mathbb{N}: \text{totBound}_\alpha(f^n)$.
- $\forall E \in \text{GR} f \exists n \in \mathbb{N}: \text{thick}_\beta(E^n) \Leftrightarrow \exists n \in \mathbb{N}: \text{totBound}_\beta(f^n)$.

**Proof.** Because every $F \in \text{GR} f^n$ is a superset of $E^n$ for some $E \in \text{GR} f$. \qed

**Proposition 12.24.**

- $\forall E \in \text{GR} f \exists n \in \mathbb{N}: \text{thick}_\alpha(E^0 \sqcup \ldots \sqcup E^n) \Leftrightarrow \exists n \in \mathbb{N}: \text{totBound}_\alpha(f^0 \sqcup \ldots \sqcup f^n)$.
- $\forall E \in \text{GR} f \exists n \in \mathbb{N}: \text{thick}_\beta(E^0 \sqcup \ldots \sqcup E^n) \Leftrightarrow \exists n \in \mathbb{N}: \text{totBound}_\beta(f^0 \sqcup \ldots \sqcup f^n)$.

**Proof.** $f^0 \sqcup \ldots \sqcup f^n = f^0 \cap \ldots \cap f^n$. Thus every $F \in \text{GR}(f^0 \cap \ldots \cap f^n)$ we have $F \in f^k$, thus $F \supseteq E_k^k$ for all $k$ for some $E_k \in \text{GR} f$ and so $F \supseteq E^0 \cup \ldots \cup E^n$ where $E = E_0 \cap \ldots \cap E_k \in \text{GR} f$. \qed

**Proposition 12.25.** All predicates in the above list are pairwise equivalent in the case if $f$ is a uniform space.

**Proof.** Because $f \circ f = f$. \qed
Chapter 13
Orderings of filters in terms of reloids

Whilst the other chapters of this book use filters to research funcoids and reloids, here the opposite thing is discussed, the theory of reloids is used to describe properties of filters.

In this chapter the word filter is used to denote a filter on a set (not on an arbitrary poset) only.

13.1 Equivalent filters

Definition 13.1. Two filters $A$ and $B$ (with possibly different base sets) are equivalent ($A \sim B$) if there exists a set $X$ such that $X \in A$ and $X \in B$ and $\mathcal{P}X \cap A = \mathcal{P}X \cap B$.

Proposition 13.2. If two filters with the same base are equivalent they are equal.

Proof. Let $A$ and $B$ be two filters and $\mathcal{P}X \cap A = \mathcal{P}X \cap B$ for some set $X$ such that $X \in A$ and $X \in B$, and Base($A$) = Base($B$). Then $A = (\mathcal{P}X \cap A) \cup \{Y \in \mathcal{P}\text{Base}(A) \mid Y \supseteq X\} = (\mathcal{P}X \cap B) \cup \{Y \in \mathcal{P}\text{Base}(B) \mid Y \supseteq X\} = B$. \hfill \Box

Proposition 13.3. $\sim$ restricted to small filters is an equivalence relation.


Symmetry. Obvious.

Transitivity. Let $A \sim B$ and $B \sim C$ for some small filters $A$, $B$, and $C$. Then exist a set $X$ such that $X \in A$ and $X \in B$ and $\mathcal{P}X \cap A = \mathcal{P}X \cap B$ and a set $Y$ such that $Y \in B$ and $Y \in C$ and $\mathcal{P}Y \cap B = \mathcal{P}Y \cap C$. So $X \cap Y \in A$ because $\mathcal{P}Y \cap \mathcal{P}X \cap A = \mathcal{P}Y \cap \mathcal{P}X \cap B = \mathcal{P}(X \cap Y) \cap B \supseteq \{X \cap Y\} \cap B \ni X \cap Y$.

Similarly we have $X \cap Y \in C$. Finally $\mathcal{P}(X \cap Y) \cap A = \mathcal{P}Y \cap \mathcal{P}X \cap A = \mathcal{P}Y \cap \mathcal{P}X \cap B = \mathcal{P}X \cap \mathcal{P}Y \cap C = \mathcal{P}(X \cap Y) \cap C$. \hfill \Box

Definition 13.4. The rebase $A \div A$ for a filter $A$ and a set $A$ (base) such that $\exists X \in A; X \subseteq A$ is defined by the formula

$$A \div A = \{X \in \mathcal{P}A \mid \exists Y \in A; Y \subseteq X\}.$$ 

Proposition 13.5. If $\exists X \in A; X \subseteq A$ then:

1. $A \div A$ is a filter;
2. $A \div A \sim A$.

Proof.

1. We need to prove that $\{X \in \mathcal{P}A \mid \exists Y \in A; Y \subseteq X\}$ is a filter on $A$. That it is an upper set is obvious. It is non-empty because $\exists Y \in A; Y \subseteq A$ and thus $A \in \{X \in \mathcal{P}A \mid \exists Y \in A; Y \subseteq X\}$. Let $P, Q \in \{X \in \mathcal{P}A \mid \exists Y \in A; Y \subseteq X\}$. Then $P, Q \subseteq A$ and $\exists P' \in A; P' \subseteq P$ and $\exists Q' \in A; Q' \subseteq Q$. So $P \cap Q \subseteq A$ and $P' \cap Q' \subseteq P \cap Q$. Thus $P \cap Q \in \{X \in \mathcal{P}A \mid \exists Y \in A; Y \subseteq X\}$. 

157
2. \((A \div A) \cap \mathcal{P}(A \cap \text{Base}(A)) = \{X \in \mathcal{P}A \mid \exists Y \in A; Y \subseteq X \} \cap \mathcal{P}(A \cap \text{Base}(A)) = \{X \in \mathcal{P}A \mid \exists Y \in A; Y \subseteq X \} \cap \mathcal{P}\text{Base}(A) = \{X \in (\mathcal{P}A \cap \text{Base}(A)) \mid X \in A\} = A \cap \mathcal{P}(A \cap \text{Base}(A))\).

Thus \(A \div A \sim A\) because \(A \cap \text{Base}(A) \supseteq X \in A\) for some \(X \in A\) and \(A \cap \text{Base}(A) \supseteq X \cap \text{Base}(A) \in \{X \in \mathcal{P}A \mid \exists Y \in A; Y \subseteq X \} = A \div A\). □

**Proposition 13.6.** \(A \in A \Rightarrow A \div A = \mathcal{P}A \cap A\).

**Proof.** Let \(A \in A\). Then \(A \div A = \{X \in \mathcal{P}A \mid \exists Y \in A; Y \subseteq X \} = \{X \in \mathcal{P}A \mid X \in A\} = \mathcal{P}A \cap A\). □

**Lemma 13.7.** If \(A \sim B\) then \(\exists Y \in A; Y \subseteq X \Leftrightarrow \exists Y \in B; Y \subseteq X\) for every filters \(A, B,\) and a set \(X\).

**Proof.** We will prove \(\exists Y \in A; Y \subseteq X \Rightarrow \exists Y \in B; Y \subseteq X\) (the other direction is similar).

We have \(\mathcal{P}K \cap A = \mathcal{P}K \cap B\) for some set \(K\) such that \(K \in A, K \in B\).

\(\exists Y \in A; Y \subseteq X \Rightarrow \exists Y \in \mathcal{P}K \cap A; Y \subseteq X \Rightarrow \exists Y \in \mathcal{P}K \cap B; Y \subseteq X \Rightarrow \exists Y \in B; Y \subseteq X\). □

**Proposition 13.8.** If \(A \sim B\) then \(B = A \div \text{Base}(B)\) for every filters \(A, B\).

**Proof.** \(\mathcal{P}Y \cap A = \mathcal{P}Y \cap B\) for some set \(Y \in A, Y \in B\). There exists a set \(X \in A\) such that \(X \in B\). Thus \(\exists X \in A; X \subseteq \text{Base}(B)\) and so \(A \sim \text{Base}(B)\) is a filter.

\(X \in A \div \text{Base}(B) \Leftrightarrow X \in \mathcal{P}\text{Base}(B) \cap \exists Y \in A; Y \subseteq X \Leftrightarrow X \in \mathcal{P}\text{Base}(B) \cap \exists Y \in B; Y \subseteq X \Leftrightarrow X \in B\) (the lemma used). □

### 13.2 Ordering of filters

Below I will define some categories having filters (with possibly different bases) as their objects and some relations having two filters (with possibly different bases) as arguments induced by these categories (defined as existence of a morphism between these two filters).

**Theorem 13.9.** \(\text{card } a = \text{card } U\) for every ultrafilter \(a\) on \(U\) if \(U\) is infinite.

**Proof.** Let \(f(X) = X\) if \(X \in a\) and \(f(X) = U \setminus X\) if \(X \notin a\). Obviously \(f\) is a surjection from \(U\) to \(a\).

Every \(X \in a\) appears as a value of \(f\) exactly twice, as \(f(X)\) and \(f(U \setminus X)\). So \(\text{card } a = \text{card } U / 2 = \text{card } U\). □

**Corollary 13.10.** Cardinality of every two ultrafilters on a set \(U\) is the same.

**Proof.** For infinite \(U\) it follows from the theorem. For finite case it is obvious. □

**Definition 13.11.** \(f \ast A = \{C \in \mathcal{P}(\text{Dst } f) \mid \langle f^{-1}\rangle C \in A\}\) for every filter \(A\) and a Set-morphism \(f\).

Below I’ll define some directed multigraphs. By an abuse of notation, I will denote these multigraphs the same as (below defined) categories based on some of these directed multigraphs with added composition of morphisms (of directed multigraphs edges). As such I will call vertices of these multigraphs objects and edges morphisms.

**Definition 13.12.** I will denote \(\text{GreFunc}_{1}\) the multigraph whose objects are filters and whose morphisms between objects \(A\) and \(B\) are Set-morphisms from \(\text{Base}(A)\) to \(\text{Base}(B)\) such that \(B \supseteq f \ast A\).

**Definition 13.13.** I will denote \(\text{GreFunc}_{2}\) the multigraph whose objects are filters and whose morphisms between objects \(A\) and \(B\) are Set-morphisms from \(\text{Base}(A)\) to \(\text{Base}(B)\) such that \(B = f \ast A\).

**Definition 13.14.** Let \(A\) be a filter on a set \(X\) and \(B\) is a filter on a set \(Y\). \(A \supseteq B\) iff \(\text{Mor}_{\text{GreFunc}_{1}}(A; B)\) is not empty.

---

13.1. We will assume that \(f \ast A\) is just a set, while it is not yet proved that it is a filter.
Definition 13.15. Let $A$ be a filter on a set $X$ and $B$ be a filter on a set $Y$. $A \trianglerighteq B$ if $\text{Mor}_{\text{GreFunc}}(A; B)$ is not empty.

Proposition 13.16.
1. $f \in \text{Mor}_{\text{GreFunc}}(A; B)$ iff $f$ is a Set-morphism from $\text{Base}(A)$ to $\text{Base}(B)$ such that
   \[ C \in B \iff (f^{-1})C \in A \]
   for every $C \in \mathcal{P}\text{Base}(B)$.
2. $f \in \text{Mor}_{\text{GreFunc}}(A; B)$ iff $f$ is a Set-morphism from $\text{Base}(A)$ to $\text{Base}(B)$ such that
   \[ C \in B \iff (f^{-1})C \in A \]
   for every $C \in \mathcal{P}\text{Base}(B)$.

Proof.
1. $f \in \text{Mor}_{\text{GreFunc}}(A; B) \iff B \supseteq f \ast A \iff \forall C \in f \ast A: C \in B \iff \forall C \in \mathcal{P}\text{Base}(B): ((f^{-1})C \in A \Rightarrow C \in B)$.
2. $f \in \text{Mor}_{\text{GreFunc}}(A; B) \iff B = f \ast A \iff \forall C: (C \in B \iff C \in f \ast A) \iff \forall C \in \mathcal{P}\text{Base}(B): (C \in B \iff (f^{-1})C \in A)$.

Definition 13.17. The directed multigraph $\text{FuncBij}$ is the directed multigraph got from $\text{GreFunc}$ by restricting to only bijective morphisms.

Definition 13.18. A filter $A$ is directly isomorphic to a filter $B$ iff there is a morphism $f \in \text{Mor}_{\text{FuncBij}}(A; B)$.

Proposition 13.19. $f \ast A = (\iota_{\text{FCD}}f)A$ for every Set-morphism $f: \text{Base}(A) \to \text{Base}(B)$.

Proof. For every set $C \in \mathcal{P}\text{Base}(B)$ we have $C \in f \ast A \iff (f^{-1})C \in A \Rightarrow \exists K \in A: (f^{-1})K = K \Rightarrow \exists K \in A: C \supseteq (f)(f^{-1})C = (f)(f^{-1})K \Rightarrow \exists K \in A: C \supseteq (\iota_{\text{FCD}}f)K = (\iota_{\text{FCD}}f)A$.

Let now $C \in (\iota_{\text{FCD}}f)A$. Then $\text{Base}(A) \subseteq (\iota_{\text{FCD}}f^{-1})(\iota_{\text{FCD}}f)A \supseteq A$ and thus $(f^{-1})C \in A$.

Corollary 13.20. $f \in \text{Mor}_{\text{GreFunc}}(A; B) \iff B \subseteq (\iota_{\text{FCD}}f)A$ for every Set-morphism $f$ from $\text{Base}(A)$ to $\text{Base}(B)$.

Corollary 13.21. $f \in \text{Mor}_{\text{GreFunc}}(A; B) \iff B = (\iota_{\text{FCD}}f)A$ for every Set-morphism $f$ from $\text{Base}(A)$ to $\text{Base}(B)$.

Corollary 13.22. $A \trianglerighteq B$ if it exists a Set-morphism $f: \text{Base}(A) \to \text{Base}(B)$ such that $B = (\iota_{\text{FCD}}f)A$.

Corollary 13.23. $A \trianglerighteq B$ if it exists a Set-morphism $f: \text{Base}(A) \to \text{Base}(B)$ such that $B \subseteq (\iota_{\text{FCD}}f)A$.

Proposition 13.24. For a bijective Set-morphism $f: \text{Base}(A) \to \text{Base}(B)$ the following are equivalent:
1. $B = f \ast A$.
2. $\forall C \in \text{Base}(B): (C \in B \iff (f^{-1})C \in A)$.
3. $\forall C \in \text{Base}(A): ((f)C \in B \iff C \in A)$.
4. $(\iota_{\text{FCD}}f)|_{A}$ is a bijection from $A$ to $B$.
5. $(\iota_{\text{FCD}}f)|_{A}$ is a function onto $B$.
6. $B = (\iota_{\text{FCD}}f)A$.
7. $f \in \text{Mor}_{\text{GreFunc}}(A; B)$. 

8. \( f \in \text{Mor}_{\text{GreFunc}_2}(A; B) \).

**Proof.**

(1) \(\Leftrightarrow\) (2). \( B = f \ast A \Leftrightarrow B = \{ C \in \mathcal{P} \text{Base}(B) \mid (f^{-1})C \in A \} \Leftrightarrow \forall C \in \text{Base}(B): (C \in B \Leftrightarrow (f^{-1})C \in A) \).

(2) \(\Leftrightarrow\) (3). Because \( f \) is a bijection.

(2) \(\Rightarrow\) (5). For every \( C \in B \) we have \( (f^{-1})C \in A \) and thus \( (\uparrow_{\text{FCD}} f) \mid A (\uparrow_{\text{FCD}} f^{-1})C = (f)(f^{-1})C = C \). Thus \( (\uparrow_{\text{FCD}} f) \mid A \) is onto \( B \).

(4) \(\Rightarrow\) (5). Obvious.

(5) \(\Rightarrow\) (4). We need to prove only that \( (\uparrow_{\text{FCD}} f) \mid A \) is an injection. But this follows from the fact that \( f \) is a bijection.

(4) \(\Rightarrow\) (3). We have \( \forall C \in \text{Base}(A): ((\uparrow_{\text{FCD}} f) \mid A)C \in B \Leftrightarrow C \in A \) and consequently \( \forall C \in \text{Base}(A): ((f)C \in B \Leftrightarrow C \in A) \).

(6) \(\Leftrightarrow\) (1). From the last corollary.

(1) \(\Leftrightarrow\) (7). Obvious.

(7) \(\Leftrightarrow\) (8). Obvious. \(\square\)

**Corollary 13.25.** The following are equivalent for every filters \( A \) and \( B \):

1. \( A \) is directly isomorphic to a filter \( B \).
2. There is a bijective Set-morphism \( f: \text{Base}(A) \rightarrow \text{Base}(B) \) such that for every \( C \in \mathcal{P} \text{Base}(B) \)
   \[ C \in B \Leftrightarrow (f^{-1})C \in A \]
3. There is a bijective Set-morphism \( f: \text{Base}(A) \rightarrow \text{Base}(B) \) such that for every \( C \in \mathcal{P} \text{Base}(B) \)
   \[ (f)C \in B \Leftrightarrow C \in A \]
4. There is a bijective Set-morphism \( f: \text{Base}(A) \rightarrow \text{Base}(B) \) such that \( (\uparrow_{\text{FCD}} f) \mid A \) is a bijection from \( A \) to \( B \).
5. There is a bijective Set-morphism \( f: \text{Base}(A) \rightarrow \text{Base}(B) \) such that \( (\uparrow_{\text{FCD}} f) \mid A \) is a function onto \( B \).
6. There is a bijective Set-morphism \( f: \text{Base}(A) \rightarrow \text{Base}(B) \) such that \( B = (\uparrow_{\text{FCD}} f) \).
7. There is a bijective morphism \( f \in \text{Mor}_{\text{GreFunc}_2}(A; B) \).

**Proposition 13.26.** GreFunc\(_1 \) and GreFunc\(_2 \) with function composition are categories.

**Proof.** Let \( f: A \rightarrow B \) and \( g: B \rightarrow C \) be morphisms of GreFunc\(_1 \). Then \( B \subseteq (\uparrow_{\text{FCD}} f)A \) and \( C \subseteq (\uparrow_{\text{FCD}} g)B \). So \( (\uparrow_{\text{FCD}} (g \circ f))A = (\uparrow_{\text{FCD}} g)(\uparrow_{\text{FCD}} f)A \supseteq (\uparrow_{\text{FCD}} g)B \supseteq C \). Thus \( g \circ f \) is a morphism of GreFunc\(_1 \). Associativity law is evident. \( \text{id}_{\text{Base}(A)} \) is the identity morphism of GreFunc\(_1 \) for every filter \( A \).

Let \( f: A \rightarrow B \) and \( g: B \rightarrow C \) be morphisms of GreFunc\(_2 \). Then \( B = (\uparrow_{\text{FCD}} f)A \) and \( C = (\uparrow_{\text{FCD}} g)B \). So \( (\uparrow_{\text{FCD}} (g \circ f))A = (\uparrow_{\text{FCD}} g)(\uparrow_{\text{FCD}} f)A = (\uparrow_{\text{FCD}} g)B = C \). Thus \( g \circ f \) is a morphism of GreFunc\(_2 \). Associativity law is evident. \( \text{id}_{\text{Base}(A)} \) is the identity morphism of GreFunc\(_2 \) for every filter \( A \). \(\square\)

**Corollary 13.27.** \( \leq_1 \) and \( \leq_2 \) are preorders.

**Theorem 13.28.** FuncBij is a groupoid.

**Proof.** First let’s prove it is a category. Let \( f: A \rightarrow B \) and \( g: B \rightarrow C \) be morphisms of FuncBij. Then \( f: \text{Base}(A) \rightarrow \text{Base}(B) \) and \( g: \text{Base}(A) \rightarrow \text{Base}(C) \) are bijections and \( B = (\uparrow_{\text{FCD}} f)A \) and \( C = (\uparrow_{\text{FCD}} g)B \). Thus \( g \circ f: \text{Base}(A) \rightarrow \text{Base}(C) \) is a bijection and \( C = (\uparrow_{\text{FCD}} (g \circ f))A \). Thus \( g \circ f \) is a morphism of FuncBij. \( \text{id}_{\text{Base}(A)} \) is the identity morphism of FuncBij for every filter \( A \). Thus it is a category.
It remains to prove only that every morphism \( f \in \text{Mor}_{\text{FuncBij}}(A; B) \) has a reverse (for every filters \( A, B \)). We have \( f \) is a bijection \( \text{Base}(A) \rightarrow \text{Base}(B) \) such that for every \( C \in \mathcal{P}\text{Base}(A) \)

\[
(f)C \in B \iff C \in A.
\]

Then \( f^{-1}: \text{Base}(B) \rightarrow \text{Base}(A) \) is a bijection such that for every \( C \in \mathcal{P}\text{Base}(A) \)

\[
(f^{-1})C \in A \iff C \in B.
\]

Thus \( f^{-1} \in \text{Mor}_{\text{FuncBij}}(B; A) \).

**Corollary 13.29.** Being directly isomorphic is an equivalence relation.

Rudin-Keisler order of ultrafilters is considered in such a book as [37].

**Obvious 13.30.** For the case of ultrafilters being directly isomorphic is the same as being Rudin-Keisler equivalent.

**Definition 13.31.** A filter \( A \) is isomorphic to a filter \( B \) iff there exist sets \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) such that \( A \triangleleft B \) is directly isomorphic to \( B \triangleleft B \).

**Obvious 13.32.** Equivalent filters are isomorphic.

**Theorem 13.33.** Being isomorphic (for filters) is an equivalence relation.

**Proof.**

**Reflexivity.** Because every filter is directly isomorphic to itself.

**Symmetry.** If filter \( A \) is isomorphic to \( B \) then there exist sets \( A \in \mathcal{A} \) and \( B \in \mathcal{B} \) such that \( A \triangleleft B \) is directly isomorphic to \( B \triangleleft B \) and thus \( B \triangleleft B \) is directly isomorphic to \( A \triangleleft A \). So \( B \) is isomorphic to \( A \).

**Transitivity.** Let \( A \) be isomorphic to \( B \) and \( B \) be isomorphic to \( C \). Then exist \( A \in \mathcal{A} \), \( B_1 \in \mathcal{B} \), \( B_2 \in \mathcal{B} \), \( C \in \mathcal{C} \) such that there are bijections \( f: A \rightarrow B_1 \) and \( g: B_2 \rightarrow C \) such that

\[
\forall X \in \mathcal{P}A: (X \in B \iff (f^{-1})X \in A) \quad \text{and} \quad \forall X \in \mathcal{P}B_2: (X \in A \iff (f)X \in B).
\]

Also \( \forall X \in \mathcal{P}B_2: (X \in B \iff (g)X \in C) \).

So \( g \circ f \) is a bijection from \( (f^{-1})(B_1 \cap B_2) \in A \) to \( (g)(B_1 \cap B_2) \in C \) such that

\[
X \in A \iff (f)X \in B \iff (g)(f)X \in C \iff (g \circ f)X \in C.
\]

Thus \( g \circ f \) establishes a bijection which proves that \( A \) is isomorphic to \( C \).

**Lemma 13.34.** Let \( \text{card} X = \text{card} Y \), \( u \) be an ultrafilter on \( X \) and \( v \) be an ultrafilter on \( Y \); let \( A \in u \) and \( B \in v \). Let \( u \triangleleft A \) and \( v \triangleleft B \) be directly isomorphic. Then if \( \text{card}(X \setminus A) = \text{card}(Y \setminus B) \) we have \( u \) and \( v \) directly isomorphic.

**Proof.** Arbitrary extend the bijection witnessing being directly isomorphic to the sets \( X \setminus A \) and \( Y \setminus B \).

**Theorem 13.35.** If \( \text{card} X = \text{card} Y \) then being isomorphic and being directly isomorphic are the same for ultrafilters \( u \) on \( X \) and \( v \) on \( Y \).

**Proof.** That if two filters are isomorphic then they are directly isomorphic is obvious.

Let ultrafilters \( u \) and \( v \) be isomorphic that is there is a bijection \( f: A \rightarrow B \) where \( A \in u, \quad B \in v \) witnessing isomorphism of \( u \) and \( v \).

If one of the filters \( u \) or \( v \) is a trivial ultrafilter then the other is also a trivial ultrafilter and as it is easy to show they are directly isomorphic. So we can assume \( u \) and \( v \) are not trivial ultrafilters.

If \( \text{card}(X \setminus A) = \text{card}(Y \setminus B) \) our statement follows from the last lemma.

Now assume without loss of generality \( \text{card}(X \setminus A) < \text{card}(Y \setminus B) \).

\[
\text{card} B = \text{card} Y \quad \text{because} \quad \text{card}(Y \setminus B) < \text{card} Y.
\]
It is easy to show that there exists $B' \supset B$ such that $\text{card}(X \setminus A) = \text{card}(Y \setminus B')$ and $\text{card} B' = \text{card} B$.

We will find a bijection $g$ from $B$ to $B'$ which witnesses direct isomorphism of $v$ to $v$ itself. Then the composition $g \circ f$ witnesses a direct isomorphism of $u \div A$ and $v \div B'$ and by the lemma $u$ and $v$ are directly isomorphic.

Let $D = B' \setminus B$. We have $D \not\in v$.

There exists a set $E \subseteq B$ such that $\text{card} E \geq \text{card} D$ and $E \not\in v$.

We have $\text{card} E = \text{card}(D \cup E)$ and thus there exists a bijection $h: E \to D \cup E$.

Let $g(x) = \begin{cases} x & \text{if } x \in B \setminus E; \\ h(x) & \text{if } x \in E. \end{cases}$

$g|_{B \setminus E}$ and $g|_E$ are bijections.

$\text{im}(g|_{B \setminus E}) = B \setminus E; \text{im}(g|_E) = \text{im} h = D \cup E$;

$(D \cup E) \cap (B \setminus E) = (D \cap (B \setminus E)) \cup (E \cap (B \setminus E)) = \emptyset \cup \emptyset = \emptyset$.

Thus $g$ is a bijection from $B$ to $(B \setminus E) \cup (D \cup E) = B \cup D = B'$.

To finish the proof it’s enough to show that $(g)v = v$. Indeed it follows from $B \setminus E \in v$. □

**Proposition 13.36.** For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $A \geq_2 B$ iff $A \div A \geq_2 B \div B$.

1. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $A \geq_1 B$ iff $A \div A \geq_1 B \div B$.

**Proof.**

1. $A \geq_2 B$ iff there exist a bijective Set-morphism $f$ such that $B = (\uparrow \text{FCD} f)_A$. The equality is obviously preserved replacing $A$ with $A \div A$ and $B$ with $B \div B$.

2. $A \geq_1 B$ iff there exist a bijective Set-morphism $f$ such that $B \subseteq (\uparrow \text{FCD} f)_A$. The equality is obviously preserved replacing $A$ with $A \div A$ and $B$ with $B \div B$. □

**Proposition 13.37.** For ultrafilters $\geq_2$ is the same as Rudin-Keisler ordering ([37]).

**Proof.** $x \geq_2 y$ iff there exist sets $A \in x$ and $B \in y$ a bijective Set-morphism $f: X \to Y$ such that $y \div B = \{ C \in \mathcal{P}Y \mid (f^{-1})C \in x \div A \}$ that is when $C \in y \div B \models (f^{-1})C \in x \div A$ what is equivalent to $C \in y \models (f^{-1})C \in x$ what is the definition of Rudin-Keisler ordering. □

**Remark 13.38.** The relation of being isomorphic for ultrafilters is traditionally called Rudin-Keisler equivalence.

**Obvious 13.39.** $(\geq_1) \supset (\geq_2)$. 

**Definition 13.40.** Let $Q$ and $R$ be binary relations on the set of filters. I will denote $\text{MonRld}_{Q,R}$ the directed multigraph with objects being filters and morphisms such monovalued reloids $f$ that $(\text{dom} f) Q A$ and $(\text{im} f) R B$.

I will also denote $\text{CoMonRld}_{Q,R}$ the directed multigraph with objects being filters and morphisms such injective reloids $f$ that $(\text{im} f) Q A$ and $(\text{dom} f) R B$. These are essentially the duals.

Some of these directed multigraphs are categories with reloid composition (see below). By abuse of notation I will denote these categories the same as these directed multigraphs.

**Theorem 13.41.** For every filters $A$ and $B$ the following are equivalent:

1. $A \geq_1 B$.

2. $\text{Mor}_{\text{MonRld}_{\leq_2}}(A; B) \neq \emptyset$.

3. $\text{Mor}_{\text{MonRld}_{\leq_1}}(A; B) \neq \emptyset$.

4. $\text{Mor}_{\text{MonRld}_{\leq_2}}(A; B) \neq \emptyset$. 


5. \( \text{Mor}_{\text{CoMonRld}}(A; B) \neq \emptyset \).
6. \( \text{Mor}_{\text{CoMonRld}}(A; B) \neq \emptyset \).
7. \( \text{Mor}_{\text{CoMonRld}}(A; B) \neq \emptyset \).

Proof.

(1) \( \Rightarrow \) (2). There exists a set-morphism \( f : \text{Base}(A) \to \text{Base}(B) \) such that \( B \subseteq (\text{FCD} f) A \). We have
\[
\text{dom}(\text{FCD} f) |_{A} = A \cap 1^{\text{Base}(A)} = A
\]
and
\[
\text{im}(\text{FCD} f) |_{A} = \text{im}(\text{FCD} f) |_{A} = \text{im}(\text{FCD} f) A \supseteq B.
\]
Thus \( \text{FCD} f |_{A} \) is a monovalued reloid such that \( \text{dom}(\text{FCD} f) |_{A} = A \) and \( \text{im}(\text{FCD} f) |_{A} \supseteq B.
\]

(2) \( \Rightarrow \) (3), (4) \( \Rightarrow \) (3), (5) \( \Rightarrow \) (6), (7) \( \Rightarrow \) (6). Obvious.

(3) \( \Rightarrow \) (1). We have \( B \subseteq (\text{FCD} f) A \) for a monovalued reloid \( f \in \text{RLD}(\text{Base}(A); \text{Base}(B)) \). Then there exists a set-morphism \( F : \text{Base}(A) \to \text{Base}(B) \) such that \( B \subseteq (\text{FCD} F) A \) that is \( A \supseteq B \).

(6) \( \Rightarrow \) (7). \( f |_{B} = B \) and \( \text{im} f |_{B} \subseteq A \).

(2) \( \Leftrightarrow \) (5), (3) \( \Leftrightarrow \) (6), (4) \( \Leftrightarrow \) (7). By duality.

Theorem 13.42. For every filters \( A \) and \( B \) the following are equivalent:
1. \( A \supseteq B \).
2. \( \text{Mor}_{\text{MonRld}}(A; B) \neq \emptyset \).
3. \( \text{Mor}_{\text{CoMonRld}}(A; B) \neq \emptyset \).

Proof.

(1) \( \Rightarrow \) (2). Let \( A \supseteq B \) that is \( B = (\text{FCD} f) A \) for some set-morphism \( f : \text{Base}(A) \to \text{Base}(B) \). Then \( \text{dom}(\text{FCD} f) |_{A} = A \) and \( \text{im}(\text{FCD} f) |_{A} = \text{im}(\text{FCD} f) |_{A} = \text{im}(\text{FCD} f) A = B \). So \( (\text{FCD} f) |_{A} \) is a sought for reloid.

(2) \( \Rightarrow \) (1). By corollary 13.78 below, there exists a set-morphism \( F : \text{Base}(A) \to \text{Base}(B) \) such that \( f = (\text{FCD} F) |_{A} \). Thus \( \text{FCD} F) A = \text{im}(\text{FCD} F) |_{A} = \text{im}(\text{FCD} F) |_{A} = \text{im}(\text{FCD} f) = \text{im} f = B \). Thus \( A \supseteq B \) is testified by the morphism \( F \).

(2) \( \Leftrightarrow \) (3). By duality.

Theorem 13.43. The following are categories (with reloid composition):
1. \( \text{MonRld}_{\subseteq} \);
2. \( \text{MonRld}_{\subseteq} \);
3. \( \text{MonRld}_{=} \);
4. \( \text{CoMonRld}_{\subseteq} \);
5. \( \text{CoMonRld}_{=} \);
6. \( \text{CoMonRld}_{=} \).

Proof. We will prove only the first three. The rest follow from duality. We need to prove only that composition of morphisms is a morphism, because associativity and existence of identity morphism are evident. We have:

1. Let \( f \in \text{Mor}_{\text{MonRld}}(A; B) \), \( g \in \text{Mor}_{\text{MonRld}}(B; C) \). Then \( \text{dom} f \subseteq A \), \( \text{im} f \supseteq B \), \( \text{dom} g \subseteq B \), \( \text{im} g \supseteq C \). So \( \text{dom}(g \circ f) \subseteq A \), \( \text{im}(g \circ f) \supseteq C \) that is \( g \circ f \in \text{Mor}_{\text{MonRld}}(A; C) \).

2. Let \( f \in \text{Mor}_{\text{MonRld}}(A; B) \), \( g \in \text{Mor}_{\text{MonRld}}(B; C) \). Then \( \text{dom} f \subseteq A \), \( \text{im} f = B \), \( \text{dom} g \subseteq B \), \( \text{im} g = C \). So \( \text{dom}(g \circ f) \subseteq A \), \( \text{im}(g \circ f) = C \) that is \( g \circ f \in \text{Mor}_{\text{MonRld}}(A; C) \).
3. Let \( f \in \text{Mor}_{\text{MonRld}}(A; B), \ g \in \text{Mor}_{\text{MonRld}}(B; C) \). Then \( \text{dom} \ f = A, \ \text{im} \ f = B, \ \text{dom} \ g = B, \ \text{im} \ g = C \). So \( \text{dom}(g \circ f) = A, \ \text{im}(g \circ f) = C \) that is \( g \circ f \in \text{Mor}_{\text{MonRld}}(A; C) \).

**Definition 13.44.** Let \( \text{BijRld} \) be the groupoid of all bijections of the category of reloid triples. Its objects are filters and its morphisms from a filter \( A \) to filter \( B \) are monovalued injective reloids \( f \) such that \( \text{dom} \ f = A \) and \( \text{im} \ f = B \).

**Theorem 13.45.** Filters \( A \) and \( B \) are isomorphic iff \( \text{Mor}_{\text{BijRld}}(A; B) \neq \emptyset \).

**Proof.**

\( \Rightarrow. \) Let \( A \) and \( B \) be isomorphic. Then there are sets \( A \in A, B \in B \) and a bijective Set-morphism \( F: A \to B \) such that \( (F') : \mathcal{P}A \cap A \to \mathcal{P}B \cap B \) is a bijection.

Obviously \( f = (F')|_{A} \) is monovalued and injective.

\[ \text{im} \ f = \bigcap \{ B \text{im} G \mid G \in (F')|_{A} \} = \bigcap \{ B \text{im} (H \cap F)|_{X} \mid H \in (F')|_{A}, X \in A \} = \bigcap \{ B \text{im} F|_{P} \mid P \in A \} = \bigcap \{ B(F|_{P}) \mid P \in \mathcal{P}A \cap A \} = \bigcap \{ B(\mathcal{P}B \cap B) \} = \bigcap \{ B \} = B. \]

Thus \( \text{dom} \ f = A \) and \( \text{im} \ f = B \).

\( \Leftarrow. \) Let \( f \) be a monovalued injective reloid such that \( \text{dom} \ f = A \) and \( \text{im} \ f = B \). Then there exist a function \( F' \) and an injective binary relation \( F'' \) such that \( F', F'' \in \text{GR} f \). Thus \( F = F' \cap F'' \) is an injection such that \( F \in \text{GR} f \). The function \( F \) is a bijection from \( A \) to \( B \) in \( F \).

The function \( F \) is an injection on \( \mathcal{P}A \cap A \) (and moreover on \( \mathcal{P}A \)). It’s simple to show that \( \forall X \in \mathcal{P}A \cap A : (F)(X) \in \mathcal{P}B \cap B \) and similarly \( \forall Y \in \mathcal{P}B \cap B : (F^{-1})(Y) \in \mathcal{P}A \cap A \). Thus \( (F)|_{\mathcal{P}A \cap A} \) is a bijection \( \mathcal{P}A \cap A \to \mathcal{P}B \cap B \). So filters \( A \) and \( B \) are isomorphic. \( \square \)

**Proposition 13.46.** \( (\geq 1) = (\geq) \circ (\geq 2) \) (where we limit to small filters).

**Proof.** \( A \geq 1 \) iff exists a function \( f : \text{Base}(A) \to \text{Base}(B) \) such that \( B \subseteq (\text{FCD} f) A \). But \( B \subseteq (\text{FCD} f) A \) is equivalent to \( \exists B' \subseteq \exists (B' \subseteq B \land B' = (\text{FCD} f) A) \). So \( A \geq 1 \) \( B \) is equivalent to existence of \( B' \subseteq \exists \) such that \( B' \subseteq B \) and existence of a function \( f : \text{Base}(A) \to \text{Base}(B) \) such that \( B' = (\text{FCD} f) A \). That is equivalent to \( A(\exists) \circ (\exists 2) B \). \( \square \)

**Proposition 13.47.** If \( a \) and \( b \) are ultrafilters then \( b \geq 1 a \iff b \geq 2 a \).

**Proof.** We need to prove only \( b \geq 1 a \Rightarrow b \geq 2 a \). If \( b \geq 1 a \) then there exists a monovalued reloid \( f : \text{Base}(b) \to \text{Base}(a) \) such that \( \text{dom} \ f = b \) and \( \text{im} \ f \subseteq a \). Then \( \text{im} f = \text{im} (\text{FCD}) f = \{ 0^{\text{FCD}}(\text{Base}(a)) \} \cup \text{atoms}^{\text{FCD}}(\text{Base}(a)) \) because \( (\text{FCD}) f \) is a monovalued funcoid. So \( \text{im} f = a \) (taken into account \( a \neq 0^{\text{FCD}}(\text{Base}(a)) \)) and thus \( b \geq 2 a \). \( \square \)

**Corollary 13.48.** For atomic filters \( \geq 1 \) is the same as \( \geq 2 \).

Thus I will write simply \( \geq \) for atomic filters.

13.2.1 Existence of no more than one monovalued injective reloid for a given pair of ultrafilters

13.2.1.1 The lemmas

The lemmas in this section were provided to me by Robert Martin Solovay in [36]. They are based on Wistar Comfort’s work.

In this section we will assume \( \mu \) is an ultrafilter on a set \( I \) and \( f : I \to I \) has the property \( X \in \mu \Leftrightarrow (f^{-1})X \in \mu \).

**Lemma 13.49.** If \( X \in \mu \) then \( X \cap (f)X \in \mu \).

**Proof.** If \( (f)X \notin \mu \) then \( X \subseteq (f^{-1})(f)X \notin \mu \) and so \( X \notin \mu \). Thus \( X \in \mu \land (f)X \in \mu \) and consequently \( X \cap (f)X \in \mu \). \( \square \)
We will say that \( x \) is periodic when \( f^n(x) = x \) for some positive integer \( n \). The least such \( n \) is called the period of \( x \).

Let’s define \( x \sim y \) iff there exist \( i, j \in \mathbb{N} \) such that \( f^i(x) = f^j(y) \). Trivially it is an equivalence relation. If \( x \) and \( y \) are periodic, then \( x \sim y \) iff exists \( n \in \mathbb{N} \) such that \( f^n(y) = x \).

Let \( A = \{ x \in I \mid x \text{ is periodic with period}\geq 1 \} \).

We will show that \( A \notin \mu \). Let’s assume \( A \in \mu \).

Let a set \( D \subseteq A \) contains (by the axiom of choice) exactly one element from each equivalence class of \( A \) defined by the relation \( \sim \).

Let \( \alpha \) be a function \( A \to \mathbb{N} \) defined as follows. Let \( x \in A \). Let \( y \) be the unique element of \( D \) such that \( x \sim y \). Let \( \alpha(x) \) be the least \( n \in \mathbb{N} \) such that \( f^n(y) = x \). Then \( \alpha(x) \) is even for odd \( n \).

Thus \( \left( \begin{array}{c} x \in B_0 \cap \langle f \rangle B_0 \end{array} \right) \subseteq B_2 \).

**Proof.** If \( x \in B_0 \cap \langle f \rangle B_0 \) then \( f^n(y) = x \) for a minimal even \( n \) and \( x = f(x') \) where \( f^m(y') = x' \) for a minimal even \( m \). Thus \( f^n(y) = f(x') \) thus \( y \) and \( x' \) lying in the same equivalence class and thus \( y = y' \). So we have \( f^n(y) = f^{m+1}(y) \). Thus \( n \leq m + 1 \) by minimality.

\( x' \) lies on an orbit and thus \( x' = f^{-1}(x) \) where by \( f^{-1} \) I mean step backward on our orbit; \( f^n(y) = f^{-1}(x) \) and thus \( x' = f^{n-1}(y) \) thus \( n - 1 \geq m \) by minimality or \( n = 0 \).

Thus \( n = m + 1 \) what is impossible for even \( n \) and \( m \). We have a contradiction what proves \( B_0 \cap \langle f \rangle B_0 \subseteq \emptyset \).

Remained the case \( n = 0 \), then \( x = f^0(y) \) and thus \( \alpha(x) = 0 \).

**Lemma 13.51.** \( B_1 \cap \langle f \rangle B_1 = \emptyset \).

**Proof.** Let \( x \in B_1 \cap \langle f \rangle B_1 \). Then \( f^n(y) = x \) for an odd \( n \) and \( x = f(x') \) where \( f^m(y') = x' \) for an odd \( m \). Thus \( f^n(y) = f(x') \) thus \( y \) and \( x' \) lying in the same equivalence class and thus \( y = y' \). So we have \( f^n(y) = f^{m+1}(y) \). Thus \( n \leq m + 1 \) by minimality.

\( x' \) lies on an orbit and thus \( x' = f^{-1}(x) \) where by \( f^{-1} \) I mean step backward on our orbit; \( f^n(y) = f^{-1}(x) \) and thus \( x' = f^{n-1}(y) \) thus \( n - 1 \geq m \) by minimality (\( n = 0 \) is impossible because \( n \) is odd).

Thus \( n = m + 1 \) what is impossible for odd \( n \) and \( m \). We have a contradiction what proves \( B_0 \cap \langle f \rangle B_0 = \emptyset \).

**Lemma 13.52.** \( B_2 \cap \langle f \rangle B_2 = \emptyset \).

**Proof.** Let \( x \in B_2 \cap \langle f \rangle B_2 \). Then \( x = y \) and \( x' = y \) where \( x = f(x') \). Thus \( x = f(x) \) and so \( x \notin A \) what is impossible.

**Lemma 13.53.** \( A \notin \mu \).

**Proof.** Suppose \( A \in \mu \).

Since \( A \in \mu \) we have \( B_0 \in \mu \) or \( B_1 \in \mu \).

So either \( B_0 \cap \langle f \rangle B_0 \subseteq B_2 \) or \( B_1 \cap \langle f \rangle B_1 \subseteq B_2 \). As such by the lemma 13.49 we have \( B_2 \in \mu \).

This is incompatible with \( B_2 \cap \langle f \rangle B_2 = \emptyset \). So we got a contradiction.

Let \( C \) be the set of points \( x \) which are not periodic but \( f^n(x) \) is periodic for some positive \( n \).

**Lemma 13.54.** \( C \notin \mu \).

**Proof.** Let \( \beta \) be a function \( C 
 arrow \mathbb{N} \) such that \( \beta(x) \) is the least \( n \in \mathbb{N} \) such that \( f^n(x) \) is periodic.

Let \( C_0 = \{ x \in C \mid \beta(x) \text{ is even} \} \) and \( C_1 = \{ x \in C \mid \beta(x) \text{ is odd} \} \).

Obviously \( C_j \cap \langle f \rangle C_j = \emptyset \) for \( j = 0, 1 \). Hence by the lemma 13.49 we have \( C_0, C_1 \notin \mu \) and thus \( C = C_0 \cup C_1 \notin \mu \).

Let \( E \) be the set of \( x \in I \) such that for no \( n \in \mathbb{N} \) we have \( f^n(x) \) periodic.
Lemma 13.55. Let \( x, y \in E \) be such that \( f^i(x) = f^j(y) \) and \( f^{i'}(x) = f^{j'}(y) \) for some \( i, j, i', j' \in \mathbb{N} \). Then \( i - j = i' - j' \).

**Proof.** \( i \mapsto f^i(x) \) is a bijection. So \( y = f^{i-j}(y) \) and \( y = f^{i'-j'}(y) \). Thus \( f^{i-j}(y) = f^{i'-j'}(y) \) and so \( i - j = i' - j' \).

Lemma 13.56. \( E \notin \mu \).

**Proof.** Let \( D' \subseteq E \) be a subset of \( E \) with exactly one element from each equivalence class of the relation \( \sim \) on \( E \).

Define the function \( \gamma: E \to \mathbb{Z} \) as follows. Let \( x \in E \). Let \( y \) be the unique element of \( D' \) such that \( x \sim y \). Choose \( i, j \in \mathbb{N} \) such that \( f^i(y) = f^j(x) \). Let \( \gamma(x) = i - j \). By the last lemma, \( \gamma \) is well-defined.

It is clear that if \( x \in E \) then \( f(x) \in E \) and moreover \( \gamma(f(x)) = \gamma(x) + 1 \).

Let \( E_0 = \{ x \in E \mid \gamma(x) \text{ is even} \} \) and \( E_1 = \{ x \in E \mid \gamma(x) \text{ is odd} \} \).

We have \( E_0 \cap \{ f \} E_0 = \emptyset \notin \mu \) and hence \( E_0 \notin \mu \).

Similarly \( E_1 \notin \mu \).

Thus \( E = E_0 \cup E_1 \notin \mu \).

Lemma 13.57. \( f \) is the identity function on a set in \( \mu \).

**Proof.** We have shown \( A, C, E \notin \mu \). But the points which lie in none of these sets are exactly points periodic with period 1 that is fixed points of \( f \). Thus the set of fixed points of \( f \) belongs to the filter \( \mu \).

13.2.1.2 The main theorem and its consequences

**Theorem 13.58.** For every ultrafilter \( a \) the morphism \( (a; a; \text{id}_a^{\text{FCD}}) \) is the only

1. monovalued morphism of the category of reloid triples from \( a \) to \( a \);
2. injective morphism of the category of reloid triples from \( a \) to \( a \);
3. bijective morphism of the category of reloid triples from \( a \) to \( a \).

**Proof.** We will prove only (1) because the rest follow from it.

Let \( f \) be a monovalued morphism from \( \text{Base}(a) \) to \( \text{Base}(a) \). Then it exists a Set-morphism \( F \) such that \( F \in xyGR f \). Trivially \( \langle F \rangle a \supseteq a \) and thus \( \langle F \rangle A \in a \) for every \( A \in a \). Thus by the lemma we have that \( F \) is the identity function on a set in \( a \) and so obviously \( f \) is an identity.

**Corollary 13.59.** For every two atomic filters (with possibly different bases) \( A \) and \( B \) there exists at most one bijective reloid triple from \( A \) to \( B \).

**Proof.** Suppose that \( f \) and \( g \) are two different bijective reloids from \( A \) to \( B \). Then \( g^{-1} \circ f \) is not the identity reloid (otherwise \( g^{-1} \circ f = \text{id}_{\text{dom} f}^{\text{RLD}} \) and so \( f = g \)). But \( g^{-1} \circ f \) is a bijective reloid (as a composition of bijective reloids) from \( A \) to \( A \) what is impossible.

13.3 Rudin-Keisler equivalence and Rudin-Keisler order

**Theorem 13.60.** Atomic filters \( a \) and \( b \) (with possibly different bases) are isomorphic if and only if \( a \gtrless b \gtrless a \).

**Proof.** Let \( a \gtrless b \gtrless a \). Then there are monovalued reloids \( f \) and \( g \) such that \( \text{dom} f = a \) and \( \text{im} f = b \) and \( \text{dom} g = b \) and \( \text{im} g = a \). Thus \( g \circ f \) and \( f \circ g \) are monovalued morphisms from \( a \) to \( a \) and from \( b \) to \( b \). By the above we have \( g \circ f = \text{id}_a^{\text{RLD}} \) and \( f \circ g = \text{id}_b^{\text{RLD}} \) so \( g = f^{-1} \) and \( f^{-1} \circ f = \text{id}_a^{\text{RLD}} \) and \( f \circ f^{-1} = \text{id}_b^{\text{RLD}} \). Thus \( f \) is an injective monovalued reloid from \( a \) to \( b \) and thus \( a \) and \( b \) are isomorphic.

The last theorem cannot be generalized from atomic filters to arbitrary filters, as it’s shown by the following example:
Example 13.61. \( A \succeq B \land B \succeq A \) but \( A \) is not isomorphic to \( B \) for some filters \( A \) and \( B \).

**Proof.** Consider \( A = \uparrow^R[0; 1] \) and \( B = \bigcap \{ \uparrow^R[0; 1 + \varepsilon] \mid \varepsilon > 0 \} \). Then the function \( f = \{(x; x/2) \mid x \in \mathbb{R}\} \) witnesses both inequalities \( A \succeq B \) and \( B \succeq A \). But these filters cannot be isomorphic because only one of them is principal. \( \square \)

Lemma 13.62. Let \( f_0 \) and \( f_1 \) be Set-morphisms. Let \( f(x; y) = (f_0x; f_1y) \) for a function \( f \). Then \( \langle \langle \mathbb{F}CD(\mathbb{Src} f_0 \times \mathbb{Src} f_1; \mathbb{Dst} f_0 \times \mathbb{Dst} f_1) f \rangle \mathbb{A} \times \mathbb{RLD} \mathbb{B} \rangle = \langle \langle \mathbb{F}CD f_0 \rangle \mathbb{A} \times \mathbb{RLD} \langle \langle \mathbb{F}CD f_1 \rangle \mathbb{B} \rangle \rangle \).

**Proof.** \( \langle \langle \mathbb{F}CD(\mathbb{Src} f_0 \times \mathbb{Src} f_1; \mathbb{Dst} f_0 \times \mathbb{Dst} f_1) f \rangle \mathbb{A} \times \mathbb{RLD} \mathbb{B} \rangle = \bigcap \{ \langle \langle \mathbb{F}CD(\mathbb{Src} f_0 \times \mathbb{Src} f_1; \mathbb{Dst} f_0 \times \mathbb{Dst} f_1) f \rangle \mathbb{A} \times \mathbb{RLD} \mathbb{B} \rangle \mid A \in \mathbb{A}, B \in \mathbb{B} \} = \bigcap \{ \langle \langle \mathbb{F}CD f_0 \rangle \mathbb{A} \times \langle \langle \mathbb{F}CD f_1 \rangle \mathbb{B} \rangle \mid A \in \mathbb{A}, B \in \mathbb{B} \} = \bigcap \{ \langle \langle \mathbb{F}CD f_0 \rangle \mathbb{A} \times \langle \langle \mathbb{F}CD f_1 \rangle \mathbb{B} \rangle \mid A \in \mathbb{A}, B \in \mathbb{B} \} = \langle \langle \mathbb{F}CD(\mathbb{Src} f_0, \mathbb{Dst} f_1) f_1 \rangle \mathbb{A} \times \mathbb{RLD} \langle \langle \mathbb{F}CD f_1 \rangle \mathbb{B} \rangle \rangle. \)

Theorem 13.63. Let \( f \) be a monovalued reloid. Then \( GR f \) is isomorphic to the filter \( dom f \).

**Proof.** Let \( f \) be a monovalued reloid. There exists a function \( F \in GR f \). Consider the bijective function \( p = \{(x; Fx) \mid x \in dom F\} \).

\[ (p)dom F = F \quad and \quad (p)dom f = \bigcap \{ \langle \langle \mathbb{F}CD(\mathbb{Src} f_0; \mathbb{Dst} f_1) f \rangle \mathbb{A} \times \mathbb{RLD} \mathbb{B} \rangle \mid K \in GR f \} = \bigcap \{ \langle \langle \mathbb{F}CD(\mathbb{Src} f_0; \mathbb{Dst} f_1) f \rangle \mathbb{A} \times \mathbb{RLD} \mathbb{B} \rangle \mid K \in GR f \} = \bigcap \{ \langle \langle \mathbb{F}CD f_0 \rangle \mathbb{A} \times \mathbb{RLD} \langle \langle \mathbb{F}CD f_1 \rangle \mathbb{B} \rangle \rangle. \]

Corollary 13.64. The graph of a monovalued reloid with atomic domain is atomic.

Corollary 13.65. \( GR \mathbb{id}_A^{\mathbb{RLD}} \) is isomorphic to \( A \) for every filter \( A \).

Theorem 13.66. There are atomic filters incomparable by Rudin-Keisler order.

**Proof.** See [13]. \( \square \)

Theorem 13.67. \( \succeq_1 \) and \( \succeq_2 \) are different relations.

**Proof.** Consider \( a \) is an arbitrary non-empty filter. Then \( a \succeq_1 0^{\mathbb{RLD}(\mathbb{Base}(a))} \) but not \( a \succeq_2 0^{\mathbb{RLD}(\mathbb{Base}(a))} \). \( \square \)

Proposition 13.68. If \( a \succeq_2 b \) where \( a \) is an ultrafilter then \( b \) is also an ultrafilter.

**Proof.** \( b = \langle \langle \mathbb{F}CD f \rangle \mathbb{A} \rangle \) for some \( f: \mathbb{Base}(a) \rightarrow \mathbb{Base}(b) \). So \( b \) is an ultrafilter since \( f \) is monovalued. \( \square \)

Corollary 13.69. If \( a \succeq_1 b \) where \( a \) is an ultrafilter then \( b \) is also an ultrafilter or \( 0^{\mathbb{RLD}(\mathbb{Base}(a))} \).

**Proof.** \( b \subseteq \langle \langle \mathbb{F}CD f \rangle \mathbb{A} \rangle \) for some \( f: \mathbb{Base}(a) \rightarrow \mathbb{Base}(b) \). Therefore \( b' = \langle \langle \mathbb{F}CD f \rangle \mathbb{A} \rangle \) is an ultrafilter. From this our statement follows. \( \square \)

Proposition 13.70. Principal filters, generated by sets of the same cardinality, are isomorphic.

**Proof.** Let \( A \) and \( B \) be sets of the same cardinality. Then there are \( f \) from \( A \) to \( B \). We have \( \langle \langle \mathbb{F}CD f \rangle \mathbb{A} = B \rangle \) and thus \( A \) and \( B \) are isomorphic. \( \square \)

Proposition 13.71. If a filter is isomorphic to a principal filter, then it is also a principal filter induced by a set with the same cardinality.

**Proof.** Let \( A \) be a principal filter and \( B \) is a filter isomorphic to \( A \). Then there are sets \( X \in A \) and \( Y \in B \) such that there is a bijection \( f: X \rightarrow Y \) such that \( \langle \langle \mathbb{F}CD f \rangle \mathbb{A} = B \rangle \).

So \( min B \) exists and \( min B = \langle \langle \mathbb{F}CD f \rangle \mathbb{A} = B \rangle \) and thus \( B \) is a principal filter (of the same cardinality as \( A \)). \( \square \)
Proposition 13.72. A filter isomorphic to a non-trivial ultrafilter is a non-trivial ultrafilter.

Proof. Let $a$ be a non-trivial ultrafilter and $a$ is isomorphic to $b$. Then $a \geq b$ and thus $b$ is an ultrafilter. The filter $b$ cannot be trivial because otherwise $a$ would be also trivial.

Theorem 13.73. For an infinite set $U$ there exist $2^{2^{\text{card } U}}$ equivalence classes of isomorphic ultrafilters.

Proof. The number of bijections between any two given subsets of $U$ is no more than $(\text{card } U)^{\text{card } U} = 2^{\text{card } U}$. The number of bijections between all pairs of subsets of $U$ is no more than $2^{2^{\text{card } U}}$. Therefore each isomorphism class contains at most $2^{\text{card } U}$ ultrafilters. But there are $2^{2^{\text{card } U}}$ ultrafilters. So there are $2^{2^{\text{card } U}}$ classes.

Remark 13.74. One of the above mentioned equivalence classes contains trivial ultrafilters.

Corollary 13.75. There exist non-isomorphic nontrivial ultrafilters on any infinite set.

13.4 Consequences

Theorem 13.76. The graph of reloid $\uparrow^A \{a\} \times^{\text{RLD}} \mathcal{F}$ is isomorphic to the filter $\mathcal{F}$ for every set $A$ and $a \in A$.

Proof. Consider $B = \{a\} \times \text{Base}(\mathcal{F})$ and $f = \{(x; (a); x) \mid x \in \text{Base}(\mathcal{F})\}$. Then $f$ is a bijection from $\text{Base}(\mathcal{F})$ to $B$.

If $X \in \mathcal{F}$ then $(f) X \subseteq B$ and $(f) X = \{a\} \times X \in \text{GR}(\uparrow^A \{a\} \times^{\text{RLD}} \mathcal{F})$.

For every $Y \in \text{GR}(\uparrow^A \{a\} \times^{\text{RLD}} \mathcal{F}) \cap \mathcal{B}$ we have $Y = \{a\} \times X$ for some $X \in \mathcal{F}$ and thus $Y = (f) X$.

So $(f)_{\mathcal{F}} \cap \mathcal{B}$ is a bijection from $\mathcal{B}$ to $\text{GR}(\uparrow^A \{a\} \times^{\text{RLD}} \mathcal{F}) \cap \mathcal{B}$.

We have $\mathcal{F} \cap \mathcal{B}$ and $\text{GR}(\uparrow^A \{a\} \times^{\text{RLD}} \mathcal{F}) \cap \mathcal{B}$ directly isomorphic and thus $\mathcal{F}$ is isomorphic to $\text{GR}(\uparrow^A \{a\} \times^{\text{RLD}} \mathcal{F})$.

Theorem 13.77. If $f$, $g$ are reloids, $f \subseteq g$ and $g$ is monovalued then $g_{|\text{dom } f} = f$.

Proof. It’s simple to show that $f = \bigcup \{f_{|a} \mid a \in \text{atoms}^{\beta} (\text{Src } f)\}$ (use the fact that $k \leq f_{|a}$ for some $a \in \text{atoms}^{\beta} (\text{Src } f)$ for every $k \leq f_{|a}$ and the fact that RLD(\text{Src } f, \text{Dst } f) is atomic).

Suppose that $g_{|\text{dom } f} \neq f$. Then there exists $a \in \text{atoms } \text{dom } f$ such that $g_{|a} \neq f_{|a}$.

Obviously $g_{|a} \equiv f_{|a}$.

If $g_{|a} \equiv f_{|a}$ then $g_{|a}$ is not atomic (because $f_{|a} \neq g_{|a}$) what contradicts to a theorem above. So $g_{|a} = f_{|a}$ what is a contradiction and thus $g_{|\text{dom } f} = f$.

Corollary 13.78. Every monovalued reloid is a restricted principal monovalued reloid.

Proof. Let $f$ be a monovalued reloid. Then there exists a function $F \in \text{GR } f$. So we have $((\uparrow^{\text{RLD}} (\text{Src } f; \text{Dst } f), F))_{|\text{dom } f} = f$.

Corollary 13.79. Every monovalued injective reloid is a restricted injective monovalued principal reloid.

Proof. Let $f$ be a monovalued injective reloid. There exists a function $F$ such that $f = ((\uparrow^{\text{RLD}} (\text{Src } f; \text{Dst } f), F))_{|\text{dom } f}$. Also there exists an injection $G \in \text{GR } f$.

Thus $f = f \cap ((\uparrow^{\text{RLD}} (\text{Src } f; \text{Dst } f), G))_{|\text{dom } f} = ((\uparrow^{\text{RLD}} (\text{Src } f; \text{Dst } f), F))_{|\text{dom } f} \cap ((\uparrow^{\text{RLD}} (\text{Src } f; \text{Dst } f), G))_{|\text{dom } f} = ((\uparrow^{\text{RLD}} (\text{Src } f; \text{Dst } f), (F \cap G)))_{|\text{dom } f}$. Obviously $F \cap G$ is an injection.

Theorem 13.80. If a reloid $f$ is monovalued and $\text{dom } f$ is a principal filter then $f$ is principal.
13.4 Consequences

Proof. $f$ is a restricted principal monovalued reloid. Thus $f = F\mid_{\text{dom } f}$ where $F$ is a principal monovalued reloid. Thus $f$ is principal. \hfill $\square$

Lemma 13.81. If a filter $\mathcal{A}$ is isomorphic to a filter $\mathcal{B}$ then if $X$ is a set then there exists a set $Y$ such that $\uparrow^{\text{Base}(\mathcal{A})}X \cap \mathcal{A}$ is a filter isomorphic to $\uparrow^{\text{Base}(\mathcal{B})}Y \cap \mathcal{B}$.

Proof. Let $f$ be a monovalued injective reloid such that $\text{dom } f = \mathcal{A}$, im $f = \mathcal{B}$.

By proposition 4.227 we have: $\uparrow^{\text{Base}(\mathcal{A})}X \cap \mathcal{A} = \mathcal{X}$ where $\mathcal{X}$ is a filter complementsive to $\mathcal{A}$. Let $\mathcal{Y} = \mathcal{A} \setminus \mathcal{X}$.

$(\mathcal{FCD})f : \mathcal{Y} \cap (\mathcal{FCD})f : \mathcal{Y} = 0 \uparrow^{\text{Base}(\mathcal{B})} \mathcal{Y}$ by injectivity of $f$.

$(\mathcal{FCD})f : \mathcal{Y} \cup (\mathcal{FCD})f : \mathcal{Y} = (\mathcal{FCD})f : (\mathcal{X} \cup \mathcal{Y}) = (\mathcal{FCD})f : \mathcal{A} = \mathcal{B}$. So $(\mathcal{FCD})f : \mathcal{X}$ is a filter complementsive to $\mathcal{B}$. So by proposition 4.227 there exists a set $Y$ such that $(\mathcal{FCD})f : \mathcal{X} = \uparrow^{\text{Base}(\mathcal{B})}Y \cap \mathcal{B}$. $f : \mathcal{X}$ is obviously a monovalued injective reloid with $\text{dom } (f : \mathcal{X}) = \uparrow^{\text{Base}(\mathcal{A})}X \cap \mathcal{A}$ and $\text{im } (f : \mathcal{X}) = \uparrow^{\text{Base}(\mathcal{B})}Y \cap \mathcal{B}$. \hfill $\square$

Example 13.82. $\mathcal{A} \geqslant \mathcal{B} \land \mathcal{B} \geqslant \mathcal{A}$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ for some filters $\mathcal{A}$ and $\mathcal{B}$.

Proof. (proof idea by Andreas Blass, rewritten using reloids by me)

Let $u_n, h_n$ with $n$ ranging over the set $\mathbb{Z}$ be sequences of ultrafilters on $\mathcal{N}$ and functions $\mathcal{N} \to \mathcal{N}$ such that $(\uparrow^{\mathcal{FCD}(\mathbb{Z} \times \mathbb{N})}h_n)_{n+1} = u_n$ and $u_n$ are pairwise non-isomorphic. (See [6] for a proof that such ultrafilters and functions exist.)

$\mathcal{A} = \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times \mathcal{RLD} u_{2n+1} \mid n \in \mathbb{Z}\}$; $\mathcal{B} = \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times \mathcal{RLD} u_{2n} \mid n \in \mathbb{Z}\}$.

Let the set-morphism $f, g : \mathbb{Z} \times \mathcal{N} \to \mathbb{Z} \times \mathcal{N}$ be defined by the formulas $f(n; x) = (n; h_n x)$ and $g(n; x) = (n - 1; h_n x - 1)$.

Using the fact that every function induces a complete funcoid and a lemma above we get:

$(\uparrow^{\mathcal{FCD}}f) : \mathcal{A} = \bigcup \{\uparrow^{\mathcal{FCD}}\mathcal{RLD} u_{2n+1} \mid n \in \mathbb{Z}\} = \bigcup \{\uparrow^{\mathcal{Z}}\{n\} \times \mathcal{RLD} u_{2n+1} \mid n \in \mathbb{Z}\} = \mathcal{B} = \bigcup \{\uparrow^{\mathcal{Z}}\{n\} \times \mathcal{RLD} u_{2n} \mid n \in \mathbb{Z}\} = \bigcup \{\uparrow^{\mathcal{Z}}\{n\} \times \mathcal{RLD} u_{2n-1} \mid n \in \mathbb{Z}\} = \mathcal{A}$.

It remains to show that $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic. Let $\mathcal{X} = \uparrow^{\mathcal{Z}}\{n\} \times \mathcal{RLD} u_{2n+1}$ for some $n \in \mathbb{Z}$. Then if $\uparrow^{\mathcal{Z} \times \mathcal{N}} X \cap \mathcal{A}$ is an ultrafilter we have $\uparrow^{\mathcal{Z} \times \mathcal{N}} X \cap \mathcal{A} = \uparrow^{\mathcal{Z}}\{n\} \times \mathcal{RLD} u_{2n+1}$ and thus by the theorem 13.76 is isomorphic to $u_{2n+1}$.

If $\mathcal{X} \not\subseteq \uparrow^{\mathcal{Z}}\{n\} \times \mathcal{RLD} u_{2n+1}$ for every $n \in \mathbb{Z}$ then $(\mathcal{Z} \times \mathcal{N}) \setminus \mathcal{X} = \uparrow^{\mathcal{Z}}\{n\} \times \mathcal{RLD} u_{2n+1}$ and thus $(\mathcal{Z} \times \mathcal{N}) \setminus \mathcal{X} \subseteq \mathcal{A}$ and thus $\uparrow^{\mathcal{Z} \times \mathcal{N}} X \cap \mathcal{A} = 0^{\mathcal{Z} \times \mathcal{N}}$.

We have also $(\uparrow^{\mathcal{Z}}\{0\} \times \mathcal{RLD} \mathcal{N}) \cap \mathcal{B} = (\uparrow^{\mathcal{Z}}\{0\} \times \mathcal{RLD} \mathcal{N}) \cap (\{\uparrow^{\mathcal{Z}}\{n\} \times \mathcal{RLD} u_{2n} \mid n \in \mathbb{Z}\} = \uparrow^{\mathcal{Z}}\{0\} \times \mathcal{RLD} \mathcal{N}) \cap (\{\uparrow^{\mathcal{Z}}\{n\} \times \mathcal{RLD} u_{2n} \mid n \in \mathbb{Z}\} = \{\uparrow^{\mathcal{Z}}\{0\} \times \mathcal{RLD} \mathcal{N}) \cap \{0\} \times \mathcal{RLD} u_{2n} \mid n \in \mathbb{Z}\} = \{\uparrow^{\mathcal{Z}}\{0\} \times \mathcal{RLD} \mathcal{N}) \cap \{0\} \times \mathcal{RLD} u_{2n} \mid n \in \mathbb{Z}\}$ is an ultrafilter.

Thus every ultrafilter generated as intersecting $\mathcal{A}$ with a principal filter $\uparrow^{\mathcal{Z} \times \mathcal{N}} X$ is isomorphic to some $u_{2n+1}$ and thus is not isomorphic to $u_0$. By the lemma it follows that $\mathcal{A}$ and $\mathcal{B}$ are non-isomorphic. \hfill $\square$

13.4.1 Metamoidalvalued reloids

Proposition 13.83. $(\bigcap \mathcal{G}) \circ f = \bigcap \{g \circ f \mid g \in \mathcal{G}\}$ for a function $f$ and a set $\mathcal{G}$ of binary relations.

Proof. $(x; z) \in (\bigcap \mathcal{G}) \circ f \Leftrightarrow \exists y : (f x = y \wedge (y; z) \in \bigcap \mathcal{G}) \Leftrightarrow (f x; z) \in \bigcap \mathcal{G} \Leftrightarrow \forall g \in \mathcal{G} : (f x; z) \in g \circ f \Leftrightarrow \forall g \in \mathcal{G} : (x; z) \in g \circ f \Leftrightarrow (x; z) \in \bigcap \{g \circ f \mid g \in \mathcal{G}\}$. \hfill $\square$

Lemma 13.84. $(\bigcap \mathcal{G}) \circ f = \bigcap \{g \circ f \mid g \in \mathcal{G}\}$ if $f$ is a monovalued principal reloid and $\mathcal{G}$ is a set of reloids (with matching sources and destinations).

Proof. Let $f = \uparrow^{\mathcal{RLD}} \varphi$ for some monovalued Rel-morphism $\varphi$.

$(\bigcap \mathcal{G}) \circ f = \bigcap \{\uparrow^{\mathcal{RLD}} (g \circ \varphi) \mid g \in \mathcal{GR} \cap \mathcal{G}\}$

$\mathcal{GR} \cap \{g \circ \varphi \mid g \in \mathcal{G}\} = \mathcal{GR} \cap \{\{\uparrow^{\mathcal{RLD}} (\Gamma \circ \varphi) \mid \Gamma \in \mathcal{P} \mathcal{GR} g\} \mid g \in \mathcal{G}\} = \mathcal{GR} \cap \{\{\uparrow^{\mathcal{RLD}} (\Gamma \circ \varphi) \mid \Gamma \in \mathcal{P} \mathcal{GR} \cap \mathcal{G}\} = \{\Gamma_0 \circ \varphi \cap \ldots \cap \Gamma_n \circ \varphi \mid \Gamma_i \in \bigcup \mathcal{G}\}$

where $i = 0, \ldots, n$ for $n \in \mathbb{N}$ = (proposition above) = $\{\Gamma_0 \cap \ldots \cap \Gamma_n \circ \varphi \mid \Gamma_i \in \bigcup \mathcal{G}\}$ where $i = 0, \ldots, n$ for $n \in \mathbb{N}$ = $\bigcap \{\Gamma \circ \varphi \mid \Gamma \in \mathcal{P} \mathcal{GR} \cap \mathcal{G}\}$. \hfill $\square$
Thus \((\prod G) \circ f = \prod \{ g \circ f \mid g \in G \}\).

\(\square\)

**Theorem 13.85.**

1. Monovalued reloids are metamonovalued.
2. Injective reloids are metainjective.

**Proof.** We will prove only the first, as the second is dual.

Let \(G\) be a set of reloids and \(f\) be a monovalued reloid.

Let \(f'\) be a principal monovalued continuation of \(f\) (so that \(f = f'|_{\text{dom} f}\)).

By the lemma \((\prod G) \circ f' = \prod \{ g \circ f' \mid g \in G \}\). Restricting this equality to \(\text{dom} f\) we get:
\((\prod G) \circ f = \prod \{ g \circ f \mid g \in G \}\).

\(\square\)

**Conjecture 13.86.** Every metamonovalued reloid is monovalued.
Chapter 14
Counter-examples about funcoids and reloids

For further examples we will use the filter defined by the formula
\[ \Delta = \bigcap \{ \uparrow^{\mathbb{F}}_{\mathbb{R}}(\varepsilon; \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \} . \]

I will denote \( \Omega(A) \) the Fréchet filter on a set \( A \).

**Example 14.1.** There exist a funcoid \( f \) and a set \( S \) of funcoids such that \( f \cap \bigcup S \neq \bigcup \{ f \cap \}_S \).

**Proof.** Let \( f = \Delta \times^{\mathbb{F}CD} \uparrow^{\mathbb{F}CD}_{\mathbb{R}}(0) \) and \( S = \{ \uparrow^{\mathbb{F}CD(R;R)}((\varepsilon; +\infty) \times \{ 0 \}) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \} \). Then \( f \cap \bigcup S = (\Delta \times^{\mathbb{F}CD} \uparrow^{\mathbb{F}CD}_{\mathbb{R}}(0) \cap \uparrow^{\mathbb{F}CD(R;R)}((0; +\infty) \times \{ 0 \})) = (\Delta \cap (0; +\infty)) \times^{\mathbb{F}CD} \uparrow^{\mathbb{F}CD}_{\mathbb{R}}(0) \neq 0^{\mathbb{F}CD(R;R)} \) while \( \bigcup \{ f \cap \}_S = \bigcup \{ 0^{\mathbb{F}CD(R;R)} \} = 0^{\mathbb{F}CD(R;R)} \). \( \Box \)

**Example 14.2.** There exist a set \( R \) of funcoids and a funcoid \( f \) such that \( f \circ \bigcup R \neq \bigcup (f \circ \_ R) \).

**Proof.** Let \( f = \Delta \times^{\mathbb{F}CD} \uparrow^{\mathbb{R}}(0) \), \( R = \{ \uparrow^{\mathbb{R}}(\varepsilon; +\infty) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \} \).

We have \( \bigcup R = \uparrow^{\mathbb{R}}(0) \times^{\mathbb{F}CD} \uparrow^{\mathbb{R}}(0; +\infty) \); \( f \circ \bigcup R = \uparrow^{\mathbb{F}CD(R;R)}((\{ 0 \} \times \{ 0 \})) \neq 0^{\mathbb{F}CD(R;R)} \) and \( \bigcup (f \circ \_ R) = \bigcup \{ 0^{\mathbb{F}CD(R;R)} \} = 0^{\mathbb{F}CD(R;R)} \). \( \Box \)

**Example 14.3.** There exist a set \( R \) of reloids and a reloid \( f \) such that \( f \circ \bigcup R \neq \bigcup (f \circ \_ R) \).

**Proof.** Let \( f = \Delta \times^{\mathbb{R}LD} \uparrow^{\mathbb{R}}(0) \), \( R = \{ \uparrow^{\mathbb{R}}(\varepsilon; +\infty) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \} \).

We have \( \bigcup R = \uparrow^{\mathbb{R}}(0) \times^{\mathbb{R}LD} \uparrow^{\mathbb{R}}(0; +\infty) \); \( f \circ \bigcup R = \uparrow^{\mathbb{R}LD(R;R)}((\{ 0 \} \times \{ 0 \})) \neq 0^{\mathbb{R}LD(R;R)} \) and \( \bigcup (f \circ \_ R) = \bigcup \{ 0^{\mathbb{R}LD(R;R)} \} = 0^{\mathbb{R}LD(R;R)} \). \( \Box \)

**Example 14.4.** There exist a set \( R \) of funcoids and filters \( \mathcal{X} \) and \( \mathcal{Y} \) such that

1. \( \mathcal{X} \cap \bigcup R \mathcal{Y} \neq \emptyset \).
2. \( \bigcup \{ (f) \mathcal{X} \mid f \in R \} \).

**Proof.**

1. Let \( \mathcal{X} = \Delta \) and \( \mathcal{Y} = 1^{\mathbb{F}}(\mathbb{R}) \). Let \( R = \{ \uparrow^{\mathbb{F}CD(R;R)}((\varepsilon; +\infty) \times \mathbb{R}) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0 \} \). Then \( \bigcup R = \uparrow^{\mathbb{F}CD(R;R)}((0; +\infty) \times \mathbb{R}) \). So \( \mathcal{X} \bigcup \bigcup R \mathcal{Y} \) and \( \forall f \in R \vdash (\mathcal{X} \mathcal{Y}) \).

2. With the same \( \mathcal{X} \) and \( R \) we have \( \bigcup \{ (f) \mathcal{X} \mid f \in R \} = 1^{\mathbb{F}}(\mathbb{R}) \) and \( (f) \mathcal{X} \mathcal{Y} = 0^{\mathbb{F}}(\mathbb{R}) \) for every \( f \in R \), thus \( \bigcup \{ (f) \mathcal{X} \mid f \in R \} = 0^{\mathbb{F}}(\mathbb{R}) \). \( \Box \)

**Example 14.5.** \( \{ A \times^{\mathbb{R}LD} B \mid B \in T \} \neq A \times^{\mathbb{R}LD} \bigcup T \) for some filter \( T \) and set of filters \( T \) (with a common base).

**Proof.** Take \( R_{+} = \{ x \in \mathbb{R} \mid x > 0 \} \), \( A = \Delta \), \( T = \{ \uparrow_{+} \mid x \} \mid x \in R_{+} \) where \( \uparrow_{+} = \uparrow^{\mathbb{R}LD} \).

\[ \bigcup_{+} T = \bigcup_{+} A \times^{\mathbb{R}LD} \bigcup_{+} T = \Delta \times^{\mathbb{R}LD} \uparrow^{\mathbb{R}LD}_{+} \cdot \bigcup_{+} \{ A \times^{\mathbb{R}LD} B \mid B \in T \} = \bigcup \{ \Delta \times^{\mathbb{R}LD} \uparrow_{+} \mid x \} \mid x \in R_{+} \} . \]

We’ll prove that \( \bigcup \{ \Delta \times^{\mathbb{R}LD} \uparrow_{+} \mid x \} \mid x \in R_{+} \} \neq \Delta \times^{\mathbb{R}LD} \uparrow^{\mathbb{R}LD}_{+} \cdot \bigcup_{+} \{ A \times^{\mathbb{R}LD} B \mid B \in T \} \). Consider \( K = \bigcup \{ \{ x \} \times (-1/x; 1/x) \mid x \in R_{+} \} \). \( K \in \Delta \times^{\mathbb{R}LD} \uparrow_{+} \) and thus \( K \supseteq \bigcup \{ \Delta \times^{\mathbb{R}LD} \uparrow_{+} \mid x \} \mid x \in R_{+} \} \). But \( K \notin \Delta \times^{\mathbb{R}LD} \uparrow^{\mathbb{R}LD}_{+} \). \( \Box \)
Theorem 14.6. For a filter \( a \) we have \( a \times \text{RLD} \subseteq \text{id} \text{RLD}^{\text{Base}(a)} \) only in the case if \( a = 0^{\text{RLD}^a} \) or \( a \) is a trivial ultrafilter.

**Proof.** If \( a \times \text{RLD} \subseteq \text{id} \text{RLD}^{\text{Base}(a)} \) then there exists \( m \in \text{GR}(a \times \text{RLD} a) \) such that \( m \subseteq \text{id} \text{Base}(a) \). Consequently there exist \( A, B \in \text{GR} a \) such that \( A \times B \subseteq \text{id} \text{Base}(a) \) what is possible only in the case when \( \text{id} \text{Base}(a) = \text{id} \text{Base}(a) B = a \) is trivial ultrafilter or the least filter.

**Corollary 14.7.** Reloidal product of a non-trivial atomic filter with itself is non-atomic.

**Proof.** Obviously \( (a \times \text{RLD} a) \cap \text{id} \text{RLD}^{\text{Base}(a)} \neq 0^{\text{RLD}^a} \) and \( (a \times \text{RLD} a) \cap \text{id} \text{RLD}^{\text{Base}(a)} \subseteq a \times \text{RLD} a \).

**Example 14.8.** There exist two atomic reloids whose composition is non-atomic and non-empty.

**Proof.** Let \( a \) be a non-trivial ultrafilter on \( N \) and \( x \in N \). Then
\[
(a \times \text{RLD } \downarrow N \{x\}) \circ (\downarrow N \{x\} \times \text{RLD} a) = \bigcup \{ (\downarrow \text{RLD}^{\downarrow N \{x\}}(A \times \{x\}) \circ (\{x\} \times A)) | A \in a \} = \bigcup \{ (\downarrow \text{RLD}^{\downarrow N \{x\}}(A \times A) | A \in a \} = a \times \text{RLD} a
\]
is non-atomic despite of \( a \times \text{RLD}^\downarrow N \{x\} \) and \( \downarrow N \{x\} \times \text{RLD} a \) are atomic.

**Example 14.9.** There exists non-monovaled atomic reloid.

**Proof.** From the previous example it follows that the atomic reloid \( \downarrow N \{x\} \times \text{RLD} a \) is not monovaled.

**Example 14.10.** Non-convex reloids exist.

**Proof.** Let \( a \) be a non-trivial ultrafilter. Then \( \text{id}^\text{RLD} a \) is non-convex. This follows from the fact that only reloidal products which are below \( \text{id} \text{RLD}^{\text{Base}(a)} \) are reloidal products of ultrafilters and \( \text{id}^\text{RLD} a \) is not their join.

**Example 14.11.** \( (\text{RLD})_{\text{in}} f \neq (\text{RLD})_{\text{out}} f \) for a funcoid \( f \).

**Proof.** Let \( f = \text{id}^{\text{FCD}(U)} \). Then \( (\text{RLD})_{\text{in}} f = \bigcup \{ a \times \text{RLD} f | \text{atom} f = a \} \) and \( (\text{RLD})_{\text{out}} f = \text{id} \text{RLD}^{\text{FCD}(U)} \). But as have shown above \( a \times \text{RLD} f = \text{id} \text{RLD}^{\text{FCD}(U)} \) for non-trivial ultrafilter \( a \), and so \( (\text{RLD})_{\text{in}} f \neq (\text{RLD})_{\text{out}} f \).

**Proposition 14.12.** \( \text{id}^{\text{FCD}(U)} \cap \text{id}^{\text{FCD}(U)}((U \times U) \setminus \text{id} U) = \text{id}^{\text{FCD}(U)}U \neq 0^{\text{FCD}(U)} U \) for every infinite set \( U \).

**Proof.** Note that \( \langle \text{id}^{\text{FCD}(U)} U \rangle x = X \cap \Omega(U) \) for every filter \( X \) on \( U \).

Let \( f = \text{id}^{\text{FCD}(U)} U \), \( g = \text{id}^{\text{FCD}(U)}((U \times U) \setminus \text{id} U) \).

Let \( x \) be a non-trivial ultrafilter on \( U \). If \( X \in x \) then card \( X \geq 2 \) (In fact, \( X \) is infinite but we don’t need this.) and consequently \( \langle g \rangle x = 1^\text{id} U \). Thus \( \langle f \rangle x = 1^\text{id} U \). Consequently
\[
\langle f \cap g \rangle x = \langle f \rangle x \cap \langle g \rangle x = x \cap 1^\text{id} U = x.
\]
Also \( \langle \text{id}^{\text{FCD}(U)} U \rangle x = x \cap \Omega(U) = x \).

Let \( x \) be a trivial ultrafilter. Then \( \langle f \rangle x = x \) and \( \langle g \rangle x = 1^\text{id} U \). So
\[
\langle f \cap g \rangle x = \langle f \rangle x \cap \langle g \rangle x = x \cap 1^\text{id} U = 0^\text{id} U.
\]
Also \( \langle \text{id}^{\text{FCD}(U)} U \rangle x = x \cap \Omega(U) = 0^\text{id} U \).

So \( \langle f \cap g \rangle x = \langle \text{id}^{\text{FCD}(U)} U \rangle x \) for every ultrafilter \( x \) on \( U \). Thus \( f \cap g = \text{id}^{\text{FCD}(U)} U \).

**Example 14.13.** There exist binary relations \( f \) and \( g \) such that \( \uparrow^\text{FCD}(A, B) f \cap \uparrow^\text{FCD}(A, B) g \neq \uparrow^\text{FCD}(A, B)(f \cap g) \) for some sets \( A, B \) such that \( f, g \subseteq A \times B \).
Counter-examples about funcoids and reloids

**Proof.** From the proposition above. □

**Example 14.14.** There exists a principal funcoid which is not a complemented element of the lattice of funcoids.

**Proof.** I will prove that quasi-complement of the funcoid id_{FCD(N)} is not its complement. We have:

$$(\text{id}_{FCD(N)\uparrow})^* = \bigcap \{ c \in FCD(N; N) \mid c \approx \text{id}_{FCD(N)} \}$$

$$= \bigcap \{ f \in FCD(N; N) \mid f \approx \text{id}_{FCD(N)} \}$$

$$= \bigcap \{ \{ \alpha \} \times FCD \uparrow N \mid \alpha, \beta \in N, \alpha \neq \beta \}$$

$$= \bigcap \{ \{ \alpha \} \times FCD \uparrow N \mid \alpha, \beta \in N, \alpha \neq \beta \}$$

(used the corollary 6.103). But by proved above

$$(\text{id}_{FCD(N)\uparrow})^* \cap \text{id}_{FCD(N)} \neq \emptyset.$$ □

**Example 14.15.** There exists a funcoid $h$ such that GR $h$ is not a filter.

**Proof.** Consider the funcoid $h = \text{id}_{FCD(N)}$. We have (from the proof of proposition 14.12) that

$$f \in \text{GR} h \text{ and } g \in \text{GR} h, \text{ but } f \cap g = \emptyset \notin \text{GR} h.$$ □

**Example 14.16.** There exists a funcoid $h \neq 0_{FCD(A; B)}$ such that $(RLD)_{\text{out}} h = 0_{RLD(A; B)}$.

**Proof.** Consider $h = \text{id}_{FCD(N)}$. By proved above $h = f \cap g$ where $f = \text{id}_{FCD(N)} = \text{id}_{FCD(N)\uparrow}$, $g = \text{id}_{FCD(N)\uparrow}(N \times N \setminus \text{id}_N)$.

We have $\text{id}_N, N \times N \setminus \text{id}_N \in \text{GR} h$.

So $(RLD)_{\text{out}} h = \bigcap \{ \{ h \} \times FCD \uparrow N \} \text{GR} h \subseteq \bigcap \{ 0_{RLD(N; N)}(\text{id}_N \cap (N \times N \setminus \text{id}_N)) = 0_{RLD(N; N)}$; and thus

$$(RLD)_{\text{out}} h = 0_{RLD(N; N)}.$$ □

**Example 14.17.** There exists a funcoid $h$ such that $(FCD)(RLD)_{\text{out}} h \neq h$.

**Proof.** It follows from the previous example. □

**Example 14.18.** $(RLD)_{\text{in}}(FCD)f \neq f$ for some convex reloid $f$.

**Proof.** Let $f = \text{id}_{RLD(N)}$. Then $(FCD)f = \text{id}_{FCD(N)}$. Let $a$ be some non-trivial ultrafilter on $N$. Then $(RLD)_{\text{in}}(FCD)f \neq a \times_{RLD} \text{id}_{RLD(N)}$ and thus $(RLD)_{\text{in}}(FCD)f \neq f$.

**Example 14.19.** There exist funcoids $f$ and $g$ such that

$$(RLD)_{\text{out}} (g \circ f) \neq (RLD)_{\text{out}} g \circ (RLD)_{\text{out}} f.$$ **Proof.** Take $f = \text{id}_{FCD(N)}$ and $g = 1_{\mathcal{S}(N)} \times FCD \uparrow N \{ \alpha \}$ for some $\alpha \in N$. Then $(RLD)_{\text{out}} f = 0_{RLD(N; N)}$ and thus

$$(RLD)_{\text{out}} g \circ (RLD)_{\text{out}} f = 0.$$ We have $g \circ f = \Omega(N) \times FCD \uparrow N \{ \alpha \}$.

Let’s prove $(RLD)_{\text{out}}(\Omega(N) \times FCD \uparrow N \{ \alpha \}) = \Omega(N) \times FCD \uparrow N \{ \alpha \}$.

**Example 14.20.** $(FCD)$ does not preserve finite meets.
Proof. \((FCD)(\text{id}^{\text{RLD}(N)} \cap (1_{\text{RLD}(N)}^{\text{RLD}(N)} \setminus \text{id}^{\text{RLD}(N)})) = (FCD)\text{id}^{\text{RLD}(N)} = 0^{\text{FCD}(N)}\).

On the other hand, \((FCD)\text{id}^{\text{RLD}(N)} \cap (FCD)(1_{\text{RLD}(N)}^{\text{RLD}(N)} \setminus \text{id}^{\text{RLD}(N)}) = \text{id}^{\text{FCD}(N) \cap \text{RLD}(N)}(N \times N \setminus \text{id}(N)) \neq 0^{\text{FCD}(N)}\) (used proposition 8.1). \(\square\)

**Corollary 14.21.** \((FCD)\) is not an upper adjoint (in general).

Considering restricting polynomials (considered as reloids) to ultrafilters, it is simple to prove that each that restriction is injective if not restricting a constant polynomial. Does this hold in general? No, see the following example:

**Example 14.22.** There exists a monovalued reloid with atomic domain which is neither injective nor constant (that is not a restriction of a constant function).

**Proof.** (based on [30]) Consider the function \(F \in N^{N \times N}\) defined by the formula \((x; y) \mapsto x\).

Let \(\omega_2\) be a non-trivial ultrafilter on the vertical line \(\{x\} \times N\) for every \(x \in N\).

Let \(T\) be the collection of such sets \(Y\) that \(Y \cap (\{x\} \times N) \in \omega_2\) for all but finitely many vertical lines. Obviously \(T\) is a filter.

Let \(\omega\) be atoms \(T\).

For every \(x \in N\) we have some \(Y \in T\) for which \((\{x\} \times N) \cap Y = \emptyset\) and thus \(\uparrow^N \times (\{x\} \times N) \cap \omega = \emptyset^{(N \times N)}\).

Let \(g = (\uparrow_{\text{RLD}(N)}^{\text{RLD}(N)} F)\). If \(g\) is constant, then there exist a constant function \(G \in \text{GR} g\) and \(F \cap G\) is also constant. Obviously dom \(\uparrow_{\text{RLD}(N \times N)}^{\text{RLD}(N \times N)} (F \cap G) \subseteq \omega\). The function \(F \cap G\) cannot be constant because otherwise \(\omega \subseteq \text{dom} \uparrow_{\text{RLD}(N \times N)}^{\text{RLD}(N \times N)} (F \cap G) \subseteq \uparrow^N \times (\{x\} \times N)\) for some \(x \in N\) what is impossible by proved above. So \(g\) is not constant.

Suppose that \(g\) is injective. Then there exists an injection \(G \in \text{GR} g\). So dom \(G\) intersects each vertical line by atmost one element that is dom \(G\) intersects every vertical line by the whole line or the line without one element. Thus dom \(G \in T \cup \omega\) and consequently dom \(G \notin \omega\) what is impossible.

Thus \(g\) is neither injective nor constant. \(\square\)

### 14.1 Second product. Oblique product

**Definition 14.23.** \(A \times_{\text{RLD}} B = (\text{RLD})_{\text{out}}(A \times FCD B)\) for every filters \(A\) and \(B\). I will call it second direct product of filters \(A\) and \(B\).

**Remark 14.24.** The letter \(F\) is the above definition is from the word “funcoid”. It signifies that it seems to be impossible to define \(A \times_{\text{RLD}} B\) directly without referring to funcoidal product.

**Definition 14.25.** Oblique products of filters \(A\) and \(B\) are defined as

\[
A \times B = \bigcap \{ \uparrow_{\text{RLD}} f | f \in \text{MorRel}(\text{Base}(A) ; \text{Base}(B)), \forall B \in B ; \uparrow_{\text{FCD}} f \subseteq A \times_{\text{FCD}} B \};
\]

\[
A \times B = \bigcap \{ \uparrow_{\text{RLD}} f | f \in \text{MorRel}(\text{Base}(A) ; \text{Base}(B)), \forall A \in A ; \uparrow_{\text{FCD}} f \subseteq B \};
\]

**Proposition 14.26.** \(A \times_{\text{RLD}} B \subseteq A \times B \subseteq A \times_{\text{RLD}} B\) for every filters \(A, B\).

**Proof.** \(A \times B \subseteq \bigcap \{ \uparrow_{\text{RLD}} f | f \in \text{MorRel}(\text{Base}(A) ; \text{Base}(B)), \forall A \in A, B \in B; \uparrow_{\text{FCD}} f \supseteq \uparrow_{\text{Base}(A)} A \times_{\text{FCD}} B \}; \)

\(A \times B \subseteq \bigcap \{ \uparrow_{\text{RLD}} f | f \in \text{MorRel}(\text{Base}(A) ; \text{Base}(B)), \uparrow_{\text{FCD}} f \supseteq A \times_{\text{FCD}} B \} = \bigcap \{ \uparrow_{\text{RLD}} f | f \in \text{xyGR}(A \times_{\text{FCD}} B) \} = (\text{RLD})_{\text{out}}(A \times_{\text{FCD}} B) = A \times_{\text{RLD}} B.\) \(\square\)

**Conjecture 14.27.** \(A \times_{\text{RLD}} B \subseteq A \times B\) for some filters \(A, B\).

A stronger conjecture:

**Conjecture 14.28.** \(A \times_{\text{RLD}} B \subseteq A \times B \subseteq A \times_{\text{RLD}} B\) for some filters \(A, B\). Particularly, is this formula true for \(A = B = \Delta \cap \uparrow_{\text{GR}} (0; +\infty)\)?
The above conjecture is similar to Fermat Last Theorem as having no value by itself but being somehow challenging to prove it.

**Example 14.29.** \( A \times B \sqsubseteq A \times \text{RLD} B \) for some filters \( A, B. \)

**Proof.** It’s enough to prove \( A \times B \neq A \times \text{RLD} B. \)

Let \( \Delta_+ = \Delta \cap \uparrow^R (0; + \infty). \) Let \( A = B = \Delta_+. \)

Let \( K = (\leq)_{\mathbb{R} \times \mathbb{R}}. \)

Obviously \( K \notin \text{GR}(A \times \text{RLD} B). \)

\( A \times B \sqsubseteq \uparrow^\text{RLD} (\text{Base}(A); \text{Base}(B)) K \) and thus \( K \in A \times B \) because \( \uparrow^\text{FCD}(\text{Base}(A); \text{Base}(B)) K \subseteq \Delta_+ \times \text{FCD} \uparrow^\text{Base}(B) B = A \times \text{FCD} \uparrow^\text{Base}(B) B \) for \( B = (0; + \infty). \)

Thus \( A \times B \neq A \times \text{RLD} B. \)

**Example 14.30.** \( A \times \text{RLD} B \sqsubseteq A \times \text{RLD} B \) for some filters \( A, B. \)

**Proof.** This follows from the above example.

**Proposition 14.31.** \( (A \times B) \cap (A \times B) = A \times \text{RLD} B \) for every filters \( A, B. \)

**Proof.** \( (A \times B) \cap (A \times B) \subseteq \cap \{ \uparrow^\text{RLD} f \mid f \in \text{Mor}_{\text{Rel}}(\text{Base}(A); \text{Base}(B)) \} = \text{FCD} f \supseteq A \times \text{FCD} B = A \times \text{RLD} B. \)

To finish the proof we need to show \( A \times B \sqsupseteq A \times \text{RLD} B \) and \( A \times B \sqsubseteq A \times \text{RLD} B. \) By symmetry it’s enough to show \( A \times B \sqsupseteq A \times \text{RLD} B \) what is proved above.

**Example 14.32.** \( (A \times B) \cup (A \times B) \sqsubseteq A \times \text{RLD} B \) for some filters \( A, B. \)

**Proof.** (based on [8]) Let \( A = B = \Omega(N). \) It’s enough to prove \( (A \times B) \cup (A \times B) \neq A \times \text{RLD} B. \)

Let \( X \in A, Y \in B \) that is \( X \in \Omega(N), Y \in \Omega(N). \)

Removing one element \( x \) from \( X \) produces a set \( P. \) Removing one element \( y \) from \( Y \) produces a set \( Q. \) Obviously \( P \in \Omega(N), Q \in \Omega(N). \)

Obviously \( (P \times N) \cup (N \times Q) \in \text{GR}((A \times B) \cup (A \times B)). \)

\( (P \times N) \cup (N \times Q) \nsubseteq X \times Y \) because \( (x, y) \notin X \times Y \) but \( (x, y) \notin (P \times N) \cup (N \times Q). \)

Thus \( (P \times N) \cup (N \times Q) \notin \text{GR}(A \times \text{RLD} B) \) by properties of filter bases.

**Example 14.33.** \( \text{RLD}_{\text{out}}(\text{FCD}) f \neq f \) for some convex reloid \( f. \)

**Proof.** Let \( f = A \times \text{RLD} B \) where \( A \) and \( B \) are from example 14.30.

\( \text{FCD}(A \times \text{RLD} B) = A \times \text{FCD} B \) by the proposition 8.9.

So \( \text{RLD}_{\text{out}}(\text{FCD})(A \times \text{RLD} B) = \text{RLD}_{\text{out}}(A \times \text{FCD} B) = A \times \text{RLD} B \neq A \times \text{RLD} B. \)
Chapter 15
Pointfree funcoids

This chapter is based on [28].

This is a routine chapter. There is almost nothing creative here. I just generalize theorems about funcoids to the maximum extent for pointfree funcoids (defined below) preserving the proof idea. The main idea behind this chapter is to find weakest theorem conditions enough for the same theorem statement as for above theorems for funcoids.

For those who know pointfree topology: Pointfree topology notions of frames and locales is a non-trivial generalization of topological spaces. Pointfree funcoids are different: I just replace the set of filters on a set with an arbitrary poset, this readily gives the definition of pointfree funcoid, almost no need of creativity here.

Pointfree funcoids are used in the below definitions of products of funcoids.

15.1 Definition

Definition 15.1. Pointfree funcoid is a quadruple \((\mathfrak{A}; \mathfrak{B}; \alpha; \beta)\) where \(\mathfrak{A}\) and \(\mathfrak{B}\) are posets, \(\alpha \in \mathfrak{B}^{\mathfrak{A}}\) and \(\beta \in \mathfrak{A}^{\mathfrak{B}}\) such that

\[\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (y \not\equiv \alpha x \iff x \not\equiv \beta y).\]

Definition 15.2. The source \(\text{Src}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{A}\) and destination \(\text{Dst}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{B}\) for every pointfree funcoid \((\mathfrak{A}; \mathfrak{B}; \alpha; \beta)\).

Definition 15.3. I will denote \(\text{FCD}(\mathfrak{A}; \mathfrak{B})\) the set of pointfree funcoids from \(\mathfrak{A}\) to \(\mathfrak{B}\) (that is with source \(\mathfrak{A}\) and destination \(\mathfrak{B}\)), for every posets \(\mathfrak{A}\) and \(\mathfrak{B}\).

Proposition 15.4. If \(\mathfrak{A}\) and \(\mathfrak{B}\) have least elements, then \(\text{FCD}(\mathfrak{A}; \mathfrak{B})\) has least element.

Proof. It is \((\mathfrak{A}; \mathfrak{B}; \emptyset \times \{\emptyset\}; \emptyset \times \{\emptyset\})\). \(\square\)

Definition 15.5. \(\langle (\mathfrak{A}; \mathfrak{B}; \alpha; \beta) \rangle \defeq \alpha\) for a pointfree funcoid \((\mathfrak{A}; \mathfrak{B}; \alpha; \beta)\).

Definition 15.6. \((\mathfrak{A}; \mathfrak{B}; \alpha; \beta)^{-1} = (\mathfrak{B}; \mathfrak{A}; \beta; \alpha)\) for a pointfree funcoid \((\mathfrak{A}; \mathfrak{B}; \alpha; \beta)\).

Proposition 15.7. If \(f\) is a pointfree funcoid then \(f^{-1}\) is also a pointfree funcoid.

Proof. It follows from symmetry in the definition of pointfree funcoid. \(\square\)

Obvious 15.8. \((f^{-1})^{-1} = f\) for a pointfree funcoid \(f\).

Definition 15.9. The relation \([f] \in \mathcal{P}(\text{Src} f \times \text{Dst} f)\) is defined by the formula (for every pointfree funcoid \(f\) and \(x \in \text{Src} f, y \in \text{Dst} f\))

\[x[f] y \defeq y \not\equiv (f)x.\]

Obvious 15.10. \(x[f] y \iff y \not\equiv (f)x \iff x \not\equiv (f^{-1})y\) for every pointfree funcoid \(f\) and \(x \in \text{Src} f, y \in \text{Dst} f\).

Obvious 15.11. \([f^{-1}] = [f]^{-1}\) for a pointfree funcoid \(f\).
**Theorem 15.12.** Let \( \mathfrak A \) and \( \mathfrak B \) be posets. Then:

1. If \( \mathfrak A \) is separable, for given value of \( \langle f \rangle \) there exists no more than one \( f \in \text{FCD}(\mathfrak A; \mathfrak B) \).
2. If \( \mathfrak A \) and \( \mathfrak B \) are separable, for given value of \( \langle f \rangle \) there exists no more than one \( f \in \text{FCD}(\mathfrak A; \mathfrak B) \).

**Proof.**

Let \( f, g \in \text{FCD}(\mathfrak A; \mathfrak B) \).

1. Let \( \langle f \rangle = \langle g \rangle \). Then for every \( x \in \mathfrak A \), \( y \in \mathfrak B \) we have \( x \neq (f^{-1})y \Leftrightarrow y \neq \langle f \rangle x \Leftrightarrow y \neq \langle g \rangle x \Leftrightarrow x \neq (g^{-1})y \) and thus by separability of \( \mathfrak A \) we have \( \langle f^{-1} \rangle y = \langle g^{-1} \rangle y \) that is \( (f^{-1}) = (g^{-1}) \) and so \( f = g \).

2. Let \( \langle f \rangle = \langle g \rangle \). Then for every \( x \in \mathfrak A \), \( y \in \mathfrak B \) we have \( y \neq (f)x \Leftrightarrow x [f] y \Leftrightarrow x [g] y \Leftrightarrow y \neq (g)x \) and thus by separability of \( \mathfrak B \) we have \( \langle f \rangle x = \langle g \rangle x \) that is \( \langle f \rangle = \langle g \rangle \). Similarly we have \( (f^{-1}) = (g^{-1}) \). Thus \( f = g \).

**Proposition 15.13.** If \( \text{Dst} f \) is separable, \( \text{Src} f \) and \( \text{Dst} f \) have least elements, then \( \langle f \rangle \text{Src} f = 0^{\text{Dst} f} \) for every pointfree funcoid \( f \).

**Proof.** \( y \neq \langle f \rangle \text{Src} f \Leftrightarrow 0^{\text{Src} f} \neq (f^{-1})y \Leftrightarrow 0 \Leftrightarrow y \neq 0^{\text{Dst} f} \). Thus by separability, \( \langle f \rangle \text{Src} f = 0^{\text{Dst} f} \).

**Proposition 15.14.** If \( \text{Dst} f \) is a separable poset then \( \langle f \rangle \) is a monotone function (for a pointfree funcoid \( f \)).

**Proof.** \( a \subseteq b \Leftrightarrow \forall x \in \text{Dst} f : (a \neq (f^{-1})x \Rightarrow b \neq (f^{-1})x) \Rightarrow \forall x \in \text{Dst} f : (x \neq \langle f \rangle a \Rightarrow x \neq \langle f \rangle b) \Rightarrow \langle f \rangle a \subseteq \langle f \rangle b \).

**Theorem 15.15.** Let \( f \) be a pointfree funcoid from a starrish join-semilattice \( \text{Src} f \) to a separable starrish join-semilattice \( \text{Dst} f \). Then \( \langle f \rangle (i \sqcup j) = \langle f \rangle i \sqcup \langle f \rangle j \) for every \( i, j \in \text{Src} f \).

**Proof.**

\[
\star(f)(i \sqcup j) = \\
\{ y \in \text{Dst} f \mid y \neq \langle f \rangle (i \sqcup j) \} = \\
\{ y \in \text{Dst} f \mid i \sqcup j \neq (f^{-1})y \} = \\
\{ y \in \text{Dst} f \mid i \neq (f^{-1})y \lor \neq (f^{-1})y \} = \\
\{ y \in \text{Dst} f \mid y \neq (f)i \lor y \neq (f)j \} = \\
\{ y \in \text{Dst} f \mid y \neq \langle f \rangle i \sqcup \langle f \rangle j \} = \\
\star((f)i \sqcup (f)j).
\]

Thus \( \langle f \rangle (i \sqcup j) = \langle f \rangle i \sqcup \langle f \rangle j \) by separability.

**Proposition 15.16.** Let \( f \) be a pointfree funcoid. Then:

1. \( k[f] i \sqcup j \Leftrightarrow k[f] i \lor k[f] j \) for every \( ij \in \text{Dst} f, k \in \text{Src} f \) if \( \text{Dst} f \) is a starrish join-semilattice.
2. \( i \sqcup j [f] k \Leftrightarrow i[f] k \lor j[f] k \) for every \( ij \in \text{Src} f, k \in \text{Dst} f \) if \( \text{Src} f \) is a starrish join-semilattice.

**Proof.**

1. \( k[f] i \sqcup j \Leftrightarrow i \sqcup j \neq (f)k \Leftrightarrow i \neq (f)k \lor j \neq (f)k \Leftrightarrow k[f] i \lor k[f] j \).
2. Similar.

---

### 15.2 Composition of pointfree funcoids

**Definition 15.17.** *Composition* of pointfree funcoids is defined by the formula

\[
(\mathfrak B; \mathfrak C; \alpha_2; \beta_2) \circ (\mathfrak A; \mathfrak B; \alpha_1; \beta_1) = (\mathfrak A; \mathfrak C; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).
\]

**Definition 15.18.** I will call funcoids \( f \) and \( g \) *composable* when \( \text{Dst} f = \text{Src} g \).

**Proposition 15.19.** If \( f, g \) are pointfree funcoids and \( \text{Dst} f = \text{Src} g \) then \( g \circ f \) is pointfree funcoid.
Proof. Let \( f = (\mathfrak{A}; \mathfrak{B}; \alpha_1; \beta_1), g = (\mathfrak{B}; \mathfrak{C}; \alpha_2; \beta_2) \). For every \( x, y \in \mathfrak{A} \) we have
\[
y \neq (\alpha_2 \circ \alpha_1) x \Leftrightarrow y \neq \alpha_2 x \Leftrightarrow \alpha_1 x \neq \beta_2 y \Leftrightarrow x \neq \beta_1 \beta_2 y \Leftrightarrow x \neq (\beta_1 \circ \beta_2) y.
\]
So \((\mathfrak{A}; \mathfrak{C}; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)\) is a pointfree funcoid.

<table>
<thead>
<tr>
<th>Obvious 15.20.</th>
<th>( (g \circ f) = (g) \circ (f) ) for every composable pointfree funcoids ( f ) and ( g ).</th>
</tr>
</thead>
<tbody>
<tr>
<td>Theorem 15.21.</td>
<td>( (g \circ f)^{-1} = f^{-1} \circ g^{-1} ) for every composable pointfree funcoids ( f ) and ( g ).</td>
</tr>
</tbody>
</table>
| Proof.        | \[
\langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle = \langle f^{-1} \circ g^{-1} \rangle;
\]
|               | \[
\langle ((g \circ f)^{-1})^{-1} \rangle = \langle (g \circ f) \rangle = \langle (f^{-1} \circ g^{-1})^{-1} \rangle.
\]

Proposition 15.22. \( (h \circ g) \circ f = h \circ (g \circ f) \) for every composable pointfree funcoids \( f, g, h \).

Proposition 15.23. Let \( f \) be a pointfree funcoid. Then for every \( x \in \text{Src} f, y \in \text{Dst} f \) we have
\[
\begin{enumerate}
  \item If \( (\text{Src} f; 3) \) is a filter with separable core then \( x[f] y \Leftrightarrow \forall X \in \text{up}^{(\text{Src} f; 3)} x : X[f] y. \)
  \item If \( (\text{Dst} f; 3) \) is a filter with separable core then \( x[f] y \Leftrightarrow \forall Y \in \text{up}^{(\text{Dst} f; 3)} y : x[f] Y. \)
\end{enumerate}

Proof. We will prove only the second because the first is similar.
\[
x[f] y \Leftrightarrow y \neq x[f] Y \Leftrightarrow \forall Y \in \text{up}(\text{Dst} f; 3) y : x[f] Y. \]

Corollary 15.24. Let \( f \) be a pointfree funcoid and \( (\text{Src} f; 3_0), (\text{Dst} f; 3_1) \) are filters with separable core. Then
\[
x[f] y \Leftrightarrow \forall X \in \text{up}^{(\text{Src} f; 3_0)} x, Y \in \text{up}^{(\text{Dst} f; 3_1)} y : x[f] Y.
\]

Proof. Apply the proposition twice.

Theorem 15.25. Let \( f \) be a pointfree funcoid. Let \( (\text{Src} f; 3_0) \) be a finitely meet-closed filter with separable core which is a meet-semilattice and \( \forall x \in \text{Src} f : \text{up}(\text{Src} f; 3_0) x \neq \emptyset \) and \( (\text{Dst} f; 3_1) \) is a primary filterator over a boolean lattice.
\[
(f)x = \bigcap_{y \in \text{Dst} f} \langle f \rangle y \in \text{up}^{(\text{Src} f; 3_0)} x.
\]

Proof. By the previous proposition for every \( y \in \text{Dst} f \):
\[
y \neq \bigcap_{y \in \text{Dst} f} \langle f \rangle y \Leftrightarrow \forall X \in \text{up}^{(\text{Src} f; 3_0)} x : X[f] y \Leftrightarrow \forall X \in \text{up}^{(\text{Src} f; 3_0)} x : y \neq \bigcap_{y \in \text{Dst} f} \langle f \rangle y.
\]
Let’s denote \( W = \{ y \in \text{Dst} f : \langle f \rangle X | X \in \text{up}^{(\text{Src} f; 3_0)} x \} \). We will prove that \( W \) is a generalized filter base over \( 3_1 \). To prove this enough to show that \( V = \{ \langle f \rangle X | X \in \text{up}^{(\text{Src} f; 3_0)} x \} \) is a generalized filter base.

Let \( \mathcal{P}, \mathcal{Q} \in V \). Then \( \mathcal{P} = \langle f \rangle A, \mathcal{Q} = \langle f \rangle B \) where \( A, B \in \text{up}^{(\text{Src} f; 3_0)} x \) and \( A \cap 3_0 B \in \text{up}^{(\text{Src} f; 3_0)} x \) (used the fact that it is a finitely meet-closed and theorem 4.44) and \( \mathcal{R} \subseteq \mathcal{P} \cap \text{Dst} f \mathcal{Q} \) for \( \mathcal{R} = \langle f \rangle (A \cap 3_0 B) \in V \) because \( \text{Dst} f \) is separable by obvious 4.136. So \( V \) is a generalized filter base and thus \( W \) is a generalized filter base.
$0^\text{Dest} f \notin W \Leftrightarrow \bigsqcap_{\text{Dest} f} W \neq 0^\text{Dest} f$ by theorem 4.121. That is

$$\forall X \in \uparrow^{(\mathcal{S}r c, 3_0)} X; y \cap_{\text{Dest} f} (f)X \neq 0^\text{Dest} f \Leftrightarrow y \cap_{\text{Dest} f} \bigcap_{\text{Dest} f} \langle (f) \rangle \uparrow^{(\mathcal{S}r c, 3_0)} X \neq 0^\text{Dest} f.$$ 

Comparing with the above, $y \cap_{\text{Dest} f} (f)x \neq 0^\text{Dest} f \Leftrightarrow y \cap_{\text{Dest} f} \bigcap_{\text{Dest} f} \langle (f) \rangle \uparrow^{(\mathcal{S}r c, 3_0)} X \neq 0^\text{Dest} f$. So $(f)x = \bigcap_{\text{Dest} f} \langle (f) \rangle \uparrow^{(\mathcal{S}r c, 3_0)} X$ because Dest $f$ is separable (obvious 4.136 and the fact that $3_1$ is a boolean lattice).

**Theorem 15.26.** Let $\langle 3; 3_0 \rangle$ and $\langle 2; 3_1 \rangle$ be primary filtrators over boolean lattices.

1. A function $\alpha \in 3^3_0$ conforming to the formulas (for every $I, J \in 3_0$)

$$\alpha 0^3_0 = 0^2_2, \quad \alpha (I \uplus J) = \alpha I \uplus \alpha J$$

can be continued to the function $(f)$ for a unique $f \in FCD(3; 2)$:

$$\langle f \rangle X = \bigcap_{\text{Dest} f} (\alpha) \uparrow^{(3; 3_0)} X$$

(15.1)

for every $X \in 3$.

2. A relation $\delta \in \mathcal{P}(3_0 \times 3_1)$ conforms to the formulas (for every $I, J, K \in 3_0$ and $I', J'$, $K' \in 3_1$)

$$\neg (0^3_0 \delta I'), \ I \uplus J \delta K' \Rightarrow I \delta K' \lor J \delta K', \quad \neg (I \delta 0^3_0), \ K \delta I' \uplus J' \Rightarrow K \delta I' \lor K \delta J'$$

(15.2)

can be continued to the relation $[f]$ for a unique $f \in FCD(3; 2)$:

$$X [f] \Rightarrow \forall X \in \uparrow^{(3; 3_0)} X, Y \in \uparrow^{(3; 3_1)} Y: X \delta Y$$

(15.3)

for every $X \in 3$, $Y \in 2$.

**Proof.** Existence of no more than one such pointfree funcoids and formulas (15.1) and (15.3) follow from two previous theorems.

2. $\{Y \in 3_1 \mid X \delta Y\}$ is obviously a free star for every $X \in 3_0$. By properties of filters on boolean lattices, there exist a unique filter $\alpha X$ such that $\partial(\alpha X) = \{Y \in 3_1 \mid X \delta Y\}$ for every $X \in 3_0$. Thus $\alpha \in 3^3_0$. Similarly it can be defined $\beta \in 3^3_1$ by the formula $\partial(\beta Y) = \{X \in 3_0 \mid X \delta Y\}$. Let’s continue the functions $\alpha$ and $\beta$ to $\alpha' \in 3^3_2$ and $\beta' \in 2^2_2$ by the formulas

$$\alpha' X = \bigcap_{\text{Dest} f} (\alpha) \uparrow^{(3; 3_0)} X \quad \text{and} \quad \beta' Y = \bigcap_{\text{Dest} f} (\beta) \uparrow^{(3; 3_1)} Y$$

and $\delta$ to $\delta' \in \mathcal{P}(3 \times 2)$ by the formula

$$X \delta' \Rightarrow \forall X \in \uparrow^{(3; 3_0)} X, Y \in \uparrow^{(3; 3_1)} Y: X \delta Y.$$ 

$Y \cap \alpha' X \neq 0^3_2 \Leftrightarrow Y \cap \bigcap_{\text{Dest} f} (\alpha) \uparrow^{(3; 3_0)} X \neq 0^3_2 \Leftrightarrow Y \cap (Y \cap (\alpha) \uparrow^{(3; 3_0)} X \neq 0^3_2$. Let’s prove that

$$W = (Y \cap (\alpha) \uparrow^{(3; 3_0)} X$$

is a generalized filter base. To prove it is enough to show that $(\alpha) \uparrow^{(3; 3_0)} X$ is a generalized filter base.

If $A, B \in (\alpha) \uparrow^{(3; 3_0)} X$ then exist $X_1, X_2 \in \uparrow^{(3; 3_0)} X$ such that $A = \alpha X_1$ and $B = \alpha X_2$. Then $\alpha (X_1 \cap X_2) \in (\alpha) \uparrow^{(3; 3_0)} X$. So $(\alpha) \uparrow^{(3; 3_0)} X$ is a generalized filter base and thus $W$ is a generalized filter base.

By properties of generalized filter bases, $\cap (Y \cap (\alpha) \uparrow^{(3; 3_0)} X \neq 0^3_2$ is equivalent to

$$\forall X \in \uparrow^{(3; 3_0)} X: Y \cap \alpha X \neq 0^3_2,$$

what is equivalent to $\forall X \in \uparrow^{(3; 3_0)} X, Y \in \uparrow^{(3; 3_1)} Y: Y \cap X \neq 0^3 \Leftrightarrow \forall X \in \uparrow^{(3; 3_0)} X, Y \in \uparrow^{(3; 3_1)} Y: Y \in \partial (\alpha X) \Leftrightarrow Y \in \uparrow^{(3; 3_0)} X, Y \in \uparrow^{(3; 3_1)} Y: Y \delta X$. Combining the equivalences we get $Y \cap Y' = X \delta Y$. Analogously $X \cap Y' \neq 0^3 \Leftrightarrow X \delta Y$. So $Y \cap X \neq 0^3 \Leftrightarrow X \cap Y \neq 0^3$, that is $(3; 2; \alpha'; \beta')$ is a pointfree funcoid. From the formula $Y \cap X \neq 0^3 \Rightarrow X \delta Y$ it follows that $[(3; 2; \alpha'; \beta')]$ is a continuation of $\delta$. 

15.4 The order of pointfree funcoids

Definition 15.28. The order of pointfree funcoids \( \text{FCD}(\mathcal{A}; \mathcal{B}) \) is defined by the formula:

\[
f \sqsubseteq g \iff \forall x \in \mathcal{A}; \langle f \rangle x \sqsubseteq \langle g \rangle x \land \forall y \in \mathcal{B}; \langle f^{-1} \rangle y \sqsubseteq \langle g^{-1} \rangle y.
\]

Proposition 15.29. It is really a partial order on the set \( \text{FCD}(\mathcal{A}; \mathcal{B}) \).

Proof.

\textbf{Reflexivity.} Obvious.

\textbf{Transitivity.} It follows from transitivity of the order relations on \( \mathcal{A} \) and \( \mathcal{B} \).

\textbf{Antisymmetry.} It follows from antisymmetry of the order relations on \( \mathcal{A} \) and \( \mathcal{B} \). \( \square \)

Remark 15.30. It is enough to define order of pointfree funcoids on every set \( \text{FCD}(\mathcal{A}; \mathcal{B}) \) where \( \mathcal{A} \) and \( \mathcal{B} \) are posets. We do not need to compare pointfree funcoids with different sources or destinations.

Obvious 15.31. \( f \sqsubseteq g \Rightarrow [f] \sqsubseteq [g] \) for every \( f, g \in \text{FCD}(\mathcal{A}; \mathcal{B}) \) for every posets \( \mathcal{A} \) and \( \mathcal{B} \).

Theorem 15.32. If \( \mathcal{A} \) and \( \mathcal{B} \) are separable posets then \( f \sqsubseteq g \Rightarrow [f] \sqsubseteq [g] \).
Proof. From the theorem 15.12. □

Theorem 15.33. Let \((\mathfrak{A}; \preceq_0)\) and \((\mathfrak{B}; \preceq_1)\) be primary filtrators over boolean lattices. Then for \(R \in \mathfrak{FCD}(\mathfrak{A}; \mathfrak{B})\) and \(X \preceq 0_0\), \(Y \preceq 0_1\) we have:

1. \(X \sqcup R \preceq Y \Leftrightarrow \exists f \in R: X [f] Y\);
2. \(\sqcup R \sqcup \{(f) [x] f \in R\} \preceq \{(f) X \preceq f \in R\}\).

Proof.

2. \(\alpha X = \{f [x] f \in R\} \) (by corollary 4.107 all joins on \(\mathfrak{B} \) exist). We have \(\alpha 0^{\mathfrak{B}} = 0^{\mathfrak{B}}\);

\[
\alpha (I \sqcup 3^{\mathfrak{B}} J) = \{f | f \in R\} \sqcup \{(f) (I \sqcup 3^{\mathfrak{B}} J) | f \in R\} \\
= \sqcup \{(f) (I \sqcup 3^{\mathfrak{B}} J) | f \in R\} \\
= \sqcup \{(f) I \sqcup 3^{\mathfrak{B}} (f) J | f \in R\} \\
= \sqcup \{(f) I | f \in R\} \sqcup 3^{\mathfrak{B}} \{(f) J | f \in R\} \\
= \alpha I \sqcup 3^{\mathfrak{B}} \alpha J
\]

(used theorem 15.15). By theorem 15.26 the function \(\alpha \) can be continued to \((h) \) for a \(h \in \mathfrak{FCD}(\mathfrak{A}; \mathfrak{B})\). Obviously

\[
\forall f \in R: h \preceq f.
\]

And \(h \) is the least element of \(\mathfrak{FCD}(\mathfrak{A}; \mathfrak{B})\) for which the condition (15.4) holds. So \(h = \sqcup R\).

1. \(X \sqcup R \preceq Y \Leftrightarrow Y \sqcap 3^{\mathfrak{B}} \sqcup R \preceq 0^{\mathfrak{B}} \Leftrightarrow Y \sqcap 3^{\mathfrak{B}} \sqcup \{(f) X \preceq f \in R\} \preceq 0^{\mathfrak{B}} \Leftrightarrow \exists f \in R: Y \sqcap 3^{\mathfrak{B}} \sqcup (f) X \preceq 0^{\mathfrak{B}} \Leftrightarrow \exists f \in R: X [f] Y\) (used the theorem 4.118).

Corollary 15.34. If \((\mathfrak{A}; \preceq_0)\) and \((\mathfrak{B}; \preceq_1)\) are primary filtrators over boolean lattices then \(\mathfrak{FCD}(\mathfrak{A}; \mathfrak{B})\) is a complete lattice.

Proof. Apply [26]. □

Theorem 15.35. Let \(\mathfrak{A}\) and \(\mathfrak{B}\) be starshir join-semilattices. Then for \(f, g \in \mathfrak{FCD}(\mathfrak{A}; \mathfrak{B})\):

1. \((f \sqcup g) x = (f) x \sqcup (g) x \) for every \(x \in \mathfrak{A}\);
2. \([f \sqcup g] = [f] \sqcup [g]\).

Proof.

1. Let \(\alpha X = (f) x \sqcup (g) x; \beta Y = (f^{-1}) y \sqcup (g^{-1}) y \) for every \(x \in \mathfrak{A}, y \in \mathfrak{B}\). Then

\[
y \preceq 3^{\mathfrak{B}} \alpha x \Leftrightarrow y \preceq (f) x \sqcup (g) x \Leftrightarrow x \preceq (f^{-1}) y \sqcup (g^{-1}) y
\]

So \(h = (\mathfrak{A}; \mathfrak{B}; \alpha; \beta)\) is a pointfree funcoid. Obviously \(h \preceq f\) and \(h \preceq g\). If \(p \sqsupseteq f\) and \(p \sqsupseteq g\) for some \(p \in \mathfrak{FCD}(\mathfrak{A}; \mathfrak{B})\) then \((p) x \sqsupseteq (f) x \sqcup (g) x = (h) x \) and \((p^{-1}) y \sqsupseteq (f^{-1}) y \sqcup (g^{-1}) y = (h^{-1}) y \) that is \(p \sqsupseteq h\).

So \(f \sqsupseteq g = h\).

2. \(x [f \sqcup g] y \Leftrightarrow y \preceq (f \sqcup g) x \Leftrightarrow y \preceq (f) x \sqcup (g) x \Leftrightarrow y \preceq (f) x \sqcup y \preceq (g) x \Leftrightarrow x [f] y \sqcup x [g] y \) for every \(x \in \mathfrak{A}, y \in \mathfrak{B}\).

Theorem 15.36. Let \(f\) be a pointfree funcoid from a separable poset \(\mathfrak{A}\) to a separable poset \(\mathfrak{B}\). If \(f\) is an injection, then \((f)\) is an order embedding \(\mathfrak{A} \to \mathfrak{B}\).

Proof. Suppose \(x \sqsupseteq y\) but \((f) x \not\preceq (f) y\).

Then by separability of \(\mathfrak{B}\) there exist \(z \preceq (f) x\) such that \(z \simeq (f) x\). Thus \((f^{-1}) z \simeq x\) and \((f^{-1}) z \not\preceq y\) what is impossible for \(x \sqsupseteq y\). □

Corollary 15.37. Let \(f\) be a pointfree funcoid from a separable poset \(\mathfrak{A}\) to a separable poset \(\mathfrak{B}\). If \(f\) is a bijection \(\mathfrak{A} \to \mathfrak{B}\), then \((f)\) is an order isomorphism \(\mathfrak{A} \to \mathfrak{B}\).
15.5 Domain and range of a pointfree funcoid

**Definition 15.38.** Let \( \mathfrak{A} \) be a poset. The identity pointfree funcoid \( \text{id}^{\text{FCD}(\mathfrak{A})} = (\mathfrak{A}; \mathfrak{A}; \text{id}_\mathfrak{A}; \text{id}_\mathfrak{A}). \)

It is trivial that identity funcoid is really a pointfree funcoid.

Let now \( \mathfrak{A} \) be a meet-semilattice.

**Definition 15.39.** Let \( a \in \mathfrak{A} \). The restricted identity pointfree funcoid \( \text{id}^{\text{FCD}(\mathfrak{A})}_a = (\mathfrak{A}; \mathfrak{A}; a \cap \mathfrak{A}; a \cap \mathfrak{A}). \)

**Proposition 15.40.** The restricted pointfree funcoid is a pointfree funcoid.

**Proof.** We need to prove that \( (a \cap \mathfrak{A}) x \neq \mathfrak{A} y \Leftrightarrow (a \cap \mathfrak{A}) y \neq \mathfrak{A} x \) what is obvious. \( \square \)

**Obvious 15.41.** \( \left( \text{id}^{\text{FCD}(\mathfrak{A})}_a \right)^{-1} = \text{id}^{\text{FCD}(\mathfrak{A})}_a. \)

**Obvious 15.42.** \( x \left[ \text{id}^{\text{FCD}(\mathfrak{A})}_a \right] y \Leftrightarrow a \neq \mathfrak{A} x \cap \mathfrak{A} y \) for every \( x, y \in \mathfrak{A}. \)

**Definition 15.43.** I will define restricting of a pointfree funcoid \( f \) to an element \( a \in \text{Src} f \) by the formula \( f |_{a} = f \circ \text{id}^{\text{FCD}(\text{Src} f)}_a. \)

**Definition 15.44.** Image of \( f \) will be defined by the formula \( \text{im} f = \bigcup \langle f \rangle \text{Src} f. \)

**Obvious 15.45.** \( \text{im} f \supseteq f x \) for every \( x \in \text{Src} f \) whenever \( \text{im} f \) is defined.

**Proposition 15.46.** \( \langle f \rangle 1 \) if \( \text{Src} f \) has greatest element 1 and \( \text{Dst} f \) is a separable poset.

**Proof.** \( \langle f \rangle 1 \) is greater than every \( f x \) (where \( x \in \text{Src} f \) ) by proposition 15.14 and thus \( \langle f \rangle 1 = \text{max} (f) \text{Src} f = \text{im} f. \quad \square \)

**Definition 15.47.** Domain of a pointfree funcoid \( f \) is defined by the formula \( \text{dom} f = \text{im} f^{-1}. \)

**Proposition 15.48.** \( \langle f \rangle \text{dom} f = \text{im} f \) if \( f \) is a pointfree funcoid and \( \text{Src} f \) has greatest element 1 and \( \text{Dst} f \) is a separable poset.

**Proof.** \( y \neq \langle f \rangle \text{dom} f \Leftrightarrow \text{dom} f \neq \langle f^{-1} \rangle y \Leftrightarrow \langle f^{-1} \rangle y \neq 0 \Leftrightarrow 1 \neq \langle f^{-1} \rangle y \Leftrightarrow y \neq \langle f \rangle 1 \) for every \( y \in \text{Dst} f. \)

So \( \langle f \rangle \text{dom} f = \langle f \rangle 1 = \text{im} f \) by separability of \( \text{Dst} f. \quad \square \)

**Proposition 15.49.** \( \langle f \rangle x = \langle f \rangle (x \sqcap \text{dom} f) \) whenever dom \( f \) is defined, for every \( x \in \text{Src} f \) for a pointfree funcoid \( f \) whose source is a meet-semilattice with least element and destination is a separable poset with least element.

**Proof.** For every \( y \in \text{Dst} f \) we have \( y \neq \langle f \rangle (x \sqcap \text{dom} f) \neq 0 \text{Dst} f \Leftrightarrow x \sqcap \text{dom} f \cap \langle f^{-1} \rangle y \neq 0 \text{Src} f \Leftrightarrow x \sqcap \text{im} f^{-1} \cap \langle f^{-1} \rangle y \neq 0 \text{Src} f \Leftrightarrow y \neq \langle f \rangle x. \) Thus \( \langle f \rangle x = \langle f \rangle (x \sqcap \text{dom} f) \) by separability of \( \text{Dst} f. \quad \square \)

**Proposition 15.50.** \( x \neq \text{dom} f \Leftrightarrow (\langle f \rangle x \text{ is not least}) \) for every pointfree funcoid \( f \) and \( x \in \text{Src} f \) if \( \text{Dst} f \) has greatest element 1 and \( \text{Src} f \) is a separable poset.

**Proof.** \( x \neq \text{dom} f \Leftrightarrow x \neq \langle f^{-1} \rangle 1 \Leftrightarrow 1 \text{Dst} f \neq \langle f \rangle x \text{ (\( \langle f \rangle x \text{ is not least}).} \quad \square \)

**Corollary 15.51.** \( \text{dom} f = \bigcup \{ a \in \text{atoms}^{\text{Src} f} \mid \langle f \rangle a \neq 0 \text{Dst} f \} \) for every pointfree funcoid \( f \) whose destination is a bounded poset and source is a separable atomistic meet-semilattice.

**Proof.** For every \( a \in \text{atoms}^{\text{Src} f} \) we have \( a \neq \text{dom} f \Leftrightarrow a \neq \text{im} f^{-1} \Leftrightarrow a \neq \langle f^{-1} \rangle 1 \text{Dst} f \Leftrightarrow 1 \text{Dst} f \neq 0 \text{Dest} f. \) So \( \text{dom} f = \bigcup \{ a \in \text{atoms}^{\text{Src} f} \mid a \neq \text{dom} f \} = \bigcup \{ a \in \text{atoms}^{\text{Src} f} \mid \langle f \rangle a \neq 0 \text{Dest} f \}. \quad \square \)
**Proposition 15.52.** \( \text{dom}(f|_a) = a \cap \text{dom} f \) for every pointfree funcoid \( f \) and \( a \in \text{Src} f \) where \( \text{Src} f \) is a separable meet-semilattice and \( \text{Dst} f \) has greatest element.

**Proof.** \( \text{dom}(f|_a) = \text{im}(\text{id}_a^{\text{FCD}(\text{Src} f)} \circ f^{-1}) = \text{im}(\text{id}_a^{\text{FCD}(\text{Src} f)})(f^{-1})1^{\text{Dst} f} = a \cap (f^{-1})1^{\text{Dst} f} = a \cap \text{dom} f. \) \( \square \)

**Proposition 15.53.** For every composable pointfree funcoids \( f \) and \( g \) where the posets \( \text{Src} f \) and \( \text{Dst} f = \text{Src} g \) have greatest elements and \( \text{Dst} f \) and \( \text{Dst} g \) are separable:
1. If \( f \not\sqsupset \text{dom} g \) then \( \text{im}(g \circ f) = \text{im} g. \)
2. If \( f \sqsubseteq \text{dom} g \) then \( \text{dom}(g \circ f) = \text{dom} g. \)

**Proof.**
1. \( \text{im}(g \circ f) = \text{im}(g \circ f)1^{\text{Src} f} = (g)(f)1^{\text{Src} f} = (g)\text{dom} g = (g)1^{\text{Src} g} = \text{im} g. \)
2. \( \text{dom}(g \circ f) = \text{dom}(f^{-1} \circ g^{-1}) \) what by the proved is equal to \( f^{-1} \text{dom} f. \) \( \square \)

### 15.6 Category of pointfree funcoids

I will define the category \( \text{pfFCD} \) of pointfree funcoids:
- The class of objects are small posets.
- The set of morphisms from \( A \) to \( B \) is \( \text{FCD}(A; B) \).
- The composition is the composition of pointfree funcoids.
- Identity morphism for an object \( A \) is \( (A; A; \text{id}_A; \text{id}_A) \).

To prove that it is really a category is trivial.

The category of pointfree funcoid triples is defined as follows:
- Objects are pairs \((A; A)\) where \( A \) is a small poset and \( A \in A \).
- The morphisms from an object \((A; A)\) to an object \((B; B)\) are tuples \((A; B; A; B; f)\) where \( f \in \text{FCD}(A; B) \) and \( \text{dom} f \subseteq A \land \text{im} f \subseteq B \).
- The composition is defined by the formula \((B; C; g) \circ (A; B; f) = (A; C; g \circ f)\).
- Identity morphism for an object \((A; A)\) is \( \text{id}^{\text{FCD}(A)} \).

To prove that it is really a category is trivial.

### 15.7 Specifying funcoids by functions or relations on atomic filters

**Theorem 15.54.** Let \( A \) be an atomic poset and \((B; 3_1)\) is a primary filtrator over a boolean lattice. Then for every \( f \in \text{FCD}(A; B) \) and \( X \in A \) we have
\[
\langle f \rangle X = \bigcup_{a \in B} \langle f \rangle \text{atoms}^a X.
\]

**Proof.** For every \( Y \in 3_1 \) we have
\[
Y \not\in B \Leftrightarrow X \not\in A (f^{-1}) Y \Leftrightarrow \exists x \in \text{atoms}^a X : x \not\in A (f^{-1}) Y \Leftrightarrow \exists x \in \text{atoms}^a X : Y \not\in B (f) x.
\]
Thus \( \partial (f) X = \bigcup \langle (f) \rangle \text{atoms}^a X = \partial \bigcup_{a \in B} \langle f \rangle \text{atoms}^a X \) (used theorem 4.132). Consequently \( \langle f \rangle X = \bigcup_{a \in B} \langle f \rangle \text{atoms}^a X \) by the corollary 4.128. \( \square \)
Proposition 15.55. Let $f$ be a pointfree funcoid. Then for every $\mathcal{X} \in \text{Src} f$ and $\mathcal{Y} \in \text{Dst} f$

1. $\mathcal{X}[f] \mathcal{Y} \iff \exists x \in \text{atoms} \mathcal{X}: x[f] \mathcal{Y}$ if $\text{Src} f$ is an atomic poset.
2. $\mathcal{X}[f] \mathcal{Y} \iff \exists y \in \text{atoms} \mathcal{Y}: \mathcal{X}[f] y$ if $\text{Dst} f$ is an atomic poset.

Proof. I will prove only the second as the first is similar.

If $\mathcal{X}[f] \mathcal{Y}$, then $\mathcal{Y} \neq (f)\mathcal{X}$, consequently exists $y \in \text{atoms} \mathcal{Y}$ such that $y \neq (f)\mathcal{X}$, $\mathcal{X}[f] y$. The reverse is obvious. □

Corollary 15.56. If $f$ is a pointfree funcoid with both source and destination being atomic posets, then for every $\mathcal{X} \in \text{Src} f$ and $\mathcal{Y} \in \text{Dst} f$

$$\mathcal{X}[f] \mathcal{Y} \iff \exists x \in \text{atoms} \mathcal{X}, y \in \text{atoms} \mathcal{Y}: x[f] y.$$  

Proof. Apply the theorem twice. □

Corollary 15.57. If $\mathfrak{A}$ is a separable atomic poset and $\mathfrak{B}$ is a separable poset then $f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$ is determined by the values of $(f)X$ for $X \in \text{atoms}^{\mathfrak{A}}$.

Proof. $y \neq (f)x \iff x \neq (f^{-1})y \iff \exists X \in \text{atoms} x: X \neq (f^{-1})y \iff \exists X \in \text{atoms} x: y \neq (f)X$.

Thus by separability of $\mathfrak{B}$ we have $(f)$ is determined by $(f)X$ for $X \in \text{atoms} x$.

By separability of $\mathfrak{A}$ we infer that $f$ can be restored from $(f)$ (Theorem 15.12). □

Theorem 15.58. Let $(\mathfrak{A}; \mathfrak{A}_0)$ and $(\mathfrak{B}; \mathfrak{B}_1)$ be primary filtrators over boolean lattices.

1. A function $\alpha \in \mathfrak{B}^{\text{atoms}^{\mathfrak{A}}}$ such that (for every $a \in \text{atoms}^{\mathfrak{A}}$)

$$\alpha a \sqsubseteq \bigsqcup \langle (\alpha) \circ \text{atoms}^{\mathfrak{A}} \rangle \uparrow^{(\mathfrak{A}; \mathfrak{B})} a$$

(15.5)

can be continued to the function $(f)$ for a unique $f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$;

$$\langle f \rangle \mathcal{X} = \bigsqcup \langle (\alpha) \text{atoms}^{\mathfrak{A}} \rangle \mathcal{X}$$

(15.6)

for every $\mathcal{X} \in \mathfrak{A}$.

2. A relation $\delta \in \mathfrak{B}(\text{atoms}^{\mathfrak{A}} \times \text{atoms}^{\mathfrak{B}})$ such that (for every $a \in \text{atoms}^{\mathfrak{A}}, b \in \text{atoms}^{\mathfrak{B}}$)

$$\forall X \in \uparrow^{(\mathfrak{A}; \mathfrak{B}, \mathfrak{J})} a, Y \in \uparrow^{(\mathfrak{B}; \mathfrak{B}, \mathfrak{J})} b \exists x \in \text{atoms}^{\mathfrak{A}} X, y \in \text{atoms}^{\mathfrak{B}} Y: x \delta y \Rightarrow a \delta b$$

(15.7)

can be continued to the relation $[f]$ for a unique $f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$;

$$\mathcal{X}[f] \mathcal{Y} \iff \exists x \in \text{atoms} \mathcal{X}, y \in \text{atoms} \mathcal{Y}: x \delta y$$

(15.8)

for every $\mathcal{X} \in \mathfrak{A}, \mathcal{Y} \in \mathfrak{B}$.

Proof. Existence of no more than one such funcoids and formulas (15.6) and (15.8) follow from the theorem 6.62 and corollary 15.24 and the fact that our filtrators are with separable core.

1. Consider the function $\alpha' \in \mathfrak{B}^{\mathfrak{A}_0}$ defined by the formula (for every $X \in \mathfrak{A}_0$)

$$\alpha' X = \bigsqcup \langle (\alpha) \text{atoms}^{\mathfrak{A}} \rangle X.$$  

Obviously $\alpha' 0^{\mathfrak{A}_0} = 0^{\mathfrak{B}}$. For every $I, J \in \mathfrak{A}_0$

$$\alpha'(I \sqcup J) = \bigsqcup \langle (\alpha') \text{atoms}^{\mathfrak{B}}(I \sqcup J) \rangle$$

$$= \bigsqcup \langle (\alpha') \text{atoms}^{\mathfrak{A}} I \sqcup \text{atoms}^{\mathfrak{B}} J \rangle$$

$$= \bigsqcup \langle (\alpha') \text{atoms}^{\mathfrak{B}} I \sqcup (\alpha') \text{atoms}^{\mathfrak{B}} J \rangle$$

$$= \bigsqcup \langle (\alpha') \text{atoms}^{\mathfrak{B}} I \sqcup \bigsqcup (\alpha') \text{atoms}^{\mathfrak{B}} J \rangle$$

$$= \alpha' I \sqcup \alpha' J.$$  

Let continue $\alpha'$ till a pointfree funcoid $f$ (by the theorem 15.26): $(f)\mathcal{X} = \bigsqcup \langle (\alpha') \uparrow^{(\mathfrak{A}; \mathfrak{A})} \rangle \mathcal{X}.$
Let’s prove the reverse of (15.5): 
\[ \bigcap \bigcup \circ \alpha \circ \text{atoms}^3 \uparrow (3;3) a = \bigcap \bigcup \circ \alpha \circ \text{atoms}^3 \uparrow (3;3) a \]
\[ \subseteq \bigcup \bigcup \circ \alpha \circ \{\{a\}\} \]
\[ = \bigcup \bigcup \circ \alpha \circ \{\{a\}\} \]
\[ = \bigcup \{\{a\}\} \]
\[ = \bigcup \{\{a\}\} = \bigcap \{\{a\}\} = a a. \]

Finally, 
\[ a a = \bigcap \bigcup \circ \alpha \circ \text{atoms}^3 \uparrow (3;3) a = \bigcap \{ (f) \uparrow (3;3) a = (f) a, \]
so \( f \) is a continuation of \( \alpha \).

2. Consider the relation \( \delta' \in \mathcal{P}(\mathfrak{z}_0 \times \mathfrak{z}_1) \) defined by the formula (for every \( X \in \mathfrak{z}_0, Y \in \mathfrak{z}_1 \))
\[ X \delta' Y \iff \exists x \in \text{atoms}^3 X, y \in \text{atoms}^3 Y : x \delta y. \]

Obviously \( \neg (X \delta' (0^3 \delta') Y) \).
\[ (I \sqcup J) \delta' Y \iff \exists x \in \text{atoms}^3 (I \sqcup J), y \in \text{atoms}^3 Y : x \delta y \]
\[ \iff \exists x \in \text{atoms}^3 I \sqcup \text{atoms}^3 J, y \in \text{atoms}^3 Y : x \delta y \]
\[ \iff \exists x \in \text{atoms}^3 I, y \in \text{atoms}^3 Y : x \delta y \vee \exists x \in \text{atoms}^3 J, y \in \text{atoms}^3 Y : x \delta y \]
\[ \iff I \delta' Y \vee J \delta' Y; \]
similarly \( X \delta' (I \sqcup J) \iff X \delta' I \vee X \delta' J \). Let’s continue \( \delta' \) till a funcoid \( f \) (by the theorem 15.26):
\[ X' (f) Y \iff \forall X \in \uparrow (3;3) X, Y \in \uparrow (3;3) Y : X \delta' Y. \]

The reverse of (15.7) implication is trivial, so
\[ \forall X \in \uparrow (3;3) a, Y \in \uparrow (3;3) b \exists x \in \text{atoms}^3 X, y \in \text{atoms}^3 Y : x \delta y \iff a \delta b. \]

\[ \forall X \in \uparrow (3;3) a, Y \in \uparrow (3;3) b \exists x \in \text{atoms}^3 X, y \in \text{atoms}^3 Y : x \delta y \iff \forall X \in \uparrow (3;3) a, Y \in \uparrow (3;3) b : X \delta' Y \iff a [f] b. \]

So \( a \delta b \iff a [f] b \), that is \( [f] \) is a continuation of \( \delta \). \( \Box \)

**Theorem 15.59.** Let \( (\mathfrak{z}; \mathfrak{z}_0) \) and \( (\mathfrak{b}; \mathfrak{z}_1) \) be primary filtrators over boolean lattices. If \( R \in \mathcal{P}_{\text{FCD}}(\mathfrak{z}; \mathfrak{b}) \) and \( x \in \text{atoms}^3, y \in \text{atoms}^3 \), then
1. \( (\bigcap R) x = \bigcap \{ (f) x \mid f \in R \}; \)
2. \( x (\bigcap R) y \iff \forall f \in R : x [f] y. \)

**Proof.**
2. Let denote \( x \delta y \iff \forall f \in R : x [f] y. \)
\[ \forall X \in \uparrow (3;3) a, Y \in \uparrow (3;3) b \exists x \in \text{atoms}^3 X, y \in \text{atoms}^3 Y : x \delta y \iff \forall f \in R, X \in \uparrow (3;3) a, Y \in \uparrow (3;3) b \exists x \in \text{atoms}^3 X, y \in \text{atoms}^3 Y : x [f] y \iff \forall f \in R, X \in \uparrow (3;3) a, Y \in \uparrow (3;3) b : X [f] Y \Rightarrow \forall f \in R : a [f] b \iff a \delta b. \]

So, by the theorem 15.58, \( \delta \) can be continued till \( [p] \) for some \( p \in \text{FCD}(\mathfrak{z}; \mathfrak{b}) \).

For every \( q \in \text{FCD}(\mathfrak{b}; \mathfrak{b}) \) such that \( \forall f \in R : q \subseteq f \) we have \( x [q] y \Rightarrow \forall f \in R : x [f] y \Rightarrow x \delta y \equiv x [p] y \), so \( q \leq p \). Consequently \( p = \bigcap R \).

From this \( x (\bigcap R) y \iff \forall f \in R : x [f] y. \)
1. From the former \( y \in \text{atoms}^3 (\bigcap R) x \iff y \cap (\bigcap R) x \neq 0^3 \iff \forall f \in R : y \cap (f) x \neq 0^3 \iff y \in \bigcap \{ (f) x \mid f \in R \} \iff y \in \text{atoms} \bigcap \{ (f) x \mid f \in R \} \) for every \( y \in \text{atoms}^3 \).

\( \mathfrak{b} \) is atomically separable by the corollary 4.138. Thus
\[ (\bigcap R) x = \bigcap \{ (f) x \mid f \in R \}. \]
15.8 More on composition of pointfree funcoids

**Proposition 15.60.** \([g \circ f] = [g] \circ [f] = (g^{-1})^{-1} \circ [f]\) for every composable pointfree funcoids \(f\) and \(g\).

**Proof.** \(x [g \circ f] y \iff y \neq (g \circ f) x \iff (g) (f) x \iff (f) x \iff x ((g) (f)) y\) for every \(x \in \mathfrak{A},\ y \in \mathfrak{B}\). Thus \([g \circ f] = [g] \circ (f)\). \([g \circ f] = [(f^{-1})^{-1}]^{-1} = (f^{-1})^{-1} = (g^{-1})^{-1} = (g^{-1})^{-1} \circ [f]\).

**Theorem 15.61.** Let \(f\) and \(g\) be pointfree funcoids and \(\mathfrak{A} = \text{Dst} \ f = \text{Src} \ g\) is an atomic poset. Then for every \(X \in \text{Src} \ f\) and \(Z \in \text{Dst} \ g\)
\[X [g \circ f] Z \iff \exists y \in \text{atoms}^{\mathfrak{A}}: (X [f] y \land g [y] Z).\]

**Proof.**
\[\exists y \in \text{atoms}^{\mathfrak{A}}: (X [f] y \land g [y] Z) \iff \exists y \in \text{atoms}^{\mathfrak{A}}: ((Z \neq (g) y \land y \neq (f) X)\]
\[\iff \exists y \in \text{atoms}^{\mathfrak{A}}: (y \neq (g^{-1}) Z \land y \neq (f) X)\]
\[\iff (g^{-1}) Z \neq (f) X\]
\[\iff X [g \circ f] Z.\]

**Theorem 15.62.** Let \(\mathfrak{A},\ \mathfrak{B},\ \mathfrak{C}\) be separable starrish join-semilattices and \(\mathfrak{B}\) is atomic. Then:
1. \(f \circ (g \sqcup h) = f \circ g \sqcup f \circ h\) for \(g, h \in \text{FCD}(\mathfrak{A}; \mathfrak{B})\) and \(f \in \text{FCD}(\mathfrak{B}; \mathfrak{C})\).
2. \((g \sqcup h) \circ f = g \circ f \sqcup h \circ f\) for \(f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})\) and \(g, h \in \text{FCD}(\mathfrak{B}; \mathfrak{C})\).

**Proof.** We will prove only the first equality because the other is analogous. We can apply theorem 15.35.
For every \(X \in \mathfrak{A},\ Y \in \mathfrak{C}\)
\[X [f \circ (g \sqcup h)] Z \iff \exists y \in \text{atoms}^{\mathfrak{A}}: (X [g \sqcup h] y \land f [y] Z)\]
\[\iff \exists y \in \text{atoms}^{\mathfrak{A}}: ((X [g] y \lor X [h] y) \land f [y] Z)\]
\[\iff \exists y \in \text{atoms}^{\mathfrak{A}}: (X [g] y \land f [y] Z) \lor (X [h] y \land f [y] Z)\]
\[\iff \exists y \in \text{atoms}^{\mathfrak{A}}: (X [g] y \land f [y] Z) \lor \exists y \in \text{atoms}^{\mathfrak{A}}: (X [h] y \land f [y] Z)\]
\[\iff X [f \circ g] Z \lor X [f \circ h] Z\]
\[\iff X [f \circ g \sqcup f \circ h] Z.\]
Thus \(f \circ (g \sqcup h) = f \circ g \sqcup f \circ h\) by theorem 15.12.

**Theorem 15.63.** Let \(\mathfrak{A},\ \mathfrak{B};\ \mathfrak{C}\) be posets of filters over some boolean lattices, \(f \in \text{FCD}(\mathfrak{A}; \mathfrak{B}),\ g \in \text{FCD}(\mathfrak{B}; \mathfrak{C}),\ h \in \text{FCD}(\mathfrak{A}; \mathfrak{C})\). Then
\[g \circ f \neq h \iff g \neq h \circ f^{-1}.\]

**Proof.**
\[g \circ f \neq h \iff \exists a \in \text{atoms}^{\mathfrak{A}},\ c \in \text{atoms}^{\mathfrak{C}}: a [(g \circ f) \cap h] c\]
\[\iff \exists a \in \text{atoms}^{\mathfrak{A}},\ c \in \text{atoms}^{\mathfrak{C}}: (a [g \circ f] c \land a [h] c)\]
\[\iff \exists a \in \text{atoms}^{\mathfrak{A}},\ b \in \text{atoms}^{\mathfrak{B}},\ c \in \text{atoms}^{\mathfrak{C}}: (a [f] b \land b [g] c \land a [h] c)\]
\[\iff \exists b \in \text{atoms}^{\mathfrak{B}},\ c \in \text{atoms}^{\mathfrak{C}}: (b [g] c \land b [h \circ f^{-1}] c)\]
\[\iff g \neq h \circ f^{-1}.\]
15.9 Direct product of elements

**Definition 15.64.** Funcoidal product $A \times^{\text{FCD}} B$ where $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ and $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements is a pointfree funcoid such that for every $X \in \mathfrak{A}$, $Y \in \mathfrak{B}$

\[
\langle A \times^{\text{FCD}} B \rangle_X = \begin{cases} B & \text{if } X \neq A; \\
0 & \text{if } X = A; \end{cases} \quad \text{and} \quad \langle (A \times^{\text{FCD}} B)^{-1} \rangle_Y = \begin{cases} A & \text{if } Y \neq B; \\
0 & \text{if } Y = B. \end{cases}
\]

**Proposition 15.65.** $A \times^{\text{FCD}} B$ is really a pointfree funcoid and

\[
X \uparrow A \times^{\text{FCD}} B \iff X \neq A \land Y \neq B.
\]

**Proof.** Obvious. □

**Proposition 15.66.** Let $\mathfrak{A}$ and $\mathfrak{B}$ be separable bounded posets, $f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$, $A \in \mathfrak{A}$, $B \in \mathfrak{B}$. Then

\[
f \subseteq A \times^{\text{FCD}} B \Rightarrow \text{dom } f \subseteq A \land \text{im } f \subseteq B.
\]

**Proof.** If $f \subseteq A \times^{\text{FCD}} B$ then dom $f \subseteq \text{dom}(A \times^{\text{FCD}} B) \subseteq A$, im $f \subseteq \text{im}(A \times^{\text{FCD}} B) \subseteq B$. If dom $f \subseteq A \land \text{im } f \subseteq B$ then $X[f] \not\supseteq Y \iff (f)X \Rightarrow Y \not\supseteq B$ and similarly $X[f] \not\supseteq X \neq A$. So $[f] \subseteq [A \times^{\text{FCD}} B]$ and thus using separability $f \subseteq A \times^{\text{FCD}} B$. □

**Theorem 15.67.** Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. For every $f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$ and $A \in \mathfrak{A}$, $B \in \mathfrak{B}$

\[
f \cap (A \times^{\text{FCD}} B) = \text{id}_B^{\text{FCD}(\mathfrak{B})} \circ f \circ \text{id}_A^{\text{FCD}(\mathfrak{A})}.
\]

**Proof.** From above FCD$(\mathfrak{A}; \mathfrak{B})$ is a (complete) lattice.

\[
h \overset{\text{def}}{=} \text{id}_B^{\text{FCD}(\mathfrak{B})} \circ f \circ \text{id}_A^{\text{FCD}(\mathfrak{A})}.
\]

For every $X \in \mathfrak{A}$

\[
\langle h \rangle X = \langle \text{id}_B^{\text{FCD}(\mathfrak{B})} \rangle \langle f \rangle \langle \text{id}_A^{\text{FCD}(\mathfrak{A})} \rangle X = B \cap (f)(A \cap X)
\]

and

\[
\langle h^{-1} \rangle X = \langle \text{id}_A^{\text{FCD}(\mathfrak{A})} \rangle \langle f^{-1} \rangle \langle \text{id}_B^{\text{FCD}(\mathfrak{B})} \rangle X = A \cap (f^{-1})(B \cap X).
\]

From this, as easy to show, $h \subseteq f$ and $h \subseteq A \times^{\text{FCD}} B$. If $g \subseteq f \land g \subseteq A \times^{\text{FCD}} B$ for a $g \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$ then dom $g \subseteq A$, im $g \subseteq B$,

\[
\langle g \rangle X = B \cap (g)(A \cap X) \subseteq B \cap (f)(A \cap X) = \langle \text{id}_B^{\text{FCD}(\mathfrak{B})} \rangle \langle f \rangle \langle \text{id}_A^{\text{FCD}(\mathfrak{A})} \rangle X = \langle h \rangle X,
\]

and similarly $\langle g^{-1} \rangle X \subseteq (h^{-1})X$.

$g \subseteq h$. So $h = f \cap (A \times^{\text{FCD}} B)$. □

**Corollary 15.68.** Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. For every $f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$ and $A \in \mathfrak{A}$ we have $f \upharpoonright A = f \cap (A \times^{\text{FCD}} 1^\mathfrak{B})$.

**Proof.** $f \cap (A \times^{\text{FCD}} 1^\mathfrak{B}) = \text{id}_1^{\text{FCD}(\mathfrak{A})} \circ f \circ \text{id}_A^{\text{FCD}(\mathfrak{A})} = f \circ \text{id}_A^{\text{FCD}(\mathfrak{A})} = f \upharpoonright A$. □

**Corollary 15.69.** Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. For every $f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$ and $A \in \mathfrak{A}$, $B \in \mathfrak{B}$ we have

\[
f \neq A \times^{\text{FCD}} B \Rightarrow f \upharpoonright (A \times^{\text{FCD}} B) \neq 0_{\text{FCD}(\mathfrak{A}; \mathfrak{B})} \iff \langle f \cap (A \times^{\text{FCD}} B) \rangle 1^\mathfrak{A} \neq 0^\mathfrak{B} \iff \langle \text{id}_B^{\text{FCD}(\mathfrak{B})} \circ f \circ \text{id}_A^{\text{FCD}(\mathfrak{A})} \rangle 1^\mathfrak{A} \neq 0^\mathfrak{B} \iff B \cap (f)(A \cap 1^\mathfrak{A}) \neq 0^\mathfrak{B} \iff B \cap (f)A \neq 0^\mathfrak{B} \iff A \upharpoonright f \neq B$. □
Theorem 15.70. Let \( \mathfrak{A}, \mathfrak{B} \) be sets of filters over boolean lattices. Then the poset \( \text{FCD}(\mathfrak{A}; \mathfrak{B}) \) is separable.

Proof. Let \( f, g \in \text{FCD}(\mathfrak{A}; \mathfrak{B}) \) and \( f \neq g \). By the theorem 15.12 \([f] \neq [g] \), That is there exist \( x, y \in \mathfrak{A} \) such that \( x[f] y \Leftrightarrow x[g] y \) that is \( f \cap (x \times \text{FCD} y) \neq \text{FCD}(\mathfrak{A}; \mathfrak{B}) \cap g \cap (x \times \text{FCD} y) \neq 0 \). Thus \( \text{FCD}(\mathfrak{A}; \mathfrak{B}) \) is separable.

Theorem 15.71. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be posets of filters over boolean lattices. If \( S \in \mathcal{P}(\mathfrak{A} \times \mathfrak{B}) \) then
\[
\bigcap \{A \times \text{FCD} B \mid (A; B) \in S\} = \bigcap \text{dom} S \times \text{FCD} \bigcap \text{im} S.
\]

Proof. If \( x \in \text{atoms}^\mathfrak{A} \) then by the theorem 15.59
\[
\langle \bigcup \{A \times \text{FCD} B \mid (A; B) \in S\} \rangle_x = \bigcup \{\langle A \times \text{FCD} B \rangle x \mid (A; B) \in S\}.
\]
If \( x \cap \cap \text{dom} S \neq 0^\mathfrak{A} \) then
\[
\forall (A; B) \in S: (x \cap A \neq 0^\mathfrak{A} \land \langle A \times \text{FCD} B \rangle x = B); \\
\{\langle A \times \text{FCD} B \rangle x \mid (A; B) \in S\} = \text{im} S;
\]
if \( x \cap \cap \text{dom} S = 0^\mathfrak{A} \) then
\[
\exists (A; B) \in S: (x \cap A = 0^\mathfrak{A} \land \langle A \times \text{FCD} B \rangle x = 0^\mathfrak{B}); \\
\{\langle A \times \text{FCD} B \rangle x \mid (A; B) \in S\} \ni 0^\mathfrak{B}.
\]

So
\[
\langle \bigcup \{A \times \text{FCD} B \mid (A; B) \in S\} \rangle_x = \begin{cases} 
\cap \text{im} S \text{ if } x \cap \cap \text{dom} S \neq 0^\mathfrak{A}; \\
0^\mathfrak{B} \text{ if } x \cap \cap \text{dom} S = 0^\mathfrak{A}.
\end{cases}
\]
From this by theorem 15.58 the statement of the theorem follows.

Corollary 15.72. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be posets of filters over boolean lattices.

For every \( \mathfrak{A}_0, \mathfrak{A}_1 \in \mathfrak{A} \) and \( \mathfrak{B}_0, \mathfrak{B}_1 \in \mathfrak{B} \)
\[
(A_0 \times \text{FCD} B_0) \cap (A_1 \times \text{FCD} B_1) = (A_0 \cap A_1) \times \text{FCD} (B_0 \cap B_1).
\]

Proof. \( (A_0 \times \text{FCD} B_0) \cap (A_1 \times \text{FCD} B_1) = \bigcap \{A_0 \times \text{FCD} B_0, A_1 \times \text{FCD} B_1\} \) what is by the last theorem equal to \( \langle A_0 \cap A_1 \rangle \times \text{FCD} (B_0 \cap B_1) \).

Theorem 15.73. Let \( (\mathfrak{A}; \mathfrak{A}_0) \) and \( (\mathfrak{B}; \mathfrak{B}_1) \) be primary filtrators over boolean lattices. If \( A \in \mathfrak{A} \) then \( A \times \text{FCD} \) is a complete homomorphism of the lattice \( \mathfrak{A} \) to a the lattice \( \text{FCD}(\mathfrak{A}; \mathfrak{B}) \), if also \( A \neq 0^\mathfrak{A} \) then it is an order embedding.

Proof. Let \( S \in \mathcal{P} \mathfrak{A}, X \in \mathfrak{A}_0, x \in \text{atoms}^\mathfrak{A} \)
\[
\langle \bigcup \langle A \times \text{FCD} \rangle S \rangle X = \bigcup \{\langle A \times \text{FCD} B \rangle X \mid B \in S\} = \bigcup \text{S if } x \cap \cap \mathfrak{A} \neq 0^\mathfrak{A} \\
0^\mathfrak{B} \text{ if } x \cap \cap \mathfrak{A} = 0^\mathfrak{A} \\
= \langle A \times \text{FCD} \cap S \rangle X.
\]
Thus \( \bigcup \langle A \times \text{FCD} \rangle S = A \times \text{FCD} \bigcup S \) by the theorem 15.25.
\[
\langle \bigcup \langle A \times \text{FCD} \rangle S \rangle x = \bigcup \{\langle A \times \text{FCD} B \rangle x \mid B \in S\} = \bigcup \text{S if } x \cap \cap \mathfrak{A} \neq 0^\mathfrak{A} \\
0^\mathfrak{B} \text{ if } x \cap \cap \mathfrak{A} = 0^\mathfrak{A} \\
= \langle A \times \text{FCD} \cap S \rangle x.
\]
Thus \( \bigcup \langle A \times \text{FCD} \rangle S = A \times \text{FCD} \bigcup S \) by the theorem 15.54.
If \( A \neq 0^\mathfrak{A} \) then obviously the function \( \mathcal{A} \times \mathcal{FCD} \) is injective.

\[\text{Proof.}\] Let \( \mathfrak{A} \) be a meet-semilatticale with least element and \( \mathfrak{B} \) be a poset with least element. If \( a \) is an atom of \( \mathfrak{A} \), \( f \in \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \) then \( f|_a = a \times \mathcal{FCD} \langle f \rangle a. \)

\[\text{Proposition 15.74.}\] Let \( \mathfrak{B} \) be a meet-semilatticale with least element and \( \mathfrak{B} \) be a poset with least element. If \( a \) is an atom of \( \mathfrak{A} \), \( f \in \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \) then \( f|_a = a \times \mathcal{FCD} \langle f \rangle a. \)

\[\text{Proof.}\] Let \( \mathcal{X} \subseteq \mathfrak{A} \).

\( \mathcal{X} \cap a \neq \emptyset \Rightarrow \langle f \rangle a \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \cap a = \emptyset \Rightarrow \langle f \rangle a \mathcal{X} = 0^\mathfrak{A}. \)

\[\text{Proposition 15.75.}\] \( f \circ (A \times \mathcal{FCD} B) = A \times \mathcal{FCD} \langle f \rangle B \) for elements \( A \in \mathfrak{A} \) and \( B \in \mathfrak{B} \) of some posets \( \mathfrak{A}, \mathfrak{B}, \mathcal{C} \) with least elements and \( f \in \mathcal{FCD}(\mathfrak{B}; \mathcal{C}). \)

\[\text{Proof.}\] Let \( \mathcal{X} \subseteq \mathfrak{A}, \mathcal{Y} \subseteq \mathfrak{B}. \)

\( (f \circ (A \times \mathcal{FCD} B)) \mathcal{X} = \left\{ \begin{array}{ll} (f) \mathcal{B} & \text{if} \ A \neq A \times A \\ 0 & \text{if} \ A = A \times A \end{array} \right\} = (A \times \mathcal{FCD} \langle f \rangle \mathcal{B}) \mathcal{X}. \)

\( ((f \circ (A \times \mathcal{FCD} B))^{-1}) \mathcal{Y} = (\mathcal{B} \times \mathcal{FCD} A) \circ f^{-1}) \mathcal{Y} = \left\{ \begin{array}{ll} A & \text{if} \ (f^{-1}) \mathcal{Y} \neq \emptyset \\ 0 & \text{if} \ (f^{-1}) \mathcal{Y} = \emptyset \end{array} \right\} = \left\{ \begin{array}{ll} A & \text{if} \ \mathcal{Y} \neq (f) \mathcal{B} \\ 0 & \text{if} \ \mathcal{Y} = (f) \mathcal{B} \end{array} \right\} = ((f) \mathcal{B} \times \mathcal{FCD} A) \mathcal{Y} = ((A \times \mathcal{FCD} \langle f \rangle \mathcal{B})^{-1}) \mathcal{Y}. \)

\[\text{15.10Atomic pointfree funcoids}\]

\[\text{Theorem 15.76.}\] Let \( \mathfrak{A}, \mathfrak{B} \) be sets of filters over boolean lattices. \( A \in \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \) is an atom of the poset \( \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \) iff there exist \( a \in \text{atoms}^\mathfrak{A} \) and \( b \in \text{atoms}^\mathfrak{B} \) such that \( f = a \times \mathcal{FCD} b. \)

\[\text{Proof.}\] \( \mathfrak{A} \) and \( \mathfrak{B} \) are atomic by the theorem 4.135.

\( \Rightarrow. \) Let \( f \) be an atom of the poset \( \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \). Let's get elements \( a \in \text{atoms} f \) and \( b \in \text{atoms} \langle f \rangle a \). Then for every \( \mathcal{X} \subseteq \mathfrak{A} \)

\( \mathcal{X} \subseteq a \Rightarrow (a \times \mathcal{FCD} b) \mathcal{X} = 0^\mathfrak{B} \subseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \supseteq a \Rightarrow (a \times \mathcal{FCD} b) \mathcal{X} = b \subseteq \langle f \rangle \mathcal{X}. \)

So \( a \times \mathcal{FCD} b \subseteq f \) because \( f \) is atomic we have \( f = a \times \mathcal{FCD} b. \)

\( \Leftarrow. \) Let \( a \in \text{atoms}^\mathfrak{A}, b \in \text{atoms}^\mathfrak{B}, f \in \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}). \) If \( b \supseteq (f) a \) then \( \neg (f) b \), \( f \cap (a \times \mathcal{FCD} b) = 0^\mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \) (because \( \mathfrak{A} \) and \( \mathfrak{B} \) are bounded meet-semilattices; if \( b \subseteq (f) a \) then \( \forall \mathcal{X} \subseteq \mathfrak{A}, (X \neq a \Rightarrow (f) \mathcal{X} \supseteq b), f \supseteq a \times \mathcal{FCD} b. \) Consequently \( f \cap (a \times \mathcal{FCD} b) = 0^\mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \cup f \supseteq a \times \mathcal{FCD} b; \)

that is \( a \times \mathcal{FCD} b \) is an atomic pointfree funcoid.

\[\text{Theorem 15.77.}\] Let \( \mathfrak{A}, \mathfrak{B} \) be sets of filters over boolean lattices. Then \( \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \) is atomic.

\[\text{Proof.}\] Let \( f \in \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \) and \( f \neq 0^\mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \). Then \( \text{dom} f \neq 0^\mathfrak{A} \), thus exists \( a \in \text{atoms} f \). So \( \langle f \rangle a \neq 0^\mathfrak{B} \) thus exists \( b \in \text{atoms} \langle f \rangle a \). Finally the atomic pointfree funcoid \( a \times \mathcal{FCD} b \subseteq f. \)

\[\text{Theorem 15.78.}\] Let \( \mathfrak{A}, \mathfrak{B} \) be sets of filters over boolean lattices. Then the poset \( \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \) is separable.

\[\text{Proof.}\] Let \( f, g \in \mathcal{FCD}(\mathfrak{A}; \mathfrak{B}), f \subseteq g. \) Then taking into account the theorem 6.62 exists \( a \in \text{atoms}^\mathfrak{A} \) such that \( \langle f \rangle a \subseteq \langle g \rangle a \). By corollary 4.138 \( \mathfrak{B} \) is atomically separable. So exists \( b \in \text{atoms}^\mathfrak{B} \) such that \( \langle f \rangle a \cap b = 0^\mathfrak{B} \) and \( b \subseteq \langle g \rangle a \). For every \( x \in \text{atoms}^\mathfrak{A} \)

\[\langle f \rangle a \cap (a \times \mathcal{FCD} b) a = \langle f \rangle a \cap b = 0^\mathfrak{B}, \quad x \neq a \Rightarrow \langle f \rangle x \cap (a \times \mathcal{FCD} b) x = \langle f \rangle x \cap 0^\mathfrak{B} = 0^\mathfrak{B}. \]

Thus \( \langle f \rangle x \cap (a \times b) x = 0^\mathfrak{B} \) and consequently \( f \cap (a \times \mathcal{FCD} b) = 0^\mathcal{FCD}(\mathfrak{A}; \mathfrak{B}) \).

\[\langle a \times \mathcal{FCD} b \rangle a = b \subseteq \langle g \rangle a, \quad x \neq a \Rightarrow \langle a \times \mathcal{FCD} b \rangle x = 0^\mathfrak{B} \subseteq \langle g \rangle x. \]

Thus \( \langle a \times \mathcal{FCD} b \rangle x \subseteq \langle g \rangle x \) and consequently \( a \times \mathcal{FCD} b \subseteq g. \)

So the lattice of pointfree funcoids is separable by the theorem 3.14.
Corollary 15.79. Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices. The poset $\text{FCD}(\mathfrak{A}; \mathfrak{B})$ is:
1. separable;
2. atomically separable;
3. conforming to Wallman’s disjunction property.

Proof. By the theorem 3.21.

Remark 15.80. For more ways to characterize (atomic) separability of the lattice of pointfree funoids see subsections “Separation subsets and full stars” and “Atomically separable lattices”.

Corollary 15.81. Let $(\mathfrak{A}; \mathfrak{B})$ and $(\mathfrak{B}; \mathfrak{A})$ be primary filtrators over boolean lattices. The poset $\text{FCD}(\mathfrak{A}; \mathfrak{B})$ is an atomistic lattice.

Proof. By the corollary 15.34 $\text{FCD}(\mathfrak{A}; \mathfrak{B})$ is a complete lattice. Let $f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$. Suppose contrary to the statement to be proved that $\bigcup a$ atoms $f \cap f$. Then there exists $a \in$ atoms $f$ such that $a \cap \bigcup a$ atoms $f = \emptyset$ what is impossible.

Proposition 15.82. Let $\mathfrak{A}$, $\mathfrak{B}$ be sets of filters over boolean lattices.
\[ \text{atoms}(f \cup g) = \text{atoms} f \cup \text{atoms} g \text{ for every } f, g \in \text{FCD}(\mathfrak{A}; \mathfrak{B}). \]

Proof. (a $\times \text{FCD} b) \cap (f \cup g) \neq \emptyset \Leftrightarrow a \{f \cup g\} b \Leftrightarrow a [f \cup g] b \Leftrightarrow (a \times \text{FCD} b) \cap f \neq \emptyset \text{FCD}(\mathfrak{A}; \mathfrak{B}) \cup (a \times \text{FCD} b) \cap g \neq \emptyset \text{FCD}(\mathfrak{A}; \mathfrak{B})$ for every $a \in$ atoms$^\mathfrak{A}$ and $b \in$ atoms$^\mathfrak{B}$ (used the corollary 15.69 and theorem 15.35).

Theorem 15.83. Let $(\mathfrak{A}; \mathfrak{B})$ and $(\mathfrak{B}; \mathfrak{A})$ be primary filtrators over boolean lattices. For every $f, g, h \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$, $R \in \mathcal{P}\text{FCD}(\mathfrak{A}; \mathfrak{B})$:
1. $f \cap (g \cup h) = (f \cap g) \cup (f \cap h)$;
2. $f \cap \bigcap R = \bigcap (f \cap R)$.

Proof. We will take into account that the lattice of funoids is an atomistic lattice (corollary 15.81).

1. $\text{atoms}(f \cap (g \cup h)) = \text{atoms} f \cap \text{atoms}(g \cup h) = \text{atoms} f \cap (\text{atoms} g \cup \text{atoms} h) = (\text{atoms} f \cap \text{atoms} g) \cup (\text{atoms} f \cap \text{atoms} h) = \text{atoms}(f \cap g) \cup \text{atoms}(f \cap h) = \text{atoms}(f \cap g) \cup (f \cap h)$.

2. $\text{atoms}(f \cup \bigcap R) = \text{atoms} f \cup \text{atoms} \bigcap R = \text{atoms} f \cup \bigcap \text{atoms} R = \bigcap (\text{atoms} f \cup \bigcap \text{atoms} R) = \bigcap (\text{atoms} f \cup \bigcap R)$. (Used the following equality.)

\[ \{(\text{atoms} f) \cup \bigcap \text{atoms} R \} = \{\text{atoms} f \} \cup \{A \in \text{atoms} R \} = \{\text{atoms} f \} \cup A \mid \exists C \in R: A = \text{atoms} C \} = \{\text{atoms} f \} \cup \{A \mid \exists C \in R: A = \text{atoms} C \} = \{\text{atoms} f \} \cup \{C \in R \} = \{\text{atoms} f \} \cup C \} = \{\text{atoms} B \mid \exists C \in R: B = f \cup C \} = \{\text{atoms} B \mid B \in (f \cup R) \} = \text{atoms}(f \cup R) \).

Corollary 15.84. Let $(\mathfrak{A}; \mathfrak{B})$ and $(\mathfrak{B}; \mathfrak{A})$ be primary filtrators over boolean lattices. Then $\text{FCD}(\mathfrak{A}; \mathfrak{B})$ is a co-brouwerian lattice.

Proposition 15.85. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be sets of filters over some boolean lattices and $f \in \text{FCD}(\mathfrak{A}; \mathfrak{B})$, $g \in \text{FCD}(\mathfrak{B}; \mathfrak{C})$. Let $\mathfrak{B}$ be an atomic poset. Then
\[ \text{atoms}(g \circ f) = \{x \times \text{FCD} z \mid x \in \text{atoms}^\mathfrak{A}, z \in \text{atoms}^\mathfrak{C}, \exists y \in \text{atoms}^\mathfrak{B}: (x \times \text{FCD} y \in \text{atoms} f \land y \times \text{FCD} z \in \text{atoms} g) \}. \]
Proof. \((x \times FCD z) \cap (g \circ f) \neq 0^{FCD(\mathfrak{A}; \mathfrak{B})} \Leftrightarrow x [g \circ f] z \Leftrightarrow \exists y \in atoms^\mathfrak{B} : (x [f] y \wedge y [g] z) \Leftrightarrow \exists y \in atoms^\mathfrak{B} : ((x \times FCD y) \cap f \neq 0^{FCD(\mathfrak{A}; \mathfrak{B})}) \wedge (y \times FCD z) \cap g \neq 0^{FCD(\mathfrak{B}; \mathfrak{C})})\) (were used the corollary 15.69 and theorem 15.61).

\[ \square \]

**Theorem 15.86.** Let \( f \) be a pointfree funcoid between sets of filters on boolean lattices.

1. \( X [f] Y \Leftrightarrow \exists F \in \text{atoms } f : X [F] Y \); 
2. \( \langle f \rangle X = \bigsqcup_{F \in \text{atoms } f} \langle F \rangle X \) for every \( X \in \mathfrak{F}(\text{Src } f) \).

Proof. 1. \( \exists F \in \text{atoms } f : X [F] Y \Leftrightarrow \exists a \in \mathfrak{F}(\text{Src } f), b \in \mathfrak{F}(\text{Dst } f) : (a \times FCD b \neq f \wedge X [a \times FCD b] Y) \Leftrightarrow \exists a \in \mathfrak{F}(\text{Src } f), b \in \mathfrak{F}(\text{Dst } f) : (a \times FCD b \neq f \wedge a \times FCD b \neq X \times FCD Y) \Leftrightarrow \exists F \in \text{atoms } f : (F \neq f \wedge F \neq X \times FCD Y) \Leftrightarrow f \neq X \times FCD Y \Leftrightarrow X [f] Y \).

2. Let \( Y \in \mathfrak{F}(\text{Dst } f) \). Suppose \( Y \neq \langle f \rangle X \). Then \( X [f] Y \); \( \exists F \in \text{atoms } f : X [F] Y \); \( \exists F \in \text{atoms } f : Y \neq \bigsqcup_{F \in \text{atoms } f} \langle F \rangle X \). So \( \langle f \rangle X \cong \bigsqcup_{F \in \text{atoms } f} \langle F \rangle X \). The contrary \( \langle f \rangle X \cong \bigsqcup_{F \in \text{atoms } f} \langle F \rangle X \) is obvious.

\[ \square \]

15.11 Complete pointfree funcoids

**Definition 15.87.** Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be posets. A pointfree funcoid \( f \in FCD(\mathfrak{A}; \mathfrak{B}) \) is complete, when for every \( S \in \mathfrak{P} \mathfrak{A} \) whenever both \( \bigsqcup S \) and \( \bigsqcup \langle f \rangle S \) are defined we have

\[ \langle f \rangle \bigsqcup S = \bigsqcup \langle f \rangle S. \]

**Proposition 15.88.** Let \( \mathfrak{A}, \mathfrak{B} \) be sets of filters over boolean lattices. A pointfree funcoid \( f \in FCD(\mathfrak{A}; \mathfrak{B}) \) is complete iff \( \langle f \rangle a = \bigsqcup \langle f \rangle \text{atoms } a \) for every \( a \in \mathfrak{A} \).

Proof. Direct implication is obvious. The reverse implication:

Let \( S \) be a set of filters.

\[ \langle f \rangle \bigsqcup S = \bigsqcup \langle f \rangle \text{atoms } S = \bigsqcup \langle f \rangle \text{atoms } S = \bigsqcup \bigsqcup \langle f \rangle \text{atomic } S = \bigsqcup \bigsqcup \langle f \rangle \text{atomic } a | a \in S = \bigsqcup \langle f \rangle a | a \in S = \bigsqcup \langle f \rangle S. \]

\[ \square \]

**Remark 15.89.** Let \( \mathfrak{A}_0 \) and \( \mathfrak{A}_1 \) be join-semilattices with least elements. I will call pointfree generalized closure such a function \( \alpha \in (\mathfrak{A}_1)^{\mathfrak{A}_0} \) that

1. \( \alpha 0^{\mathfrak{A}_0} = 0^{\mathfrak{A}_1} \); 
2. \( \forall I, J \in \mathfrak{A}_0 : \alpha (I \cup J) = \alpha I \cup \alpha J \).

**Definition 15.90.** Let \( (\mathfrak{A}; \mathfrak{A}_0) \) and \( (\mathfrak{B}; \mathfrak{A}_1) \) be primary filtrators over boolean lattices. I will call a co-complete pointfree funcoid a pointfree funcoid \( f \in FCD(\mathfrak{A}; \mathfrak{B}) \) such that \( \langle f \rangle |_{\mathfrak{A}_0} \) is a pointfree generalized closure.

**Proposition 15.91.** Let \( (\mathfrak{A}; \mathfrak{A}_0) \) and \( (\mathfrak{B}; \mathfrak{A}_1) \) be primary filtrators over boolean lattices. Co-complete pointfree funcoids \( FCD(\mathfrak{A}; \mathfrak{B}) \) bijectively correspond to pointfree generalized closures \( \mathfrak{A}_1^{\mathfrak{A}_0} \), where the bijection is \( f \mapsto \langle f \rangle |_{\mathfrak{A}_0} \).

Proof. It follows from the theorem 15.26.

**Theorem 15.92.** Let \( (\mathfrak{A}; \mathfrak{A}_0) \) be semifiltered, star-separable, down-aligned filtrator with finitely meet closed, join-closed, and separable core, where \( \mathfrak{A}_0 \) is a complete boolean lattice and both \( \mathfrak{A}_0 \) and \( \mathfrak{A} \) are atomistic lattices.

Let \( (\mathfrak{B}; \mathfrak{A}_1) \) be a star-separable filtrator.

The following conditions are equivalent for every pointfree funcoid \( f \in FCD(\mathfrak{A}; \mathfrak{B}) \):

1. \( f^{-1} \) is co-complete;
2. \( \forall S \in \mathfrak{P} \mathfrak{A}, J \in \mathfrak{A}_1 : (\bigcup^\mathfrak{A} S [f] J) \Rightarrow \exists I \in S: I [f] J \).

\[ \square \]
3. \( \forall S \in \mathcal{P}_3, J \in \mathcal{J}_1 : (\bigsqcup^3 S) [f] J \Rightarrow \exists I \in S : I [f] J \); 
4. \( f \) is complete; 
5. \( \forall S \in \mathcal{P}_3 : (f)\bigsqcup^3 S = \bigsqcup^3 (f)S \).

**Proof.** First note that the theorem 4.53 applies to the filtrator \((\mathfrak{A}; \mathfrak{J})\).

\((3) \Rightarrow (1)\). For every \( S \in \mathcal{P}_3, J \in \mathcal{J}_1 \)

\[
\bigsqcup^3 S \cap^{\mathfrak{A}} \langle f^{-1} \rangle J \neq 0^\mathfrak{A} \Rightarrow \exists I \in S : I \cap^{\mathfrak{A}} \langle f^{-1} \rangle J \neq 0^\mathfrak{A},
\]

consequently by the theorem 4.53 we have \( \langle f^{-1} \rangle J \in \mathfrak{J}_0 \).

\((1) \Rightarrow (2)\). For every \( S \in \mathcal{P}_\mathfrak{A}, J \in \mathcal{J}_1 \) we have \( \langle f^{-1} \rangle J \in \mathfrak{J}_0 \), consequently

\[
\forall S \in \mathcal{P}_\mathfrak{A}, J \in \mathcal{J}_1 : \left( \bigsqcup^3 S \neq \langle f^{-1} \rangle J \Rightarrow \exists I \in S : I \neq \langle f^{-1} \rangle J \right).
\]

From this follows (2).

\((2) \Rightarrow (4)\). Let \( \langle f \rangle \bigsqcup^3 S \) and \( \bigsqcup^3 (\langle f \rangle)S \) are defined. We have \( \langle f \rangle \bigsqcup^3 S = \langle (f) \rangle \bigsqcup^3 S \).

\[
J \cap^{\mathfrak{A}} \langle f \rangle \bigsqcup^3 S \neq 0^\mathfrak{A} \Leftrightarrow \exists S \in \mathcal{I} : J \cap^{\mathfrak{A}} S \neq 0^\mathfrak{A} \Rightarrow \exists I \in S : I \neq \langle f \rangle J.
\]

Thus \( \langle f \rangle \bigsqcup^3 S = \bigsqcup^3 (\langle f \rangle)S \) by star-separability of \((\mathfrak{A}; \mathfrak{J})\).

\((5) \Rightarrow (3)\). Let \( \langle f \rangle \bigsqcup^3 S \) is defined. Then \( \bigsqcup^3 (\langle f \rangle)S \) is also defined because \( \langle f \rangle \bigsqcup^3 S = \bigsqcup^3 (\langle f \rangle)S \). Then \( \bigsqcup^3 S \cap^{\mathfrak{A}} \langle f \rangle J \Rightarrow J \cap^{\mathfrak{A}} \langle f \rangle \bigsqcup^3 S \neq 0^\mathfrak{A} \Leftrightarrow J \cap^{\mathfrak{A}} \langle f \rangle S \neq 0^\mathfrak{A} \) what by the theorem 4.53 equivalent to \( \exists I \in S : J \cap^{\mathfrak{A}} \langle f \rangle I \neq 0^\mathfrak{A} \) that is \( \exists I \in S : I \neq \langle f \rangle J \).

\((2) \Rightarrow (3)\), \((4) \Rightarrow (5)\). By join-closedness of the core of \((\mathfrak{A}; \mathfrak{J})\).

\[\square\]

**Theorem 15.93.** Let \((\mathfrak{A}; \mathfrak{J}_0)\) and \((\mathfrak{B}; \mathfrak{J}_1)\) be primary filtrators over boolean lattices. If \( R \) is a set of co-complete pointfree funcoids in \( \text{FCD}(\mathfrak{A}; \mathfrak{B}) \) then \( \bigsqcup R \) is a co-complete pointfree funcoid.

**Proof.** Let \( R \) be a set of co-complete pointfree funcoids. Then for every \( X \in \mathfrak{J}_0 \)

\[
\langle \bigsqcup R \rangle X = \bigsqcup \langle (f)X \mid f \in R \rangle = \bigsqcup \langle (f)X \mid f \in R \rangle \in \mathfrak{J}_1
\]

(used the theorem 15.33 and corollary 4.96).

\[\square\]

Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be posets with least elements. I will denote \( \text{ComplFCD}(\mathfrak{A}; \mathfrak{B}) \) and \( \text{CoComplFCD}(\mathfrak{A}; \mathfrak{B}) \) the sets of complete and co-complete funcoids correspondingly from a poset \( \mathfrak{A} \) to a poset \( \mathfrak{B} \).

**Proposition 15.94.**

1. Let \( f \in \text{ComplFCD}(\mathfrak{A}; \mathfrak{B}) \) and \( g \in \text{ComplFCD}(\mathfrak{B}; \mathfrak{C}) \) where \( \mathfrak{A} \) and \( \mathfrak{C} \) are posets with least elements and \( \mathfrak{B} \) is a complete lattice. Then \( g \circ f \in \text{ComplFCD}(\mathfrak{A}; \mathfrak{C}) \).

2. Let \( f \in \text{CoComplFCD}(\mathfrak{A}; \mathfrak{B}) \) and \( g \in \text{CoComplFCD}(\mathfrak{B}; \mathfrak{C}) \) where \( \mathfrak{A} \), \( \mathfrak{B} \) and \( \mathfrak{C} \) are posets with least elements and \( (\mathfrak{A}; \mathfrak{J}_0) \), \( (\mathfrak{B}; \mathfrak{J}_1) \), \( (\mathfrak{C}; \mathfrak{J}_2) \) are filtrators. Then \( g \circ f \in \text{CoComplFCD}(\mathfrak{A}; \mathfrak{C}) \).

**Proof.**

1. Let \( \bigsqcup S \) and \( \bigsqcup (\langle g \circ f \rangle)S \) are defined. Then

\[
\langle g \circ f \rangle \bigsqcup S = \langle g \rangle \langle f \rangle \bigsqcup S = \langle g \rangle \bigsqcup \langle (f) \rangle S = \bigsqcup \langle (g) \rangle \langle (f) \rangle S = \bigsqcup \langle (g \circ f) \rangle S.
\]

2. \( \langle g \circ f \rangle \mathfrak{J}_0 = \langle g \rangle \langle f \rangle \mathfrak{J}_0 \in \mathfrak{J}_2 \) because \( \langle f \rangle \mathfrak{J}_0 \in \mathfrak{J}_1 \).

\[\square\]

**Proposition 15.95.** Let \((\mathfrak{A}; \mathfrak{J}_0)\) and \((\mathfrak{B}; \mathfrak{J}_1)\) be primary filtrators over boolean lattices. Then \( \text{CoComplFCD}(\mathfrak{A}; \mathfrak{B}) \) (with induced order) is a complete lattice.
Proposition 15.103. \[ \text{Denition 15.102.} \]

15.13 Monovalued and injective pointfree funcoids

Proof. Follows from the theorem 15.93. \[ \square \]

Theorem 15.96. Let (\( \mathfrak{A} ; \mathfrak{J}_0 \)) and (\( \mathfrak{B} ; \mathfrak{J}_1 \)) be primary filtrators where \( \mathfrak{J}_0 \) and \( \mathfrak{J}_1 \) are boolean lattices. Let \( R \) be a set of pointfree funcoids from \( \mathfrak{A} \) to \( \mathfrak{B} \).

\( g \circ (\bigsqcup R) = \bigsqcup \{ g \circ f \mid g \in R \} = \bigsqcup \{ g \circ f \mid f \in R \} \) if \( g \) is a complete pointfree funcoid from \( \mathfrak{B} \).

Proof. \( \langle g \circ (\bigsqcup R) \rangle X = \langle g \rangle (\bigsqcup R) X = \langle g \rangle \bigsqcup \{ (f) X \mid f \in R \} = \bigsqcup \{ (g \circ f) X \mid f \in R \} = \bigsqcup \{ (g \circ f) R \} X \) for every \( X \in \mathfrak{A} \). So \( g \circ (\bigsqcup R) = \bigsqcup \{ g \circ f \} \). \[ \square \]

15.12 Completion and co-completion

Definition 15.97. Let (\( \mathfrak{A} ; \mathfrak{J}_0 \)) and (\( \mathfrak{B} ; \mathfrak{J}_1 \)) be primary filtrators over boolean lattices and \( \mathfrak{J}_1 \) is a complete atomistic lattice.

Co-completion of a pointfree funcoid \( f \in \text{FCD} (\mathfrak{A} ; \mathfrak{B}) \) is pointfree funcoid \( \text{CoCompl} f \) defined by the formula (for every \( X \in \mathfrak{J}_0 \))

\[ (\text{CoCompl} f) X = \text{Cor} (f) X. \]

Proposition 15.98. Above defined co-completion always exists.

Proof. Existence of \( \text{Cor} (f) X \) follows from completeness of \( \mathfrak{J}_1 \).

We may apply the theorem 15.26 because

\[ \text{Cor} (f) (X \sqcup \mathfrak{J}_0 Y) = \text{Cor}(f) X \sqcup \mathfrak{J}_B (f) Y = \text{Cor} (f) X \sqcup \mathfrak{J}_1 \text{Cor} (f) Y \]

by proposition 4.156. \[ \square \]

Obvious 15.99. Co-completion is always co-complete.

Proposition 15.100. For above defined always \( \text{CoCompl} f \subseteq f \).

Proof. By proposition 4.101. \[ \square \]

Proposition 15.101. Monovalued pointfree funcoids between sets of filters on boolean lattices are metamonovalued.

Proof. \( \langle \prod G \rangle \circ f = \prod G \langle f \rangle x = \prod_{g \in G} \langle g \rangle (f) x = \prod_{g \in G} (g \circ f) x = \langle \prod_{g \in G} (g \circ f) \rangle x \) for every ultrafilter \( x \in \text{atoms}^{1 \text{Src} f} \). Thus \( \langle \prod G \rangle \circ f = \prod_{g \in G} (g \circ f) \). \[ \square \]

15.13 Monovalued and injective pointfree funcoids

Definition 15.102. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be posets. Let \( f \in \text{FCD}(\mathfrak{A}; \mathfrak{B}) \).

The pointfree funcoid \( f \) is:

- \textit{monovalued} when \( f \circ f^{-1} \sqsubseteq \text{id}_{\text{FCD}(\mathfrak{B})} \).
- \textit{injective} when \( f^{-1} \circ f \sqsubseteq \text{id}_{\text{FCD}(\mathfrak{A})} \).

Monovaluedness is dual of injectivity.

Proposition 15.103. Let \( \mathfrak{A} \) and \( \mathfrak{B} \) be posets. Let \( f \in \text{FCD}(\mathfrak{A}; \mathfrak{B}) \).

The pointfree funcoid \( f \) is:

- \textit{monovalued} iff \( f \circ f^{-1} \sqsubseteq \text{id}_{\text{im} f} \), if \( \text{im} f \) is defined and \( \mathfrak{B} \) is a meet-semilattice;
- \textit{injective} iff \( f^{-1} \circ f \sqsubseteq \text{id}_{\text{dom} f} \), if \( \text{dom} f \) is defined and \( \mathfrak{A} \) is a meet-semilattice.
15.13 Monovalued and injective pointfree funcoids

Proof. It’s enough to prove \( f \circ f^{-1} \subseteq \text{id}_{\text{FCD}^{(3)}} \Leftrightarrow f \circ f^{-1} \subseteq \text{id}_{\text{im}f}^{\text{FCD}^{(3)}} \).

\( \Leftarrow \). Obvious.

\( \Rightarrow \). Let \( f \circ f^{-1} \subseteq \text{id}_{\text{FCD}^{(3)}} \). Then \( (f \circ f^{-1})x \subseteq x \) and \( (f \circ f^{-1})x \subseteq \text{im} f \). Thus \( (f \circ f^{-1})x \subseteq x \cap f = \langle \text{id}_{\text{im} f}^{\text{FCD}^{(3)}} \rangle x \).

\[ \langle (f \circ f^{-1})x \subseteq x \text{ and } ((f \circ f^{-1})x = (f \circ f^{-1})x \subseteq \text{im} f \text{. Thus } ((f \circ f^{-1})x \subseteq x \cap f = \langle \text{id}_{\text{im} f}^{\text{FCD}^{(3)}} \rangle x \text{.} \]

Thus \( f \circ f^{-1} \subseteq \text{id}_{\text{im} f}^{\text{FCD}^{(3)}} \). \( \square \)

Theorem 15.104. Let \( A \) be an atomistic meet-semilattice with least element, \( B \) is an atomistic bounded meet-semilattice. The following statements are equivalent for every \( f \in \text{FCD}(A; B) \):

1. \( f \) is monovalued.
2. \( \forall a \in \text{atoms}^A; (f)a \in \text{atoms}^B \cup \{0^B\} \).
3. \( \forall i, j \in A; (f^{-1})(i \cap j) = (f^{-1}i) \cap (f^{-1}j) \).

Proof.

\( (2) \Rightarrow (3) \). Let \( a \in \text{atoms}^A, (f)a = b \). Then because \( b \in \text{atoms}^B \cup \{0^B\} \)

\[ (i \cap j) \cap b \neq 0^B \Rightarrow i \cap b \neq 0^B \land j \cap b \neq 0^B; \]

\[ a[f]i \cap j \Rightarrow a[f]i \land a[f]j; \]

\[ i \cap j[f^{-1}]a \Rightarrow i[f^{-1}]a \land j[f^{-1}]a; \]

\[ a \cap B^{-1}(i \cap j) \neq 0^B \Rightarrow a \cap (f^{-1}i) \neq 0^B \land a \cap (f^{-1}j) \neq 0^B; \]

\[ a \cap (f^{-1}i) \cap (f^{-1}j) \neq 0^B \Rightarrow a \cap (f^{-1}i) \cap (f^{-1}j) \neq 0^B; \]

\[ (f^{-1})(i \cap j) = (f^{-1}i) \cap (f^{-1}j). \]

\( (3) \Rightarrow (1) \). \( (f^{-1}a) \cap (f^{-1}b) = (f^{-1}(a \cap b)) = (f^{-1}0^B) = 0^A \) (by proposition 15.13 because \( A \) is separable by proposition 3.22) for every two distinct \( a, b \in \text{atoms}^A \). This is equivalent to \( (f^{-1})a \neq [f]b; b \cap (f^{-1})a = 0^B; b \cap (f \circ f^{-1})a = 0^B; (f^{-1})(f \circ f^{-1}) \). So \( a[f \circ f^{-1}]b \Rightarrow a = b \) for every \( a, b \in \text{atoms}^A \). This is possible only (corollary 15.56 and the fact that \( B \) is atomic) when \( f \circ f^{-1} \subseteq \text{id}_{\text{FCD}^{(3)}} \).

\( (2) \Rightarrow (1) \). Suppose \( (f)a \notin \text{atoms}^B \cup \{0^B\} \) for some \( a \in \text{atoms}^A \). Then there exist two atoms \( p \neq q \) such that \( (f)a \sqsubseteq p \land (f)a \sqsubseteq q \). Consequently \( p \cap (f)a \neq 0^B; a \cap (f^{-1}p) \neq 0^B; a \sqsubseteq (f^{-1}p); (f \circ f^{-1})p = (f^{-1})(f \circ f^{-1})p \sqsubseteq (f)a \sqsubseteq q \) (by proposition 3.22); \( (f \circ f^{-1})p \neq 0^B \) and \( (f \circ f^{-1})p \neq 0^B \). So it cannot be \( f \circ f^{-1} \subseteq \text{id}_{\text{FCD}^{(3)}} \). \( \square \)

Theorem 15.105. Let \( (A; 3_0) \) and \( (B; 3_1) \) be primary filtrators over a boolean lattice. A pointfree funcoid \( f \in \text{FCD}(A; B) \) is monovalued iff

\[ \forall I, J \in 3_1; (f^{-1})(I \cap 3_1, J) = (f^{-1}I) \cap (f^{-1}J). \]

Proof. \( A \) and \( B \) are complete lattices (corollary 4.107).

\( (B; 3_1) \) is a filtrator with separable core by the theorem 4.112.

\( (B; 3_1) \) is finitely meet-closed by the theorem 4.97.

\( A \) and \( B \) are starrish by corollary 4.114.

\( A \) is separable by obvious 4.136.

We are under conditions of the theorem 15.25.

\( \Rightarrow \). Obvious (taking into account that \( (B; 3_1) \) is finitely meet-closed).

\( \Leftarrow \). \( (f^{-1})(I \cap J) = \bigcap (f^{-1})(I \cap J) \sqsubseteq B; 3_1 \) \( I \cap J = \bigcap (f^{-1})(I \cap J) \sqsubseteq 3_1, J \mid I \in \text{up} I, J \in \text{up} J = \bigcap (f^{-1})(I \cap J) \sqsubseteq 3_1, J \mid I \in \text{up} I, J \in \text{up} J = \bigcap (f^{-1})(I \cap J) \sqsubseteq 3_1, J \mid J \in \text{up} J \sqsubseteq (f^{-1})(I \cap J) \) (used theorem 15.25, theorem 4.110, theorem 15.15). \( \square \)
15.14 Elements closed regarding a pointfree funcoid

Let $\mathfrak{A}$ be a poset. Let $f \in \text{FCD}(\mathfrak{A}; \mathfrak{A})$.

**Definition 15.106.** Let’s call closed regarding a pointfree funcoid $f$ such element $a \in \mathfrak{A}$ that $\langle f \rangle a \subseteq a$.

**Proposition 15.107.** If $i$ and $j$ are closed (regarding a pointfree funcoid $f \in \text{FCD}(\mathfrak{A}; \mathfrak{A})$), $S$ is a set of closed elements (regarding $f$), then

1. $i \sqcup j$ is a closed element, if $\mathfrak{A}$ is a separable starish join-semilattice;
2. $\bigsqcup S$ is a closed element if $\mathfrak{A}$ is a separable complete lattice.

**Proof.** $\langle f \rangle (i \sqcup j) = \langle f \rangle i \sqcup \langle f \rangle j \subseteq i \sqcup j$ (theorem 15.15), $\langle f \rangle \bigsqcup S \subseteq \bigsqcup \langle f \rangle S \subseteq \bigsqcup S$ (used separability of $\mathfrak{A}$ twice). Consequently the elements $i \sqcup j$ and $\bigsqcup S$ are closed.

**Proposition 15.108.** If $S$ is a set of elements closed regarding a complete pointfree funcoid $f$ with separable destination which is a complete lattice, then the element $\bigsqcup S$ is also closed regarding our funcoid.

**Proof.** $\langle f \rangle \bigsqcup S = \bigsqcup \langle f \rangle S \subseteq \bigsqcup S$.

15.15 Connectedness regarding a pointfree funcoid

Let $\mathfrak{A}$ be a poset with least element. Let $\mu \in \text{FCD}(\mathfrak{A}; \mathfrak{A})$.

**Definition 15.109.** An element $a \in \mathfrak{A}$ is called connected regarding a pointfree funcoid $\mu$ over $\mathfrak{A}$ when

$$\forall x, y \in \mathfrak{A} \setminus \{0\}; (x \sqcup y = a \Rightarrow x \upharpoonright y).$$

**Proposition 15.110.** Let $(\mathfrak{A}; 3)$ be a co-separable filtrator with join-closed core. An $A \in 3$ is connected regarding a funcoid $\mu$ iff

$$\forall X, Y \in 3 \setminus \{0\}; (X \sqcup Y = A \Rightarrow X \upharpoonright Y).$$

**Proof.**

$\Rightarrow$. Obvious.

$\Leftarrow$. Follows from co-separability.

**Obvious 15.111.** For $\mathfrak{A}$ being a set of filters over a boolean lattice, an element $a \in \mathfrak{A}$ is connected regarding a pointfree funcoid $\mu$ if it is connected regarding the funcoid $\mu \cap (a \times \text{FCD} a)$.
Chapter 16
Convergence of funcoids

16.1 Convergence

The following generalizes the well-known notion of a filter convergent to a point or to a set:

**Definition 16.1.** A filter $\mathcal{F} \in \mathfrak{F}(\text{Dst } \mu)$ converges to a filter $\mathcal{A} \in \mathfrak{F}(\text{Src } \mu)$ regarding a funcoid $\mu$ $(\mathcal{F} \xrightarrow{\mu} \mathcal{A})$ iff $\mathcal{F} \subseteq \langle \mu \rangle \mathcal{A}$.

**Definition 16.2.** A funcoid $f$ converges to a filter $\mathcal{A} \in \mathfrak{F}(\text{Src } \mu)$ regarding a funcoid $\mu$ where $\text{Dst } f = \text{Dst } \mu$ (denoted $f \xrightarrow{\mu} \mathcal{A}$) iff $\text{im } f \subseteq \langle \mu \rangle \mathcal{A}$ that is iff $\text{im } f \subseteq \langle \mu \rangle \mathcal{A}$.

**Definition 16.3.** A funcoid $f$ converges to a filter $\mathcal{A} \in \mathfrak{F}(\text{Src } \mu)$ on a filter $\mathcal{B} \in \mathfrak{F}(\text{Src } f)$ regarding a funcoid $\mu$ where $\text{Dst } f = \text{Dst } \mu$ iff $f \xrightarrow{\mu} \mathcal{A}$.

**Remark 16.4.** We can define also convergence for a reloid $f$: $f \xrightarrow{\mu} \mathcal{A} \iff \text{im } f \subseteq \langle \mu \rangle \mathcal{A}$ or what is the same $f \xrightarrow{\mu} \mathcal{A} \leftrightarrow (\text{FCD}) f \xrightarrow{\mu} \mathcal{A}$.

**Theorem 16.5.** Let $f$, $g$ be funcoids, $\mu$, $\nu$ be endofuncoids, $\text{Dst } f = \text{Src } g = \text{Ob } \mu$, $\text{Dst } g = \text{Ob } \nu$, $\mathcal{A} \in \mathfrak{F}(\text{Ob } \mu)$. If $f \xrightarrow{\mu} \mathcal{A}$,

$$g \circ f \subseteq \langle \langle \mu \rangle \mathcal{A} \times \text{FCD} \langle \mu \rangle \mathcal{A} \rangle \nu,$$

and $\langle \mu \rangle \mathcal{A} \supseteq \mathcal{A}$, then $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$.

**Proof.** $\text{im } f \subseteq \langle \mu \rangle \mathcal{A}$; $\langle g \rangle \text{im } f \subseteq \langle g \rangle \langle \mu \rangle \mathcal{A}$; $\text{im } (g \circ f) \subseteq \langle g \circ f \rangle \langle \mu \rangle \mathcal{A}$; $\text{im } (g \circ f) \subseteq \langle g \rangle \langle \mu \rangle \mathcal{A}$; $\text{im } (g \circ f) \subseteq \langle g \rangle \langle \mu \rangle \mathcal{A}$; $\text{im } (g \circ f) \subseteq \langle g \rangle \langle \mu \rangle \mathcal{A}$; $\text{im } (g \circ f) \subseteq \langle g \rangle \langle \mu \rangle \mathcal{A}$. □

**Corollary 16.6.** Let $f$, $g$ be funcoids, $\mu$, $\nu$ be endofuncoids, $\text{Dst } f = \text{Src } g = \text{Ob } \mu$, $\text{Dst } g = \text{Ob } \nu$, $\mathcal{A} \in \mathfrak{F}(\text{Ob } \mu)$. If $f \xrightarrow{\mu} \mathcal{A}$, $g \in C(\mu; \nu)$, and $\langle \mu \rangle \mathcal{A} \supseteq \mathcal{A}$ then $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$.

**Proof.** From the last theorem and theorem 10.7. □

16.2 Relationships between convergence and continuity

**Lemma 16.7.** Let $\mu$, $\nu$ be endofuncoids, $f \in \text{FCD}(\text{Ob } \mu; \text{Ob } \nu)$, $\mathcal{A} \in \mathfrak{F}(\text{Ob } \mu)$, $\text{Src } f = \text{Ob } \mu$, $\text{Dst } f = \text{Ob } \nu$. If $f \in C(\langle \mu \rangle \mathcal{A} \nu)$ then

$$f \big|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A} \subseteq \langle f \circ \mu \rangle \mathcal{A} \subseteq \langle \nu \circ f \rangle \mathcal{A}.$$

**Proof.** $f \big|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A} \iff \text{im } f \big|_{\langle \mu \rangle \mathcal{A}} \subseteq \langle \nu \rangle \langle f \rangle \mathcal{A} \subseteq \langle f \rangle \mathcal{A} \subseteq \langle \nu \circ f \rangle \mathcal{A} \iff \langle f \circ \mu \rangle \mathcal{A} \subseteq \langle \nu \circ f \rangle \mathcal{A}$. □

**Theorem 16.8.** Let $\mu$, $\nu$ be endofuncoids, $f \in \text{FCD}(\text{Ob } \mu; \text{Ob } \nu)$, $\mathcal{A} \in \mathfrak{F}(\text{Ob } \mu)$, $\text{Src } f = \text{Ob } \mu$, $\text{Dst } f = \text{Ob } \nu$. If $f \in C(\langle \mu \rangle \mathcal{A} \nu)$ then $f \big|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A}$. 197
Proof. \( f|_{\mu \cdot A} \uparrow (f) A \Leftrightarrow (by \ the \ lemma) \Leftrightarrow (f \circ \mu|_{A}) A \subseteq (\nu \circ f) A \Leftrightarrow f \circ \mu|_{A} \subseteq \nu \circ f \Leftrightarrow f \in C(\mu|_{A}; \nu) \). \hfill \Box

Corollary 16.9. Let \( \mu, \nu \) be endofuncoids, \( f \in FCD(Ob \mu; Ob \nu), A \in \mathcal{F}(Ob \mu) \), Src \( f = Ob \mu \), Dst \( f = Ob \nu \). If \( f \in C(\mu; \nu) \) then \( f|_{\mu \cdot A} \uparrow (f) A \).

Theorem 16.10. Let \( \mu, \nu \) be endofuncoids, \( f \in FCD(Ob \mu; Ob \nu), A \in \mathcal{F}(Ob \mu) \) be an ultrafilter, Src \( f = Ob \mu \), Dst \( f = Ob \nu \). If \( f \in C(\mu|_{A}; \nu) \) iff \( f|_{\mu \cdot A} \uparrow (f) A \).

Proof. \( f|_{\mu \cdot A} \uparrow (f) A \Leftrightarrow (by \ the \ lemma) \Leftrightarrow (f \circ \mu|_{A}) A \subseteq (\nu \circ f) A \Leftrightarrow (used \ the \ fact \ that \ A \ is \ an \ ultrafilter) \Leftrightarrow f \circ \mu|_{A} \subseteq \nu \circ f \Leftrightarrow f \in C(\mu|_{A}; \nu) \). \hfill \Box

16.3 Limit

Definition 16.11. \( \lim^{\mu} f = a \) iff \( f|_{\mu} \uparrow \text{Src} \mu \{a\} \) for a \( T_{2} \)-separable funcoid \( \mu \) and a non-empty funcoid \( f \) such that Dst \( f = Dst \mu \).

It is defined correctly, that is \( f \) has no more than one limit.

Proof. Let \( \lim^{\mu} f = a \) and \( \lim^{\mu} f = b \). Then \( \text{im} f \subseteq \langle \mu \rangle^{*} \{a\} \) and \( \text{im} f \subseteq \langle \mu \rangle^{*} \{b\} \).

Because \( f \not\in \mathcal{FCD}(\text{Src} f; \text{Dst} f) \) we have \( f \notin \mathcal{F}(\text{Src} f) \subseteq \langle \mu \rangle^{*} \{a\} \cap \langle \mu \rangle^{*} \{b\} \not\subseteq \mathcal{F}(\text{Dst} f) \); \( \uparrow \text{Src} \mu \{b\} \cap \langle \mu^{-1}\rangle^{*} \langle \mu \rangle^{*} \{a\} \not\subseteq \mathcal{F}(\text{Src} \mu) \); \( \uparrow \text{Src} \mu \{b\} \cap \langle \mu^{-1}\rangle^{*} \langle \mu \rangle^{*} \{a\} \not\subseteq \mathcal{F}(\text{Src} \mu) \); \( \{a\} \not\subseteq \mathcal{F}(\text{Src} \mu) \); \( \{a\} \not\subseteq \mathcal{F}(\text{Src} \mu) \). Because \( \mu \) is \( T_{2} \)-separable we have \( a = b \). \hfill \Box

Definition 16.12. \( \lim^{B} \mu f = \lim^{\mu} (f|_{B}) \).

Remark 16.13. We can also in an obvious way define limit of a reloid.

16.4 Generalized limit

16.4.1 Definition

Let \( \mu \) and \( \nu \) be endofuncoids. Let \( G \) be a transitive permutation group on Ob \( \mu \).

For an element \( r \in G \) we will denote \( \uparrow r \uparrow FCD(Ob \mu; Ob \nu)_{r} \).

We require that \( \mu \) and every \( r \in G \) commute, that is

\[ \mu \circ \uparrow r = \uparrow r \circ \mu. \]

We require for every \( y \in \text{Ob} \nu \)

\[ \nu \sqsupset \langle \nu \rangle^{*}\{y\} \times \text{FCD} \langle \nu \rangle^{*}\{y\}. \]

Proposition 16.14. Formula (16.1) follows from \( \nu \sqsupset \nu \circ \nu^{-1} \).

Proof. Let \( \nu \sqsupset \nu \circ \nu^{-1} \). Then \( \langle \nu \rangle^{*}\{y\} \times \text{FCD} \langle \nu \rangle^{*}\{y\} = \langle \nu \rangle^{\text{Ob} \nu \{y\}} \times \text{FCD} \langle \nu \rangle^{\text{Ob} \nu \{y\}} = \nu \circ (\langle \text{Ob} \nu \{y\} \times \text{Ob} \nu \{y\}) \circ \nu^{-1} = \nu \circ \text{FCD}(\text{Ob} \nu; \text{Ob} \nu)(\{y\} \times \{y\}) \circ \nu^{-1} = \nu \circ \text{id}(\text{FCD}(\text{Ob} \nu) \circ \nu^{-1} = \nu \circ \nu^{-1} \subseteq \nu. \)

Remark 16.15. The formula (16.1) usually works if \( \nu \) is a proximity. It does not work if \( \mu \) is a pretopology or preclosure.

We are going to consider (generalized) limits of arbitrary functions acting from Ob \( \mu \) to Ob \( \nu \). (The functions in consideration are not required to be continuous.)

Remark 16.16. Most typically \( G \) is the group of translations of some topological vector space.

198
**Generalized limit** is defined by the following formula:

**Definition 16.17.** $\text{xlim} f \overset{\text{def}}{=} \{ \nu \circ f \circ \triangleright \mid r \in G \}$ for any funcoid $f$.

**Remark 16.18.** Generalized limit technically is a set of funcoids.

We will assume that the $\text{dom} f \supseteq (\mu)^*\{x\}$.

**Definition 16.19.** $\text{xlim}_x f \overset{\text{def}}{=} \text{xlim} f_{\mid (\mu)^*\{x\}}$.

**Obvious 16.20.** $\text{xlim}_x f = \{ \nu \circ f_{\mid (\mu)^*\{x\}} \circ \triangleright \mid r \in G \}$.

**Remark 16.21.** $\text{xlim}_x f$ is the same for funcoids $\mu$ and $\text{Compl} \mu$.

The function $\triangleright$ will define an injection from the set of points of the space $\nu$ ("numbers", "points", or "vectors") to the set of all (generalized) limits (i.e., values which $\text{xlim}_x f$ may take).

**Definition 16.22.** $\tau(y) \overset{\text{def}}{=} \{ (\mu)^*\{x\} \times \text{FCD} (\nu)^*\{y\} \mid x \in D \}$.

**Proposition 16.23.** $\tau(y) = \{ (\mu)^*\{x\} \times \text{FCD} (\nu)^*\{y\} \circ \triangleright \mid r \in G \}$ for every $x \in D$.

**Proof.** $(\mu)^*\{x\} \times \text{FCD} (\nu)^*\{y\} \circ \triangleright = (\triangleright^{-1})((\mu)^*\{x\} \times \text{FCD} (\nu)^*\{y\}) = (\mu)^*\{r^{-1} x\} \times \text{FCD} (\nu)^*\{y\} = \{ (\mu)^*\{x\} \times \text{FCD} (\nu)^*\{y\} \mid x \in D \}$.

Reversely $(\mu)^*\{x\} \times \text{FCD} (\nu)^*\{y\} = (\mu)^*\{x\} \times \text{FCD} (\nu)^*\{y\}) \circ \triangleright e$ where $e$ is the identify element of $G$.

**Proposition 16.24.** $\tau(y) = \text{xlim} ((\mu)^*\{x\} \times \text{FCD Base}(\text{dom} \nu)^*\{y\}))$ (for every $x$). Informally: Every $\tau(y)$ is a generalized limit of a constant funcoid.

**Proof.** $\text{xlim} ((\mu)^*\{x\} \times \text{FCD Base}(\text{dom} \nu)^*\{y\}) = \{ \nu \circ ((\mu)^*\{x\} \times \text{FCD Base}(\text{dom} \nu)^*\{y\}) \circ \triangleright r \mid r \in G \} = ((\mu)^*\{x\} \times \text{FCD} (\nu)^*\{y\}) \circ \triangleright e = \tau(y)$.

**Theorem 16.25.** If $f_{\mid (\mu)^*\{x\}} \in C(\mu; \nu)$ and $(\mu)^*\{x\} \supseteq \text{Ob} (\mu)^*\{x\}$ then $\text{xlim}_x f = \tau(f(x))$.

**Proof.** $f_{\mid (\mu)^*\{x\}} \circ \mu \subseteq \nu \circ f_{\mid (\mu)^*\{x\}} \subseteq \nu \circ f$; thus $(\nu)(\mu)^*\{x\} \subseteq (\nu)(f)^*\{x\}$; consequently we have $\nu \supseteq (\nu)(f)^*\{x\} \times \text{FCD} (\nu)(f)^*\{x\} \supseteq (\nu)(f)^*\{x\} \times \text{FCD} (\nu)(f)^*\{x\}$.

$$\nu \circ f_{\mid (\mu)^*\{x\}} \supseteq ((\nu)(f)^*\{x\} \times \text{FCD} (\nu)(f)^*\{x\}) \circ f_{\mid (\mu)^*\{x\}} = (\nu)(f)^*\{x\} \times \text{FCD} (\nu)(f)^*\{x\} \supseteq (\nu)(f)^*\{x\} \times \text{FCD} (\nu)(f)^*\{x\}$$

$$\text{dom} f_{\mid (\mu)^*\{x\}} \times \text{FCD} (\nu)(f)^*\{x\} = (\mu)^*\{x\} \times \text{FCD} (\nu)(f)^*\{x\}.$$
Proof. \( \text{im } f|_{\langle \mu \rangle^*\{x\}} \subseteq \langle \nu \rangle^*\{y\} \); \( \langle f \rangle \langle \mu \rangle^*\{x\} \subseteq \langle \nu \rangle^*\{y\} \); 

\[ \nu \circ f|_{\langle \mu \rangle^*\{x\}} \subseteq \] 

\[ ((\langle \nu \rangle^*\{y\}) \times_{\text{FCD}} (\langle \nu \rangle^*\{y\})) \circ f|_{\langle \mu \rangle^*\{x\}} = \] 

\[ ((f|_{\langle \mu \rangle^*\{x\}})^{-1}) (\langle \nu \rangle^*\{y\}) \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}) = \] 

\[ \langle f|_{\langle \mu \rangle^*\{x\}} \circ f^{-1} \rangle (\langle \nu \rangle^*\{y\}) \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}) \subseteq \] 

\[ \langle \text{id}_{\langle \mu \rangle^*\{x\}} \circ f^{-1} \rangle (\langle \mu \rangle^*\{x\}) \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}) = \] 

\[ \langle \text{id}_{\langle \mu \rangle^*\{x\}} \rangle (\langle \mu \rangle^*\{x\}) \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}) = \] 

\[ \langle \mu \rangle^*\{x\} \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}). \] 

On the other hand, \( f|_{\langle \mu \rangle^*\{x\}} \subseteq \langle \mu \rangle^*\{x\} \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}) \); 

\[ \nu \circ f|_{\langle \mu \rangle^*\{x\}} \subseteq \langle \mu \rangle^*\{x\} \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}) \subseteq \langle \mu \rangle^*\{x\} \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}). \] 

So \( \nu \circ f|_{\langle \mu \rangle^*\{x\}} = \langle \mu \rangle^*\{x\} \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}) \). 

\[ \text{xlim}_x f = \{ \nu \circ f|_{\langle \mu \rangle^*\{x\}} \uparrow r \mid r \in G \} = \{ ((\langle \mu \rangle^*\{x\}) \times_{\text{FCD}} (\langle \nu \rangle^*\{y\}) \circ \uparrow r \mid r \in G \} = \tau(y). \] 

**Corollary 16.28.** If \( \lim_{\langle \mu \rangle^*\{x\}}^\nu f = y \) then \( \text{xlim}_x f = \tau(y) \).

We have injective \( \tau \) if \( \langle \nu \rangle^*\{y_1\} \cap \langle \nu \rangle^*\{y_2\} = 0_{\text{Ob} \nu} \) for every distinct \( y_1, y_2 \in \text{Ob} \nu \) that is if \( \nu \) is \( T_2 \)-separable.
Chapter 17
Multifuncoids and staroids

17.1 Product of two funcoids

17.1.1 Lemmas

Lemma 17.1. Let $A$, $B$, $C$ be sets, $f \in \text{FCD}(A; B)$, $g \in \text{FCD}(B; C)$, $h \in \text{FCD}(A; C)$. Then

$$g \circ f \neq h \iff g \neq h \circ f^{-1}.$$ 

Proof. Proposition 6.66.

Lemma 17.2. Let $A$, $B$, $C$ be sets, $f \in \text{RLD}(A; B)$, $g \in \text{RLD}(B; C)$, $h \in \text{RLD}(A; C)$. Then

$$g \circ f \neq h \iff g \neq h \circ f^{-1}.$$ 

Proof. Theorem 7.16.

17.1.2 Definition

Definition 17.3. I will call a quasi-invertible category a partially ordered dagger category such that it holds

$$g \circ f \neq h \iff g \neq h \circ f^\dagger$$ (17.1)

for every morphisms $f \in \text{Mor}(A; B)$, $g \in \text{Mor}(B; C)$, $h \in \text{Mor}(A; C)$, where $A$, $B$, $C$ are objects of this category.

Inverting this formula, we get $f^\dagger \circ g^\dagger \neq h^\dagger \iff g^\dagger \neq f \circ h^\dagger$. After replacement of variables, this gives: $f^\dagger \circ g^\dagger \neq h^\dagger \iff g^\dagger \neq f \circ h$.

As it follows from above, the category of funcoids and the category of reloids are quasi-invertible (taking $f^\dagger = f^{-1}$). Moreover the category of pointfree funcoids between lattices of filters on boolean lattices is quasi-invertible (theorem 15.63).

Definition 17.4. The cross-composition product of morphisms $f$ and $g$ of a quasi-invertible category is the pointfree funcoid $\text{Mor}(\text{Src } f; \text{Src } g) \to \text{Mor}(\text{Dst } f; \text{Dst } g)$ defined by the formulas (for every $a \in \text{Mor}(\text{Src } f; \text{Src } g)$ and $b \in \text{Mor}(\text{Dst } f; \text{Dst } g)$):

$$\langle f \times (C) g \rangle a = g \circ a \circ f^\dagger \quad \text{and} \quad \langle (f \times (C) g)^{-1} \rangle b = g^\dagger \circ b \circ f.$$ 

The cross-composition product is a pointfree funcoid from $\text{Mor}(\text{Src } f; \text{Src } g)$ to $\text{Mor}(\text{Dst } f; \text{Dst } g)$.

We need to prove that it is really a pointfree funcoid that is that

$$b \neq \langle f \times (C) g \rangle a \iff a \neq \langle (f \times (C) g)^{-1} \rangle b.$$ 

This formula means $b \neq g \circ a \circ f^\dagger \iff a \neq g^\dagger \circ b \circ f$ and can be easily proved applying the formula (17.1) two times.
Proposition 17.15. \( a \left[ f \times^{(C)} g \right] \) \( b \Leftrightarrow a \circ f \neq g \circ b \).

Proof. From the definition. \( \square \)

Proposition 17.16. \( a \left[ f \times^{(C)} g \right] \) \( b \Leftrightarrow f \left[ a \times^{(C)} b \right] g \).

Proof. \( f \left[ a \times^{(C)} b \right] g \Leftrightarrow f \circ a \neq b \circ g \Leftrightarrow a \circ f \neq g \circ b \Leftrightarrow a \left[ f \times^{(C)} g \right] b \). \( \square \)

Theorem 17.7. \( (f \times^{(C)} g)^{-1} = f^\dagger \times^{(C)} g^\dagger \).

Proof. For every funcoids \( a \in \text{Mor}(\text{Src} f; \text{Src} g) \) and \( b \in \text{Mor}(\text{Dst} f; \text{Dst} g) \) we have:
\[
\langle \left( (f \times^{(C)} g)^{-1} \right)^{-1} \rangle = \langle (f \times^{(C)} g) \rangle a = g \circ a \circ f = \langle \left( f^\dagger \times^{(C)} g^\dagger \right)^{-1} \rangle a.
\]

Theorem 17.8. Let \( f, g \) be pointfree funcoids between filters on boolean lattices. Then for every filters \( A_0 \in \mathcal{B}(\text{Src} f), B_0 \in \mathcal{B}(\text{Src} g) \)
\[\langle f \times^{(C)} g \rangle (A_0 \times^{\text{FCD}} B_0) = (f)A_0 \times^{\text{FCD}} (g)B_0.\]

Proof. For every atom \( a_1 \times^{\text{FCD}} b_1 \) (\( a_1 \in \text{atoms}^{\text{Dst} f}, b_1 \in \text{atoms}^{\text{Dst} g} \)) (see theorem 15.76) of the lattice of funcoids we have:
\[
a_1 \times^{\text{FCD}} b_1 \neq \langle f \times^{(C)} g \rangle (A_0 \times^{\text{FCD}} B_0) \Leftrightarrow A_0 \times^{\text{FCD}} B_0 \left[ f \times^{(C)} g \right] a_1 \times^{\text{FCD}} b_1 \Leftrightarrow (A_0 \times^{\text{FCD}} B_0) \circ f \neq g^{-1} \circ b_1 \circ f = \langle f \times^{(C)} g \rangle \;
\]

Corollary 17.9. \( A_0 \times^{\text{FCD}} B_0 \left[ f \times^{(C)} g \right] A_1 \times^{\text{FCD}} B_1 \Leftrightarrow A_0 \left[ f \right] A_1 \land B_0 \left[ g \right] B_1 \) for every \( A_0 \in \mathcal{B}(\text{Src} f), A_1 \in \mathcal{B}(\text{Dst} f), B_0 \in \mathcal{B}(\text{Src} g), B_1 \in \mathcal{B}(\text{Dst} g) \) where \( \text{Src} f, \text{Dst} f, \text{Src} g, \text{Dst} g \) are boolean lattices.

Proof. \( A_0 \times^{\text{FCD}} B_0 \left[ f \times^{(C)} g \right] A_1 \times^{\text{FCD}} B_1 \Leftrightarrow A_1 \times^{\text{FCD}} B_1 \neq \langle f \times^{(C)} g \rangle (A_0 \times^{\text{FCD}} B_0) \Leftrightarrow A_1 \times^{\text{FCD}} B_1 \neq \langle f \rangle A_0 \times^{\text{FCD}} (g)B_0 \Leftrightarrow a_1 \neq \langle f \rangle A_0 \land B_1 \neq \langle g \rangle B_0 \Leftrightarrow A_0 \left[ f \right] A_1 \land B_0 \left[ g \right] B_1. \]

17.2 Function spaces of posets

Definition 17.10. Let \( \mathfrak{A} \) be a family of posets indexed by some set \( \text{dom} \mathfrak{A} \). We will define order of families of posets by the formula
\[a \sqsubseteq b \Leftrightarrow \forall i \in \text{dom} \mathfrak{A} : a_i \sqsubseteq b_i.\]
I will call this new poset \( \mathfrak{A} = \prod \mathfrak{A} \) the function space of posets and the above order product order.

Proposition 17.11. The function space for posets is also a poset.

Proof.

Reflexivity. Obvious.

Antisymmetry. Obvious.

Transitivity. Obvious. \( \square \)

Obvious 17.12. \( \mathfrak{A} \) has least element iff each \( \mathfrak{A}_i \) has a least element. In this case
\[\text{Least}(\mathfrak{A}) = \prod_{i \in \text{dom} \mathfrak{A}} \text{Least}(\mathfrak{A}_i).\]

Proposition 17.13. \( a \neq b \Leftrightarrow \exists i \in \text{dom} \mathfrak{A} : a_i \neq b_i \) for every \( a, b \in \prod \mathfrak{A} \) if every \( \mathfrak{A}_i \) has least element.
**Proposition 17.14.**

1. If \( \mathfrak{A}_i \) are join-semilattices then \( \mathfrak{A} \) is a join-semilattice and
   \[ A \cup B = \lambda i \in \text{dom} \mathfrak{A}: A_i \cup B_i. \quad (17.2) \]

2. If \( \mathfrak{A}_i \) are meet-semilattices then \( \mathfrak{A} \) is a meet-semilattice and
   \[ A \cap B = \lambda i \in \text{dom} \mathfrak{A}: A_i \cap B_i. \quad (17.3) \]

**Proof.** It is enough to prove the formula (17.2).

It’s obvious that \( \lambda i \in \text{dom} \mathfrak{A}: A_i \cup B_i \supseteq A, B. \)

Let \( C \supseteq A, B \). Then (for every \( i \in \text{dom} \mathfrak{A} \)) \( C_i \supseteq A_i \) and \( C_i \supseteq B_i \). Thus \( C_i \supseteq A_i \cup B_i \) that is \( C \supseteq \lambda i \in \text{dom} \mathfrak{A}: A_i \cup B_i. \)

**Corollary 17.15.** If \( \mathfrak{A}_i \) are lattices then \( \mathfrak{A} \) is a lattice.

**Obvious 17.16.** If \( \mathfrak{A}_i \) are distributive lattices then \( \mathfrak{A} \) is a distributive lattice.

**Proposition 17.17.** If \( \mathfrak{A}_i \) are boolean lattices then \( \prod \mathfrak{A} \) is a boolean lattice.

**Proof.** We need to prove only that every element \( a \in \prod \mathfrak{A} \) has a complement. But this complement is evidently \( \lambda i \in \text{dom} \mathfrak{A}: \bar{a}_i \).

**Proposition 17.18.** If every \( \mathfrak{A}_i \) is a poset then for every \( S \in \mathcal{P}\prod \mathfrak{A} \)

1. \( \bigcup S = \lambda i \in \text{dom} \mathfrak{A}: \bigcup \{x_i \mid x \in S\} \) whenever every \( \bigcup \{x_i \mid x \in S\} \) exists;

2. \( \bigcap S = \lambda i \in \text{dom} \mathfrak{A}: \bigcap \{x_i \mid x \in S\} \) whenever every \( \bigcap \{x_i \mid x \in S\} \) exists.

**Proof.** It’s enough to prove the first formula.

\( (\lambda i \in \text{dom} \mathfrak{A}: \bigcup \{x_i \mid x \in S\})_i = \bigcup \{x_i \mid x \in S\} \) for every \( x \in S \) and \( i \in \text{dom} \mathfrak{A} \).

Let \( y \supset x \) for every \( x \in S \). Then \( y_i \supset x_i \) for every \( i \in \text{dom} \mathfrak{A} \) and thus \( y_i \supset \bigcup \{x_i \mid x \in S\} \) (\( \lambda i \in \text{dom} \mathfrak{A}: \bigcup \{x_i \mid x \in S\} \)), that is \( y \supset \lambda i \in \text{dom} \mathfrak{A}: \bigcup \{x_i \mid x \in S\} \).

Thus \( \bigcup S = \lambda i \in \text{dom} \mathfrak{A}: \bigcup \{x_i \mid x \in S\} \) by the definition of join.

**Corollary 17.19.** If \( \mathfrak{A}_i \) are posets then for every \( S \in \mathcal{P}\prod \mathfrak{A} \)

1. \( \bigcup S = \lambda i \in \text{dom} \mathfrak{A}: \bigcup \{x_i \mid x \in S\} \) whenever every \( \bigcup S \) exists;

2. \( \bigcap S = \lambda i \in \text{dom} \mathfrak{A}: \bigcap \{x_i \mid x \in S\} \) whenever every \( \bigcap S \) exists.

**Proof.** It is enough to prove that (for every \( i \)) \( \bigcup \{x_i \mid x \in S\} \) exists whenever \( \bigcup S \) exists.

Fix \( i \in \text{dom} \mathfrak{A} \).

Take \( y_i = (\bigcup S)_i \) and let prove that \( y_i \) is the least upper bound of \( \{x_i \mid x \in S\} \).

\( y_i \) is an upper bound of \( \{x_i \mid x \in S\} \) because \( \bigcup S \supset x \) and thus \( (\bigcup S)_i \supset x_i \) for every \( x \in S \).

Let \( x \in S \) and for some \( t \in \mathfrak{A}_i \)

\[ T(t) = \lambda j \in \text{dom} \mathfrak{A} \left\{ \begin{array}{ll} t & \text{if } i = j \smallskip \\
 x_i & \text{if } i \neq j \end{array} \right. \]

Let \( t \supset x_i \). Then \( T(t) \supset x \) for every \( x \in S \). So \( T(t) \supset \bigcup S \) and consequently \( t = T(t)_i \supset y_i \).

So \( y_i \) is the least upper bound of \( \{x_i \mid x \in S\} \).

**Corollary 17.20.** If \( \mathfrak{A}_i \) are complete lattices then \( \mathfrak{A} \) is a complete lattice.

**Obvious 17.21.** If \( \mathfrak{A}_i \) are complete (co-)brouwerian lattices then \( \mathfrak{A} \) is a (co-)brouwerian lattice.
Proposition 17.22. If each $\mathfrak{A}_i$ is a separable poset with least element (for some index set $n$) then $\prod_i \mathfrak{A}_i$ is a separable poset.

Proof. Let $a \neq b$. Then $\exists i \in \text{dom } \mathfrak{A}_i; a_i \neq b_i$. So $\exists x \in \mathfrak{A}_i; (x \neq a_i \land x \neq b_i)$ (or vice versa).

Let $y = (((\text{dom } \mathfrak{A}_i) \setminus \{i\}) \times \{0\}) \cup \{(i; x)\}$. Then $y \neq a$ and $y \neq b$. \hfill \square

Obvious 17.23. If every $\mathfrak{A}_i$ is a poset with least element $0_i$, then the set of atoms of $\prod_i \mathfrak{A}_i$ is $\{(\{k\} \times \text{atoms}^{\mathfrak{A}_k} ) \cup (\lambda i \in (\text{dom } \mathfrak{A}_i) \setminus \{k\}; 0_i) \mid k \in \text{dom } \mathfrak{A}_i\}$.

Proposition 17.24. If every $\mathfrak{A}_i$ is an atomistic poset with least element $0_i$, then $\prod_i \mathfrak{A}_i$ is an atomistic poset.

Proof. Let every $\mathfrak{A}_i \neq 0_i$. Thus

$$x = \lambda i \in \text{dom } x; x_i = \lambda i \in \text{dom } x; \bigcup_{i \in \text{dom } x} \lambda j \in \text{dom } x; \left\{ x_i \text{ if } j = i, 0_i \text{ if } j \neq i \right\}.$$

Take join two times. \hfill \square

Corollary 17.25. If $\mathfrak{A}_i$ are atomistic posets with least elements, then $\prod_i \mathfrak{A}_i$ is atomically separable.

Proof. Proposition 3.19. \hfill \square

Proposition 17.26. Let $(\mathfrak{A}_i; 3_i; \mathfrak{B}_i; \mathfrak{C}_i) \in n$ be a family of filtrators. Then $(\prod_i \mathfrak{A}_i; \prod_i 3_i)$ is a filtrator.

Proof. We need to prove that $\prod_i 3_i$ is a sub-poset of $\prod_i \mathfrak{A}_i$. First $\prod_i 3_i \subseteq \prod_i \mathfrak{A}_i$ because $3_i \subseteq \mathfrak{A}_i$ for each $i \in n$.

Let $A, B \in \prod_i 3_i$ and $A \sqcup \prod_i B$. Then $\forall i \in n; A_i \sqcup 3_i B_i$; consequently $\forall i \in n; A_i \sqcup \mathfrak{B}_i B_i$ that is $A \sqcup \prod_i \mathfrak{B}_i B$.

Proposition 17.27. Let $(\mathfrak{A}_i; 3_i; \mathfrak{B}_i; \mathfrak{C}_i) \in n$ be a family of filtrators.

1. The filtrator $(\prod_i \mathfrak{A}_i; \prod_i 3_i)$ is (finitely) join-closed if every $(\mathfrak{A}_i; 3_i)$ is (finitely) join-closed.

2. The filtrator $(\prod_i \mathfrak{A}_i; \prod_i 3_i)$ is (finitely) meet-closed if every $(\mathfrak{A}_i; 3_i)$ is (finitely) meet-closed.

Proof. Let every $(\mathfrak{A}_i; 3_i)$ be finitely join-closed. Let $A, B \in \prod_i 3_i$ and $A \sqcup \prod_i B$ exist. Then (by corollary 17.19) $A \sqcup 3_i B = \lambda i \in n; A_i \sqcup 3_i B_i = \lambda i \in n; A_i \sqcup 3_i B_i = A \sqcup \prod_i B$.

Let now every $(\mathfrak{A}_i; 3_i)$ be join-closed. Let $S \subseteq \mathfrak{A}_i \prod_i 3_i$ and $\bigcup_i 3_i S \in n$. Then (by corollary 17.19) $\bigcup_i 3_i S = \lambda i \in \text{dom } \mathfrak{A}_i; \bigcup_i 3_i \lambda x \in S = \lambda x \in \text{dom } \mathfrak{A}_i; \bigcup_i 3_i \lambda x \in S = \bigcup_i 3_i S$.

The rest follows from symmetry. \hfill \square

Proposition 17.28. If each $(\mathfrak{A}_i; 3_i)$ where $i \in n$ (for some index set $n$) is a down-aligned filtrator with separable core then $(\prod_i \mathfrak{A}_i; \prod_i 3_i)$ is with separable core.

Proof. Let $a \neq b$. Then $\exists i \in n; a_i \neq b_i$. So $\exists x \in \mathfrak{A}_i; (x \neq a_i \land x \neq b_i)$ (or vice versa).

Take $y = ((\text{dom } \mathfrak{A}_i) \setminus \{i\}) \times \{0\}) \cup \{(i; x)\}$. Then we have $y \neq a$ and $y \neq b$ and $y \in 3_i$. \hfill \square

Proposition 17.29. Let every $\mathfrak{A}_i$ be a bounded lattice. Every $(\mathfrak{A}_i; 3_i)$ is a central filtrator if $(\prod_i \mathfrak{A}_i; \prod_i 3_i)$ is a central filtrator.

Proof. $x \in Z(\prod_i \mathfrak{A}_i)$ \iff $\exists y \in \prod_i \mathfrak{A}_i; (x \sqcap y = 0^{\prod_i} \land x \sqcup y = 1^{\prod_i}) \iff \forall y \in \prod_i \mathfrak{A}_i \forall i \in \text{dom } \mathfrak{A}_i \exists y \in \mathfrak{A}_i; (x_i \sqcap y = 0^{\mathfrak{A}_i} \land x_i \sqcup y = 1^{\mathfrak{A}_i}) \iff \forall i \in \text{dom } \mathfrak{A}_i x_i \in Z(\mathfrak{A}_i)$.

Proposition 17.30. For every element $a$ of a product filtrator $(\prod_i \mathfrak{A}_i; \prod_i 3_i)$:

1. $\up a = \prod_i \text{dom } a \up a_i$;
2. $\down a = \prod_i \text{dom } a \down a_i$.\hfill \square
Proof. We will prove only the first as the second is dual.

\[ \text{up } a = \{ c \in \prod 3 \mid c \sqsupseteq a \} = \{ c \in \prod 3 \mid \forall i \in \text{dom } a: c_i \sqsupseteq a_i \} = \{ c \in \prod 3 \mid \forall i \in \text{dom } a: c_i = \text{up } a_i \} = \prod_{i \in \text{dom } a} \text{up } a_i. \]

Proposition 17.31. If every \((\mathfrak{A}_i; \mathfrak{J}_i)\) is a filtered complete lattice filtrator, then \((\prod \mathfrak{A}; \prod 3)\) is a filtered complete lattice filtrator.

Proof. That \(\prod \mathfrak{A}\) is a complete lattice is already proved above. We have for every \(a \in \prod \mathfrak{A}\)

\[ \prod \mathfrak{A} \text{ up } a = \lambda i \in \text{dom } \mathfrak{A}: \prod \mathfrak{J} \{ x_i \mid x \in \text{up } a \} = \lambda i \in \text{dom } \mathfrak{A}: \prod \mathfrak{J} \{ x \mid x \in \text{up } a_i \} = \lambda i \in \text{dom } \mathfrak{A}: \text{up } a_i = \lambda i \in \text{dom } \mathfrak{A}: a_i = a. \]

Proposition 17.32. If every \((\mathfrak{A}_i; \mathfrak{J}_i)\) is a prefilered complete lattice filtrator with \(\text{up } x \neq \emptyset\) for every \(x \in \mathfrak{A}_i\) (for every \(i \in n\)), then \((\prod \mathfrak{A}; \prod 3)\) is a prefilered complete lattice filtrator.

Proof. Let \(a, b \in \prod \mathfrak{A}\) and \(a \neq b\). Then there exists \(i \in n\) such that \(a_i \neq b_i\) and so \(\text{up } a_i \neq \text{up } b_i\). Consequently \(\prod_{i \in \text{dom } a} \text{up } a_i \neq \prod_{i \in \text{dom } a} \text{up } b_i\) (taken into account that \(\text{up } x \neq \emptyset\) for every \(x \in \mathfrak{A}_i\)) that is \(\text{up } a \neq \text{up } b\).

Proposition 17.33. Let every \((\mathfrak{A}_i; \mathfrak{J}_i)\) be a semifiltered filtrator with \(\text{up } x \neq \emptyset\) for every \(x \in \mathfrak{A}_i\) (for every \(i \in n\)). Then \((\prod \mathfrak{A}; \prod 3)\) is a semifiltered filtrator.

Proof. Let every \((\mathfrak{A}_i; \mathfrak{J}_i)\) be a semifiltered filtrator. Let \(a \uparrow b \in \prod \mathfrak{A}\) for some \(a, b \in \prod \mathfrak{A}\). Then \(\prod_{i \in \text{dom } a} \text{up } a_i \uparrow \prod_{i \in \text{dom } a} \text{up } b_i\) and consequently (taking into account that \(\text{up } x \neq \emptyset\) for every \(x \in \mathfrak{A}_i\)) \(\text{up } a_i \uparrow \text{up } b_i\) for every \(i \in n\). Then \(\forall i \in n: a_i \subseteq b_i\). Really, \(\lambda i \in n: a_i \cap b_i = \lambda i \in n: (a_i \cap b_i) \cap b_i = 0\) and \(b \uparrow (\lambda i \in n: a_i \cap b_i) = \lambda i \in n: b_i \cap (a_i \cup b_i) = \lambda i \in n: b_i \cup a_i = a \cup b\).

Proposition 17.34. Let \((\mathfrak{A}_i; \mathfrak{J}_i)\) be filtrators and each \(\mathfrak{J}_i\) is a complete lattice with \(\text{up } x \neq \emptyset\) for every \(x \in \mathfrak{A}_i\) (for every \(i \in n\)). For \(a \in \prod \mathfrak{A}\):

1. \(\text{Cor } a = \lambda i \in \text{dom } a: \text{Cor } a_i;\)
2. \(\text{Cor’}_a = \lambda i \in \text{dom } a: \text{Cor’}_a_i.\)

Proof. We will prove only the first, because the second is dual.

\[ \text{Cor } a = \prod 3 \text{ up } a = \lambda i \in \text{dom } a: \prod 3 \{ x_i \mid x \in \text{up } a \} = (\text{up } x \neq \emptyset \text{ taken into account}) = \lambda i \in \text{dom } a: \prod 3 \text{up } a_i = \lambda i \in \text{dom } a: \text{Cor } a_i. \]

Proposition 17.35. If each \((\mathfrak{A}_i; \mathfrak{J}_i)\) is a filtrator with (co)-separable core and each \(\mathfrak{A}_i\) has a least (greatest) element, then \((\prod \mathfrak{A}; \prod 3)\) is a filtrator with (co)-separable core.

Proof. We will prove only for separable core, as co-separable core is dual.

\[ x \uparrow \prod 3 \text{ y } \Leftrightarrow (\text{used the fact that } \mathfrak{A}_i \text{ has a least element}) \Leftrightarrow \forall i \in \text{dom } \mathfrak{A}_i: x_i \uparrow \mathfrak{J}_i \Rightarrow \forall i \in \text{dom } \mathfrak{A}_i: \exists X \ni x_i \in \text{up x } \Rightarrow \exists X \ni x \in \text{up } x \in \text{dom } \mathfrak{A}_i: X_i \uparrow \mathfrak{J}_i \Rightarrow \exists X \ni x \in \text{up } x \ni \mathfrak{A}_i: X \uparrow \prod 3 \text{ y } \text{ for every } x, y \in \prod \mathfrak{A} . \]

Obvious 17.36.

1. If each \((\mathfrak{A}_i; \mathfrak{J}_i)\) is a down-aligned filtrator, then \((\prod \mathfrak{A}; \prod 3)\) is a down-aligned filtrator.
2. If each \((\mathfrak{A}_i; \mathfrak{J}_i)\) is an up-aligned filtrator, then \((\prod \mathfrak{A}; \prod 3)\) is an up-aligned filtrator.

Proposition 17.37. If every \(b_i\) is substructive from \(a_i\) where \(a\) and \(b\) are \(n\)-indexed families of distributive lattices with least elements (where \(n\) is an index set), then \(a \setminus b = \lambda i \in n: a_i \setminus b_i\).

Proof. We need to prove \((\lambda i \in n: a_i \setminus b_i) \cap b = 0\) and \(a \cup b = b \cup (\lambda i \in n: a_i \setminus b_i)\).

Really, \(\lambda i \in n: a_i \setminus b_i \cap b = \lambda i \in n: (a_i \setminus b_i) \cap b_i = 0\) and \(b \cup (\lambda i \in n: a_i \setminus b_i) = \lambda i \in n: b_i \cup (a_i \setminus b_i) = \lambda i \in n: b_i \cup a_i = a \cup b.\)

Proposition 17.38. If every \(\mathfrak{A}_i\) is a distributive lattice, then \(a \setminus^* b = \lambda i \in \text{dom } \mathfrak{A}: a_i \setminus^* b_i\) for every \(a, b \in \prod \mathfrak{A}\) whenever every \(a_i, b_i\) is defined.

Proof. If some \(\mathfrak{A}_i\) is empty, our statement is obvious. Let’s assume \(\mathfrak{A}_i \neq \emptyset\).
We need to prove that \( \lambda i \in \text{dom } \mathfrak{A}; a_i \setminus^* b_i = \bigcap \{ z \in \prod \mathfrak{A} | \ a \sqsubseteq b \cup z \} \).

To prove it is enough to show \( a_i \setminus^* b_i = \bigcap \{ z_i | z \in \prod \mathfrak{A}, a \sqsubseteq b \cup z \} \) that is \( a_i \setminus^* b_i = \bigcap \{ z \in \mathfrak{A}_i | a_i \sqsubseteq b \cup z \} \) (for the reverse implication take \( z_j = a_i \) for \( j \neq i \)) what is true by definition. \( \square \)

**Proposition 17.39.** If every \( \mathfrak{A}_i \) is a distributive lattice with least element, then \( a \# b = \lambda i \in \text{dom } \mathfrak{A}; a_i \# b_i \) for every \( a, b \in \prod \mathfrak{A} \) whenever every \( a \# b_i \) is defined.

**Proof.** We need to prove that \( \lambda i \in \text{dom } \mathfrak{A}; a_i \# b_i = \bigcup \{ z \in \prod \mathfrak{A} | z \sqsubseteq a \wedge z \sqsupseteq b \} \).

To prove it is enough to show \( a_i \# b_i = \bigcup \{ z_i | z \in \prod \mathfrak{A}, z \sqsubseteq a \wedge z \sqsupseteq b \} \) that is \( a_i \# b_i = \bigcup \{ z \in \mathfrak{A}_i | z \sqsubseteq a_i \wedge z \sqsupseteq b_i \} \) (take \( z_j = 0 \) for \( j \neq i \)) what is true by definition. \( \square \)

**Proposition 17.40.** Let every \( \mathfrak{A}_i \) be a poset with least element and \( a^*_i \) is defined. Then \( a^* = \lambda i \in \text{dom } \mathfrak{A}; a^*_i \).

**Proof.** We need to prove that \( \lambda i \in \text{dom } \mathfrak{A}; a^*_i = \bigcup \{ c \in \mathfrak{A} | c \gg a \} \). To prove this it is enough to show that \( a^*_i = \bigcup \{ c_i | c \in \prod \mathfrak{A}, c \gg a \} \) that is \( a^*_i = \bigcup \{ c_i | c \in \prod \mathfrak{A}, \forall j \in \text{dom } \mathfrak{A}: c_j \gg a_j \} \) that is \( a^*_i = \bigcup \{ c_i | c \in \prod \mathfrak{A}, c_i \gg a_i \} \) (take \( c_j = 0 \) for \( j \neq i \)) that is \( a^*_i = \bigcup \{ c \in \mathfrak{A}_i | c \gg a_i \} \) what is true by definition. \( \square \)

**Corollary 17.41.** Let every \( \mathfrak{A}_i \) be a poset with greatest element and \( a^+_i \) is defined. Then \( a^+ = \lambda i \in \text{dom } \mathfrak{A}; a^+_i \).

**Proof.** By duality. \( \square \)

### 17.3 Definition of staroids

Let \( n \) be a set. As an example, \( n \) may be an ordinal, \( n \) may be a natural number, considered as a set by the formula \( n = \{ 0, \ldots, n - 1 \} \). Let \( \mathfrak{A} = \mathfrak{A}_{i\in n} \) be a family of posets indexed by the set \( n \).

**Definition 17.42.** I will call an anchored relation a pair \( f = (\text{form } f; \text{GR } f) \) of a family \( \text{form } f \) of sets indexed by the same index set and a relation \( \text{GR } f \in \mathcal{P} \prod \text{form } f \). I call \( \text{GR } f \) the graph of the anchored relation \( f \). I denote \( \text{Anch } (\mathfrak{A}) \) the set of small anchored relations of the form \( \mathfrak{A} \).

**Definition 17.43.** An anchored relation on powersets is an anchored relation \( f \) such that every \( (\text{form } f)_i \), is a powerset.

I will denote \( \text{arity } f = \text{dom } \text{form } f \).

**Definition 17.44.** Every set of anchored relations of the same form constitutes a poset by the formula \( f \subseteq g \iff \text{GR } f \subseteq \text{GR } g \).

**Definition 17.45.** An anchored relation is an anchored relation between posets when every \( (\text{form } f)_i \), is a poset.

**Definition 17.46.** Let \( f \) be an anchored relation. For every \( i \in \text{arity } f \) and \( L \in \prod ((\text{form } f)_{\text{arity } f \setminus \{ i \}}) \)

\[
(\text{val } f)_i L = \{ X \in (\text{form } f)_i | L \cup \{(i; X)\} \in \text{GR } f \}
\]

("val" is an abbreviation of the word "value").

**Obvious 17.47.** \( X \in (\text{val } f)_i L \iff L \cup \{(i; X)\} \in \text{GR } f \).

**Proposition 17.48.** \( f \) can be restored knowing \( \text{form } f \) and \( (\text{val } f)_i \), for some \( i \in n \).

**Proof.** \( \text{GR } f = \{ K \in \prod \text{form } f | K \in \text{GR } f \} = \{ L \cup \{(i; X)\} | L \in \prod ((\text{form } f)_{\text{arity } f \setminus \{ i \}}), X \in (\text{form } f)_i, L \cup \{(i; X)\} \in \text{GR } f \} = \{ L \cup \{(i; X)\} | L \in \prod ((\text{form } f)_{\text{arity } f \setminus \{ i \}}), X \in (\text{val } f)_i L \}. \square \)
**Definition 17.49.** A **prestaroid** is an anchored relation \( f \) between posets such that \((\text{val } f), L\) is a free star for every \( i \in \text{arity } f \), \( L \in \prod (\text{form } f)|_{(\text{arity } f) \setminus \{i\}}\).

**Definition 17.50.** A **staroid** is a prestaroid whose graph is an upper set (on the poset \( \prod \text{form}(f) \)).

**Proposition 17.51.** If \( L \in \prod \text{form } f \) and \( L_i = 0(\text{form } f)_i \) for some \( i \in \text{arity } f \) then \( L \notin f \) if \( f \) is a prestaroid.

**Proof.** Let \( K = L|_{(\text{arity } f) \setminus \{i\}} \). We have \( 0 \notin (\text{val } f)_i K \cup \{i; 0\} \notin f \); \( L \notin f \).

**Definition 17.52.** An **infinitary prestaroid** is such a prestaroid whose arity is infinite; **finitary prestaroid** is such a of prestaroid whose arity is finite.

Next we will define **completary staroids.** First goes the general case, next simpler case for the special case of join-semilattices instead of arbitrary posets.

**Definition 17.53.** A **completary staroid** is an anchored relation between posets conforming to the formulas:

1. \( \forall K \in \prod \text{form } f ; (K \supseteq L_0 \wedge K \supseteq L_1 \Rightarrow K \in \text{GR } f) \leftrightarrow \exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)}i) \in \text{GR } f \) for every \( L_0, L_1 \in \prod \text{form } f \).

2. If \( L \in \prod \text{form } f \) and \( L_i = 0(\text{form } f)_i \) for some \( i \in \text{arity } f \) then \( L \notin \text{GR } f \).

**Lemma 17.54.** Every graph of completary staroid is an upper set.

**Proof.** Let \( f \) be a completary staroid. Let \( L_0 \subseteq L_1 \) for some \( L_0, L_1 \in \prod \text{form } f \) and \( L_0 \in \text{GR } f \). Then taking \( c = n \times \{0\} \) we get \( \lambda i \in n : L_{c(i)}i = \lambda i \in n : L_0i = L_0 \in \text{GR } f \) and thus \( L_1 \in \text{GR } f \) because \( L_1 \supseteq L_0 \wedge L_1 \supseteq L_1 \).

**Proposition 17.55.** A relation between posets whose form is a family of join-semilattices is a completary staroid iff both:

1. \( L_0 \cup L_1 \in \text{GR } f \Leftrightarrow \exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)}i) \in \text{GR } f \) for every \( L_0, L_1 \in \prod \text{form } f \).

2. If \( L \in \prod \text{form } f \) and \( L_i = 0(\text{form } f)_i \) for some \( i \in \text{arity } f \) then \( L \notin \text{GR } f \).

**Proof.** Let the formulas (1) and (2) hold. Then \( f \) is an upper set: Let \( L_0 \subseteq L_1 \) for some \( L_0, L_1 \in \prod \text{form } f \) and \( L_0 \in \text{GR } f \). Then taking \( c = n \times \{0\} \) we get \( \lambda i \in n : L_{c(i)}i = \lambda i \in n : L_0i = L_0 \in \text{GR } f \) and thus \( L_1 = L_0 \cup L_1 \in f \).

Thus to finish the proof it is enough to show that

\[
L_0 \cup L_1 \in \text{GR } f \Leftrightarrow \forall K \in \prod \text{form } f ; (K \supseteq L_0 \wedge K \supseteq L_1 \Rightarrow K \in \text{GR } f)
\]

under condition that \( \text{GR } f \) is an upper set. But this is obvious.

**Proposition 17.56.** Every completary staroid is a staroid.

**Proof.** Let \( f \) be a completary staroid.

Let \( i \in \text{arity } f \), \( K \in \prod_{i \notin (\text{arity } f) \setminus \{i\}} \text{form } f \). Let \( L_0 = K \cup \{(i; X_0)\}, L_1 = K \cup \{(i; X_1)\} \) for some \( X_0, X_1 \in \mathcal{A}_i \).

Let

\[
\forall Z \in \mathcal{A}_i ; (Z \supseteq X_0 \wedge Z \supseteq X_1 \Rightarrow Z \in (\text{val } f)_i K);
\]

then

\[
\forall Z \in \mathcal{A}_i ; (Z \supseteq X_0 \wedge Z \supseteq X_1 \Rightarrow K \cup \{(i; Z)\} \in \text{GR } f).
\]

If \( z \supseteq L_0 \wedge z \supseteq L_1 \) then \( z \supseteq K \cup \{(i; z_i)\} \), thus taking into account that \( \text{GR } f \) is an upper set,

\[
\forall z \in \prod \mathcal{A}_i ; (z \supseteq L_0 \wedge z \supseteq L_1 \Rightarrow K \cup \{(i; z_i)\} \in \text{GR } f).
\]

\[
\forall z \in \prod \mathcal{A}_i ; (z \supseteq L_0 \wedge z \supseteq L_1 \Rightarrow z \in \text{GR } f).
\]
Thus, by the definition of completary staroid, \( L_0 \in \text{GR} f \lor L_1 \in \text{GR} f \) that is
\[
X_0 \in (\text{val} f)_i K \lor X_1 \in (\text{val} f)_i K.
\]
So \((\text{val} f)_i K\) is a free star (taken into account that \( z_i = 0^{(\text{form} f)} \Rightarrow z \notin \text{GR} f \) and that \((\text{val} f)_i K\) is an upper set).

**Exercise 17.1.** Find a simplified proof for the case if every \((\text{form})_i\) is a join-semilattice.

**Lemma 17.57.** Every finitary prestaroid is completary.

**Proof.** \( \exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)}i) \in \text{GR} f \Leftrightarrow \exists c \in \{0, 1\}^{n-1} : \{(n - 1; L_{0(n - 1)}); (n - 1; L_{1(n - 1)})\} \cup \{(\lambda i \in n - 1 : L_{c(i)}i) : \in \text{GR} f \Leftrightarrow \exists c \in \{0, 1\}^{n-1} : L_0(n - 1) \in (\text{val} f)_n - 1(\lambda i \in n - 1 : L_{c(i)}i) \lor L_1(n - 1) \in (\text{val} f)_n - 1(\lambda i \in n - 1 : L_{c(i)}i) \Leftrightarrow \exists c \in \{0, 1\}^{n-1} : L_0(n - 1) \in (\text{form} f)_n - 1(\lambda i \in n - 1 : L_{c(i)}i) \lor L_1(n - 1) \in (\text{form} f)_n - 1(\lambda i \in n - 1 : L_{c(i)}i) \Rightarrow \exists c \in \{0, 1\}^{n-1} : K \subseteq L_0(n - 1) \lor L_1(n - 1) \Rightarrow K \in (\text{val} f)_n - 1(\lambda i \in n - 1 : L_{c(i)}i) \Leftrightarrow \exists c \in \{0, 1\}^{n-1} : (K_n - 1 \subseteq L_0(n - 1) \lor K_n - 1 \subseteq L_1(n - 1) \Rightarrow \{(n - 1; K) \cup (\lambda i \in n - 1 : L_{c(i)}i) : \in \text{GR} f \Leftrightarrow \exists c \in \{0, 1\}^{n-1} : K \subseteq L_0(n - 1) \lor L_1(n - 1) \Rightarrow K \in (\text{val} f)_n - 1(\lambda i \in n - 1 : L_{c(i)}i) \Rightarrow \exists c \in \{0, 1\}^{n-1} : K_n - 1 \subseteq L_0(n - 1) \lor K_n - 1 \subseteq L_1(n - 1) \Rightarrow \{(n - 1; K) \cup (\lambda i \in n - 1 : L_{c(i)}i) : \in \text{GR} f \).

**Exercise 17.2.** Prove the simpler special case of the above theorem when the form is a family of join-semilattices.

**Theorem 17.58.** For finite arity the following are the same:

1. prestaroids;
2. staroids;
3. completary staroids.

**Proof.** \( f \) is a finitary prestaroid \( \Rightarrow f \) is a finitary completary staroid.

\( f \) is a finitary completary staroid \( \Rightarrow f \) is a finitary staroid.

\( f \) is a finitary staroid \( \Rightarrow f \) is a finitary prestaroid.

**Definition 17.59.** We will denote the set of staroids, prestaroids, and completary staroids of a form \( \mathfrak{A} \) correspondingly as \( \text{Strd}(\mathfrak{A}) \), \( \text{pStrd}(\mathfrak{A}) \), and \( \text{cStrd}(\mathfrak{A}) \).

### 17.4 Upgrading and downgrading a set regarding a filtrator

Let fix a filtrator \( (\mathfrak{A}; \mathfrak{3}) \).

**Definition 17.60.** \( \| f = f \cap \mathfrak{3} \) for every \( f \in \mathcal{P} \mathfrak{A} \) (downgrading \( f \)).

**Definition 17.61.** \( \| f = \{ L \in \mathfrak{A} : \text{up} L \subseteq f \} \) for every \( f \in \mathcal{P} \mathfrak{3} \) (upgrading \( f \)).

**Obvious 17.62.** \( a \in \| f \Leftrightarrow \text{up} a \subseteq f \) for every \( f \in \mathcal{P} \mathfrak{3} \) and \( a \in \mathfrak{A} \).

**Proposition 17.63.** \( \| \| f = f \) if \( f \) is an upper set.

**Proof.** \( \| \| f = \| f \cap \mathfrak{3} = \{ L \in \mathfrak{3} : \text{up} L \subseteq f \} = \{ L \in \mathfrak{3} : \text{up} L \subseteq f \} = f \cap \mathfrak{3} = f \).

### 17.4.1 Upgrading and downgrading staroids

Let fix a family \( (\mathfrak{A}; \mathfrak{3}) \) of filtrators.

For a graph \( f \) of a staroid define \( \| f \) and \( \| f \) taking the filtrator of \( (\mathfrak{A}; \mathfrak{3}; \mathfrak{3}) \).

For a staroid \( f \) define:

\begin{align*}
\text{form} \| f = \mathfrak{3} & \quad \text{and} \quad \text{GR} \| f = \| \text{GR} f; \\
\text{form} \| f = \mathfrak{A} & \quad \text{and} \quad \text{GR} \| f = \| \text{GR} f.
\end{align*}
Proposition 17.64. \((\val \uplus f).L = (\val f).L \cap 3\) for every \(L \in \prod 3_{(\arity f)}\).

Proof. \((\val \uplus f).L = \{X \in 3_i \mid L \cup \{(i; X)\} \in \GR f \cap \prod 3\} = \{X \in 3_i \mid L \cup \{(i; X)\} \in \GR f\} = (\val f).L \cap 3_i.\)

Proposition 17.65. Let \((A; 3_i)\) be finitely join-closed filtrators with both the base and the core being join-semilattices. If \(f\) is a staroid of the form \(A\), then \(\uplus f\) is a staroid of the form \(3_i\).

Proof. Let \(f\) be a staroid.

We need to prove that \((\val \uplus f).L\) is a free star. It follows from the last proposition and the fact that it is finitely join-closed.

17.5 Principal staroids

Definition 17.66. The staroid generated by an anchored relation \(F\) is the staroid \(f = \uparrow^{\Strd} F\) on powersets such that \(\uparrow \circ L \in \GR f \Leftrightarrow \prod L \neq F\) and \((\form f)_i = \mathcal{P}(\form f)_i\) for every \(L \in \prod_{i \in \arity f} \mathcal{P}(\form f)_i.\)

Definition 17.67. A principal staroid is a staroid generated by some anchored relation.

Proposition 17.68. Every principal staroid is a completary staroid.

Proof. That \(L \not\in f\) if \(L_i = (\form f)_i\) for some \(i \in \arity f\) is obvious. It remains to prove

\[\prod_{i \in \arity f} (L_0 \cup L_1) \neq F \Leftrightarrow \exists c \in \{0, 1\}^{\arity f}; \prod_{i \in \arity f} L_c(i) \neq F.\]

Really

\[\prod_{i \in \arity f} (L_0 \cup L_1) \neq F \Leftrightarrow \exists x \in \prod (L_0 \cup L_1): x \in F \Leftrightarrow \exists x \in \prod_{i \in \arity f} (\form f)_i \forall i \in \arity f: (x_i \in L_0 i \cup L_1 i \land x \in F) \Leftrightarrow \exists x \in \prod_{i \in \arity f} (\form f)_i (\exists c \in \{0, 1\}^{\arity f}; x \in \prod_{i \in \arity f} L_c(i) \land x \in F) \Leftrightarrow \exists c \in \{0, 1\}^{\arity f}; \prod_{i \in \arity f} L_c(i) \neq F.\]

Definition 17.69. The upgraded staroid generated by an anchored relation \(F\) is the staroid \(\uplus \uplus \uparrow^{\Strd} F.\)

Proposition 17.70. \(\uparrow^{\Strd} F = \uplus \uplus \uparrow^{\Strd} F.\)

Proof. Because \(\GR \uparrow^{\Strd} F\) is an upper set.

Conjecture 17.71. Every upgraded principal staroid is a completary staroid.

Conjecture 17.72. Filtrators of staroids on powersets are join-closed.

17.6 Multifuncoids

Definition 17.73. Let \((A; 3_i)\) (where \(i \in n\) for an index set \(n\)) be an indexed family of filtrators. I call a premultifuncoid sketch \(f\) of the form \((A; 3_i)\) the \(n\)-indexed family \(\alpha\) of functions where for every \(i \in n\)

\[\alpha_i: \prod 3_{(\dom A)} \rightarrow A_i.\]

I denote \((f) = \alpha.\)

Definition 17.74. A premultifuncoid sketch on powersets is a premultifuncoid sketch such that every \((A; 3_i)\) is the primary filtrator of filters on a powerset.
Theorem 17.8. Let $\mathcal{A}$ be an indexed family of starrish posets. The prestaroid corresponding to a premultifuncoid is really a prestaroid.

Proof. By the definition of starrish posets.

Definition 17.78. I will call a multifuncoid a premultifuncoid to which corresponds a staroid.

Definition 17.79. I will call a completary multifuncoid a premultifuncoid to which corresponds a completary staroid.

Theorem 17.80. Fix some indexed family $\mathcal{A}$ of boolean lattices. The the set of premultifuncoids $g$ for the filtrator $(\mathcal{F}_i; \mathcal{P}_i)$ bijectively corresponds to set of prestaroids $f$ of form $\mathcal{P} = \lambda i \in \mathcal{A}: \mathcal{P}_i$ by the formulas:

1. $f = [g]$;
2. $\partial(g).L = (\text{val } f).L$ for every $i \in \mathcal{A}$, $L \in \prod_{i \in \mathcal{A}} \mathcal{P}_i(|dom A|)$. 

Proof. Let $f$ be a prestaroid of the form $\mathcal{P}$. If $\alpha$ is defined by the formula $\alpha_i.L = (\text{val } f)_i.L$ then $\partial \alpha_i.L = (\text{val } f)_i.L$. Then

$$L_i \neq \alpha_i.L|_{\text{dom } L \setminus \{i\}} \Leftrightarrow L \in f \Leftrightarrow L_j \neq \alpha_j.L|_{\text{dom } L \setminus \{i\}}.$$ 

For the prestaroid $f'$ defined by the formula $L \in f' \Leftrightarrow L_i \neq \alpha_i.L|_{\text{dom } L \setminus \{i\}}$ we have:

$$L \in f' \Leftrightarrow L_i \in \partial \alpha_i.L|_{\text{dom } L \setminus \{i\}} \Leftrightarrow L_i \in (\text{val } f)_i.L|_{\text{dom } L \setminus \{i\}} \Leftrightarrow L \in f;$$

thus $f' = f$.

Let now $\alpha$ be an indexed family of functions $\alpha_i \in \mathcal{F}_i(|\text{dom } \mathcal{A}| \setminus \{i\})$ conforming to the formula (17.4). Let relation $f$ between posets be defined by the formula $L \in f \Leftrightarrow L_i \neq \alpha_i.L|_{\text{dom } L \setminus \{i\}}$. Then

$$(\text{val } f).L = \{K \in \mathcal{P}_i | K \neq \alpha_i.L|_{\text{dom } L \setminus \{i\}} = \partial \alpha_i.L|_{\text{dom } L \setminus \{i\}}$$

and thus $(\text{val } f).L$ is a core star that is $f$ is a prestaroid. For the indexed family $\alpha'$ defined by the formula $\alpha'_i.L = (\text{val } f)_i.L$ we have

$$\partial \alpha'_i.L = \partial f_i.L = \{K \in \mathcal{P}_i | K \neq \alpha_i.L = \partial \alpha_i.L;$$

thus $\alpha' = \alpha$ (taking into account that $\mathcal{P}_i$ is a boolean lattice).

We have shown that these are bijections.

Definition 17.81. I will denote $\Lambda f$ the premultifuncoid corresponding to a prestaroid $f$ (for an indexed family of boolean lattices) by the above theorem.

Theorem 17.82. Fix some indexed family $\mathcal{F}$ of boolean lattices. $(f)_j(L \cup \{(i; X \cup Y)\}) = (f)_j(L \cup \{(i; X)\} \cup (f)_j(L \cup \{(i; Y)\}))$ for every premultifuncoid $f$ for the family $(\mathcal{F}_i; \mathcal{P}_i)$ of filtrators and $i, j \in \text{arity } f$, $i \neq j$, $L \in \prod_{l \in L \setminus \{i,j\}} \mathcal{F}_l$, $X, Y \in \mathcal{A}_i$.

Proof. Let $i \in \text{arity } f$ and $L \in \prod_{l \in L \setminus \{i,j\}} \mathcal{F}_l$. Let $Z \in \mathcal{F}_i$.

$Z \neq (f)_j(L \cup \{(i; X \cup Y)\}) \Leftrightarrow L \cup \{(i; X \cup Y), (j; Z)\} \in f \Leftrightarrow X \cup Y \in (\text{val } f)_j(L \cup \{(i; Z)\}) \Leftrightarrow X \in (\text{val } f)_j(L \cup \{(j; Z)\}) \Leftrightarrow X \cup Y \in (\text{val } f)_j(L \cup \{(i; Z)\}) \Leftrightarrow X \cup Y \in (\text{val } f)_j(L \cup \{(j; Z)\}) \Leftrightarrow X \cup Y \in (\text{val } f)_j(L \cup \{(i; X, Y)\}) \Leftrightarrow Z \neq (f)_j(L \cup \{(i; X, Y)\})$.

Thus $(f)_j(L \cup \{(i; X \cup Y)\}) = (f)_j(L \cup \{(i; X)\} \cup (f)_j(L \cup \{(i; Y)\})$. 

□
Let us consider the filtrator \( \prod_{i \in \text{arity}} f \mathcal{F}(\text{form } f)_i; \prod_{i \in \text{arity}} f (\text{form } f)_i \).

**Theorem 17.83.** Let \( (\mathcal{A}_i; \mathcal{B}_i) \) be a family of join-closed down-aligned filtrators whose both base and core are join-semilattices. Let \( f \) be a staroid of the form \( \mathcal{B}_i \). Then \( ||f|| \) is a staroid of the form \( \mathcal{A}_i \).

**Proof.** First prove that \( ||f|| \) is a prestaroid. We need to prove that \( 0 \notin (\mathcal{GR}) ||f||_i \) (that is up \( 0 \nsubseteq (\mathcal{GR}) f_i \), that is \( 0 \notin (\mathcal{GR}) f_i \) what is true by the theorem conditions) and that for every \( X, Y \in \mathcal{A}_i \) and \( \mathcal{L} \in \prod_{i \in \text{arity}} (f)_i \mathcal{A}_i \), where \( i \in \text{arity} f \)
\[
\mathcal{L} \cup \{(i; X \cup Y)\} \in \mathcal{GR} ||f|| \iff \mathcal{L} \cup \{(i; X)\} \in \mathcal{GR} ||f|| \vee \mathcal{L} \cup \{(i; Y)\} \in \mathcal{GR} ||f||.
\]
The reverse implication is obvious. Let \( \mathcal{L} \cup \{(i; X \cup Y)\} \in \mathcal{GR} ||f|| \). Then for every \( L \in \mathcal{L} \) and \( X \in X \cup Y \), \( Y \in \mathcal{Y} \) we have and \( X \uparrow Y \supseteq X \cup Y \) thus \( L \cup \{(i; X \cup Y)\} \in \mathcal{GR} f \) and thus
\[
L \cup \{(i; X)\} \in \mathcal{GR} f \vee L \cup \{(i; Y)\} \in \mathcal{GR} f
\]
consequently \( \mathcal{L} \cup \{(i; X)\} \in \mathcal{GR} ||f|| \vee \mathcal{L} \cup \{(i; Y)\} \in \mathcal{GR} ||f|| \). It is left to prove that \( ||f|| \) is an upper set, but this is obvious. \( \square \)

There is a conjecture similar to the above theorems:

**Conjecture 17.84.** \( L \in ||[f]|| \Rightarrow ||[f]|| \prod_{i \in \text{arity } \mathcal{A}} \text{atoms } L_i \neq \emptyset \) for every multifuncoid \( f \) for the filtrator \( (\mathcal{F}^n; \mathcal{P}^n) \).

**Conjecture 17.85.** Let \( \mathcal{U} \) be a set, \( \mathcal{F} \) be the set of filters on \( \mathcal{U} \), \( \mathcal{P} \) be the set of principal filters on \( \mathcal{U} \), let \( n \) be an index set. Consider the filtrator \( (\mathcal{F}^n; \mathcal{P}^n) \). Then if \( f \) is a completary staroid of the form \( \mathcal{P}^n \), then \( ||f|| \) is a completary staroid of the form \( \mathcal{F}^n \).

**Obvious 17.86.** \( (\bigsqcup F)K = \bigsqcup_{f \in F} FK \) for every set \( F \) of premultifuncoid sketches of the same form \( \mathcal{A} \) and \( K \in \prod \mathcal{A} \) whenever \( \bigsqcup_{f \in F} FK \) is defined.

### 17.7 Join of multifuncoids

Premultifuncoid sketches are ordered by the formula \( f \subseteq g \iff (f) \subseteq (g) \) where \( \subseteq \) in the right part of this formula is the product order. I will denote \( \cap, \sqcup, \prod, \bigsqcup \) (without an index) the order poset operations on the poset of premultifuncoid sketches.

**Remark 17.87.** To describe this, the definition of product order is used twice. Let \( f \) and \( g \) be premultifuncoid sketches of the same form \( \mathcal{A} \)
\[
(f) \subseteq (g) \iff \forall i \in \text{dom } \mathcal{A}; (f)_i \subseteq (g)_i \quad \text{and} \quad (f) \subseteq (g) \iff \forall L \in \mathcal{F}(\text{dom } \mathcal{A}) \setminus \{(f)_i \subseteq (g)_i \}.
\]

**Theorem 17.88.** \( f \sqcup^\text{pFCD}(\mathcal{A}) g = f \sqcup g \) for every premultifuncoids \( f \) and \( g \) for the same indexed family of starish join-semilattices filtrators.

**Proof.** \( \alpha_i x = f_i x \sqcup g_i x \). It is enough to prove that \( \alpha \) is a premultifuncoid.

We need to prove:
\[
L_i \neq \alpha_i L_{\{\text{dom } L\} \setminus \{i\}} \iff L_j \neq \alpha_j L_{\{\text{dom } L\} \setminus \{j\}}.
\]

Really, \( L_i \neq \alpha_i L_{\{\text{dom } L\} \setminus \{i\}} \iff L_i \neq f_i L_{\{\text{dom } L\} \setminus \{i\}} \sqcup g_i L_{\{\text{dom } L\} \setminus \{i\}} \iff L_i \neq f_i L_{\{\text{dom } L\} \setminus \{i\}} \vee L_i \neq g_i L_{\{\text{dom } L\} \setminus \{i\}} \iff L_j \neq f_j L_{\{\text{dom } L\} \setminus \{j\}} \sqcup g_j L_{\{\text{dom } L\} \setminus \{j\}} \iff L_j \neq f_j L_{\{\text{dom } L\} \setminus \{j\}} \vee L_j \neq g_j L_{\{\text{dom } L\} \setminus \{j\}}. \quad \square
\]

**Theorem 17.89.** \( \bigsqcup^\text{pFCD}(\mathcal{A}) F = \bigsqcup F \) for every set \( F \) of premultifuncoids for the same indexed family of join infinite distributive complete lattices filtrators.
Proof. $\alpha_i x \overset{\text{def}}{=} \bigcup_{f \in F} f_i x$. It is enough to prove that $\alpha$ is a premultifuncoid.

We need to prove:

\[ L_i \neq \alpha_i L|_{(\text{dom } L) \setminus \{i\}} \iff L_j \neq \alpha_j L|_{(\text{dom } L) \setminus \{j\}}. \]

Really, $L_i \neq \alpha_i L|_{(\text{dom } L) \setminus \{i\}} \iff L_i \neq \bigcup_{f \in F} f_i L|_{(\text{dom } L) \setminus \{i\}} \iff \exists f \in F: L_i \neq f_i L|_{(\text{dom } L) \setminus \{i\}} \iff \exists f \in F: L_j \neq f_j L|_{(\text{dom } L) \setminus \{j\}} \iff L_j \neq \alpha_j L|_{(\text{dom } L) \setminus \{j\}}. \]

\[ \square \]

Proposition 17.90. The mapping $f \mapsto [f]$ is an order embedding, for multifuncoids for indexed families $(\mathfrak{A}_i: \mathfrak{A}_i)$ of down-aligned staroidal filtrators with separable finitely meet-closed core.

Proof. The mapping $f \mapsto [f]$ is defined because $\mathfrak{A}_i$ are staroidal posets and $(\mathfrak{A}_i: \mathfrak{A}_i)$ is with finitely meet-closed core and down-aligned. The mapping is injective because the filtrators are with separable cores $(\{X \in \mathfrak{A}_i \mid X \neq (f)A\} = \{X \in \mathfrak{A}_i \mid X \neq (f)B\})$. Thus $f \mapsto [f]$ is a monotone function is obvious.

\[ \square \]

Remark 17.91. This order embedding is useful to describe properties of posets of prestaroids.

Theorem 17.92. If $f, g$ are multifuncoids for the filtrator $(\mathfrak{F}; \mathfrak{Y}_i)$ where $\mathfrak{Y}_i$ are separable staroidal posets, then $\bigcup_{\mathfrak{F} \in \mathfrak{Y}'(\mathfrak{A})} g \in FCD(\mathfrak{A})$.

Proof. Let $A \in \bigcup_{\mathfrak{F} \in \mathfrak{Y}'(\mathfrak{A})} g$ and $B \supseteq A$. Then for every $k \in \text{dom } \mathfrak{A}$

\[ A_k \neq (f \cup g) A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}; A_k \neq f(A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}) \cup g(A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}). \]

Thus $A_k \neq f(A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}) \cup g(A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}})$.

\[ \square \]

Remark 17.93. If $F$ is a set of multifuncoids for the same indexed family of join infinite distributive complete lattice filtrators, then $\bigcup_{\mathfrak{F} \in \mathfrak{Y}'(\mathfrak{A})} F \in FCD(\mathfrak{A})$.

Proof. Let $A \in \bigcup_{\mathfrak{F} \in \mathfrak{Y}'(\mathfrak{A})} F$ and $B \supseteq A$. Then for every $k \in \text{dom } \mathfrak{A}$

\[ A_k \neq \bigcup_{f \in F} f A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}} \bigcup_{f \in F} f A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}} = \bigcup_{f \in F} f A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}. \]

Thus $\exists f \in F: A_k \neq f A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}; A \in [f]$ for some $f \in F$; $B \in [f]$ for some $f \in F$; $B \neq f A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}} \bigcup_{f \in F} f A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}} \bigcup_{f \in F} f A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}} = \bigcup_{f \in F} f A|_{(\text{dom } \mathfrak{A}) \setminus \{k\}}$. Thus $B \in \bigcup_{\mathfrak{F} \in \mathfrak{Y}'(\mathfrak{A})} F$.

\[ \square \]

Conjecture 17.94. The formula $\bigcup_{\mathfrak{F} \in \mathfrak{Y}'(\mathfrak{A})} g \in cFCD(\mathfrak{A})$ is not true in general for completary multifuncoids (even for completary multifuncoids on powersets) $f$ and $g$ of the same form $\mathfrak{A}$.

17.8 Infinite product of elements and filters

Definition 17.95. Let $\mathcal{A}_i$ be a family of elements of a family $\mathfrak{A}$, of posets. The staroidal product $\prod_{\mathfrak{A}} A_i$ is defined by the formula (for every $L \in \prod_{\mathfrak{A}} A$

\[ \prod_{\mathfrak{A}} A = \mathfrak{A} \quad \text{and} \quad L \in \text{GR} \prod_{\mathfrak{A}} A \iff \forall i \in \text{dom } \mathfrak{A}: A_i \neq L_i. \]

Proposition 17.96. If $\mathfrak{A}_i$ are powerset algebras, staroidal product of principal filters is essentially equivalent to Cartesian product. More precisely, $\prod_{i \in \text{dom } \mathfrak{A}} A_i = \bigcup_{\text{posets } A} A$ for an indexed family $A$ of sets.
Proposition 17.104. Proof. Theorem 17.103. Proof. Corollary 17.102. Proposition 17.101. Definition 17.100. Let \( (\mathfrak{A}_n)_n \) be an indexed family of down-aligned filtrators. Then for every \( A \in \prod \mathfrak{A} \) \( \funcoidal \prod\mathfrak{A} \) \( A \) defined by the formula (for every \( L \in \prod 3 \)): 

\[
\left( \prod_{k} A \right)_{k} = \begin{cases} A_{k} & \text{if } \forall i \in (\text{dom } \mathfrak{A}) \setminus \{k\}: A_{i} \neq L_{i} \\ 0 & \text{otherwise.} \end{cases}
\]

Proposition 17.101. \( \prod^{\Strd(3)} A = [\prod^{\funcoidal(3)} A] \).

Proof. \( L \in \GR \prod^{\Strd(3)} A \Leftrightarrow \forall i \in \text{dom } \mathfrak{A}: A_{i} \neq L_{i} \Leftrightarrow \forall i \in (\text{dom } \mathfrak{A}) \setminus \{k\}: A_{i} \neq L_{i} \land L_{k} \neq A_{k} \Leftrightarrow L_{k} \neq \left( \prod_{k}^{\funcoidal(3)} A \right)_{k} \Leftrightarrow L \in \GR \left[ \prod^{\funcoidal(3)} A \right] \).

Corollary 17.102. Funcoidal product is a compleetary multifuncoid.

Proof. It is enough to prove that funcoidal product is a premultifuncoid. Really, 

\[
L_{i} \neq \left( \prod_{i}^{\funcoidal(3)} A \right)_{i} \Leftrightarrow \forall i \in (\text{dom } \mathfrak{A}) \setminus \{i\}: A_{i} \neq L_{i} \Leftrightarrow \left( \prod_{j}^{\funcoidal(3)} A \right)_{j} \Leftrightarrow L \in \GR \left[ \prod^{\funcoidal(3)} A \right]_{j} \setminus \{i\}.
\]

Theorem 17.103. If our filtrator \( (\prod \mathfrak{A}: 3) \) is with separable core and \( A \in \prod 3 \), then \( \prod^{\Strd(3)} A \) is a down-aligned filtrator. 

Proof. \( \GR \prod^{\Strd(3)} A = \{ L \in \prod \mathfrak{A} | \text{up } L \subseteq \prod^{\Strd(3)} A \} = \{ L \in \prod \mathfrak{A} | \forall K \in \text{up } L, i \in \text{dom } \mathfrak{A}: A_{i} \neq K_{i} \} = \{ L \in \prod \mathfrak{A} | \forall i \in \text{dom } \mathfrak{A}: A_{i} \neq L_{i} \} = \GR \prod^{\Strd(3)} A \).

Proposition 17.104. Let \( (\prod \mathfrak{A}: 3) \) be a meet-closed filtrator, \( A \in \prod 3 \). Then \( \prod^{\Strd(3)} A = \prod^{\Strd(3)} A \).

Proof. \( \GR \prod^{\Strd(3)} A = \GR \prod^{\Strd(3)} A = \{ L \in \prod \mathfrak{A} | \forall i \in \text{dom } \mathfrak{A}: A_{i} \neq L_{i} \} = \{ L \in \prod \mathfrak{A} | \forall i \in \text{dom } \mathfrak{A}: A_{i} \neq L_{i} \} \cap \prod 3 = \{ L \in \prod 3 | \forall i \in \text{dom } \mathfrak{A}: A_{i} \neq L_{i} \} = \GR \prod^{\Strd(3)} A \).
Corollary 17.105. If each \( (\mathfrak{F}_i; \mathcal{P}_i) \) is a powerset filtrator and \( A \in \prod \mathcal{P}_i \), then \( \prod_{\{ \text{Strd}(\mathfrak{F}) \}} A \) is a principal staroid.

Proof. Use the “obvious” fact above. \( \square \)

Theorem 17.106. Let \( \mathfrak{F} \) be a family of sets of filters on distributive lattices with least elements. Let \( a \in \prod \mathfrak{F} \), \( S \in \mathcal{P} \prod \mathfrak{F} \), and every \( \text{Pr}_i, S_i \) be a generalized filter base, \( \prod S = a \). Then

\[
\prod_{\{ \text{Strd}(\mathfrak{F}) \}} a = \prod \left\{ (\text{Strd}(\mathfrak{F})) A \mid A \in S \right\}.
\]

Proof. That \( \prod_{\{ \text{Strd}(\mathfrak{F}) \}} a \) is a lower bound for \( \left\{ (\text{Strd}(\mathfrak{F})) A \mid A \in S \right\} \) is obvious.

Let \( f \) be a lower bound for \( \left\{ (\text{Strd}(\mathfrak{F})) A \mid A \in S \right\} \). Thus \( \forall A \in S : \text{GR} f \subseteq \text{GR} \prod_{\{ \text{Strd}(\mathfrak{F}) \}} A \). Thus for every \( A \in S \) we have \( L \in \text{GR} f \) implies \( \forall i \in \text{dom} \mathfrak{F} ; A_i \neq L_i \). Then, by properties of generalized filter bases, \( \forall i \in \text{dom} \mathfrak{F} ; a_i \neq L_i \) that is \( L \in \text{GR} \prod_{\{ \text{Strd}(\mathfrak{F}) \}} a \).

So \( f \subseteq \prod_{\{ \text{Strd}(\mathfrak{F}) \}} a \). \( \square \)

Conjecture 17.107. Let \( \mathfrak{F} \) be a family of sets of filters on distributive lattices with least elements. Let \( a \in \prod \mathfrak{F} \), \( S \in \mathcal{P} \prod \mathfrak{F} \) be a generalized filter base, \( \prod S = a \), \( f \) is a staroid of the form \( \prod \mathfrak{F} \). Then

\[
\prod_{\{ \text{Strd}(\mathfrak{F}) \}} a \neq f \Leftrightarrow \forall A \in S : \prod_{\{ \text{Strd}(\mathfrak{F}) \}} A \neq f.
\]

17.8.1 On products of staroids

Definition 17.108. \( \prod^{(D)} F = \{ \text{uncurry } z \mid z \in \prod F \} \) (reindexation product) for every indexed family \( F \) of relations.

Definition 17.109. Reindexation product of an indexed family \( F \) of anchored relations is defined by the formulas:

\[
\text{form}^{(D)} F = \text{uncurry}(\text{form} F) \quad \text{and} \quad \text{GR}^{(D)} F = \prod^{(D)} (\text{GR} \circ F).
\]

Obvious 110.

1. \( \text{form}^{(D)} F = \{ (i; j) ; (\text{form} F_i)_j \mid i \in \text{dom} F, j \in \text{arity} F_i \} \);

2. \( \text{GR}^{(D)} F = \{ (i; j) ; (z_i)_j \mid i \in \text{dom} F, j \in \text{arity} F_i \} \mid z \in \prod (\text{GR} \circ F) \} \).

Proposition 111. \( \prod^{(D)} F \) is an anchored relation if every \( F_i \) is an anchored relation.

Proof. We need to prove \( \text{GR}^{(D)} F \subseteq \prod \text{form}(\prod^{(D)} F) \) that is

\[
\{ (i; j) ; (z_i)_j \mid i \in \text{dom} F, j \in \text{arity} F_i \} \mid z \in \prod (\text{GR} \circ F) \} \subseteq \prod \{ (i; j) ; (\text{form} F_i)_j \mid i \in \text{dom} F, j \in \text{arity} F_i \} ;
\]

\( \forall z \in \prod (\text{GR} \circ F), i \in \text{dom} F, j \in \text{arity} F_i ; (z_i)_j \in (\text{form} F_i)_j \); Actually, \( z_i \in \text{GR} F_i \subseteq \prod (\text{form} F_i) \) and thus \( (z_i)_j \in (\text{form} F_i)_j \). \( \square \)

Obvious 112. \( \text{arity}^{(D)} F = \prod_{i \in \text{dom} F} \text{arity} F_i = \{ (i; j) \mid i \in \text{dom} F, j \in \text{arity} F_i \} \).

Definition 113. \( f \times^{(D)} g = \prod^{(D)} [f; g] \).

Lemma 114. \( \prod^{(D)} F \) is an upper set if every \( F_i \) is an upper set.
Proposition 17.115. Let $F$ be an indexed family of anchored relations and every $(\text{form } F)_i$ is a join-semilattice.

1. $\prod^{(D)} F$ is a prestaroid if every $F_i$ is a prestaroid.
2. $\prod^{(D)} F$ is a staroid if every $F_i$ is a staroid.
3. $\prod^{(D)} F$ is a completable staroid if every $F_i$ is a completable staroid.

Proof.

1. Let $q \in \text{arity } \prod^{(D)} F$ that is $q = (i; j)$ where $i \in \text{dom } F$, $j \in \text{arity } F_i$; let

$$ L \in \prod \left( (\text{form } \prod^{(D)} F)_{(\text{arity } \prod^{(D)} F) \setminus \{q\}} \right) $$

that is $L_{(i'; j')} \in \left( \text{form } \prod^{(D)} F \right)_{(i'; j')}$ for every $(i'; j') \in \left( \text{arity } \prod^{(D)} F \right) \setminus \{q\}$, that is $L_{(i'; j')} \in (\text{form } F_i)$. We have $X \in \left( \text{form } \prod^{(D)} F \right)_{(i; j)} \Leftrightarrow X \in (\text{form } F_i)_j$. So

$$ \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\text{form } F_i)_j \mid L \cup \{(i; j); X\} \in \text{GR } \prod^{(D)} F \right\}; $$

$$ \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\text{form } F_i)_j \mid \exists z \in \prod \left( \text{GR } \circ F \right)_{\{(i; j); X\}} = \text{uncurry } z \right\}; $$

$$ \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\text{form } F_i)_j \mid \exists z \in \prod \left( \left( \text{GR } \circ F \right)_{\{(i; j); X\}} \right) : v \in \text{GR } F_i; \right. \left. \begin{array}{l} (L = \text{uncurry } z \land v_j = X) \end{array} \right\}; $$

If $\exists z \in \prod \left( \left( \text{GR } \circ F \right)_{\{(i; j); X\}} \right) : v \in \text{GR } F_i; v_j = X$ then

$$ \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L = \emptyset $$

is a free star. We can assume it is true. So

$$ \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\text{form } F_i)_j \mid \exists v \in \text{GR } F_i; v_j = X \right\}. $$

Thus

$$ \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\text{form } F_i)_j \mid \exists K \in (\text{form } F_i)_{\{(i; j)\}}; K \cup \{(j; X)\} \in \text{GR } F_i \right\} = \left\{ X \in (\text{form } F_i)_j \mid \exists K \in (\text{form } F_i)_{\{(i; j)\}}; X \in (\text{val } F_i)_j; K \right\}. $$

Thus $A \cup B \in \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L \Leftrightarrow \exists K \in (\text{form } F_i)_{\{(i; j)\}} ; A \cup B \in (\text{val } F_i)_j K \Leftrightarrow \exists K \in (\text{form } F_i)_{\{(i; j)\}} ; A \cup B \in (\text{val } F_i)_j L \cup B \in \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L \Leftrightarrow \exists K \in (\text{form } F_i)_{\{(i; j)\}} ; B \in (\text{val } F_i)_j K \Leftrightarrow A \in \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L \cup B \in \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L \Leftrightarrow A \in \left( \text{val } \prod^{(D)} F \right)_{(i; j)} L \Leftrightarrow K \cup \{(j; X)\} \not\in \text{GR } F_i. $
2. From the lemma.
3. We need to prove

\[ L_0 \sqcup L_1 \in \text{GR} \prod^{(D)}_i \! F \Leftrightarrow \exists c \in \{0, 1\}^{\text{arity} \prod^{(D)}_i \! F} : \left( \lambda i \in \text{arity} \prod^{(D)}_i \! F : L_{c(i)} \right) \in \text{GR} \prod^{(D)}_i \! F \]

for every \( L_0, L_1 \in \prod \text{form} \prod^{(D)}_i \! F \) that is \( L_0, L_1 \in \prod \text{uncurry}(\text{form} \circ F) \).

Really \( L_0 \sqcup L_1 \in \text{GR} \prod^{(D)}_i \! F \Leftrightarrow L_0 \sqcup L_1 \in \{\text{uncurry} z \mid z \in \prod (\text{GR} \circ F)\} \).

\[ \exists c \in \{0, 1\}^{\text{arity} \prod^{(D)}_i \! F} : \left( \lambda i \in \text{arity} \prod^{(D)}_i \! F : L_{c(i)} \right) \in \text{GR} \prod^{(D)}_i \! F \Leftrightarrow \exists c \in \{0, 1\}^{\text{arity} \prod^{(D)}_i \! F} : \text{curry} \left( \lambda i \in \text{arity} \prod^{(D)}_i \! F : L_{c(i)} \right) \in \prod (\text{GR} \circ F) \Leftrightarrow \exists c \in \{0, 1\}^{\text{arity} \prod^{(D)}_i \! F} : \text{curry} \left( \lambda i \in \text{arity} \prod^{(D)}_i \! F : L_{c(i)} \right) \in \prod (\text{GR} \circ F) \]

For staroids it is defined \textit{ordinated product} \( \prod^{(\text{ord})}_i \! F \) as defined in the section “Ordinated product” above.

\textbf{Obvious 17.116.} If \( f \) and \( g \) are anchored relations and there exists a bijection \( \varphi \) from \( \text{arity} \, f \) to \( \text{arity} \, g \) such that \( \{F \circ \varphi \mid F \in \text{GR} \, f\} = \text{GR} \, g \), then:

1. \( f \) is a prestaroid iff \( g \) is a prestaroid.
2. \( f \) is a staroid iff \( g \) is a staroid.
3. \( f \) is a completary staroid iff \( g \) is a completary staroid.

\textbf{Corollary 17.117.} Let \( F \) be an indexed family of anchored relations and every \((\text{form} \, F)_i\) is a join-semi-lattice.

1. \( \prod^{(\text{ord})}_i \! F \) is a prestaroid if every \( F_i \) is a prestaroid.
2. \( \prod^{(\text{ord})}_i \! F \) is a staroid if every \( F_i \) is a staroid.
3. \( \prod^{(\text{ord})}_i \! F \) is a completary staroid if every \( F_i \) is a completary staroid.

\textbf{Proof.} Use the fact that \( \text{GR} \prod^{(\text{ord})}_i \! F = \{F \circ (\bigoplus (\text{dom} \circ F))^{-1} \mid F \in \text{GR} \prod^{(D)}_i \! F\} \).

\textbf{Definition 17.118.} \( f \times^{(\text{ord})}_i \! g = \prod^{(\text{ord})}_i \! [f; g] \).

\textbf{Remark 17.119.} If \( f \) and \( g \) are binary funcoids, then \( f \times^{(\text{ord})}_i \! g \) is ternary.

\textbf{Proposition 17.120.} \( \prod^{\text{Strd}}_i \! a = \prod \prod^{\text{Strd}} \! a \) if each \( a_i \in \mathfrak{A}_i \) (for \( i \in n \) where \( n \) is some index set) where each \((\mathfrak{A}_i \in n; \mathfrak{3}_i \in n)\) is a down aligned filtrator with separable core.

\textbf{Proof.} \( \text{GR} \prod \prod^{\text{Strd}}_i \! a = \{L \in \prod \mathfrak{A} \mid \text{up} \, L \subseteq \mathfrak{3}_i \cap \prod^{\text{Strd}} \! a\} = \{L \in \prod \mathfrak{A} \mid \text{up} \, L \subseteq \prod^{\text{Strd}} \! a\} = \{L \in \prod \mathfrak{A} \mid \forall K \in \text{up} \, L : K \in \prod^{\text{Strd}}_i \! a\} = \{L \in \prod \mathfrak{A} \mid \forall K \in \text{up} \, L, i \in n : K_i \neq a_i\} = \{L \in \prod \mathfrak{A} \mid \forall i \in n, K \in \text{up} \, L : K_i \neq a_i\} = \{L \in \prod \mathfrak{A} \mid \forall i \in n : L_i \neq a_i\} = \text{GR} \prod^{\text{Strd}}_i \! a \) (taken into account that our filtrator is with separable core).
17.9 Star categories

**Definition 17.121.** A precategory with star-morphisms consists of
1. a precategory \( C \) (the base precategory);
2. a set \( M \) (star-morphisms);
3. a function “arity” defined on \( M \) (how many objects are connected by this star-morphism);
4. a function \( \text{Obj}_m; \) arity \( m \mapsto \text{Obj}(C) \) defined for every \( m \in M \);
5. a function \((\text{star composition}) \) \( (m; f) \mapsto \text{StarComp}(m; f) \) defined for \( m \in M \) and \( f \) being an (arity \( m \))-indexed family of morphisms of \( C \) such that \( \forall i \in \text{arity } m \colon \text{Src } f_i = \text{Obj}_m i \) (\( \text{Src } f_i \) is the source object of the morphism \( f_i \)) such that \( \text{arity } \text{StarComp}(m; f) = \text{arity } m \)
such that it holds:
1. \( \text{StarComp}(m; f) \in M \);
2. \((\text{associativity law})\)

\[ \text{StarComp}(\text{StarComp}(m; f); g) = \text{StarComp}(m; \lambda i \in \text{arity } m \colon g_i \circ f_i). \]

(Here by definition \( \lambda x \in D \colon F(x) = \{(x; F(x)) \mid x \in D\}. \)

The meaning of the set \( M \) is an extension of \( C \) having as morphisms things with arbitrary (possibly infinite) indexed set \( \text{Obj}_m \) of objects, not just two objects as morphisms of \( C \) have only source and destination.

**Definition 17.122.** I will call \( \text{Obj}_m \) the form of the star-morphism \( m \).

(Having fixed a precategory with star-morphisms) I will denote \( \text{StarMor}(P) \) the set of star-morphisms of the form \( P \).

**Proposition 17.123.** The sets \( \text{StarMor}(P) \) are disjoint (for different \( P \)).

**Proof.** If two star-morphisms have different forms, they are clearly not equal. \( \square \)

**Definition 17.124.** A category with star-morphisms is a precategory with star-morphisms whose base is a category and the following equality \((\text{the law of composition with identity})\) holds for every star-morphism \( m \):

\[ \text{StarComp}(m; \lambda i \in \text{arity } m \colon \text{id}_{\text{Obj}_m i}) = m. \]

**Definition 17.125.** A partially ordered precategory with star-morphisms is a category with star-morphisms, whose base precategory is a partially ordered precategory and every set

\[ \{m \in M \mid \text{Obj}_m = X\} \]

is partially ordered for every \( X \), such that:
1. \( m_0 \subseteq m_1 \land f_0 \subseteq f_1 \Rightarrow \text{StarComp}(m_0; f_0) \subseteq \text{StarComp}(m_1; f_1) \) for every \( m_0, m_1 \in M \) such that \( \text{Obj}_{m_0} = \text{Obj}_{m_1} \) and indexed families \( f_0 \) and \( f_1 \) of morphisms such that

\[ \forall i \in \text{arity } m \colon \text{Src } f_0 i = \text{Src } f_1 i = \text{Obj}_{m_0} i = \text{Obj}_{m_1} i \quad \text{and} \quad \forall i \in \text{arity } m \colon \text{Dst } f_0 i = \text{Dst } f_1 i. \]

**Definition 17.126.** A quasi-invertible precategory with star-morphisms is a partially ordered precategory with star-morphisms whose base precategory is a quasi-invertible precategory, such that for every index set \( n \), star-morphisms \( a \) and \( b \) of arity \( n \), and an \( n \)-indexed family \( f \) of morphisms of the base precategory it holds

\[ b \neq \text{StarComp}(a; f) \iff a \neq \text{StarComp}(b; f^\dagger). \]

(Here \( f^\dagger = \lambda i \in \text{dom } f \colon (f_i)^{\dagger}. \))
Definition 17.127. A quasi-invertible category with star-morphisms is a quasi-invertible precategory with star-morphisms which is a category with star-morphisms. Each category with star-morphisms gives rise to a category (abrupt category, see a remark below why I call it “abrupt”), as described below. Below for simplicity I assume that the set \( M \) and the set of our indexed families of functions are disjoint. The general case (when they are not necessarily disjoint) may be easily elaborated by the reader.

- Objects are indexed (by arity \( m \) for some \( m \in M \)) families of objects of the category \( C \) and an (arbitrarily chosen) object \( \text{None} \) not in this set.
- There are the following disjoint sets of morphisms:
  1. indexed (by arity \( m \) for some \( m \in M \)) families of morphisms of \( C \);
  2. elements of \( M \);
  3. the identity morphism \( \text{id}_{\text{None}} \) on \( \text{None} \).

- Source and destination of morphisms are defined by the formulas:
  \[
  \text{Src} f = \lambda i \in \text{dom} f : \text{Src} f_i ;
  \]
  \[
  \text{Dst} f = \lambda i \in \text{dom} f : \text{Dst} f_i ;
  \]
  \[
  \text{Src} m = \text{None} ;
  \]
  \[
  \text{Dst} m = \text{Obj} m .
  \]

- Compositions of morphisms are defined by the formulas:
  \[
  g \circ f = \lambda i \in \text{dom} f : g_i \circ f_i \text{ for our indexed families } f \text{ and } g \text{ of morphisms} ;
  \]
  \[
  f \circ m = \text{StarComp}(m; f) \text{ for } m \in M \text{ and a composable indexed family } f ;
  \]
  \[
  m \circ \text{id}_{\text{None}} = m \text{ for } m \in M ;
  \]
  \[
  \text{id}_{\text{None}} \circ \text{id}_{\text{None}} = \text{id}_{\text{None}} .
  \]

- Identity morphisms for an object \( X \) are:
  \[
  \lambda i \in X : \text{id}_X \text{ if } X \neq \text{None} ;
  \]
  \[
  \text{id}_{\text{None}} \text{ if } X = \text{None} .
  \]

We need to prove it is really a category.

Proof. We need to prove:

1. Composition is associative.
2. Composition with identities complies with the identity law.

Really:

1. \((h \circ g) \circ f = \lambda i \in \text{dom} f : (h_i \circ g_i) \circ f_i = \lambda i \in \text{dom} f : h_i \circ (g_i \circ f_i) = h \circ (g \circ f) ;
   g \circ (f \circ m) = \text{StarComp}(\text{StarComp}(m; f); g) = \text{StarComp}(m; \lambda i \in \text{arity} m : g_i \circ f_i) =
   \text{StarComp}(m; g \circ f) = (g \circ f) \circ m ;
   f \circ (m \circ \text{id}_{\text{None}}) = f \circ m = (f \circ m) \circ \text{id}_{\text{None}} .
   \]
2. \(m \circ \text{id}_{\text{None}} = m ; \text{id}_{\text{Dst} m} \circ m = \text{StarComp}(m; \lambda i \in \text{arity} m : \text{id}_{\text{Obj} m . i}) = m .
   \]

Remark 17.128. I call the above defined category abrupt category because (excluding identity morphisms) it allows composition with an \( m \in M \) only on the left (not on the right) so that the morphism \( m \) is “abrupt” on the right.

By \([x_0 ; \ldots ; x_{n-1}]\) I denote an \( n \)-tuple.

Definition 17.129. Precategory with star morphisms induced by a dagger precategory \( C \) is:

- The base category is \( C \).
- Star-morphisms are morphisms of \( C \).
• arity \( f = \{0, 1\} \).
• \( \text{Obj}_m = [\text{Src} \ m; \text{Dst} \ m] \).
• \( \text{StarComp}(m; [f; g]) = g \circ m \circ f^\dagger \).

Let prove it is really a precategory with star-morphisms.

**Proof.** We need to prove the associativity law:

\[
\text{StarComp}(\text{StarComp}(m; [f; g]); [p; q]) = \text{StarComp}(m; [p \circ f; q \circ g]).
\]

Really,
\[
\text{StarComp}(\text{StarComp}(m; [f; g]); [p; q]) = \text{StarComp}(g \circ m \circ f^\dagger; [p; q]) = q \circ g \circ m \circ f^\dagger \circ p^\dagger = q \circ g \circ m \circ (p \circ f)^\dagger = \text{StarComp}(m; [p \circ f; q \circ g]). \tag*{\square}
\]

**Definition 17.130.** Category with star morphisms *induced* by a dagger category \( C \) is the above defined precategory with star-morphisms.

That it is a category (the law of composition with identity) is trivial.

**Remark 17.131.** We can carry definitions (such as below defined cross-composition product) from categories with star-morphisms into plain dagger categories. This allows us to research properties of cross-composition product of indexed families of morphisms for categories with star-morphisms without separately considering the special case of dagger categories and just binary star-composition product.

### 17.9.1 Abrupt of quasi-invertible categories with star-morphisms

**Definition 17.132.** The abrupt partially ordered precategory of a partially ordered precategory with star-morphisms is the abrupt precategory with the following order of morphisms:

- Indexed (by arity \( m \) for some \( m \in M \)) families of morphisms of \( C \) are ordered as function spaces of posets.
- Star-morphisms (which are morphisms \( \text{None} \to \text{Obj}_m \) for some \( m \in M \)) are ordered in the same order as in the precategory with star-morphisms.
- Morphisms \( \text{None} \to \text{None} \) which are only the identity morphism ordered by the unique order on this one-element set.

We need to prove it is a partially ordered precategory.

**Proof.** It trivially follows from the definition of partially ordered precategory with star-morphisms.

### 17.10 Product of an arbitrary number of funcoids

In this section it will be defined a product of an arbitrary (possibly infinite) family of funcoids.

#### 17.10.1 Mapping a morphism into a pointfree funcoid

**Definition 17.133.** Let’s define the pointfree funcoid \( \chi f \) for every morphism \( f \) or a quasi-invertible category:

\[
(\chi f)a = f \circ a \quad \text{and} \quad ((\chi f)^{-1})b = f^\dagger \circ b.
\]

We need to prove it is really a pointfree funcoid.
Proof. \(b \neq (\chi f)a \iff b \neq f \circ a \iff a \neq f^\dagger \circ b \iff a \neq ((\chi f)^{-1})b.\) \(\square\)

Remark 17.134. \((\chi f) = (f \circ -)\) is the \(\text{Mor-}\)functor \(^{\text{17.1}}\) \(\text{Mor}(f, -)\) and we can apply Yoneda lemma to it. (See any category theory book for definitions of these terms.)

Obvious 17.135. \((\chi (g \circ f))a = g \circ f \circ a\) for composable morphisms \(f\) and \(g\) or a quasi-invertible category.

17.10.2 General cross-composition product

Let fix a quasi-invertible category with with star-morphisms. If \(f\) is an indexed family of morphisms from its base category, then the pointfree funcoid \(\prod (f)\) from \(\text{StarMor}(\mathcal{C}, \alpha)\) to \(\text{StarMor}(\mathcal{C}, \alpha)\) is defined by the formulas (for all star-morphisms \(a, b\) of these forms):

\[
\begin{align*}
\prod (f) a &= \text{StarComp}(a; f) \quad \text{and} \\
\left(\prod (f)^{-1}\right) b &= \text{StarComp}(b; f^\dagger).
\end{align*}
\]

It is really a pointfree funcoid by the definition of quasi-invertible category with star-morphisms.

Theorem 17.136. \((\prod (f) g) \circ (\prod (f) f) = \prod (f^\dagger) (g, f_i)\) for every \(n\)-indexed families \(f\) and \(g\) of composable morphisms of a quasi-invertible category with star-morphisms.

Proof. \((\prod (f) g) \circ (\prod (f) f) = \prod (f) (g, f_i) = \text{StarComp}(\text{StarComp}(a; f); g)\) and \((\prod (f) g) \circ (\prod (f) f) = \prod (f) (g, f_i) = \text{StarComp}(\text{StarComp}(a; f); g)\).

The rest follows from symmetry. \(\square\)

Corollary 17.137. \((\prod (f) f_0 ... f_n) = \prod (f) (f_0 ... f_n)\) for every \(n\)-indexed families \(f_0, ..., f_{n-1}\) of composable morphisms of a quasi-invertible category with star-morphisms.

Proof. By math induction. \(\square\)

17.10.3 Star composition of binary relations

First define star composition for an \(n\)-ary relation \(a\) and an \(n\)-indexed family \(f\) of binary relations as an \(n\)-ary relation complying with the formulas:

\[
\begin{align*}
\text{ObjStarComp}(a; f) &= \{\ast\}^n; \\
L \in \text{StarComp}(a; f) &\iff \exists y \in a \forall i \in n: y_i f_i L_i
\end{align*}
\]

where \(\ast\) is a unique object of the group of small binary relations considered as a category.

Proposition 17.138. \(b \neq \text{StarComp}(a; f) \iff \exists x \in a, y \in b \forall j \in n: x_j f_j y_j.\)

Proof. \(b \neq \text{StarComp}(a; f) \iff \exists y: (y \in b \land y \in \text{StarComp}(a; f)) \iff \exists y: (y \in b \land \exists x \in a \forall j \in n: x_j f_j y_j) \iff \exists x \in a, y \in b \forall j \in n: x_j f_j y_j.\) \(\square\)

Theorem 17.139. The group of small binary relations considered as a category together with the set of all \(n\)-ary relations (for every small \(n\)) and the above defined star-composition form a quasi-invertible category with star-morphisms.

Proof. We need to prove:

1. \(\text{StarComp}(\text{StarComp}(m; f); g) = \text{StarComp}(m; a \in n: g_i f_i);\)

\(^{17.1}\) Also called Hom-functor.
2. StarComp\((m; \lambda i \in \text{arity } m; \text{id}_{\text{Obj}_m} i) = m\);

3. \(b \neq \text{StarComp}(a; f) \Leftrightarrow a \neq \text{StarComp}(b; f')\)

(the rest is obvious).

Really,

1. \(L \in \text{StarComp}(a; f) \Leftrightarrow \exists y \in a \forall i \in n: y_i f_i L_i\).

Define the relation \(R(f)\) by the formula \(x R(f) y \Leftrightarrow \forall i \in n: x_i f_i y_i\). Obviously

\(R(\lambda i \in n: g_i \circ f_i) = R(g) \circ R(f)\).

\(L \in \text{StarComp}(a; f) \Leftrightarrow \exists y \in a: y R(f) L\).

\(L \in \text{StarComp}(\text{StarComp}(a; f); g) \Leftrightarrow \exists p \in \text{StarComp}(a; f): p R(g) L \Leftrightarrow \exists p, y \in a:\ (y R(f) p \land p R(g) L) \Leftrightarrow \exists y \in a: y(R(g) \circ R(f)) L \Leftrightarrow \exists y \in a: y R(\lambda i \in n: g_i \circ f_i) L \Leftrightarrow \)

\(L \in \text{StarComp}(a; \lambda i \in n: g_i \circ f_i)\) because \(p \in \text{StarComp}(a; f) \Leftrightarrow \exists y \in a: y R(f) p\).

2. Obvious.

3. It follows from the proposition above. \(\square\)

**Obvious 17.140.** \(\text{StarComp}(a \cup b; f) = \text{StarComp}(a; f) \cup \text{StarComp}(b; f)\) for \(n\)-ary relations \(a, b\) and an \(n\)-indexed family \(f\) of binary relations.

**Theorem 17.141.** \(\prod_{i \in C} f_i \prod_{i \in n} (f_i) a_i \) for every family \(f = f_i \in n\) of binary relations and \(a = a_i \in n\) where \(a_i\) is a small set (for each \(i \in n\)).

**Proof.** \(L \in \prod_{i \in C} f_i \prod_{i \in n} (f_i) a_i \Leftrightarrow L \in \text{StarComp}(\prod_{i \in C} f_i \prod_{i \in n} (f_i) a_i) \Leftrightarrow \exists y \in \prod_{i \in n} a_i \forall i \in n: y_i f_i L_i \Leftrightarrow \exists y \in \prod_{i \in n} a_i \forall i \in n: \{y_i\} \neq \{f_i^{-1}\} \{L_i\} \Leftrightarrow \forall i \in n: a_i = f_i^{-1}(L_i) \Leftrightarrow \forall i \in n: \{L_i\} \neq \{f_i\} a_i \Leftrightarrow L_i = (f_i) a_i \Leftrightarrow L \in \prod_{i \in n} (f_i) a_i. \square\)

**17.10.4 Star composition of Rel-morphisms**

Define **star composition** for an \(n\)-ary anchored relation \(a\) and an \(n\)-indexed family \(f\) of Rel-morphisms as an \(n\)-ary anchored relation complying with the formulas:

\[
\text{Obj}_{\text{StarComp}(a; f)} = \lambda i \in \text{arity } a: \text{Dst } f_i; \\
\text{arity } \text{StarComp}(a; f) = \text{arity } a; \\
L \in \text{GR } \text{StarComp}(a; f) \Leftrightarrow L \in \text{StarComp}(\text{GR } a; \text{GR } f).
\]

(Here I denote \(\text{GR}(A; B; f) = f\) for every Rel-morphism \(f\).)

**Proposition 17.142.** \(b \neq \text{StarComp}(a; f) \Leftrightarrow \exists x \in a, y \in b \forall j \in n: x_j f_j y_j\).

**Proof.** From the previous section. \(\square\)

**Theorem 17.143.** Relations with above defined compositions form a quasi-invertible category with star-morphisms.

**Proof.** We need to prove:

1. \(\text{StarComp}(\text{StarComp}(m; f); g) = \text{StarComp}(m; \lambda i \in \text{arity } m: g_i \circ f_i)\);

2. \(\text{StarComp}(m; \lambda i \in \text{arity } m: \text{id}_{\text{Obj}_m} i) = m\);

3. \(b \neq \text{StarComp}(a; f) \Leftrightarrow a \neq \text{StarComp}(b; f')\)

(the rest is obvious).

It follows from the previous section. \(\square\)

**Proposition 17.144.** \(\text{StarComp}(a \cup b; f) = \text{StarComp}(a; f) \cup \text{StarComp}(b; f)\) for an \(n\)-ary anchored relations \(a, b\) and an \(n\)-indexed family \(f\) of Rel-morphisms.
Proof. It follows from the previous section. \(\Box\)

**Theorem 17.145.** Cross-composition product of a family of Rel-morphisms is a principal funcoid.

**Proof.** By the proposition and symmetry \(\prod^{(C)} f\) is a pointfree funcoid. Obviously it is a funcoid \(\prod_{i \in n} \text{Src} f_i \rightarrow \prod_{i \in n} \text{Dst} f_i\). Its completeness (and dually co-completeness) is obvious. \(\Box\)

### 17.10.5 Cross-composition product of funcoids

Let \(a\) be an an anchored relation of the form \(\mathbb{A}\) and \(\text{dom} \mathbb{A} = n\).

Let every \(f_i\) (for all \(i \in n\)) be a pointfree funoid with \(\text{Src} f_i = \mathbb{A}_i\).

The star-composition of \(a\) with \(f\) is an anchored relation of the form \(\lambda i \in \text{dom} \mathbb{A}: \text{Dst} f_i\) defined by the formula

\[
L \in \text{GR StarComp}(a; f) \iff \exists y \in \text{GR} a \cap \prod_{i \in n} \text{atoms}^\mathbb{A}_i \forall i \in n: y_i [f_i] L_i.
\]

**Theorem 17.146.** Let \(\text{Dst} f_i\) be a starrish join-semilattice for every \(i \in n\).

1. If \(a\) is a staroid then \(\text{StarComp}(a; f)\) is a staroid.
2. If \(a\) is a completary staroid and then \(\text{StarComp}(a; f)\) is a completary staroid.

**Proof.**

1. First prove that \(\text{StarComp}(a; f)\) is a staroid. We need to prove that \((\text{val StarComp}(a; f))_L\) (for every \(j \in n\)) is a free star, that is

\[
\{ X \in (\text{form } f)_j \mid L \cup \{(j; X)\} \in \text{GR StarComp}(a; f) \}
\]

is a free star, that is the following is a free star

\[
\{ X \in (\text{form } f)_j \mid R(X) \}
\]

where \(R(X) \iff \exists y \in \prod_{i \in n} \text{atoms}^\mathbb{A}_i: (\forall i \in n \setminus \{j\}: y_i [f_i] L_i \land y_j [f_j] X \land y \in \text{GR} a).

\[
R(X) \iff \exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^\mathbb{A}_i: (\forall i \in n \setminus \{j\}: y_i [f_i] L_i \land y_j [f_j] X \land y \in (\text{val } a)_j(y_{i \setminus \{j\}})) \iff \exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^\mathbb{A}_i: (\forall i \in n \setminus \{j\}: y_i [f_i] L_i \land y' [f_j] X \land y' \in (\text{val } a)_j(y_{i \setminus \{j\}})).
\]

If \(\exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^\mathbb{A}_i: \forall i \in n \setminus \{j\}: y_i [f_i] L_i\) is false our statement is obvious. We can assume it is true.

So it is enough to prove that

\[
\left\{ X \in (\text{form } f)_j \mid \exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^\mathbb{A}_i, y' \in \text{atoms}^\mathbb{A}_j: (y' [f_j] X \land y' \in (\text{val } a)_j(y_{i \setminus \{j\}})) \right\}
\]

is a free star. That is

\[
Q = \left\{ X \in (\text{form } f)_j \mid \exists y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^\mathbb{A}_i, y' \in \text{atoms}^\mathbb{A}_j \cap (\text{val } a)_j(y_{i \setminus \{j\}}): y' [f_j] X \right\}
\]

is a free star. \((\text{form } f)_j \notin Q\) is obvious. That \(Q\) is an upper set is obvious. It remains to prove that \(X_0 \cup X_1 \in Q \Rightarrow X_0 \in Q \lor X_1 \in Q\) for every \(X_0, X_1 \in (\text{form } f)_j\). Let \(X_0 \cup X_1 \in Q\). Then there exist \(y \in \prod_{i \in n \setminus \{j\}} \text{atoms}^\mathbb{A}_i, y' \in \text{atoms}^\mathbb{A}_j \cap (\text{val } a)_j(y_{i \setminus \{j\}})\) such that \(y' [f_j] X_0 \lor y' [f_j] X_1\). Consequently (proposition 15.16) \(y' [f_j] X_0 \lor y' [f_j] X_1\). But then \(X_0 \in Q \lor X_1 \in Q\).

To finish the proof we need to show that \(\text{GR StarComp}(a; f)\) is an upper set, but this is obvious.
2. Let $a$ be a completary staroid. Let $L_0 \sqcup L_1 \in \text{GR StarComp}(a; f)$ that is $\exists y \in \prod_{i \in n} \text{atoms } \mathfrak{A}_i$: $(\forall i \in n: y_i [f_i] L_0 i \sqcup L_1 i \wedge y \in a)$ that is $\exists c \in \{0, 1\}^n, y \in \prod_{i \in n} \text{atoms } \mathfrak{A}_i$: $(\forall i \in n: y_i [f_i] L_{c(i)} i \wedge y \in a)$ (taken into account that Dst $f_i$ is starrish) that is $\exists c \in \{0, 1\}^n$: $(\lambda i \in n: L_{c(i)} i) \in \text{GR StarComp}(a; f)$. So StarComp$(a; f)$ is a completary staroid.

Lemma 17.147. $b \not\in \text{Anch}(\mathfrak{A}) \text{ StarComp}(a; f) \Leftrightarrow \forall A \in \text{GR } a, B \in \text{GR } b, i \in n: A_i [f_i] B_i$ for anchored relations $a$ and $b$, provided that Src $f_i$ are atomic posets.

Proof.

$$b \not\in \text{Anch}(\mathfrak{A}) \text{ StarComp}(a; f) \Leftrightarrow$$

$$\exists x \in \text{Anch}(\mathfrak{A}) \setminus \{0\}: (x \sqsubseteq b \wedge x \not\subseteq \text{ StarComp}(a; f)) \Leftrightarrow$$

$$\exists x \in \text{Anch}(\mathfrak{A}) \setminus \{0\}: (x \sqsubseteq b \wedge \forall B \in \text{GR } x: B \in \text{GR StarComp}(a; f)) \Leftrightarrow$$

$$\exists x \in \text{Anch}(\mathfrak{A}) \setminus \{0\}: (x \sqsubseteq b \wedge \forall B \in \text{GR } x \exists A \in \prod_{i \in \text{dom } \mathfrak{A}} \text{ atoms } \mathfrak{A}_i: (\forall i \in n: A_i [f_i] B_i \wedge A \in \text{GR } a)) \Leftrightarrow$$

$$\exists x \in \text{Anch}(\mathfrak{A}) \setminus \{0\}: (x \sqsubseteq b \wedge \forall B \in \text{GR } x, A \in \text{GR } a, i \in n: A_i [f_i] B_i) \Leftrightarrow$$

$$\forall B \in \text{GR } b, A \in \text{GR } a, i \in n: A_i [f_i] B_i.$$  

Theorem 17.148. $a \left[ \prod_{i \in n} (C_f) \right] b \Leftrightarrow \forall A \in a, B \in b, i \in n: A_i [f_i] B_i$ for anchored relations $a$ and $b$, provided that Src $f_i$ and Dst $f_i$ are atomic posets.

Proof. From the lemma.

Conjecture 17.149. $b \not\in \text{pStrd}(\mathfrak{A}) \text{ StarComp}(a; f) \Leftrightarrow b \not\in \text{pStrd}(\mathfrak{A}) \text{ StarComp}(a; f)$ for staroids $a$ and $b$.

Theorem 17.150. Anchored relations with objects being atomic posets and above defined compositions form a quasi-invertible category with star-morphisms.

Proof. We need to prove:

1. StarComp$(\text{StarComp}(m; f); g) = \text{StarComp}(m; \lambda i \in \text{arity } m: g_i \circ f_i)$;

2. StarComp$(m; \lambda i \in \text{arity } m: \text{id}_{\text{Obj}(m)} i) = m$;

3. $b \not\in \text{StarComp}(a; f) \Leftrightarrow a \not\in \text{StarComp}(b; f^1)$

(the rest is obvious).

Really,

1. $L \in \text{GR StarComp}(a; f) \Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{ atoms } \mathfrak{A}_i: y_i [f_i] L_i$.

Define the relation $R(f)$ by the formula $x R(f) y \Leftrightarrow \forall i \in n: x_i [f_i] y_i$. Obviously

$$R(\lambda i \in n: g_i \circ f_i) = R(g) \circ R(f).$$

$L \in \text{GR StarComp}(a; f) \Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{ atoms } \mathfrak{A}_i: y R(f) L \Leftrightarrow$

$L \in \text{GR StarComp}(\text{StarComp}(a; f); g) \Leftrightarrow \exists p \in \text{GR StarComp}(a; f) \cap \prod_{i \in n} \text{ atoms } \mathfrak{A}_i: p R(g) L \Leftrightarrow \exists p \in \text{GR } a \cap \prod_{i \in n} \text{ atoms } \mathfrak{A}_i: (y R(f)) p \wedge p R(g) L \Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{ atoms } \mathfrak{A}_i: y R(\lambda i \in n: g_i \circ f_i) L \Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{ atoms } \mathfrak{A}_i: y R(g) \circ R(f) L \Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{ atoms } \mathfrak{A}_i: y R(\lambda i \in n: f_i \circ f_i) L \Leftrightarrow \exists y \in \text{GR } a \cap \prod_{i \in n} \text{ atoms } \mathfrak{A}_i: y R(f) p$.

2. Obvious.

3. It follows from the lemma above.

Theorem 17.151. $\left[ \prod_{i \in n} (C_f) \right] \prod_{i \in n} \text{Strd } a = \prod_{i \in n} \text{Strd } (f_i) a_i$ for every families $f = f_i \in n$ of pointfree funcoids between atomic posets and $a = a_i \in n$ where $a_i \in \text{Src } f_i$. 

**Proof.** $L \in \text{GR}\left( \prod_{i \in \text{dom} f}^{(C)} f \right) \prod_{a \in \text{Dom} a}^{\text{Std}} a \Leftrightarrow L \in \text{GR StarComp}\left( \prod_{a}^{\text{Std}} a ; f \right) \Leftrightarrow \exists y \in \prod_{i \in \text{dom} \mathfrak{X}} \text{atoms}^{\mathfrak{X}} ; \forall i \in n: (y_i \mid f_i \mid L_i \wedge y_i \neq a_i) \Leftrightarrow \forall i \in n: a_i \mid f_i \mid L_i \Leftrightarrow \forall i \in n: L_i \neq (f_i)_{a_i} \Leftrightarrow L \in \text{GR} \prod_{i \in \text{dom}}^{\text{Std}} (f_i)_{a_i}.$

**Conjecture 17.152.** StarComp($a \cup b ; f$) = StarComp($a ; f$) $\cup$ StarComp($b ; f$) for anchored relations $a, b$ of a form $\mathfrak{X}$, where every $\mathfrak{X}_i$ is a distributive lattice, and an indexed family $f$ of pointfree funcoids with Src $f_i = \mathfrak{X}_i$.

### 17.10.6 Simple product of pointfree funcoids

**Definition 17.153.** Let $f$ be an indexed family of pointfree funcoids with every Src $f_i$ and Dst $f_i$ (for all $i \in \text{dom} f$) being a poset with least element. **Simple product of $f$** is

$$\prod_{i \in \text{dom} f}^{(S)} f = \left( \lambda x \in \prod_{i \in \text{dom} f} \text{Src} f_i ; \lambda i \in \text{dom} f : (f_i)_{x_i} ; \lambda y \in \prod_{i \in \text{dom} f} \text{Dst} f_i ; \lambda i \in \text{dom} f : (f_i^{-1})_{y_i} \right).$$

**Proposition 17.154.** Simple product is a pointfree funcoid

$$\prod_{i \in \text{dom} f}^{(S)} f \in \text{FCD}\left( \prod_{i \in \text{dom} f} \text{Src} f_i ; \prod_{i \in \text{dom} f} \text{Dst} f_i \right).$$

**Proof.** Let $x \in \prod_{i \in \text{dom} f} \text{Src} f_i$ and $y \in \prod_{i \in \text{dom} f} \text{Dst} f_i$. Then (take into account that Src $f_i$ and Dst $f_i$ are posets with least elements) $y \neq \left( \lambda x \in \prod_{i \in \text{dom} f} \text{Src} f_i ; \lambda i \in \text{dom} f : (f_i)_{x_i} \right) \Leftrightarrow y \neq (f_i)_{x_i}$.

**Obvious 17.155.** $(\prod_{i \in \text{dom} f}^{(S)} f)_x = (f_i)_{x_i}$ for $x \in \prod_{i \in \text{dom} f} \text{Src} f_i$.

**Obvious 17.156.** $(\prod_{i \in \text{dom} f}^{(S)} f)_x = (f_i)_{x_i}$ for $x \in \prod_{i \in \text{dom} f} \text{Src} f_i$.

**Proposition 17.157.** $f_i$ can be restored if we know $\prod_{i \in \text{dom} f}^{(S)} f$ if $f_i$ is a family of pointfree funcoids between posets with least elements.

**Proof.** Let’s restore the value of $(f_i)_{x_i}$ where $i \in \text{dom} f$ and $x \in f_i$. Let $x'_i = x$ and $x'_j = 0$ for $j \neq i$.

Then $(f_i)_{x_i} = (f_i)_{x'_i} = (\prod_{i \in \text{dom} f}^{(S)} f)_{x'_i}$.

We have restored the value of $(f_i)_{x_i}$. Restoring the value of $(f_i)_{x'_i}$ is similar.

**Remark 17.158.** In the above proposition it is not required that $f_i$ are non-zero.

**Proposition 17.159.** $(\prod_{i \in \text{dom} f}^{(S)} g) \circ (\prod_{i \in \text{dom} f}^{(S)} f) = \prod_{i \in \text{dom} f}^{(S)} (g_i \circ f_i)$ for $n$-indexed families $f$ and $g$ of composable pointfree funcoids between posets with least elements.

**Proof.** $(\prod_{i \in \text{dom} f}^{(S)} (g_i \circ f_i))_{x} = (f_i)_{x_i} = (f_i)_{x'_i} = (\prod_{i \in \text{dom} f}^{(S)} f)_{x'_i}$.

Thus $(\prod_{i \in \text{dom} f}^{(S)} (g_i \circ f_i)) = (\prod_{i \in \text{dom} f}^{(S)} g \circ (\prod_{i \in \text{dom} f}^{(S)} f))$.

$(\prod_{i \in \text{dom} f}^{(S)} (g_i \circ f_i))^{-1} = (\prod_{i \in \text{dom} f}^{(S)} g \circ (\prod_{i \in \text{dom} f}^{(S)} f))^{-1}$ is similar.

**Corollary 17.160.** $(\prod_{i \in \text{dom} f_i}^{(S)} f_{k-1} \circ \ldots \circ (\prod_{i \in \text{dom} f_i}^{(S)} f_0) = \prod_{i \in \text{dom} f_i}^{(S)} (f_{k-1} \circ \ldots \circ f_0)$ for every $n$-indexed families $f_0, \ldots, f_{n-1}$ of composable pointfree funcoids between posets with least elements.
17.11 Multireloids

Definition 17.161. I will call a multireloid of the form \( A = A_i \in \alpha \), where every each \( A_i \) is a set, a pair \( (f; A) \) where \( f \) is a filter on the set \( \prod A \).

Definition 17.162. I will denote \( \text{Obj}(f; A) = A \) and \( \text{GR}(f; A) = f \) for every multireloid \( (f; A) \).

I will denote \( \text{RLD}(A) \) the set of multireloids of the form \( A \).

The multireloid \( \uparrow^{\text{RLD}(A)} F \) for a relation \( F \) is defined by the formulas:

\[
\text{Obj} \uparrow^{\text{RLD}(A)} F = A \quad \text{and} \quad \text{GR} \uparrow^{\text{RLD}(A)} F = \uparrow \prod A F.
\]

Let \( a \) be a multireloid of the form \( A \) and \( \text{dom} A = n \).

Let every \( f_i \) be a reloid with \( \text{Src} f_i = A_i \).

The star-composition of \( a \) with \( f \) is a multireloid of the form \( \lambda \in \text{dom} A; \text{Dst} f_i \), defined by the formulas:

\[
\text{arity StarComp}(a; f) = n; \\
\text{GR StarComp}(a; f) = \prod \left\{ \uparrow^{\text{RLD}(A)} \text{GR StarComp}(A; F) \mid A \in \text{GR} a, F \in \prod_{i \in n} \text{GR} f_i \right\}; \\
\text{Obj}_{\lambda \in n; \text{Dst} f_i} \text{StarComp}(a; f) = \lambda \in n; \text{Dst} f_i.
\]

Theorem 17.163. Multireloids with above defined compositions form a quasi-invertible category with star-morphisms.

Proof. We need to prove:

1. \( \text{StarComp}(\text{StarComp}(m; f); g) = \text{StarComp}(m; \lambda \in \text{arity} m; g_i \circ f_i) \);
2. \( \text{StarComp}(m; \lambda \in \text{arity} m; \text{id}_{\text{Obj}_{\lambda \in n} i} = m) \);
3. \( b \neq \text{StarComp}(a; f) \iff a \neq \text{StarComp}(b; f^\top) \)

(the rest is obvious).

Really,

1. Using properties of generalized filter bases, \( \text{StarComp}(\text{StarComp}(a; f); g) = \prod \left\{ \uparrow^{\text{RLD}(A)} \text{StarComp}(B; G) \mid \forall B \in \text{GR} \text{StarComp}(a; f), G \in \prod_{i \in n} g_i \right\} = \prod \left\{ \uparrow^{\text{RLD}(A)} \text{StarComp}(\text{StarComp}(A; F); G) \mid A \in \text{GR} a, F \in \prod_{i \in n} f_i, G \in \prod_{i \in n} g_i \right\} = \prod \left\{ \uparrow^{\text{RLD}(A)} \text{StarComp}(A; G \circ F) \mid A \in \text{GR} a, F \in \prod_{i \in n} f_i, G \in \prod_{i \in n} g_i \right\} = \prod \left\{ \uparrow^{\text{RLD}(A)} \text{StarComp}(A; H) \mid A \in \text{GR} a, H \in \prod_{\lambda \in n; \text{Dom} f_i} \lambda \in n; g_i \circ f_i \right\} = \text{StarComp}(a; \lambda \in n; g_i \circ f_i). \)

2. \( \text{StarComp}(m; \lambda \in \text{arity} m; \text{id}_{\text{Obj}_{\lambda \in n} i} = \prod \left\{ \uparrow^{\text{RLD}(A)} \text{StarComp}(A; H) \mid A \in \text{GR} m, H \in \prod_{i \in n \in \text{arity} m} \text{GR} \text{id}_{\text{Obj}_{\lambda \in n} i} \right\} = \prod \left\{ \uparrow^{\text{RLD}(A)} \text{StarComp}(A; \lambda \in \text{arity} m; H_i) \mid A \in \text{GR} m, H \in \prod_{i \in n \in \text{arity} m} \text{GR} \text{id}_{\text{Obj}_{\lambda \in n} i} \right\} = \prod \left\{ \uparrow^{\text{RLD}(A)} \text{StarComp}(A; \lambda \in \text{arity} m; \text{id}_{X_i}) \mid A \in \text{GR} m, X \in \prod_{i \in n \in \text{arity} m} \text{Obj}_{\lambda \in n} i \right\} = \prod \left\{ \uparrow^{\text{RLD}(A)} (A \cap X) \mid A \in \text{GR} m, X \in \prod_{i \in n \in \text{arity} m} \text{Obj}_{\lambda \in n} i \right\} = \prod \left\{ \uparrow^{\text{RLD}(A)} A \mid A \in \text{GR} m \right\} = m. \)

3. Using properties of generalized filter bases,

\( b \neq \text{StarComp}(a; f) \iff \forall A \in \text{GR} a, B \in \text{GR} b, F \in \prod_{i \in n \in \text{arity} m} \text{GR} f_i; B \neq \text{StarComp}(A; F) \iff \forall A \in \text{GR} a, B \in \text{GR} b, F \in \prod_{i \in n \in \text{arity} m} \text{GR} f_i; B \neq \left( \prod_{i \in n \in \text{arity} m} \text{GR} f_i \right)^A \iff \forall A \in \text{GR} a, B \in \text{GR} b, F \in \prod_{i \in n \in \text{arity} m} \text{GR} f_i; A \neq \left( \prod_{i \in n \in \text{arity} m} \text{GR} f_i \right)^A \iff a \neq \text{StarComp}(b; f^\top). \) \( \square \)

Definition 17.164. Let \( f \) be a multireloid of the form \( A \). Then for \( i \in \text{dom} A \)

\[
\text{Pr}_i^{\text{RLD}} f = \prod \left( \uparrow^{A_i} \text{Pr}_i \right) \text{GR} f.
\]
Proposition 17.165. \( \Pr_i^{RLD} f = (\Pr_i)_{GR} a \) for every multireloid \( a \) and \( i \) arity \( a \), given a set \( A \supseteq (\Pr_i)a \).

Proof. It’s enough to show that \( (\Pr_i)_{GR} f \) is a filter.

That \( (\Pr_i)_{GR} f \) is an upper set is obvious.

Let \( X, Y \in (\Pr_i)_{GR} f \). Then there exist \( F, G \in GR f \) such that \( X = \Pr_i F \), \( Y = \Pr_i G \). Then \( X \cap Y \supseteq \Pr_i (F \cap G) \in (\Pr_i)_{GR} f \). Thus \( X \cap Y \in (\Pr_i)_{GR} f \). \( \square \)

Definition 17.166. \( \prod_{i \in \Delta} X_i = \prod \{ X_i \mid X_i \in \prod X \} \) for every indexed family \( \mathcal{X} \) of filters on powersets.

Proposition 17.167. \( \Pr_k^{RLD} \prod_{i \in \Delta} X_i = x_k \) for every indexed family \( x \) of proper filters.

Proof. \( \Pr_k^{RLD} \prod_{i \in \Delta} X_i = (\Pr_k) \prod_{i \in \Delta} X_i \). \( \square \)

Conjecture 17.168. \( GR StarComp(a \sqcup b; f) = GR StarComp(a; f) \sqcup GR StarComp(b; f) \) if \( f \) is a reloid and \( a, b \) are multireloids of the same form, composable with \( f \).

Theorem 17.169. \( \prod_{i \in \Delta} A = \bigcup \{ \prod_{i \in \Delta} a \mid a \in \prod_{i \in \Delta} A \text{ atoms } A_i \} \) for every indexed family \( A \) of filters on powersets.

Proof. Obviously \( \prod_{i \in \Delta} A \supseteq \bigcup \{ \prod_{i \in \Delta} a \mid a \in \prod_{i \in \Delta} A \text{ atoms } A_i \} \).

Reversely, let \( K \in GR \bigcup \{ \prod_{i \in \Delta} a \mid a \in \prod_{i \in \Delta} A \text{ atoms } A_i \} \).

Consequently \( K \in GR \prod_{i \in \Delta} a \) for every \( a \in \prod_{i \in \Delta} A \text{ atoms } A_i \); \( K \supseteq \prod X \) for every \( X \) \( a \in \Delta \).

But \( \bigcup_{X \in \prod a} X = \prod_{i \in \Delta} A \bigcup (\Pr_i) X \supseteq \prod_{j \in \Delta} A_j \) for some \( Z_j \in A_j \) because \( (\Pr_i) X \in A_i \) and our lattice is atomistic. So \( K \in GR \prod_{i \in \Delta} A \). \( \square \)

Theorem 17.170. Let \( a, b \) be indexed families of filters on powersets of the same form \( \mathfrak{A} \). Then
\[
\prod_{i \in \Delta} (a \cap b) = \prod_{i \in \Delta} (a_i \cap b_i).
\]

Proof.
\[
\begin{align*}
\prod_{i \in \Delta} (a \cap b) &= \left\{ \prod_{i \in \Delta} (P \cap Q) \mid P \in \prod_{i \in \Delta} a, Q \in \prod_{i \in \Delta} b \right\} = \\
&= \left\{ \prod_{i \in \Delta} (p_i \cap q_i) \mid p \in \prod_{i \in \Delta} a, q \in \prod_{i \in \Delta} b \right\} = \\
&= \left\{ \prod_{i \in \Delta} (a_i \cap b_i) \right\} = \\
&= \prod_{i \in \Delta} (a_i \cap b_i). \quad \square
\end{align*}
\]

Theorem 17.171. If \( S \in \mathcal{P} \prod_{i \in \Delta} \mathfrak{A}(a_i) \) where \( \mathfrak{A} \) is an indexed family of sets, then
\[
\bigcap \left\{ \prod_{i \in \Delta} a \mid a \in S \right\} = \prod_{i \in \Delta} \left( \prod_{i \in \Delta} (a_i) \right) \prod_{i \in \Delta} (\mathfrak{A}(a_i)) \prod_{i \in \Delta} S.
\]
Proof. If $S = \emptyset$ then $\bigcap \{ \prod_{i \in \text{dom } A}^{\text{RLD}} a \mid a \in S \} = \bigcap \emptyset = 1^{\text{RLD}(A)}$ and $\prod_{i \in \text{dom } A}^{\text{RLD}} \langle \hat{t}^{(\text{RLD})} \rangle a \in S = \prod_{i \in \text{dom } A}^{\text{RLD}} \emptyset = \prod_{i \in \text{dom } A}^{\text{RLD}} \langle \hat{t}^{(\text{RLD})} \rangle \emptyset = \prod_{i \in \text{dom } A}^{\text{RLD}} 1^{\text{RLD}(A)}$, thus $\bigcap \{ \prod_{i \in \text{dom } A}^{\text{RLD}} a \mid a \in S \} = \prod_{i \in \text{dom } A}^{\text{RLD}} \langle \hat{t}^{(\text{RLD})} \rangle a$. Let $S \neq \emptyset$.

$\bigcap \{ \langle \hat{t}^{(\text{RLD})} \rangle a \mid a \in S \} \subseteq \bigcap \bigcup \{ \langle \hat{t}^{(\text{RLD})} \rangle a \mid a \in S \}$ for every $a \in S$ because $a_i \in \text{Pr}_i S$. Thus $\prod_{i \in \text{dom } A}^{\text{RLD}} \langle \hat{t}^{(\text{RLD})} \rangle a$, $\prod_{i \in \text{dom } A}^{\text{RLD}} \langle \hat{t}^{(\text{RLD})} \rangle \text{Pr}_i S \subseteq \prod_{i \in \text{dom } A}^{\text{RLD}} a$.

Now suppose $F \in \text{GR} \prod_{i \in \text{dom } A}^{\text{RLD}} \langle \hat{t}^{(\text{RLD})} \rangle \text{Pr}_i S$. Then there exist $X \in \prod_{i \in \text{dom } A}^{\text{RLD}} \langle \hat{t}^{(\text{RLD})} \rangle \text{Pr}_i S$ such that $F \supseteq \prod_{i \in \text{dom } A}^{\text{RLD}} a$. It is enough to prove that there exist $a \in S$ such that $F \in \text{GR} \prod_{i \in \text{dom } A}^{\text{RLD}} a$. For this it is enough $\prod_{i \in \text{dom } A}^{\text{RLD}} X \in \text{GR} \prod_{i \in \text{dom } A}^{\text{RLD}} a$.

Really, $X_i \in \prod_{i \in \text{dom } A}^{\text{RLD}} a$, for every $a_i \in S$. Thus $\prod_{i \in \text{dom } A}^{\text{RLD}} a$.

\begin{definition}
I call a multireloid \emph{principal} iff its graph is a principal filter.
\end{definition}

\begin{definition}
I call a multireloid \emph{convex} iff it is a join of reloidal products.
\end{definition}

\begin{theorem}
StarComp(a \cup b; f) = StarComp(a; f) \sqcup StarComp(b; f) for multireloids $a$, $b$ and an indexed family $f$ of reloids with $\text{Src} f_i = (\text{form } a_i) = (\text{form } b_i)$.
\end{theorem}

\begin{proof}
$\text{GR} \left( \text{StarComp}(a; f) \sqcup \text{StarComp}(b; f) \right) = \bigcap \left\{ \langle \hat{t}^{(\text{RLD}(\text{form } a))} \text{StarComp}(A; F) \mid A \in \text{GR } a \right\}$, $F \in \prod_{i \in \text{dom } A}^{\text{GR} f_i}$, $\bigcap \left\{ \langle \hat{t}^{(\text{RLD}(\text{form } b))} \text{StarComp}(B; F) \mid B \in \text{GR } b \right\}$, $F \in \prod_{i \in \text{dom } B}^{\text{GR} f_i}$, $A \in \text{GR } a$, $B \in \text{GR } b$, $F \in \prod_{i \in \text{dom } A}^{\text{GR} f_i}$ and $\text{GR} \left( \text{StarComp}(A; F) \sqcup \text{StarComp}(B; F) \right) = \bigcap \left\{ \langle \hat{t}^{(\text{RLD}(\text{form } a))} \text{StarComp}(C; F) \mid C \in \text{GR } a \cup b \right\}$, $F \in \prod_{i \in \text{dom } A}^{\text{GR} f_i}$.
\end{proof}

\begin{conjecture}
$f \subseteq \prod_{i \in \text{dom } A}^{\text{RLD} a} \Leftrightarrow \forall i \in \text{arity } f: \text{Pr}_i^{\text{RLD} a} f \subseteq a_i$ for every multireloid $f$ and $a_i \in \mathfrak{S}(\text{form } f_i)$ for every $i \in \text{arity } f$.
\end{conjecture}

\subsection{Starred reloidal product}

Tychonoff product of topological spaces inspired me the following definition, which seems possibly useful just like Tychonoff product:

\begin{definition}
Let $a$ be an $n$-indexed ($n$ is an arbitrary index set) family of filters on sets. $\prod_{i \in \text{dom } A}^{\text{RLD} a}$ is the reloid of the form $\prod_{i \in \text{dom } A}^{\text{RLD} Base(a_i)}$ induced by the filter base \[ \left\{ \prod_{i \in \text{dom } A}^{\text{RLD} \left( \begin{array}{l}
A_i \\
\text{Base}(a_i)
\end{array} \right)} \mid m \text{ is a finite subset of } n, A \in \prod_{i \in \text{dom } A}^{\left( a_i \right)} \right\} \]

\end{definition}

\begin{obvious}
It is really a filter base.
\end{obvious}

\begin{obvious}
$\prod_{i \in \text{dom } A}^{\text{RLD} a}$ is a filter base.
\end{obvious}

\begin{proposition}
$\prod_{i \in \text{dom } A}^{\text{RLD} a} = \prod_{i \in \text{dom } A}^{\text{RLD} a}$ if $n$ is finite.
\end{proposition}

\begin{proof}
Take $m = n$ to show that $\prod_{i \in \text{dom } A}^{\text{RLD} a} \subseteq \prod_{i \in \text{dom } A}^{\text{RLD} a}$.
\end{proof}

\begin{proposition}
$\prod_{i \in \text{dom } A}^{\text{RLD} a} = 0^{\text{RLD}(\text{Base}(a_i))}$ if $a_i$ is the non-proper filter for some $i \in n$.
\end{proposition}
Proof. Take $A_i = \emptyset$ and $m = \{i\}$. Then $\prod_{i \in n} \left( \left\{ A_i^{\text{Base}(a_i)} \mid if i \in m \right\} \right) = \emptyset$. 

Example 17.181. There exists an indexed family $a$ of principal filters such that $\prod^{\text{RLD}^*} a$ is non-principal.

Proof. Let $n = 2$. Let $\text{Base}(a_i) = \mathbb{N}$ and each $a_i$ is a principal filter corresponding to a two-element set.

Every $\prod_{i \in n} \left( \left\{ A_i^{\text{Base}(a_i)} \mid if i \in m \right\} \right)$ is at least $\mathfrak{c}^n \geq \mathfrak{c}$ elements.

There are elements $\prod_{i \in n}^{\text{RLD}^*} a$ with cardinality $2^n = n$. They can’t be elements of $\prod_{i \in n}^{\text{RLD}^*} a$ because $n = \omega < \mathfrak{c}$. 

Corollary 17.182. There exists an indexed family $a$ of principal filters such that $\prod_{i \in n}^{\text{RLD}^*} a \neq \prod_{i \in n}^{\text{RLD}^*} a$.

Proof. Because $\prod_{i \in n}^{\text{RLD}^*} a$ is principal.

Proposition 17.183. $\text{Pr}_{i \in n}^{\text{RLD}^*} x = x_i$ for every indexed family $x$ of proper filters.

Proof. $\text{Pr}_{i \in n}^{\text{RLD}^*} x = (\text{Pr}_{i \in n}^{\text{GR}^*} x) \prod_{i \in n}^{\text{RLD}^*} x = x_i$.

17.12 Subatomic product of funcoids

Definition 17.184. Let $f$ be an indexed family of funcoids. Then $\prod_{i \in n}^{(A)} f$ (subatomic product) is a funcoid $\prod_{i \in n}^{(A)} f \subseteq \prod_{i \in n}^{\text{dom} f} \text{Src} f_i \rightarrow \prod_{i \in n}^{\text{dom} f} \text{Dst} f_i$ such that for every $a \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Src} f_i), b \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Dst} f_i)}}$, $a \left[ \prod_{i \in n}^{(A)} f \right] b \Leftrightarrow \forall i \in \text{dom} f : \text{Pr}_{i \in n}^{\text{RLD}^*} a[f_i] \prod_{i \in n}^{\text{RLD}^*} b$.

Proposition 17.185. The funcoid $\prod_{i \in n}^{(A)} f$ exists.

Proof. To prove that $\prod_{i \in n}^{(A)} f$ exists we need to prove (for every $a \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Src} f_i), b \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Dst} f_i)})$, $\forall X \in \text{GR} a, Y \in \text{GR} b \exists x \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Src} f_i)} X, y \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Dst} f_i)} Y : x \left[ \prod_{i \in n}^{(A)} f \right] y \Rightarrow a \left[ \prod_{i \in n}^{(A)} f \right] b$.

Let $\forall X \in \text{GR} a, Y \in \text{GR} b \exists x \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Src} f_i)} X, y \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Dst} f_i)} Y : x \left[ \prod_{i \in n}^{(A)} f \right] y$.

Then $\forall X \in \text{GR} a, Y \in \text{GR} b \exists x \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Src} f_i)} X, y \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Dst} f_i)} Y \forall i \in \text{dom} f : \text{Pr}_{i \in n}^{\text{RLD}^*} x[f_i] \prod_{i \in n}^{\text{RLD}^*} b$.

Then because $\text{Pr}_{i \in n}^{\text{RLD}^*} x[f_i] \prod_{i \in n}^{\text{RLD}^*} b$ and likewise for $y$: $\forall X \in \text{GR} a, Y \in \text{GR} b \forall i \in \text{dom} f \exists x \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Src} f_i)} X, y \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Dst} f_i)} Y : x[f_i] y$.

Thus $\forall X \in \text{GR} a, Y \in \text{GR} b \forall i \in \text{dom} f : x[f_i] \text{Pr}_{i \in n}^{\text{RLD}^*} X, y \in \text{atoms}^{\text{RLD}(\lambda \in \text{dom} f : \text{Src} f_i)} Y$.

Then $\forall X \in \text{GR} a, Y \in \text{GR} b \forall i \in \text{dom} f : \text{Pr}_{i \in n}^{\text{RLD}^*} x[f_i] \prod_{i \in n}^{\text{RLD}^*} b$.

Thus $\forall X \in \text{GR} a, Y \in \text{GR} b \forall i \in \text{dom} f : \text{Pr}_{i \in n}^{\text{RLD}^*} x[f_i] \prod_{i \in n}^{\text{RLD}^*} b$.

So $\forall i \in \text{dom} f : \text{Pr}_{i \in n}^{\text{RLD}^*} a[f_i] \prod_{i \in n}^{\text{RLD}^*} b$.
and thus \( a \left[ f \times^{(A)} g \right] b \).

\[ \square \]

**Remark 17.186.** It seems that the proof of the above theorem can be simplified using cross-composition product.

**Theorem 17.187.** \( \prod^{(A)}_{i \in \mathbb{N}} (g_i \circ f_i) = \prod^{(A)} g \circ \prod^{(A)} f \) for indexed (by an index set \( n \)) families \( f \) and \( g \) of funcoids such that \( \forall i \in n: \text{Dst} f_i = \text{Src} g_i \).

**Proof.** Let \( a, b \) be ultrafilters on \( \prod_{i \in \mathbb{N}} \text{Src} f_i \) and \( \prod_{i \in \mathbb{N}} \text{Dst} g_i \) correspondingly,

\[
a \left[ \prod^{(A)} (g_i \circ f_i) \right] b \Leftrightarrow \forall i \in \text{dom} f: (\text{Pr}_i a [g_i \circ f_i] (\text{Pr}_i) b) \Leftrightarrow \forall i \in \text{dom} f \exists C \in \text{atoms}^{\text{Src} f_i} ((\text{Pr}_i) a [f_i] C \land C [g_i] (\text{Pr}_i) b) \Leftrightarrow \forall i \in \text{dom} f \exists c \in \text{atoms}^{\text{Dst} f_i} ((\text{Pr}_i) a [f_i] (\text{Pr}_i) c \land (\text{Pr}_i) c [g_i] (\text{Pr}_i) b) \Leftrightarrow \exists c \in \text{atoms}^{\text{Dst} f_i}.
\]

Let

\[
\forall i \in \text{dom} f \exists c \in \text{atoms}^{\text{Dst} f_i} ((\text{Pr}_i) a [f_i] (\text{Pr}_i) c \land (\text{Pr}_i) c [g_i] (\text{Pr}_i) b).
\]

Then there exists \( c' \in \{ \text{atoms}^{\text{Dst} f_i} \}^n \) such that

\[
\forall i \in \text{dom} f: (\text{Pr}_i) a [f_i] (\text{Pr}_i) c' \land (\text{Pr}_i) c' [g_i] (\text{Pr}_i) b).
\]

Then take \( c'' = \prod^{\text{RLD}} c' \). Then \( \forall i \in \text{dom} f: (\text{Pr}_i) a [f_i] (\text{Pr}_i) c'' \land (\text{Pr}_i) c'' [g_i] (\text{Pr}_i) b) \). Thus

\[
\exists c \in \text{atoms}^{\text{Dst} f_i} ((\text{Pr}_i) a [f_i] (\text{Pr}_i) c \land (\text{Pr}_i) c [g_i] (\text{Pr}_i) b).
\]

We have

\[
\prod^{(A)} (g_i \circ f_i) \left[ \prod^{(A)} f \right] b \Leftrightarrow \prod^{(A)} g \circ \prod^{(A)} f \left[ \prod^{(A)} f \right] b.
\]

\[ \square \]

**Corollary 17.188.** \( \left( \prod^{(A)} f_{k-1} \right) \circ \ldots \circ \left( \prod^{(A)} f_0 \right) = \prod^{(A)} (f_{k-1} \circ \ldots \circ f_0) \) for every \( n \)-indexed families \( f_0, \ldots, f_{n-1} \) of composable funcoids.

**Proposition 17.189.** \( \prod^{\text{RLD}} a \left[ \prod^{(A)} f \right] \prod^{\text{RLD}} b \Leftrightarrow \forall i \in \text{dom} f: a_i [f_i] b_i \) for an indexed family \( f \) of funcoids and indexed families \( a \) and \( b \) of filters where \( a_i \in \mathcal{F} (\text{Src} f_i) \), \( b_i \in \mathcal{F} (\text{Dst} f_i) \) for every \( i \in \text{dom} f \).

**Proof.** If \( a_i = 0 \) or \( b_i = 0 \) for some \( i \) our theorem is obvious. We will take \( a_i \neq 0 \) and \( b_i \neq 0 \), thus there exist

\[
x \in \text{atoms} \prod^{\text{RLD}} a, \quad y \in \text{atoms} \prod^{\text{RLD}} b.
\]

\[
\prod^{\text{RLD}} a \left[ \prod^{(A)} f \right] \prod^{\text{RLD}} b \Leftrightarrow \exists x \in \text{atoms} \prod^{\text{RLD}} a, \quad y \in \text{atoms} \prod^{\text{RLD}} b: x \left[ \prod^{(A)} f \right] y \Leftrightarrow \exists x \in \text{atoms} \prod^{\text{RLD}} a, \quad y \in \text{atoms} \prod^{\text{RLD}} b: \forall i \in \text{dom} f: a_i [f_i] b_i.
\]

\[ \square \]

**Theorem 17.190.** \( \left( \prod^{(A)} f \right) x = \prod^{\text{RLD}} \left( \left( f_i \right) \prod^{\text{RLD}} x \right) \) for an indexed family \( f \) of funcoids and \( x \in \text{atoms}^{\text{Src} f_i} \) for every \( n \in \text{dom} f \).

**Proof.** For every ultrafilter \( y \in \mathcal{F} \left( \prod_{i \in \text{dom} f} \text{Dst} f_i \right) \) we have:

\[
y \neq \prod^{\text{RLD}} \left( \left( f_i \right) \prod^{\text{RLD}} x \right) \Leftrightarrow \forall i \in \text{dom} f: \prod^{\text{RLD}} a_i [f_i] b_i \Leftrightarrow \forall i \in \text{dom} f: \prod^{\text{RLD}} a_i [f_i] b_i \Leftrightarrow \prod^{\text{RLD}} x [f_i] b_i.
\]

Thus

\[
\left( \prod^{(A)} f \right) x = \prod^{\text{RLD}} \left( \left( f_i \right) \prod^{\text{RLD}} x \right).
\]

\[ \square \]
Corollary 17.191. \( (f \times^A g)x = (f)(\text{dom } x) \times^{\text{RLD}} (g)(\text{im } x) \).

17.13 On products and projections

Conjecture 17.192. For principal funcoids \( \prod^{(C)} x \) and \( \prod^{(A)} x \) coincide with the conventional product of binary relations.

17.13.1 Staroidal product

Let \( f \) be a staroid, whose form components are boolean lattices.

Definition 17.193. Staroidal projection of a staroid \( f \) is the filter \( \text{Pr}_{k}^{\text{Strd}} x \) corresponding to the free star

\[
(\text{val } f)_{k}(\lambda i \in (\text{arity } f) \setminus \{k\} ; 1_{(\text{form } f)_i}).
\]

Proposition 17.194. \( \text{Pr}_{k} \text{ GR } \prod^{\text{Strd}} x = *x_{k} \) if \( x \) is an indexed family of proper filters, and \( k \in \text{dom } x \).

Proof. \( \text{Pr}_{k} \text{ GR } \prod^{\text{Strd}} x = \text{Pr}_{k} \{ L \in \prod_{i \in \text{dom } x} \text{form } x_i \mid \forall i \in \text{dom } x : x_i \neq L_i \} = (\text{used the fact that } x_i \text{ are proper filters}) = \{ l \in \text{form } x_k \mid x_k \neq l \} = *x_{k}. \)

Proposition 17.195. \( \text{Pr}_{k}^{\text{Strd}} \prod^{\text{Strd}} x = x_{k} \) if \( x \) is an indexed family of proper filters, and \( k \in \text{dom } x \).

Proof. \( \partial \text{ Pr}_{k}^{\text{Strd}} \prod^{\text{Strd}} x = (\text{val } \prod^{\text{Strd}} x)_{k}(\lambda i \in (\text{dom } x) \setminus \{k\} ; 1_{(\text{form } x)_i}) = \{ X \in (\text{form } \prod^{\text{Strd}} x)_{k} \mid (\lambda i \in (\text{dom } x) \setminus \{k\} ; 1_{(\text{form } x)_i}) \cup \{ (k, X) \} \in \text{GR } \prod^{\text{Strd}} x \} = \{ X \in \text{Base } x_{k} \mid (\forall i \in (\text{dom } x) \setminus \{k\} ; 1_{(\text{form } x)_i}) \neq x_{i} \wedge X \neq x_{k} \} = \{ X \in \text{Base } x_{k} \mid X \neq x_{k} \} = \partial x_{k}. \)

Consequently \( \text{Pr}_{k}^{\text{Strd}} \prod^{\text{Strd}} x = x_{k}. \)

17.13.2 Cross-composition product of pointfree funcoids

Definition 17.196. Zero pointfree funcoid from a poset \( A \) to to a poset \( B \) is such a pointfree funcoid \( A \rightarrow B \) that \( (f)x = 0 \) for every \( x \in A \).

Proposition 17.197. A pointfree funcoid \( f \) is zero iff \( [f]=\emptyset \).

Proof. Direct implication is obvious.

Let now \( [f]=\emptyset \). Then \( (f)x \preceq y \) for every \( x \in \text{Src } f, y \in \text{Dst } f \) and thus \( (f)x \preceq (f)x \). It is possible only when \( (f)x = 0^{\text{Dset } f} \).

Corollary 17.198. A pointfree funcoid is zero iff its reverse is zero.

Proposition 17.199. Values \( x_{i} \) (for every \( i \in \text{dom } x \)) can be restored from the value of \( \prod^{(C)} x \) provided that \( x \) is an indexed family of non-zero pointfree funcoids if \( \text{Src } f_{i} \) (for every \( i \in n \)) is an atomic lattice and every \( \text{Dst } f_{i} \) is an atomic poset with greatest element.

Proof. \( \prod^{(C)} x \prod^{\text{Strd}} p = \prod_{i \in n}^{\text{Strd}} (x_{i})p_{i} \) by the theorem 17.151.

Since \( x_{i} \) is non-zero there exist \( p \) such that \( (x_{i})p_{i} \) is non-zero. Take \( k \in n, p'_{i} = p_{i} \) for \( i \neq k \) and \( p'_{k} = q \) for an arbitrary value \( q \); then (using the staroidal projections from the previous subsection)

\[
(x_{k})q = \text{Pr}_{k}^{\text{Strd}} \prod_{i \in n}^{\text{Strd}} (x_{i})p'_{i} = \text{Pr}_{k}^{\text{Strd}} \prod^{(C)} x \prod^{\text{Strd}} p'.
\]
So the value of $x$ can be restored from $\prod^{(C)} x$ by this formula. □

17.13.3 Subatomic product

**Proposition 17.200.** Values $x_i$ (for every $i \in \text{dom } x$) can be restored from the value of $\prod^{(A)} x$ provided that $x$ is an indexed family of non-zero funcoids.

**Proof.** Fix $k \in \text{dom } f$. Let for some filters $x$ and $y$

$$a = \begin{cases} 1^{\delta}(\text{Base}(x)) & \text{if } i \neq k; \\ x & \text{if } i = k \end{cases} \quad \text{and} \quad b = \begin{cases} 1^{\delta}(\text{Base}(y)) & \text{if } i \neq k; \\ y & \text{if } i = k. \end{cases}$$

Then $x [x_k] y \Leftrightarrow a_k [x_k] b_k \Leftrightarrow \forall i \in \text{dom } f; a_i [x_i] b_i \Leftrightarrow \prod^{\text{RLD}} a \left[ \prod^{(A)} x \right] \prod^{\text{RLD}} b$. So we have restored $x_k$ from $\prod^{(A)} x$. □

**Definition 17.201.** For every funcoid $f \colon \prod A \to \prod B$ (where $A$ and $B$ are indexed families of sets) consider the funcoid $\Pr^{(A)}_k f$ defined by the formula

$$X \left[ \Pr^{(A)}_k f \right]^* Y \Leftrightarrow \prod^{\text{RLD}}_{i \in \text{dom } A} \left( \begin{cases} 1^{\delta}(A_i) & \text{if } i \neq k; \\ \uparrow^A J_i & \text{if } i = k \end{cases} \right) \left[ f \right] \prod^{\text{RLD}}_{i \in \text{dom } B} \left( \begin{cases} 1^{\delta}(B_i) & \text{if } i \neq k; \\ \uparrow^B Y_i & \text{if } i = k \end{cases} \right).$$

**Proposition 17.202.** $\Pr^{(A)}_k f$ is really a funcoid.

**Proof.** $\neg \left( \emptyset \left[ \Pr^{(A)}_k f \right]^* Y \right)$ is obvious.

$$\prod^{\text{RLD}}_{i \in \text{dom } A} \left( \begin{cases} 1^{\delta}(A_i) & \text{if } i \neq k; \\ \uparrow^A J_i & \text{if } i = k \end{cases} \right) \Leftrightarrow \prod^{\text{RLD}}_{i \in \text{dom } B} \left( \begin{cases} 1^{\delta}(B_i) & \text{if } i \neq k; \\ \uparrow^B Y_i & \text{if } i = k \end{cases} \right) \Leftrightarrow \prod^{\text{RLD}}_{i \in \text{dom } \text{Src } f} \left( \begin{cases} 1^{\delta}(A_i) & \text{if } i \neq k; \\ \uparrow^A J_i & \text{if } i = k \end{cases} \right) \cup \prod^{\text{RLD}}_{i \in \text{dom } \text{Dst } f} \left( \begin{cases} 1^{\delta}(B_i) & \text{if } i \neq k; \\ \uparrow^B Y_i & \text{if } i = k \end{cases} \right) \Leftrightarrow \prod^{\text{RLD}}_{i \in \text{dom } \text{Src } f} \left( \begin{cases} 1^{\delta}(A_i) & \text{if } i \neq k; \\ \uparrow^A J_i & \text{if } i = k \end{cases} \right) \left[ f \right] \prod^{\text{RLD}}_{i \in \text{dom } \text{Dst } f} \left( \begin{cases} 1^{\delta}(B_i) & \text{if } i \neq k; \\ \uparrow^B Y_i & \text{if } i = k \end{cases} \right) \Leftrightarrow \prod^{\text{RLD}}_{i \in \text{dom } \text{Src } f} \left( \begin{cases} 1^{\delta}(A_i) & \text{if } i \neq k; \\ \uparrow^A J_i & \text{if } i = k \end{cases} \right) \Leftrightarrow I \left[ \Pr^{(A)}_k f \right]^* Y \lor J \left[ \Pr^{(A)}_k f \right]^* Y.$$

The rest follows from symmetry. □

**Proposition 17.203.** For every funcoid $f \colon \prod A \to \prod B$ (where $A$ and $B$ are indexed families of sets) there exists a funcoid $\Pr^{(A)}_k f$ defined by the formula

$$X \left[ \Pr^{(A)}_k f \right] Y \Leftrightarrow \prod^{\text{RLD}}_{i \in \text{dom } A} \left( \begin{cases} 1^{\delta}(A_i) & \text{if } i \neq k; \\ X & \text{if } i = k \end{cases} \right) \left[ f \right] \prod^{\text{RLD}}_{i \in \text{dom } B} \left( \begin{cases} 1^{\delta}(B_i) & \text{if } i \neq k; \\ Y & \text{if } i = k \end{cases} \right).$$
Proof.

\[ \forall X \in \mathcal{X}, Y \in \mathcal{Y}: \prod_{i \in \operatorname{dom} \text{Src} \ f} \left( \left\{ \begin{array}{ll} 1^\delta (A_i) & \text{if } i \neq k; \\ \uparrow A_i & \text{if } i = k \end{array} \right\} \right) [f] \prod_{i \in \operatorname{dom} \text{Dst} \ f} \left( \left\{ \begin{array}{ll} 1^\delta (B_i) & \text{if } i \neq k; \\ \uparrow B_i & \text{if } i = k \end{array} \right\} \right) \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y}: \left[ \mathcal{X} \left[ \prod_{i \in \operatorname{dom} \ f} \left( \left\{ \begin{array}{ll} 1^\delta (A_i) & \text{if } i \neq k; \\ \uparrow A_i & \text{if } i = k \end{array} \right\} \right) \right] Y \right] \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y}: \left[ \mathcal{X} \left[ \prod_{i \in \operatorname{dom} \ f} \left( \left\{ \begin{array}{ll} 1^\delta (B_i) & \text{if } i \neq k; \\ \uparrow B_i & \text{if } i = k \end{array} \right\} \right) \right] Y \right]. \]

\[ \Box \]

Remark 17.204. Reloidal product above can be replaced with starred reloidal product, because of finite number of non-maximal multipliers in the products.

Obvious 17.205. \( \prod_{i \in \operatorname{dom} \ f} \left( \left\{ \begin{array}{ll} 1^\delta (A_i) & \text{if } i \neq k; \\ \uparrow A_i & \text{if } i = k \end{array} \right\} \right) \) provided that \( x \) is an indexed family of non-zero funcoids.

17.13.4 Other

Conjecture 17.206. Values \( x_i \) (for every \( i \in \operatorname{dom} \ f \)) can be restored from the value of \( \prod^{(C)} \) \( f \) provided that \( x \) is an indexed family of non-zero reloids.

Definition 17.207. Displaced product \( \prod^{(DP)} \ f = \prod^{(C)} \ f \) for every indexed family of pointfree funcoids, where downgrading is defined for the filtrator

\[ \left( \text{FCD}(\text{StarMor}(\lambda i \in \operatorname{dom} \ f: \text{Src} \ f_i); \text{StarMor}(\lambda i \in \operatorname{dom} \ f: \text{Dst} \ f_i)); \langle \uparrow \text{FCD} \rangle \mathcal{P} \left( \prod_{i \in \operatorname{dom} \ f} \text{Src} \ f_i \times \prod_{i \in \operatorname{dom} \ f} \text{Dst} \ f_i \right) \right). \]

Remark 17.208. Displaced product is a funcoid (not just a pointfree funcoid).

Conjecture 17.209. Values \( x_i \) (for every \( i \in \operatorname{dom} \ f \)) can be restored from the value of \( \prod^{(DP)} \ f \) provided that \( x \) is an indexed family of non-zero funcoids.

Definition 17.210. Let \( f \in \mathcal{P} \ Z^{Y} \) where \( Z \) is a set and \( Y \) is a function.

\[ \text{Pr}^{(D)}_{k} f = \text{Pr} \left\{ z : z \in f \right\}. \]

Proposition 17.211. \( \text{Pr}^{(D)}_{k} \prod^{(D)} \ f = F_k \) for every indexed family \( F \) of non-empty relations.

Proof. Obvious. \( \Box \)

Corollary 17.212. \( \operatorname{GR} \text{Pr}^{(D)}_{k} \prod^{(D)} \ f = \operatorname{GR} F_k \) and \( \operatorname{form} \text{Pr}^{(D)}_{k} \prod^{(D)} \ f = \operatorname{form} F_k \) for every indexed family \( F \) of non-empty anchored relations.

17.14 Relationships between cross-composition and sub-atomic products

Proposition 17.213. \( a \left[ f \times^{(C)} g \right] b \Leftrightarrow \operatorname{dom} a \left[ f \right] \operatorname{dom} b \land \operatorname{im} a \left[ g \right] \operatorname{im} b \) for funcoids \( f \) and \( g \) and atomic funcoids \( a \in \text{FCD}(\text{Src} \ f; \text{Src} \ g) \) and \( b \in \text{FCD}(\text{Dst} \ f; \text{Dst} \ g) \).
Proof. \( a \left[ f \times^{(C)} g \right] b \iff a \circ f^{-1} \neq g^{-1} \circ b \iff (\text{dom } a \times^{\text{FCD}} \text{im } a) \circ f^{-1} \neq g^{-1} \circ (\text{dom } b \times^{\text{FCD}} \text{im } b) \iff \langle f \rangle \text{dom } a \times^{\text{FCD}} \text{im } a \neq b \times^{\text{FCD}} \langle g^{-1} \rangle \text{im } b \iff \langle f \rangle \text{dom } a \neq \text{dom } b \text{ and } a \neq (g^{-1}) \text{im } b \iff \text{dom } a \left[ f \right] \text{dom } b \land \text{im } a \left[ g \right] \text{im } b. \)

**Proposition 17.214.** \( \mathcal{X} \left[ \prod^{(A)} f \right] \mathcal{Y} \iff \forall i \in \text{dom } f: \text{Pr}_{i}^{\text{RLD}} \mathcal{X} \left[ f_{i} \right] \text{Pr}_{i}^{\text{RLD}} \mathcal{Y} \) for every indexed family \( f \) of funcoids and \( \mathcal{X} \in \text{RLD}(\lambda i \in \text{dom } f; \text{Src } f_{i}), \mathcal{Y} \in \text{RLD}(\lambda i \in \text{dom } f; \text{Dst } f_{i}). \)

**Proof.** \( \mathcal{X} \left[ \prod^{(A)} f \right] \mathcal{Y} \iff \exists a \in \text{atoms } \mathcal{X}, b \in \text{atoms } \mathcal{Y}: \forall i \in \text{dom } f: \text{Pr}_{i}^{\text{RLD}} a \left[ f_{i} \right] \text{Pr}_{i}^{\text{RLD}} b \iff \forall i \in \text{dom } f \exists x \in \text{atoms } \text{Pr}_{i}^{\text{RLD}} \mathcal{X}, y \in \text{atoms } \text{Pr}_{i}^{\text{RLD}} \mathcal{Y}: x_{i} \left[ f_{i} \right] y_{i} \iff \forall i \in \text{dom } f: \text{Pr}_{i}^{\text{RLD}} \mathcal{X} \left[ f_{i} \right] \text{Pr}_{i}^{\text{RLD}} \mathcal{Y}. \)

**Corollary 17.215.** \( \mathcal{X} \left[ f \times^{(A)} g \right] \mathcal{Y} \iff \text{dom } \mathcal{X} \left[ f \right] \mathcal{Y} \land \text{im } \mathcal{X} \left[ g \right] \mathcal{Y} \) for funcoids \( f, g \) and reloids \( \mathcal{X} \in \text{RLD}(\text{Src } f; \text{Src } g), \) and \( \mathcal{Y} \in \text{RLD}(\text{Dst } f; \text{Dst } g). \)

**Lemma 17.216.** For every \( A \in \text{Rel}(X; Y) \) (for every sets \( X, Y \)) we have:

\[
\{ \text{dom } a; \text{im } a \mid a \in \text{atoms } A \} = \{ \text{dom } a; \text{im } a \mid a \in \text{atoms } A \}.
\]

**Proof.** Let \( x \in \{ \text{dom } a; \text{im } a \mid a \in \text{atoms } A \}. \) Then \( x_{0} = \text{dom } a \) and \( x_{1} = \text{im } a \) where \( a \in \text{atoms } A. \)

Then \( x_{0} = \text{dom } (\text{FCD})a \) and \( x_{1} = \text{im } (\text{FCD})a \) and obviously \( (\text{FCD})a \in \text{atoms } A. \) So \( x \in \{ \text{dom } a; \text{im } a \mid a \in \text{atoms } A \}. \)

Let now \( x \in \{ \text{dom } a; \text{im } a \mid a \in \text{atoms } A \}. \) Then \( x_{0} = \text{dom } a \) and \( x_{1} = \text{im } a \) where \( a \in \text{atoms } A. \)

\[
x_{0} \left[ \text{FCD} A \right] x_{1} \iff x_{0} \left[ (\text{FCD})A \right] x_{1} \iff x_{0} \times^{\text{RLD}} x_{1} = x_{0} \times^{\text{RLD}} x_{1} \neq x_{0} \times^{\text{RLD}} x_{1} \neq \text{RLD } A. \quad \text{Thus there exists atomic reloid } x' \text{ such that } x' \in \text{atoms } A \text{ and } \text{dom } x' = x_{0}, \text{im } x' = x_{1}. \]

So \( x \in \{ \text{dom } a; \text{im } a \mid a \neq \text{atoms } A \}. \)

**Theorem 17.217.** \( \text{FCD } A \left[ f \times^{(C)} g \right] \text{FCD } B \iff \text{RLD } A \left[ f \times^{(A)} g \right] \text{RLD } B \) for funcoids \( f, g, \) and Relmorphisms \( A: \text{Src } f \to \text{Src } g, \) and \( B: \text{Dst } f \to \text{Dst } g. \)

**Proof.** \( \text{FCD } A \left[ f \times^{(C)} g \right] \text{FCD } B \iff \exists a \in \text{atoms } A \text{FCD } A, b \in \text{atoms } A \text{FCD } B: a \left[ f \times^{(C)} g \right] b \iff \exists a \in \text{atoms } A \text{FCD } A, b \in \text{atoms } A \text{FCD } B: (\text{dom } a \left[ f \right] \text{dom } b \land \text{im } a \left[ g \right] \text{im } b) \Rightarrow \exists a \in \text{atoms } A \text{FCD } A, a_{1} \in \text{atoms } A \text{FCD } A, b_{0} \in \text{atoms } A \text{FCD } B, b_{1} \in \text{atoms } A \text{FCD } B: (a_{0} \left[ f_{0} \right] b_{0} \land a_{1} \left[ g_{1} \right] b_{1}). \)

On the other hand:

\[
\forall a \in \text{atoms } A \exists a_{0} \in \text{atoms } A \text{FCD } A, a_{1} \in \text{atoms } A \text{FCD } B, a_{0} \left[ f_{0} \right] b_{0} \land a_{1} \left[ g_{1} \right] b_{1} \Rightarrow \exists a \in \text{atoms } A \text{FCD } A, b_{0} \in \text{atoms } A \text{FCD } B, b_{1} \in \text{atoms } A \text{FCD } B: (a_{0} \times^{\text{FCD}} b_{0} \neq f \land a_{1} \times^{\text{FCD}} b_{1} \neq g) \Rightarrow \exists a \in \text{atoms } A \text{FCD } A, b \in \text{atoms } A \text{FCD } B: (\text{dom } a \left[ f \right] \text{dom } b \land \text{im } a \left[ g \right] \text{im } b).
\]

Also using the lemma we have \( \exists a \in \text{atoms } A \text{FCD } A, b \in \text{atoms } A \text{FCD } B: (\text{dom } a \left[ f \right] \text{dom } b \land \text{im } a \left[ g \right] \text{im } b) \Rightarrow \exists a \in \text{atoms } A \text{FCD } A, b \in \text{atoms } A \text{FCD } B: (\text{dom } a \left[ f \right] \text{dom } b \land \text{im } a \left[ g \right] \text{im } b). \)

So \( \text{FCD } A \left[ f \times^{(C)} g \right] \text{FCD } B \iff \exists a \in \text{atoms } A \text{FCD } A, b \in \text{atoms } A \text{FCD } B: a \left[ f \times^{(A)} g \right] b \iff \exists a \in \text{atoms } A \text{FCD } A, b \in \text{atoms } A \text{FCD } B: (a \left[ f \right] b \land \text{im } a \left[ g \right] \text{im } b) \Rightarrow \exists a \in \text{atoms } A \text{FCD } A, b \in \text{atoms } A \text{FCD } B: a \left[ f \times^{(C)} g \right] b \iff \exists a \in \text{atoms } A \text{FCD } A, b \in \text{atoms } A \text{FCD } B. \)

**Corollary 17.218.** \( f \times^{(A)} g = \sqcup \left( f \times^{(C)} g \right) \) where downgrading is taken on the filtrator

\[
\left( \text{FCD } (\text{RLD})(\lambda i \in \text{dom } f; \text{Src } f_{i}); \text{FCD } (\lambda i \in \text{dom } f; \text{Dst } f_{i}) \right) = \text{FCD } \left( \lambda i \in \text{dom } f; \text{Src } f_{i} \right);
\]

and upgrading is taken on the filtrator

\[
\left( \text{FCD } (\text{RLD})(\lambda i \in \text{dom } f; \text{Src } f_{i}); \text{FCD } (\lambda i \in \text{dom } f; \text{Dst } f_{i}) \right).
\]
where we equate \( n \)-ary relations with corresponding principal multifuncoids and principal multireloidists, when appropriate.

**Proof.** Leave as an exercise for the reader. \( \square \)

**Conjecture 17.219.** \( \uparrow^{\text{FCD}}A \left[ \prod_i (C_f) \right] \uparrow^{\text{FCD}}B \iff \uparrow^{\text{RLD}}A \left[ \prod_i (A_f) \right] \uparrow^{\text{RLD}}B \) for every indexed family \( f \) of funcoids and \( A \in \mathcal{P} \prod_{i \in \text{dom} f} \text{Src} f_i, \ B \in \mathcal{P} \prod_{i \in \text{dom} f} \text{Dst} f_i \).

**Theorem 17.220.** For every filters \( a_0, a_1, b_0, b_1 \) we have
\[
a_0 \times^{\text{FCD}} b_0 \left[ f \times^{(C)} g \right] a_1 \times^{\text{FCD}} b_1 \iff a_0 \times^{\text{RLD}} b_0 \left[ f \times^{(A)} g \right] a_1 \times^{\text{RLD}} b_1.
\]

**Proof.** \( a_0 \times^{\text{RLD}} b_0 \left[ f \times^{(A)} g \right] a_1 \times^{\text{RLD}} b_1 \iff \forall A_0 \in a_0, B_0 \in b_0, A_1 \in a_1, B_1 \in b_1: A_0 \times B_0 \left[ f \times^{(A)} g \right]^* A_1 \times B_1.
\]

Thus it is equivalent to \( a_0 \left[ f \right] a_1 \land b_0 \left[ g \right] b_1 \) that is \( a_0 \times^{\text{FCD}} b_0 \left[ f \times^{(C)} g \right]^* a_1 \times^{\text{FCD}} b_1. \) (It was used the theorem 17.148.) \( \square \)

Can the above theorem be generalized for the infinitary case?

### 17.15 Coordinate-wise continuity

**Theorem 17.221.** Let \( \mu \) and \( \nu \) be indexed (by some index set \( n \)) families of endomorphisms for a quasi-invertible dagger category with star-morphisms, and \( f_i \in \text{Mor}(\text{Ob} \mu_i; \text{Ob} \nu_i) \) for every \( i \in n \). Then:

1. \( \forall i \in n: f_i \in C(\mu_i; \nu_i) \Rightarrow \prod_i (C_f) f \in C \left( \prod_i (C) \mu; \prod_i (C) \nu \right); \)
2. \( \forall i \in n: f_i \in C'(\mu_i; \nu_i) \Rightarrow \prod_i (C') f \in C' \left( \prod_i (C') \mu; \prod_i (C') \nu \right); \)
3. \( \forall i \in n: f_i \in C''(\mu_i; \nu_i) \Rightarrow \prod_i (C'') f \in C'' \left( \prod_i (C'') \mu; \prod_i (C'') \nu \right). \)

**Proof.** Using the corollary 17.137:

1. \( \forall i \in n: f_i \in C(\mu_i; \nu_i) \Rightarrow \forall i \in n: f_i \circ \mu_i \subseteq \nu_i \circ f_i \Rightarrow \prod_i (C_f) \circ (f_i \circ \mu_i) \subseteq \prod_i (C_f) \circ (\nu_i \circ f_i) \Rightarrow \left( \prod_i (C_f) \right) \circ (\prod_i (C) \mu) \subseteq \left( \prod_i (C_f) \right) \circ (\prod_i (C) \nu \circ (\prod_i (C_f) \circ (\prod_i (C) \mu) \cap (\prod_i (C) \nu)). \)
2. \( \forall i \in n: f_i \in C'(\mu_i; \nu_i) \Rightarrow \forall i \in n: \mu_i \subseteq (f_i \circ \mu_i) \subseteq \prod_i (C_f) \circ (f_i \circ \mu_i) \subseteq \prod_i (C_f) \circ (\nu_i \circ f_i) \Rightarrow \left( \prod_i (C_f) \right) \circ (\prod_i (C) \mu) \subseteq \left( \prod_i (C_f) \right) \circ (\prod_i (C) \nu \circ (\prod_i (C_f) \circ (\prod_i (C) \mu) \cap (\prod_i (C) \nu)) \right). \)
3. \( \forall i \in n: f_i \in C''(\mu_i; \nu_i) \Rightarrow \forall i \in n: f_i \circ \mu_i \circ f_i \subseteq \nu_i \Rightarrow \prod_i (C_f) \circ (f_i \circ \mu_i \circ f_i) \subseteq \prod_i (C_f) \circ (\nu_i \circ f_i) \Rightarrow \left( \prod_i (C_f) \right) \circ (\prod_i (C) \mu) \subseteq \left( \prod_i (C_f) \right) \circ (\prod_i (C) \nu \circ (\prod_i (C_f) \circ (\prod_i (C) \mu) \cap (\prod_i (C) \nu)) \right). \) \( \square \)

**Theorem 17.222.** Let \( \mu \) and \( \nu \) be indexed (by some index set \( n \)) families of endofuncoids, and \( f_i \in \text{FCD}(\text{Ob} \mu_i; \text{Ob} \nu_i) \) for every \( i \in n \). Then:

1. \( \forall i \in n: f_i \in C(\mu_i; \nu_i) \Rightarrow \prod_i (A_f) f \in C \left( \prod_i (A) \mu; \prod_i (A) \nu \right); \)
2. \( \forall i \in n: f_i \in C'(\mu_i; \nu_i) \Rightarrow \prod_i (A_f) f \in C' \left( \prod_i (A) \mu; \prod_i (A) \nu \right); \)
3. \( \forall i \in n: f_i \in C''(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in C''(\prod^{(A)} \mu; \prod^{(A)} \nu). \)

**Proof.** Similar to the previous theorem.

**Theorem 17.223.** Let \( \mu \) and \( \nu \) be indexed (by some index set \( n \)) families of pointfree endofuncoids between posets with least elements, and \( f_i \in FCD(\text{Ob} \mu_i; \text{Ob} \nu_i) \) for every \( i \in n \). Then:

1. \( \forall i \in n: f_i \in C(\mu_i; \nu_i) \Rightarrow \prod^{(S)} f \in C\left( \prod^{(S)} \mu; \prod^{(S)} \nu \right) \);
2. \( \forall i \in n: f_i \in C'(\mu_i; \nu_i) \Rightarrow \prod^{(S)} f \in C'\left( \prod^{(S)} \mu; \prod^{(S)} \nu \right) \);
3. \( \forall i \in n: f_i \in C''(\mu_i; \nu_i) \Rightarrow \prod^{(S)} f \in C''\left( \prod^{(S)} \mu; \prod^{(S)} \nu \right) \).

**Proof.** Similar to the previous theorem.

### 17.16 Counter-examples

**Example 17.224.** \( || || f \neq f \) for some staroid \( f \) whose form is an indexed family of filters on a set.

**Proof.** Let \( f = \{ A \in \mathfrak{F}(\mathfrak{U}) \mid \uparrow\mathfrak{U} \text{Cor} A \neq \Delta \} \) for some infinite set \( \mathfrak{U} \) where \( \Delta \) is some non-principal filter on \( \mathfrak{U} \).

\[ A \cup B \in f \iff \uparrow\mathfrak{U} \text{Cor} (A \cup B) \neq \Delta \iff \uparrow\mathfrak{U} \text{Cor} A \cup \uparrow\mathfrak{U} \text{Cor} B \neq \Delta \iff \uparrow\mathfrak{U} \text{Cor} A \cap \Delta \neq 0^{\mathfrak{U}(\mathfrak{U})} \vee \uparrow\mathfrak{U} \text{Cor} B \cap \Delta \neq 0^{\mathfrak{U}(\mathfrak{U})} \Rightarrow A \cup B \in f. \]

Obviously \( \emptyset^{\mathfrak{U}(\mathfrak{U})} \notin f \). So \( f \) is a free star. But free stars are essentially the same as 1-staroids.

For the below counter-examples we will define a staroid \( \vartheta \) with arity \( \vartheta = N \) and \( \text{GR} \vartheta \in \mathcal{P}(\mathcal{N}^N) \) (based on a suggestion by Andreas Blass):

\[ A \in \text{GR} \vartheta \iff \sup_{i \in N} \text{card}(A_i \cap i) = N \land \forall i \in N: A_i \neq \emptyset. \]

**Proposition 17.225.** \( \vartheta \) is a staroid.

**Proof.** \((\text{val } \vartheta)_i L = \mathcal{P}N \setminus \{ \emptyset \} \) for every \( L \in (\mathcal{P}N)^{N \setminus \{ i \}} \) if

\[ \sup_{i \in N \setminus \{ i \}} \text{card}(A_j \cap i) = N \land \forall j \in N \setminus \{ i \}: L_j \neq \emptyset. \]

Otherwise \((\text{val } \vartheta)_i L = \emptyset \). Thus \((\text{val } \vartheta)_i L \) is a free star. So \( \vartheta \) is a staroid.

**Proposition 17.226.** \( \vartheta \) is a completary staroid.

**Proof.** \( A_0 \cup A_1 \in \text{GR } \vartheta \iff A_0 \cup A_1 \in \text{GR } \vartheta \iff \sup_{i \in N} \text{card}((A_{0i} \cup A_{1i}) \cap i) = N \land \forall i \in N: A_{0i} \cup A_{1i} \neq \emptyset \iff \sup_{i \in N} \text{card}((A_{0i} \cap i) \cup (A_{1i} \cap i)) = N \land \forall i \in N: A_{0i} \cup A_{1i} \neq \emptyset \).

If \( A_0 \cap i = \emptyset \) then \( A_0 \cap i = \emptyset \) and thus \( A_{0i} \cap i \supseteq A_{0i} \cap i \). Thus we can select \( c(i) \in \{0, 1\} \) in such a way that \( \forall d \in \{0, 1\}: c(A_d \cap i \cap i) \supseteq c(A_{d \cap i} \cap i) \) and \( A_{c(i) \cap i} \neq \emptyset \). (Consider the case \( A_0 \cap i, A_1 \cap i \neq \emptyset \) and the similar cases \( A_0 \cap i = \emptyset \) and \( A_1 \cap i = \emptyset \).)

So \( A_0 \cup A_1 \in \text{GR } \vartheta \iff \sup_{i \in N} \text{card}(A_{c(i) \cap i} \cap i) = N \land \forall i \in N: A_{c(i) \cap i} \neq \emptyset \iff (\lambda i \in N: A_{c(i)} \cap i) \in \text{GR } \vartheta, \)

Thus \( \vartheta \) is completary.

**Obvious 17.227.** \( \vartheta \) is non-zero.

**Example 17.228.** For every family \( a = a_i \in N \) of ultrafilters \( \prod^{\text{Strd}} a \) is not an atom nor of the poset of staroids neither of the poset of completary staroids of the form \( \lambda i \in N: \text{Base}(a_i) \).

**Proof.** It’s enough to prove \( \vartheta \notin \prod^{\text{Strd}} a \).
Let $\uparrow^N R_i = a_i$ if $a_i$ is principal and $R_i = N \setminus i$ if $a_i$ is non-principal.

We have $\forall i \in N: R_i \in a_i$.

We have $R \notin GR \vartheta$ because $\sup_{i \in N} \text{card}(R_i \cap i) \neq N$.

$R \in \prod^{\text{Std}} a$ because $\forall X \in a_i: X \cap R_i \neq \emptyset$.

So $\vartheta \notin \prod^{\text{Std}} a$. 

\begin{remark}
At http://mathoverflow.net/questions/60925/special-infinite-relations-and-ultrafilters there are a proof for arbitrary infinite form, not just for $N$.
\end{remark}

\begin{conjecture}
There exists a non-completary staroid.
\end{conjecture}

\begin{conjecture}
There exists a prestaroid which is not a staroid.
\end{conjecture}

\begin{conjecture}
The set of staroids of the form $A^B$ where $A$ and $B$ are sets is atomic.
\end{conjecture}

\begin{conjecture}
The set of staroids of the form $A^B$ where $A$ and $B$ are sets is atomistic.
\end{conjecture}

\begin{conjecture}
The set of completary staroids of the form $A^B$ where $A$ and $B$ are sets is atomic.
\end{conjecture}

\begin{conjecture}
The set of completary staroids of the form $A^B$ where $A$ and $B$ are sets is atomistic.
\end{conjecture}

\begin{example}
StarComp(a; f \sqcup g) \neq StarComp(a; f) \sqcup StarComp(a; g)$ in the category of binary relations with star-morphisms for some $n$-ary relation $a$ and an $n$-indexed families $f$ and $g$ of functions.
\end{example}

\begin{proof}
Let $n = \{0, 1\}$. Let $GR a = \{(0; 1), (1; 0)\}$ and $f = \{(1; 0), (1; 1)\}$. For every $0, 1$-indexed family of $\mu$ of functions:

$L \in \text{StarComp}(a; \mu) \iff \exists g \in a: (y_0 \mu_0 L_0 \land y_1 \mu_1 L_1) \iff \exists y_0 \in \text{dom} \mu_0, y_1 \in \text{dom} \mu_1: (y_0 \mu_0 L_0 \land y_1 \mu_1 L_1)$

for every $n$-ary relation $\mu$.

Consequently

$L \in \text{StarComp}(a; f) \iff L_0 = 1 \land L_1 = 0 \iff L = (0; 1)$

that is $\text{StarComp}(a; f) = \{(1; 0)\}$. Similarly

$\text{StarComp}(a; g) = \{(0; 1)\}$.

Also

$L \in \text{StarComp}(a; f \sqcup g) \iff \exists y_0, y_1 \in \{0, 1\}: ((y_0 f_0 L_0 \lor y_0 g_0 L_0) \land (y_1 f_1 L_1 \lor y_1 g_1 L_1))$.

Thus

$\text{StarComp}(a; f \sqcup g) = \{(0; 1), (1; 0), (0; 0), (1; 1)\}$. 

\end{proof}

\begin{corollary}
The above inequality is possible also for star-morphisms of funcoids and star-morphisms of reloids.
\end{corollary}

\begin{proof}
Because finitary funcoids and reloids between finite sets are essentially the same as finitary relations and our proof above works for binary relations.
\end{proof}

17.17 Conjectures

\begin{remark}
Below I present special cases of possible theorems. The theorems may be generalized after the below special cases are proved.
\end{remark}

\begin{conjecture}
For every two $a$. funcoids; $b$. of reloids $f$ and $g$ we have:

1. $(\text{RLD})_{ab} [f \times^{(DP)} g] (\text{RLD})_{ab} b \iff a [f \times^{(C)} g] b$ for every funcoids $a \in \text{FCD}(\text{Src} f; \text{Src} g)$,

$b \in \text{FCD}(\text{Dst} f; \text{Dst} g)$;

\end{conjecture}
2. \((\text{RLD})_{\text{out}} a \left[ f \times (\text{DP}) \right] \) \((\text{RLD})_{\text{out}} b \leftrightarrow a \left[ f \times (C) \right] b\) for every funcoids \(a \in \text{FCD}(\text{Src} f; \text{Src} g), b \in \text{FCD}(\text{Dst} f; \text{Dst} g)\);

3. \((\text{FCD}) a \left[ f \times (C) \right] \) \((\text{FCD}) b \leftrightarrow a \left[ f \times (\text{DP}) \right] b\) for every reloids \(a \in \text{RLD}(\text{Src} f; \text{Src} g), b \in \text{RLD}(\text{Dst} f; \text{Dst} g)\).

**Definition 17.240.** A staroid on power sets is such a staroid \(f\) that every (form \(f\)) is a lattice of all subsets of some set.

**Conjecture 17.241.** \(\prod^\text{Strd} a \neq \prod^\text{Strd} b \leftrightarrow b \in \prod^\text{Strd} a \leftrightarrow a \in \prod^\text{Strd} b \leftrightarrow \forall i \in n: a_i \neq b_i\) for every \(n\)-indexed families \(a\) and \(b\) of filters on powersets.

**Conjecture 17.242.** Let \(f\) be a staroid on powersets and \(a \in \prod_{i \in \text{arity} f} \text{Src} f, b \in \prod_{i \in \text{arity} f} \text{Dst} f\). Then

\[
\prod^\text{Strd} a \left[ \prod^{(C)} \right] \prod^\text{Strd} b \leftrightarrow \forall i \in n: a_i [f_i] b_i.
\]

**Proposition 17.243.** The conjecture 17.242 is a consequence of the conjecture 17.241.

**Proof.** \(\prod^\text{Strd} a \left[ \prod^{(C)} \right] \prod^\text{Strd} b \leftrightarrow \prod^\text{Strd} b \neq \prod_{i \in n} (f_i) a_i \leftrightarrow \forall i \in n: b_i \neq (f_i) a_i \leftrightarrow \forall i \in n: a_i [f_i] b_i\).

\(
\square
\)

**Conjecture 17.244.** For every indexed families \(a\) and \(b\) of filters and an indexed family \(f\) of pointfree funcoids we have

\[
\prod^\text{Strd} a \left[ \prod^{(C)} \right] \prod^\text{Strd} b \leftrightarrow \prod^\text{RLD} a \left[ \prod^{(\text{DP})} \right] \prod^\text{RLD} b.
\]

Strengthening of an above result:

**Conjecture 17.245.** If \(a\) is a completary staroid and \(\text{Dst} f_i\) is a starrish poset for every \(i \in n\) then \(\text{StarComp}(a; f)\) is a completary staroid.

Straightening of above results:

**Conjecture 17.246.**

1. \(\prod^{(D)} f\) is a prestaroid if every \(F_i\) is a prestaroid.
2. \(\prod^{(D)} f\) is a completary staroid if every \(F_i\) is a completary staroid.

**Conjecture 17.247.** If \(f_1\) and \(f_2\) are funcoids, then there exists a pointfree funcoid \(f_1 \times f_2\) such that

\[
(f_1 \times f_2)x = \bigcup \{(f_1)X \times \text{FCD}(f_2)X \mid X \in \text{atoms} x\}
\]

for every ultrafilter \(x\).

**Conjecture 17.248.** Let \(\mathfrak{A} = \mathfrak{A}_{i \in n}\) be a family of boolean lattices.

A relation \(\delta \in \mathcal{P}\prod\text{atoms}^{\uparrow \mathfrak{A}}\mathfrak{A}\) such that for every \(a \in \prod\text{atoms}^{\uparrow \mathfrak{A}}\mathfrak{A}\)

\[
\forall A \in a: \delta \cap \prod_{i \in n} \text{atoms}^{\uparrow \mathfrak{A}} A_i \neq \emptyset \Rightarrow a \in \delta
\]

\(17.5\)

can be continued till the function \(\| f \) for a unique staroid \(f\) of the form \(\lambda i \in n: \mathcal{P}(\mathfrak{A}_i)\). The funcoid \(f\) is completary.

For every \(\mathcal{X} \in \prod_{i \in n} \mathfrak{A}\)

\[
\mathcal{X} \in \text{GR} \| f \leftrightarrow \delta \cap \prod_{i \in n} \text{atoms} \mathcal{X}_i \neq \emptyset,
\]

\(17.6\)
**Conjecture 17.249.** Let $R$ be a set of staroids of the form $\lambda i \in n: \mathfrak{F}(\mathfrak{A}_i)$ where every $\mathfrak{A}_i$ is a boolean lattice. If $x \in \prod_{i \in n} \text{atoms}^\mathfrak{F}(\mathfrak{A}_i)$ then $x \in \text{GR} \iff \forall f \in R: x \in \text{GR}$. 

17.17.1 Informal questions

Do products of funcoids and reloids coincide with Tychonov topology?
- Limit and generalized limit for multiple arguments.
- Is product of connected spaces connected?
- Product of $T_0$-separable is $T_0$, of $T_1$ is $T_1$?
- Relationships between multireloids and staroids.
- Generalize the section “Specifying funcoids by functions or relations on atomic filters” from [28].
- Generalize “Relationships between funcoids and reloids”.
- Explicitly describe the set of complemented funcoids.
Chapter 18
Identity staroids

18.1 Additional propositions

Proposition 18.1. \( \{ (f)_k X \mid X \in \operatorname{up} \left( \prod_{i \in \mathbb{N} \setminus \{k\}} A_i, \prod_{i \in \mathbb{N} \setminus \{k\}} \mathcal{B}_i \right) \mathcal{X} \} \) is a filter base on \( \mathfrak{A}_k \) for every family \( (\mathfrak{A}_i; \mathcal{B}_i) \) of filtrators where \( i \in n \) for some index set \( n \) (provided that \( f \) is a multifuncoid of the form \( \mathfrak{A} \) and \( k \in n \) and \( \mathcal{X} \in \prod_{i \in \mathbb{N} \setminus \{k\}} \mathfrak{A}_i \)).

Proof. Let \( \mathcal{K}, \mathcal{L} \in \{ (f)_k X \mid X \in \operatorname{up} \mathcal{X} \} \). Then there exist \( X, Y \in \operatorname{up} \mathcal{X} \) such that \( \mathcal{K} = (f)_k X, \mathcal{L} = (f)_k Y \). We can take \( Z \in \operatorname{up} \mathcal{X} \) such that \( Z \subseteq X, Y \). Then evidently \( (f)_k Z \subseteq \mathcal{K} \) and \( (f)_k Z \subseteq \mathcal{L} \) and \( (f)_k Z \in \{ (f)_k X \mid X \in \operatorname{up} \mathcal{X} \} \). □

[FIXME: The following theorem seems erroneous and even a nonsense. Should remove it after checking that nothing below depends on it.]

Proposition 18.2. \( \langle \prod f_k \rangle \mathcal{X} = \prod_{X \in \operatorname{up} \mathcal{X}} (f)_k X \) for a filterator \( \left( \prod_{i \in \mathbb{N} \setminus \{k\}} \mathfrak{A}_i, \prod_{i \in \mathbb{N} \setminus \{k\}} \mathcal{B}_i \right) (i \in n \) for some index set \( n \) where every \( \mathfrak{A}_i \) is a boolean lattice, \( k \in n \), and \( \mathcal{X} \in \prod_{i \in \mathbb{N} \setminus \{k\}} \mathfrak{A}_i \).

Proof. \( \mathfrak{A}_k \) is separable by obvious 4.136. \( (\mathfrak{A}_k; \mathcal{B}_k) \) is with separable core by theorem 4.112.

Proposition 18.3. Pseudofuncoid from a set \( A \) to a set \( B \) is a relation \( f \) between filters on \( A \) and \( B \) such that:

\( \neg \left( f \circ 0 \right), \ \mathcal{I} \cap \mathcal{J} f \mathcal{K} \Leftrightarrow \mathcal{I} f \mathcal{K} \cap \mathcal{J} f \mathcal{K} \) (for every \( \mathcal{I}, \mathcal{J} \in \mathfrak{A}(A), \mathcal{K} \in \mathfrak{B}(B) \)),

\( \neg \left( f \circ I \right), \ \mathcal{K} f \mathcal{I} \cup \mathcal{J} \Leftrightarrow \mathcal{K} f \mathcal{I} \cup \mathcal{K} f \mathcal{J} \) (for every \( \mathcal{I}, \mathcal{J} \in \mathfrak{B}(B), \mathcal{K} \in \mathfrak{A}(A) \)).

Obvious 18.4. Pseudofuncoid is just a staroid of the form \( (\mathfrak{A}(A); \mathfrak{B}(B)) \).

Obvious 18.5. \( [f] \) is a pseudofuncoid for every funcoid \( f \).

Example 18.6. If \( A \) and \( B \) are infinite sets, then there exist two distinct pseudofuncoids \( f \) and \( g \) from \( A \) to \( B \) such that \( f \cap (\mathfrak{F} \times \mathfrak{F}) = g \cap (\mathfrak{F} \times \mathfrak{F}) = [c] \cap (\mathfrak{F} \times \mathfrak{F}) \) for some funcoid \( c \).

Remark 18.7. Considering a pseudofuncoid \( f \) as a staroid, we get \( f \cap (\mathfrak{F} \times \mathfrak{F}) = \|reak f \|reak \).

Proof. Take \( f = \{ (\mathcal{X}, \mathcal{Y}) \mid \mathcal{X} \in \mathfrak{A}(A), \mathcal{Y} \in \mathfrak{B}(B), \bigcap \mathcal{X} \) and \( \bigcap \mathcal{Y} \) are infinite \} \)

239
and
\[ g = f \cup \{(X; Y) \mid X \in \mathcal{F}(A), Y \in \mathcal{F}(B), X \supseteq a, Y \supseteq b\} \]

where \( a \) and \( b \) are nontrivial ultrafilters on \( A \) and \( B \) correspondingly. \( c \) is the funcoid defined by the relation
\[ [c]^* = \delta = \{(X;Y) \mid X \in \mathcal{P}A, Y \in \mathcal{P}B, X \text{ and } Y \text{ are infinite}\}. \]

First prove that \( f \) is a pseudofuncoid. The formulas \( (f \not\in \mathcal{P}) \) and \( \neg f \not\in \mathcal{P} \) are obvious. We have 
\[ \mathcal{I} \sqcup J f K \Leftrightarrow \bigcap (\mathcal{I} \sqcup J) \text{ and } \bigcap \mathcal{J} \text{ are infinite} \Leftrightarrow \bigcap \mathcal{I} \text{ or } \bigcap \mathcal{J} \text{ is infinite} \wedge \bigcap \mathcal{Y} \text{ are infinite} \Leftrightarrow \bigcap (\mathcal{I} \text{ or } \mathcal{J}) \text{ is infinite} \wedge \bigcap \mathcal{Y} \text{ are infinite} \vee (\bigcap \mathcal{I} \text{ and } \bigcap \mathcal{Y} \text{ are infinite}) \Leftrightarrow \mathcal{I} f K \vee J f K. \]

Similarly, \( K f \mathcal{I} \sqcup J \Leftrightarrow K f \mathcal{I} \vee K f J. \) So \( f \) is a pseudofuncoid.

Let now prove that \( g \) is a pseudofuncoid. The formulas \( (g \not\in \mathcal{P}) \) and \( \neg (g \not\in \mathcal{P}) \) are obvious. Let 
\[ \mathcal{I} \sqcup J g K. \]
Then either \( \mathcal{I} \sqcup J g K \) and then \( \mathcal{I} \sqcup J g K \) or \( \mathcal{I} \sqcup J \supseteq a \) and then \( \mathcal{I} \supseteq a \vee J \supseteq a \) thus having \( \mathcal{I} g K \vee J g K. \) The reverse implication is obvious. We have 
\[ \mathcal{I} \sqcup J g K \Leftrightarrow \mathcal{I} g K \vee J g K \text{ and similarly } K g \mathcal{I} \sqcup J \Leftrightarrow K g \mathcal{I} \vee K g J. \]

So \( g \) is a pseudofuncoid.

Obviously \( f \neq g \) (\( a, g, b \) but not \( a, f, b \)).

It remains to prove \( f \cap (\mathcal{P} \times \mathcal{Q}) = g \cap (\mathcal{P} \times \mathcal{Q}) = [c] \cap (\mathcal{P} \times \mathcal{Q}) \). Really, \( f \cap (\mathcal{P} \times \mathcal{Q}) = [c] \cap (\mathcal{P} \times \mathcal{Q}) \) is obvious. If \( (\uparrow A; \uparrow Y) \in g \cap (\mathcal{P} \times \mathcal{Q}) \) then either \( (\uparrow A; \uparrow Y) \in f \cap (\mathcal{P} \times \mathcal{Q}) \) or \( X \in \text{ up } a, Y \in \text{ up } b, \) so \( X \) and \( Y \) are infinite and thus \( (\uparrow A; \uparrow Y) \in f \cap (\mathcal{P} \times \mathcal{Q}) \). So \( g \cap (\mathcal{P} \times \mathcal{Q}) = f \cap (\mathcal{P} \times \mathcal{Q}) \). \( \square \)

**Remark 18.8.** The above counter-example shows that pseudofuncoids (and more generally, any staroids on filters) are “second class” objects, they are not full-fledged because they don’t bijectively correspond to funcoids and the elegant funcoid theory does not apply to them.

From the above it follows that staroids on filters do not correspond (by restriction) to staroids on principal filters (or staroids on sets).

### 18.3 Complete staroids and multifuncoids

#### 18.3.1 Complete free stars

**Definition 18.9.** Let \( \mathfrak{A} \) be a poset. **Complete free stars** on \( \mathfrak{A} \) are such \( S \in \mathcal{P}\mathfrak{A} \) that the least element (if it exists) is not in \( S \) and for every \( T \in \mathcal{P}\mathfrak{A} \)
\[ \forall Z \in \mathfrak{A}: (\forall X \in T; Z \supseteq X \Rightarrow Z \subseteq S) \Leftrightarrow T \cap S \neq \emptyset. \]

**Obvious 18.10.** Every complete free star is a free star.

**Proposition 18.11.** \( S \in \mathcal{P}\mathfrak{A} \) where \( \mathfrak{A} \) is a poset is a complete free star iff all the following:

1. The least element (if it exists) is not in \( S \).
2. \( \forall Z \in \mathfrak{A}: (\forall X \in T; Z \supseteq X \Rightarrow Z \subseteq S) \Rightarrow T \cap S \neq \emptyset. \)
3. \( S \) is an upper set.

**Proof.**

\( \Rightarrow. \) (1) and (2) are obvious. \( S \) is an upper set because \( S \) is a free star.

\( \Leftarrow. \) We need to prove that
\[ \forall Z \in \mathfrak{A}: (\forall X \in T; Z \supseteq X \Rightarrow Z \subseteq S) \Leftarrow T \cap S \neq \emptyset. \]

Let \( X' \in T \cap S \). Then \( \forall X \in T; Z \supseteq X \Rightarrow Z \subseteq S \) because \( S \) is an upper set.

**Proposition 18.12.** Let \( S \) be a complete lattice. \( S \in \mathcal{P}\mathfrak{A} \) is a complete free star iff all the following:

1. The least element (if it exists) is not in \( S \).
2. \( \bigcup T \in S \Rightarrow T \cap S \neq \emptyset \) for every \( T \in \mathcal{P}S \).
3. \( S \) is an upper set.

**Proof.**

\( \Rightarrow \). We need to prove only \( \bigcup T \in S \Rightarrow T \cap S \neq \emptyset \). Let \( \bigcup T \in S \). Because \( S \) is an upper set, we have \( \forall X \in T : Z \supseteq X \Rightarrow Z \supseteq \bigcup T \Rightarrow Z \in S \) from which we conclude \( T \cap S \neq \emptyset \).

\( \Leftarrow \). We need to prove only \( \forall Z \in \mathfrak{A} : (\forall X \in T : Z \supseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset \).

Really, if \( \forall Z \in \mathfrak{A} : (\forall X \in T : Z \supseteq X \Rightarrow Z \in S) \), then \( \bigcup T \in S \) and thus \( \bigcup T \in S \Rightarrow T \cap S \neq \emptyset \). \( \square \)

**Proposition 18.13.** Let \( \mathfrak{A} \) be a complete lattice. \( S \in \mathcal{P}\mathfrak{A} \) is a complete free star if the least element (if it exists) is not in \( S \) and for every \( T \in \mathcal{P}S \)

\[
\bigcup T \in S \Leftrightarrow T \cap S \neq \emptyset.
\]

**Proof.**

\( \Rightarrow \). We need to prove only \( \bigcup T \in S \Rightarrow T \cap S \neq \emptyset \) what follows from that \( S \) is an upper set.

\( \Leftarrow \). We need to prove only that \( S \) is an upper set. To prove this we can use the fact that \( S \) is a free star. \( \square \)

### 18.3.1.1 Completely starrish posets

**Definition 18.14.** I will call a poset **completely starrish** when the full star \( \star a \) is a complete free star for every element \( a \) of this poset.

**Obvious 18.15.** Every completely starrish poset is starrish.

**Proposition 18.16.** Every complete join infinite distributive lattice is starrish.

**Proof.** Let \( \mathfrak{A} \) be a join infinite distributive lattice, \( a \in \mathfrak{A} \). Obviously \( 0 \notin \star a \) (if 0 exists); obviously \( \star a \) is an upper set. If \( \bigcup T \in \star a \), then \( (\bigcup T) \cap a \) is non-least that is \( \bigcup (a \cap T) \) is non-least what is equivalent to \( a \cap x \) being non-least for some \( x \in T \) that is \( x \in \star a \). \( \square \)

**Theorem 18.17.** If \( \mathfrak{A} \) is a completely starrish complete lattice lattice then

\[
\text{atoms} \bigcup T = \bigcup \langle \text{atoms} \rangle T.
\]

for every \( T \in \mathcal{P}\mathfrak{A} \).

**Proof.** For every atom \( c \) we have: \( c \in \text{atoms} \bigcup T \Leftrightarrow c \notin \bigcup T \Leftrightarrow \bigcup T \in \star c \Leftrightarrow \exists X \in T : X \in \star c \Leftrightarrow \exists X \in T : X \neq c \Leftrightarrow \exists X \in T : c \in \text{atoms} X \Leftrightarrow c \in \bigcup \langle \text{atoms} \rangle T \). \( \square \)

### 18.3.2 More on free stars and complete free stars

**Obvious 18.18.** \( \partial F = \bigcup \star F \) for an element \( F \) of down-aligned finitely meet closed filtrator.

**Corollary 18.19.** \( \partial F = \bigcup \star F \) for every filter \( F \) on a poset.

**Proposition 18.20.** \( \star F = \bigcup \partial F \) for an element \( F \) of a filtrator with separable core.

**Proof.** \( X \in \bigcup \partial F \Leftrightarrow X \subseteq \partial F \Leftrightarrow \forall X \in \cup X : X \neq \partial F \Leftrightarrow \partial X \neq F \Leftrightarrow X \in \star F \). \( \square \)

**Corollary 18.21.** \( \star F = \bigcup \partial F \) for every filter \( F \) on a distributive lattice with least element.

**Proposition 18.22.** For a semifiltered, star-separable, down-aligned filtrator \( (\mathfrak{A}; \mathfrak{B}) \) with finitely meet closed and separable core where \( \mathfrak{B} \) is a complete boolean lattice and both \( \mathfrak{B} \) and \( \mathfrak{A} \) are atomistic lattices the following conditions are equivalent for any \( F \in \mathfrak{A} \):

1. \( F \in \mathfrak{B} \).
2. $\partial F$ is a complete free star on $\mathfrak{F}$.

3. $\star F$ is a complete free star on $\mathfrak{F}$.

**Proof.**

(1)$\Rightarrow$(2). That $\partial F$ does not contain the least element is obvious. That $\partial F$ is an upper set is obvious. So it remains to apply theorem 4.53.

(2)$\Rightarrow$(3). That $\star F$ does not contain the least element is obvious. That $\star F$ is an upper set is obvious. So it remains to apply theorem 4.53.

(3)$\Rightarrow$(1). Apply theorem 4.53. \hfill $\square$

**Corollary 18.23.** For a filter $F \in \mathfrak{F}$ on a complete atomic boolean lattice the following conditions are equivalent:

1. $F \in \mathfrak{F}$.

2. $\partial F$ is a complete free star on $\mathfrak{F}$.

3. $\star F$ is a complete free star on $\mathfrak{F}$.

**Theorem 18.24.** Let $\mathfrak{F}$ be a boolean lattice. For any set $S \in \mathcal{P}\mathfrak{F}$ there exists a principal filter $\mathcal{A}$ such that $\partial \mathcal{A} = S$ iff $S$ is a complete free star (on $\mathfrak{F}$).

**Proof.**

$\Rightarrow$. From the previous theorem.

$\Leftarrow$. $0^\mathfrak{F} \not\in S$ and $\bigcup T \in S \Leftrightarrow T \cap S \neq \emptyset \Leftrightarrow \exists X \in T: X \in S$. Take $\mathcal{A} = \langle X \mid X \in \mathfrak{F} \setminus S \rangle$. We will prove that $\mathcal{A}$ is a principal filter. That $\mathcal{A}$ is a filter follows from properties of free stars. It remains to show that $\mathcal{A}$ is a principal filter. It follows from the following equivalence:

\[ \bigcup^\mathfrak{F} A \in \mathcal{A} \Leftrightarrow \bigcup^\mathfrak{F} (\neg)A \in \mathcal{A} \Leftrightarrow \bigcup^\mathfrak{F} (\neg)A \not\in S \Leftrightarrow \neg \exists X \in (\neg)A: X \in S \Leftrightarrow \forall X \in (\neg)A: X \not\in S \Leftrightarrow \forall X \in A: X \in A \Leftrightarrow 1. \]

$\square$

**Proposition 18.25.**

1. If $S$ is a free star on $\mathfrak{F}$ then $\| S$ is a free star on $\mathfrak{F}$, provided that $\mathfrak{F}$ is a join-semilattice and the filtrator $(\mathfrak{F}; \mathfrak{F})$ is down-aligned and with finitely join-closed core.

2. If $S$ is a free star on $\mathfrak{F}$ then $\| S$ is a free star on $\mathfrak{F}$, provided that $\mathfrak{F}$ is a boolean lattice.

**Proof.**

1. $X \cup^\mathfrak{F} Y \in \| S \Leftrightarrow X \cup^\mathfrak{F} Y \in S \Leftrightarrow X \cup^\mathfrak{F} Y \in S \Leftrightarrow X \in S \cap Y \in S \Leftrightarrow X \in \| S \vee Y \in \| S$ for every $X, Y \in \mathfrak{F}$; $0 \notin \| S$ is obvious.

2. There exists a filter $F$ such that $S = \partial F$. For every filters $X, Y \in \mathfrak{F}$:

$X \cup^\mathfrak{F} Y \in \| S \Leftrightarrow \up(X \cup^\mathfrak{F} Y) \subseteq S \Leftrightarrow \forall K \in \up(X \cup^\mathfrak{F} Y): K \notin \partial F \Leftrightarrow \forall K \in \up(X \cup^\mathfrak{F} Y): K \notin \partial F \Leftrightarrow \forall K \in \up(X \cup^\mathfrak{F} Y): K \notin \partial F \Leftrightarrow \forall X \in \up Y: X \notin \partial F \vee \forall X \in \up Y: X \notin \partial F \vee \up X \subseteq S \Leftrightarrow X \in \| S \forall Y \in \| S:

$0 \notin \| S \Leftrightarrow \up 0 \subseteq S \Leftrightarrow 0 \in S$ what is false. \hfill $\square$

**Corollary 18.26.** If $S$ is a free star on $\mathfrak{F}$ then $\| S$ is a free star on $\mathfrak{F}$, provided that $\mathfrak{F}$ is a join-semilattice.

**Proposition 18.27.**

1. If $S$ is a complete free star on $\mathfrak{F}$ then $\| S$ is a complete free star on $\mathfrak{F}$, provided that $\mathfrak{F}$ is a complete lattice and the filtrator $(\mathfrak{F}; \mathfrak{F})$ is down-aligned and with join-closed core.

2. If $S$ is a complete free star on $\mathfrak{F}$ then $\| S$ is a complete free star on $\mathfrak{F}$, provided that $\mathfrak{F}$ is a boolean lattice.
18.4 Complete staroids and multifuncoids

Proof.

1. \( \bigcup^3 T \in \bigcup S \iff \bigcup^3 T \in S \iff \bigcup^3 T \in S \iff T \cap S \neq \emptyset \iff T \cap S \neq \emptyset \) for every \( T \in \mathcal{P}^3 \); \( 0 \notin \bigcup S \) is obvious.

2. There exists a principal filter \( F \) such that \( S = \partial F \).

\[ \bigcup^3 T \in \bigcup S \iff \text{up } \bigcup^3 T \in S \iff \forall K \in \text{up } \bigcup^3 T: K \in \partial F \iff \forall K \in \text{up } \bigcup^3 T: K \neq F \iff \exists K \in T: K \in \partial F \iff \exists K \in T: K \neq F \iff \exists K \in T: \forall K \in \text{up } F. \]

\( 0 \notin \bigcup S \iff \text{up } 0 \subseteq S \iff 0 \in S \) what is false. \( \Box \)

Corollary 18.28. If \( S \) is a complete free star on \( \mathfrak{F} \) then \( \bigcup S \) is a complete free star on \( \mathfrak{F} \), provided that \( \mathfrak{F} \) is a complete lattice.

18.4 Complete staroids and multifuncoids

Definition 18.29. Consider an indexed family \( \mathfrak{Z} \) of posets. A pre-staroid \( f \) of the form \( \mathfrak{Z} \) is complete in argument \( k \in \text{arity } f \) when \( (\text{val } f)_k L \) is a complete free star for every \( L \in \prod_{i \in (\text{arity } f) \setminus \{k\}} \mathfrak{Z}_i \).

Definition 18.30. Consider an indexed family \( (\mathfrak{A}_i; \mathfrak{Z}_i) \) of filters and pre-multifuncoid \( f \) is of the form \( \prod \mathfrak{Z} \). Then \( f \) is complete in argument \( k \) if \( (f)_k L \in \mathfrak{Z}_k \) for every family \( L \in \prod_{i \in (\text{arity } f) \setminus \{k\}} \mathfrak{Z}_i \).

Proposition 18.31. Consider an indexed family \( (\mathfrak{F}; \mathfrak{Z}_i) \) of primary filters over boolean lattices. Let \( f \) be a pre-multifuncoid of the form \( \mathfrak{F} \) and \( k \in \text{arity } f \). The following are equivalent:

1. Pre-multifuncoid \( f \) is complete in argument \( k \).

2. Pre-staroid \( \bigcup f \) is complete in argument \( k \).

Proof. Let \( L \in \prod \mathfrak{Z} \). We have \( L \in \text{GR } \bigcup f \iff L_k \neq (f)_k L \mid (\text{dom } L) \setminus \{1\} \); \( (\text{val } f)_k L = \partial (f)_k L \) by the theorem 17.80.

So \( (\text{val } f)_k L \) is a complete free star when \( (f)_k L \in \mathfrak{Z}_k \) (proposition 18.22) for every \( L \in \prod_{i \in (\text{arity } f) \setminus \{k\}} \mathfrak{Z}_i \). \( \Box \)

Example 18.32. Consider funcoid \( f = \text{id}^{\mathcal{FCD}(U)} \). It is obviously complete in each of its two arguments. Then \( f \) is not complete in each of its two arguments because \( (\mathcal{X}; \mathcal{Y}) \in [f] \iff \mathcal{X} \neq \mathcal{Y} \) what does not generate a complete free star if one of the arguments (say \( \mathcal{X} \)) is a fixed nonprincipal filter.

Theorem 18.33. Consider a semi-filtered, star-separable, down-aligned filter \((\mathfrak{A}; \mathfrak{Z})\) with finitely meet closed and separable core where \( \mathfrak{Z} \) is a complete boolean lattice and both \( \mathfrak{A} \) and \( \mathfrak{Z} \) are atomistic lattices.

Let \( f \) be a multifuncoid of the aforementioned form. Let \( k, l \in \text{arity } f \) and \( k \neq l \). The following are equivalent:

1. \( f \) is complete in the argument \( k \).

2. \( \langle f \rangle_l (L \cup \{(k; x)\}) = \bigcup_{x \in X} \langle f \rangle_l (L \cup \{(k; x)\}) \) for every \( X \in \mathcal{P} \mathfrak{Z}_k, L \in \prod_{i \in (\text{arity } f) \setminus \{k, l\}} \mathfrak{Z}_i \).

3. \( \langle f \rangle_l (L \cup \{(k; x)\}) = \bigcup_{x \in X} \langle f \rangle_l (L \cup \{(k; x)\}) \) for every \( X \in \mathcal{P} \mathfrak{A}_k, L \in \prod_{i \in (\text{arity } f) \setminus \{k, l\}} \mathfrak{Z}_i \).

Proof.

(3) \( \Rightarrow \) (2). Obvious.

(2) \( \Rightarrow \) (1). Let \( Y \in \mathfrak{Z} \).

\[ \bigcup X \neq (f)_k (L \cup \{(l; Y)\}) \iff Y \neq (f)_l (L \cup \{(k; X)\}) \iff Y \neq \bigcup_{x \in X} (f)_l (L \cup \{(k; x)\}) \iff \exists x \in X: x \neq (f)_k (L \cup \{(l; Y)\}). \]

It is equivalent (proposition 18.22 and the fact that \( [f] \) is an upper set) to \( (f)_k (L \cup \{(l; Y)\}) \) being a principal filter and thus \( (\text{val } f)_l L \) being a complete free star.
18.5 Identity staroids and multifuncoids

18.5.1 Identity relations

Denote \( \text{id}_{A[n]} = \{ (\lambda i \in n : x) \mid x \in A \} = \{ n \times \{ x \} \mid x \in A \} \) the \( n \)-ary identity relation on a set \( A \) (for each index set \( n \)).

**Proposition 18.34.** \( \prod X \neq \text{id}_{A[n]} \Leftrightarrow \bigcap_{i \in n} X_i \cap A \neq \emptyset \).

**Proof.** \( \prod X \neq \text{id}_{A[n]} \Leftrightarrow \exists \exists \in A : n \times \{ \ell \} \in \prod X \Leftrightarrow \exists \exists \in A \forall \ell \in n : \ell \in X_i \Rightarrow \bigcap_{i \in n} X_i \cap A \neq \emptyset . \)

18.5.2 Universal definitions of identity staroids

Consider a filtrator \((\mathfrak{A}; \mathfrak{Z})\) and \( A \in \mathfrak{A} \).

I will define below small identity staroids \( \text{id}_{A[n]}^{\text{Std}} \) and big identity staroids \( \text{ID}_{A[n]}^{\text{Std}} \). That they are really staroids and even completary staroids (under certain conditions) is proved below.

**Definition 18.35.** Consider a filtrator \((\mathfrak{A}; \mathfrak{Z})\). Let \( \mathfrak{Z} \) be a complete lattice. Let \( A \in \mathfrak{A} \), let \( n \) be an index set.

\[
\text{form} \text{id}_{A[n]}^{\text{Std}} = \mathfrak{Z}^n ; \quad L \in \text{GR} \text{id}_{A[n]}^{\text{Std}} \Leftrightarrow \bigcap_{i \in n} L_i \in \partial A.
\]

**Obvious 18.36.** \( X \in \text{GR} \text{id}_{A[n]}^{\text{Std}} \Leftrightarrow \forall A \in \text{up} A \cap \mathfrak{Z}^3 \bigcap_{i \in n} X_i \cap A = \emptyset \) if our filtrator is with separable core.

**Definition 18.37.** The subset \( X \) of a poset \( \mathfrak{A} \) has a nontrivial lower bound \((I \text{ denote this predicate as } \text{MEET}(X))\) iff there is nonleast \( a \in \mathfrak{A} \) such that \( \forall x \in X : a \sqsubseteq x \).

**Definition 18.38.** Staroid \( \text{ID}_{A[n]}^{\text{Std}} \) (for any \( A \in \mathfrak{A} \) where \( \mathfrak{A} \) is a poset) is defined by the formulas:

\[
\text{form} \text{ID}_{A[n]}^{\text{Std}} = \mathfrak{A}^n ; \quad L \in \text{GR} \text{ID}_{A[n]}^{\text{Std}} \Leftrightarrow \text{MEET} \{ L_i \mid i \in n \} \cup \{ A \} \}
\]

**Obvious 18.39.** If \( \mathfrak{A} \) is complete lattice, then \( L \in \text{GR} \text{ID}_{A[n]}^{\text{Std}} \Leftrightarrow \bigcap L \Leftrightarrow A . \)

**Obvious 18.40.** If \( \mathfrak{A} \) is complete lattice and \( a \) is an atom, then \( L \in \text{GR} \text{ID}_{A[n]}^{\text{Std}} \Leftrightarrow \bigcap L \sqcup a . \)

**Obvious 18.41.** If \( \mathfrak{A} \) is a complete lattice then there exists a multifuncoid \( \Lambda \text{ID}_{A[n]}^{\text{Std}} \) such that \( \langle \Lambda \text{ID}_{A[n]}^{\text{Std}} \rangle_k L = \bigcap_{i \in n} L_i \cap A \) for every \( k \in n, L \in \mathfrak{A}^n \setminus \{ k \} \).

**Proposition 18.42.** If \((\mathfrak{A}; \mathfrak{Z})\) is a meet-closed filtrator and \( \mathfrak{Z} \) is a complete lattice and \( \mathfrak{A} \) is a meet-semilattice. There exists a multifuncoid \( \Lambda \text{id}_{A[n]}^{\text{Std}} \) such that \( \langle \Lambda \text{id}_{A[n]}^{\text{Std}} \rangle_k L = \bigcap_{i \in n} L_i \cap A \) for every \( k \in n, L \in \mathfrak{A}^n \setminus \{ k \} \).

**Proof.** We need to prove that \( L \cup \{ (k; X) \} \in \text{id}_{A[n]}^{\text{Std}} \Leftrightarrow \bigcap_{i \in n} L_i \cap A \neq \mathfrak{Z} X \). But

\[
\bigcap_{i \in n} L_i \cap A \neq \mathfrak{Z} X \Leftrightarrow \bigcap_{i \in n} L_i \cap A \neq \mathfrak{Z} L \Leftrightarrow \bigcap_{i \in n} \left( L \cup \{ (k; X) \} \right) \neq \mathfrak{Z} A \Leftrightarrow L \cup \{ (k; X) \} \in \text{id}_{A[n]}^{\text{Std}} . \)

18.5.3 Identities are staroids

**Proposition 18.43.** Let \( \mathfrak{A} \) be a complete distributive lattice and \( A \in \mathfrak{A} \). Then \( \text{ID}_{A[n]}^{\text{Std}} \) is a staroid.
18.5 Identity staroids and multifuncoids

**Proposition 18.50.** Let \( L \notin \text{GR} \text{ID}_{A[n]}^{\text{Strd}} \) if \( L_i = 0 \) for some \( k \in n \) is obvious. It remains to prove

\[
L \cup \{(k; X \cup Y)\} \in \text{GR} \text{ID}_{A[n]}^{\text{Strd}} \iff L \cup \{(k; X)\} \in \text{GR} \text{ID}_{A[n]}^{\text{Strd}} \lor L \cup \{(k; Y)\} \in \text{GR} \text{ID}_{A[n]}^{\text{Strd}}.
\]

It is equivalent to

\[
\bigcap_{i \in n \setminus \{k\}} L_i \cap (X \cup Y) \neq A \iff \bigcap_{i \in n \setminus \{k\}} L_i \cap X \neq A \lor \bigcap_{i \in n \setminus \{k\}} L_i \cap Y \neq A.
\]

Really, \( \bigcap_{i \in n \setminus \{k\}} L_i \cap (X \cup Y) \neq A \iff \left( \bigcap_{i \in n \setminus \{k\}} L_i \cap X \right) \cup \left( \bigcap_{i \in n \setminus \{k\}} L_i \cap Y \right) \neq A \iff \bigcap_{i \in n \setminus \{k\}} L_i \cap X \neq A \lor \bigcap_{i \in n \setminus \{k\}} L_i \cap Y \neq A. \)

\( \square \)

**Proposition 18.44.** Let \( (\mathfrak{A}; 3) \) be a starroid over a complete meet infinite distributive lattice and \( A \in \mathfrak{A} \). Then \( \text{id}_{A[n]}^{\text{Strd}} \) is a staroid.

**Proof.** That \( L \notin \text{GR} \text{id}_{A[n]}^{\text{Strd}} \) if \( L_i = 0 \) for some \( k \in n \) is obviously. It remains to prove

\[
L \cup \{(k; X \cup Y)\} \in \text{GR} \text{id}_{A[n]}^{\text{Strd}} \iff L \cup \{(k; X)\} \in \text{GR} \text{id}_{A[n]}^{\text{Strd}} \lor L \cup \{(k; Y)\} \in \text{GR} \text{id}_{A[n]}^{\text{Strd}}.
\]

It is equivalent to

\[
\bigcap_{i \in n \setminus \{k\}} L_i \cap (X \cup Y) \neq A \iff \bigcap_{i \in n \setminus \{k\}} L_i \cap X \neq A \lor \bigcap_{i \in n \setminus \{k\}} L_i \cap Y \neq A.
\]

Really, \( \bigcap_{i \in n \setminus \{k\}} L_i \cap (X \cup Y) \neq A \iff \left( \bigcap_{i \in n \setminus \{k\}} L_i \cap X \right) \cup \left( \bigcap_{i \in n \setminus \{k\}} L_i \cap Y \right) \neq A \iff \bigcap_{i \in n \setminus \{k\}} L_i \cap X \neq A \lor \bigcap_{i \in n \setminus \{k\}} L_i \cap Y \neq A. \)

\( \square \)

**Proposition 18.45.** Let \( (\mathfrak{A}; 3) \) be a distributive lattice staroid with least element and finitely join-closed core which is a join semilattice. \( \text{ID}_{A[n]}^{\text{Strd}} \) is a complete staroid for every \( A \in \mathfrak{A} \).

**Proof.** \( \partial \mathcal{A} \) is a free star by theorem 4.47.

\[
L_0 \cup L_i \in \text{GR} \text{ID}_{A[n]}^{\text{Strd}} \iff \forall i \in n: (L_0 \cup L_i) \subseteq \partial \mathcal{A} \iff \forall i \in n: L_0 \cup L_i \subseteq \partial \mathcal{A} \iff \forall i \in n: (L_0 \cup L_i) \subseteq \partial \mathcal{A} \iff \exists \mathcal{C} \subseteq \{0, 1\}^n \forall i \in n: L_{c(i)} \subseteq \partial \mathcal{A} \iff \exists \mathcal{C} \subseteq \{0, 1\}^n: (\lambda i \in n: L_{c(i)} \subseteq \partial \mathcal{A}) \iff \exists \mathcal{C} \subseteq \{0, 1\}^n: (\lambda i \in n: L_{c(i)} \subseteq \partial \mathcal{A}) \iff \exists \mathcal{C} \subseteq \{0, 1\}^n: (\lambda i \in n: L_{c(i)} \subseteq \partial \mathcal{A}). \)

\( \square \)

**Lemma 18.46.** \( X \in \text{GR} \text{id}_{A[n]}^{\text{Strd}} \Rightarrow \text{Cor}^t \prod_{i \in n} X_i \neq A \) for a join-closed filter \( (\mathfrak{A}; 3) \) such that both \( \mathfrak{A} \) and \( 3 \) are complete lattices, provided that \( A \in \mathfrak{A} \).

**Proof.** \( X \in \text{GR} \text{id}_{A[n]}^{\text{Strd}} \Rightarrow \prod_{i \in n} X_i \neq A \Rightarrow \text{Cor}^t \prod_{i \in n} X_i \neq A. \)

\( \square \)

**Conjecture 18.47.** \( \text{id}_{A[n]}^{\text{Strd}} \) is a complete staroid for every set-theoretic filter \( \mathcal{A} \).

**Proposition 18.48.** Let each \( (\mathfrak{A}_i; 3_i) \) for \( i \in n \) (where \( n \) is an index set) is a finitely join-closed staroid, such that each \( \mathfrak{A}_i \) and each \( 3_i \) are join-semilattices. If \( f \) is a complete staroid of the form \( \mathfrak{A} \) then \( \| f \| \) is a complete staroid of the form \( 3 \). [TODO: Move this proposition (and note its corollary).]

**Proof.** \( L_0 \cup L_1 \in \text{GR} \| f \| \Rightarrow L_0 \cup L_1 \in \text{GR} \| f \| \Rightarrow L_0 \cup L_1 \in \text{GR} f \Rightarrow L_0 \cup L_1 \in \text{GR} f \Rightarrow \exists c \subseteq \{0, 1\}^n: (\lambda i \in n: L_{c(i)} \subseteq \text{GR} f) \Rightarrow \exists c \subseteq \{0, 1\}^n: (\lambda i \in n: L_{c(i)} \subseteq \text{GR} f) \Rightarrow \text{GR} f \text{ for every } L_0, L_1 \in \prod_{i \in n} 3. \)

\( \square \)

**Conjecture 18.49.** \( \| \text{id}_{A[n]}^{\text{Strd}} \| \) is a complete staroid if \( \mathcal{A} \) is a filter on a set and \( n \) is an index set.

18.5.4 Special case of sets and filters

**Proposition 18.50.** \( \cap_{i \in n} X \in \text{GR} \text{id}_{A[n]}^{\text{Strd}} \Rightarrow \forall A \in a: \prod_{i \in n} X \neq \text{id}_{A[n]} \) for every filter \( a \) on a powerset and index set \( n \).
Proof. \( \forall A \in a: \prod X \neq \text{id}_A[n] \Leftrightarrow \forall A \in a: \prod_{i \in n} X_i \cap A \neq \emptyset \Leftrightarrow \forall A \in a: \prod_{i \in n} \exists^3 X_i \neq \exists A \Leftrightarrow \prod_{i \in n} (\exists^3 X_i) \neq a \Leftrightarrow \prod_{i \in n} (\exists^3 X_i) \neq a, X \in \text{GR id}_A[n]. \) \( \square \)

Proposition 18.51. \( Y \in \text{GR id}_A[n] \Leftrightarrow \forall A \in \mathcal{A}: Y \in \text{GR id}_A[n] \) for every filter \( \mathcal{A} \) on a powerset and \( Y \in \mathbb{P}^n. \)

Proof. Take \( Y = \exists^3 X. \)

\( \forall A \in \mathcal{A}: Y \in \text{GR id}_A[n] \Leftrightarrow \forall A \in \mathcal{A}: \exists^3 X \in \text{GR id}_A[n] \Leftrightarrow \forall A \in \mathcal{A}: \prod X \neq \text{id}_A[n] \Leftrightarrow \exists^3 X \in \text{GR id}_A[n] \Leftrightarrow \exists^3 X \in \text{GR id}_A[n] \). \( \square \)

Proposition 18.52. \( \exists^3 X \in \text{GR id}_A[n] \Leftrightarrow \forall A \in \mathcal{A} \exists t \in A \forall i \in n: t \in X_i. \)

Proof. \( \exists^3 X \in \text{GR id}_A[n] \Leftrightarrow \forall A \in \mathcal{A} \exists t \in A \forall i \in n: t \in X_i. \)

18.5.5 Relationships between big and small identity staroids

Definition 18.53. \( a_{\text{Strd}}^n = \prod_{i \in n} a \) for every element \( a \) of a poset and an index set \( n. \)

Proposition 18.54. \( \prod \text{id}_A[n] \subseteq \text{ID}_A[n] \subseteq a_{\text{Strd}}^n \) for every filter \( a \) (on any distributive lattice) and an index set \( n. \)

Proof. \( \text{GR} \prod \text{id}_A[n] \subseteq \text{GR ID}_A[n]. \) \( L \in \text{GR} \prod \text{id}_A[n] \Leftrightarrow \up L \subseteq \text{GR id}_A[n] \Leftrightarrow \forall L \in \text{GR} \text{id}_A[n] \Leftrightarrow \forall \mathcal{L} \in \text{GR} \text{ID}_A[n] \Leftrightarrow \) (proposition 4.112) \( \Rightarrow \forall \mathcal{L} \in \text{GR} \text{id}_A[n] \Leftrightarrow \forall \mathcal{L} \in \text{GR} \text{ID}_A[n]. \) \( \square \)

Proposition 18.55. \( \prod \text{id}_A[n] \subseteq \text{ID}_A[n] = a_{\text{Strd}}^n \) for every nontrivial ultrafilter \( a \) on a set.

Proof. \( \text{GR} \prod \text{id}_A[n] \neq \text{GR ID}_A[n]. \) Let \( \forall L \in \text{GR} \prod \text{id}_A[n] \) it’s enough to show \( L \notin \text{GR id}_A[n] \) for some \( L \in \text{GR} \text{id}_A[n]. \) Really, take \( L_i = L \) for some \( L \in \text{GR ID}_A[n]. \) Then \( L \in \text{GR} \text{id}_A[n] \Leftrightarrow \forall A \in a \exists t \in A \forall i \in a: t \in i \) what is clearly false (we can always take \( i \notin a \) such that \( t \notin i \) for any point \( t \)).

\( \text{GR ID}_A[n] = \text{GR} a_{\text{Strd}}^n. \) \( L \in \text{GR ID}_A[n] \Leftrightarrow \forall i \in n: L \ni a \Leftrightarrow \forall i \in a: L_i \neq a \Leftrightarrow L \in \text{GR} a_{\text{Strd}}^n. \) \( \square \)

Corollary 18.56. \( a_{\text{Strd}}^n \) isn’t an atom when \( a \) is a nontrivial ultrafilter.

Corollary 18.57. Staroidal product of an infinite indexed family of ultrafilters may be non-atomic.

Proposition 18.58. \( \text{id}_A[n] \) is determined by the value of \( \prod \text{id}_A[n] \). Moreover \( \text{id}_A[n] = \prod \text{id}_A[n]. \)

Proof. Use general properties of upgrading and downgrading (proposition 17.63). \( \square \)

Lemma 18.59. \( L \in \text{GR ID}_A[n] \) if \( \bigcup_{i \in n} L_i \cup a \) has finite intersection property (for primary filtrators).

Proof. \( L \in \text{GR ID}_A[n] \Leftrightarrow \bigcap_{i \in n} L_i \cup a \neq 0 \Leftrightarrow \forall X \in \bigcap_{i \in n} L_i \cup a: X \neq \emptyset \) what is equivalent of \( \bigcup_{i \in n} L_i \cup a \) having finite intersection property. \( \square \)

Proposition 18.60. \( \text{ID}_A[n] \) is determined by the value of \( \prod \text{id}_A[n] \) moreover \( \text{ID}_A[n] = \prod \text{id}_A[n] \) (for primary filtrators).
Proposition 18.65. $\uparrow\text{id}^{\text{Strd}}_{A[n]} \subseteq \uparrow\uparrow\text{id}^{\text{Strd}}_{A[n]}$ for every principal $A$ and an index set $n$.

Proof. $L \in \uparrow\uparrow\text{id}^{\text{Strd}}_{A[n]} \Leftrightarrow \exists L \subseteq \uparrow\text{id}^{\text{Strd}}_{A[n]}$. Therefore, $L \subseteq \uparrow\text{id}^{\text{Strd}}_{A[n]} \Leftrightarrow \forall L \in \uparrow\text{id}^{\text{Strd}}_{A[n]} \Leftrightarrow \forall L \in \uparrow\text{id}^{\text{Strd}}_{A[n]} \Leftrightarrow \forall L \in \uparrow\text{id}^{\text{Strd}}_{A[n]} \Leftrightarrow \forall L \in \text{GR} \text{id}^{\text{Strd}}_{A[n]}$.

Proposition 18.66. $\uparrow\text{id}^{\text{Strd}}_{A[n]} \subseteq \text{id}^{\text{Strd}}_{A[n]}$ for every nontrivial ultrafilter $A$.

Proof. Suppose $\text{id}^{\text{Strd}}_{A[n]} = \uparrow\uparrow\text{id}^{\text{Strd}}_{A[n]}$. Then $\text{id}^{\text{Strd}}_{A[n]} = \text{id}^{\text{Strd}}_{A[n]}$ for every nontrivial ultrafilter $A$.

Proposition 18.67. $L \in \text{GR} \text{id}^{\text{Strd}}_{A[n]} \Leftrightarrow a \cap \bigcap_{i \in n} L_i \neq \emptyset$ if $a$ is an element of a complete lattice.

Proposition 18.68. $L \in \text{GR} \text{id}^{\text{Strd}}_{A[n]} \Leftrightarrow \forall i \in n: L_i \supseteq a \Leftrightarrow \forall i \in n: L_i \neq a$ if $a$ is an ultrafilter on $\mathbb{A}$.

18.5.6 Identity staroids on principal filters

For principal filter $\uparrow A$ (where $A$ is a set) the above definitions coincide with $n$-ary identity relation, as formulated in the below propositions:

Proposition 18.69. $\uparrow\uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]} = \text{id}^{\text{Strd}}_{\uparrow A[n]}$.

Proof. $L \in \text{GR} \uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]} \Leftrightarrow \exists L \subseteq \text{id}^{\text{Strd}}_{\uparrow A[n]} \Leftrightarrow \exists L \in \text{GR} \text{id}^{\text{Strd}}_{\uparrow A[n]}$.

Proposition 18.70. $\uparrow\uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]} = \text{id}^{\text{Strd}}_{\uparrow A[n]}$.

Proof. $\uparrow\uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]} = \text{id}^{\text{Strd}}_{\uparrow A[n]}$ is obvious from the above.

Proposition 18.71. $\uparrow\uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]} \subseteq \uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]}$.

Proof. $X \in \text{GR} \uparrow\uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]} \Leftrightarrow \exists X \subseteq \text{id}^{\text{Strd}}_{\uparrow A[n]} \Leftrightarrow \forall Y \in \text{up} X: Y \in \text{GR} \uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]} \Leftrightarrow \forall Y \in \text{up} X: Y \in \text{id}^{\text{Strd}}_{\uparrow A[n]} \Leftrightarrow \forall Y \in \text{up} X: Y \cap \uparrow A \neq \emptyset \Leftrightarrow \bigcap_{i \in n} Y_i \cap \uparrow A \neq \emptyset \Leftrightarrow X \in \text{GR} \text{id}^{\text{Strd}}_{\uparrow A[n]}$.

Proposition 18.72. $\uparrow\uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]} \subseteq \text{id}^{\text{Strd}}_{\uparrow A[n]}$ for some set $A$.

Proof. We need to prove $\uparrow\uparrow\text{id}^{\text{Strd}}_{\uparrow A[n]} \neq \text{id}^{\text{Strd}}_{\uparrow A[n]}$ that is it's enough to prove (see the above proof) that $\forall Y \in \text{up} X: \bigcap_{i \in n} Y_i \cap \uparrow A \neq \emptyset \Leftrightarrow \bigcap_{i \in n} X_i \cap \uparrow A \neq \emptyset$. A counter-example follows:
but Proposition 18.76. \( \forall L \in \text{form } f: (\text{MEET}(\{L_i \mid i \in n \} \cup \{a\}) \Rightarrow L \in \text{GR } f) \).

But \( \exists a \in \text{atoms } A: \text{MEET}(\{L_i \mid i \in n \} \cup \{a\}) \Leftrightarrow \exists a \in \text{atoms } A: \prod_{i \in n}^A L_i \neq a \Leftrightarrow \prod_{i \in n}^A L_i \neq A \Leftrightarrow L \in \text{ID}^\text{Strd}_a[n]. \)

So \( L \in \text{ID}^\text{Strd}_a[n] \Rightarrow L \in \text{GR } f. \) Thus \( f \not\supseteq \text{ID}^\text{Strd}_a[n]. \)

Then use the fact that \( \text{ID}^\text{Strd}_a[n] = a^\text{Strd}_a[n]. \)

Proposition 18.76. \( \text{id}^\text{Strd}_a[n] \not\supseteq \text{id}^\text{Strd}_a[n] \) as every \( a \in \text{atoms } A \) is obvious.

Let \( f \not\supseteq \text{id}^\text{Strd}_a[n] \) for every \( a \in \text{atoms } A \). Then \( \forall L \in \text{GR } \text{id}^\text{Strd}_a[n]: L \in \text{GR } f \) that is

\[ \forall L \in \text{form } f: \left( \prod_{i \in n}^3 L_i \neq a \Rightarrow L \in \text{GR } f \right). \]

But \( \exists a \in \text{atoms } A: \prod_{i \in n}^3 L_i \neq a \Leftrightarrow \prod_{i \in n}^3 L_i \neq A \Leftrightarrow L \in \text{id}^\text{Strd}_a[n]. \)
18.6 Finite case

**Theorem 18.77.** Let \( n \) be a finite set.

1. \( \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \models \| \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \| \) if \( \mathfrak{A} \) and \( \mathfrak{F} \) are meet-semilattices and \((\mathfrak{A}; \mathfrak{F})\) is a finitely meet-closed filtrator.

2. \( \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \models \| \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \| \) if \((\mathfrak{A}; \mathfrak{F})\) is a principal filtrator over a distributive lattice.

**Proof.**

1. \( L \in \text{GR} \implies L \in \text{GR} \implies \text{MEET}(\{L_i \mid i \in n \} \cup \{A\}) \implies \bigcap_{i \in n} L_i \cap A \neq 0 \implies \text{by finiteness} \implies \bigcap_{i \in n} L_i \cap A \neq 0 \implies L \in \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \) for every \( L \in \prod \mathfrak{A} \).

2. \( L \in \text{GR} \implies \| \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \| \implies \text{up} L \subseteq \text{GR} \implies \forall K \in \text{up} L: K \in \text{GR} \implies \forall K \in \text{up} L: \bigcap_{i \in n} K_i \neq 0 \implies A \subseteq \bigcap_{i \in n} K_i \implies \forall K \in \text{up} L: \bigcap_{i \in n} K_i \neq 0 \implies A \subseteq \bigcap_{i \in n} K_i \implies \text{by the formula for finite meet of filters, theorem 4.111} \implies A \subseteq \bigcap_{i \in n} K_i \implies \forall K \in \bigcap_{i \in n} K_i: A \subseteq K \implies \forall K \in \bigcap_{i \in n} K_i: A \neq 0 \implies \text{by separability of core, theorem 4.112} \implies \bigcap_{i \in n} L_i \not= A \implies L \in \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \). \qed

**Proposition 18.78.** Let \((\mathfrak{A}; \mathfrak{F})\) be a finitely meet closed filtrator. \( \| \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \| \) and \( \| \text{id}_{\mathcal{A}[n]}^{\text{Strd}} \| \) are the same for finite \( n \).

**Proof.** Because \( \bigcap_{i \in \text{dom} \mathfrak{L}} L_i = \bigcap_{i \in \text{dom} \mathfrak{L}} L_i \) for finitary \( L \). \qed

18.7 Counter-examples and conjectures

The following example shows that the theorem 18.33 can’t be strenghtened:

**Example 18.79.** For some multifunucoid \( f \) on powersets complete in argument \( k \) the following formula is false:

\[
(f)_{\{L \cup \{(k; \{X\})\}} = \bigcup_{X \in \mathcal{P}^{\mathfrak{L}}} f(L \cup \{(k; x)\}) \text{ for every } X \in \mathcal{P}^{\mathfrak{L}}, \ L \in \prod_{k \in \text{arity} \mathfrak{L} \setminus \{k, l\}} \mathfrak{F}.
\]

**Proof.** Consider multifunucoid \( f = \text{id}^{\text{Strd}}_{U[3]} \) where \( U \) is an infinite set (of the form \( \mathcal{P}^{\mathfrak{L}} \)) and \( L = \{Y\} \) where \( Y \) is a nonprincipal filter on \( U \).

\[
(f)_{\{L \cup \{(k; \{X\})\}} = Y \setminus \bigcup_{X \in \mathcal{P}^{\mathfrak{L}}} \{(k; x)\} = \bigcup_{X \in \mathcal{P}^{\mathfrak{L}}} \{(k; x)\} \implies \bigcup_{X \in \mathcal{P}^{\mathfrak{L}}} \{(k; x)\} \text{ only if } Y \text{ is principal: Really: } Y \cap \bigcup_{X \in \mathcal{P}^{\mathfrak{L}}} \{(k; x)\} \implies Y \not= \bigcup_{X \in \mathcal{P}^{\mathfrak{L}}} \{(k; x)\} \text{ only if } Y \cap \bigcup_{X \in \mathcal{P}^{\mathfrak{L}}} \{(k; x)\} \not= 0 \implies \exists \exists X \in \mathcal{P}^{\mathfrak{L}}: Y \neq x \text{ and thus } Y \text{ is principal. But we claimed above that it is nonprincipal.} \]

**Example 18.80.** There exists a staroid \( f \) and an indexed family \( X \) of principal filters (with arity \( f = \text{dom} X \) and \( \text{form} \ f_i = \text{Base}(X_i) \) for every \( i \in \text{arity} f \)), such that \( f \not\in \prod^{\text{Strd}} X \) and \( Y \cap X \not\in \text{GR} f \) for some \( Y \in \text{GR} f \).

**Remark 18.81.** Such examples obviously do not exist if both \( f \) is a principal staroid and \( X \) and \( Y \) are indexed families of principal filters (because for powerset algebras stalkoidal product is equivalent to Cartesian product). This makes the above example inspired.

**Proof.** (Monroe Eskew) Let \( a \) be any (trivial or nontrivial) ultrafilter on an infinite set \( U \). Let \( A, B \in a \) be such that \( A \cap B \not\in a \). In other words, \( A, B \) are arbitrary nonempty sets such that \( \emptyset \not\in A \cap B \not\in A, B \) and \( a \) be an ultrafilter on \( A \cap B \).
Let $f$ be the staroid whose graph consists of functions $p: U \to a$ such that either $p(n) \geq A$ for all but finitely many $n$ or $p(n) \geq B$ for all but finitely many $n$. Let’s prove $f$ is really a staroid.

It’s obvious $p x \neq \emptyset$ for every $x \in U$. Let $k \in U$, $L \in a^U \setminus \{k\}$. It is enough (taking symmetry into account) to prove that

$$L \cup \{(k; x \cup y)\} \in \text{GR } f \Leftrightarrow L \cup \{(k; x)\} \in \text{GR } f \lor L \cup \{(k; y)\} \in \text{GR } f.$$  \hspace{1cm} (18.1)

Really, $L \cup \{(k; x \cup y)\} \in \text{GR } f$ iff $x \cup y \in a$ and $L(n) \geq A$ for all but finitely many $n$ or $L(n) \geq B$ for all but finitely many $n$; $L \cup \{(k; x)\} \in \text{GR } f$ iff $x \in a$ and $L(n) \geq A$ for all but finitely many $n$ or $L(n) \geq B$; and similarly for $y$.

But $x \cup y \in a \Leftrightarrow x \in a \lor y \in a$ because $a$ is an ultrafilter. So, the formula (18.1) holds, and we have proved that $f$ is really a staroid.

Take $X$ be the constant function with value $A$ and $Y$ be the constant function with value $B$. For every $p \in \text{GR } f: p \neq X$ because $p \cap X_i \in a$; so $\text{GR } f \subseteq \text{GR } \prod^\text{Strd} X$ that is $f \subseteq \prod^\text{Strd} X$.

Finally, $Y \cap X \notin \text{GR } f$ because $X \cap Y = \lambda i \in U: A \cap B$.

Some conjectures similar to the above example:

**Conjecture 18.82.** There exists a completary staroid $f$ and an indexed family $X$ of principal filters (with arity $f = \text{dom } X$ and (form $f) = \text{Base}(X_i)$ for every $i \in \text{arity } f$), such that $f \subseteq \prod^\text{Strd} X$ and $Y \cap X \notin \text{GR } f$ for some $Y \in \text{GR } f$.

**Conjecture 18.83.** There exists a staroid $f$ and an indexed family $x$ of ultrafilters (with arity $f = \text{dom } x$ and (form $f) = \text{Base}(x_i)$ for every $i \in \text{arity } f$), such that $f \subseteq \prod^\text{Strd} x$ and $Y \cap x \notin \text{GR } f$ for some $Y \in \text{GR } f$.

Other conjectures:

**Conjecture 18.84.** If staroid $0 \neq f \subseteq a^\text{Strd}$ for an ultrafilter $a$ and an index set $n$, then $n \times \{a\} \in \text{GR } f$. (Can it be generalized for arbitrary staroidal products?)

**Conjecture 18.85.** The following posets are atomic:

1. anchored relations on powersets;
2. staroids on powersets;
3. completary staroids on powersets.

**Conjecture 18.86.** The following posets are atomistic:

1. anchored relations on powersets;
2. staroids on powersets;
3. completary staroids on powersets.

The above conjectures seem difficult, because we know almost nothing about structure of atomic staroids.

**Conjecture 18.87.** A staroid on powersets is principal iff it is complete in every argument.

**Conjecture 18.88.** If $a$ is an ultrafilter, then $id^\text{Strd}_a$ is an atom of the lattice of:

1. anchored relations of the form $(\mathcal{P}\text{Base}(a))^n$;
2. staroids of the form $(\mathcal{P}\text{Base}(a))^n$;
3. completary staroids of the form $(\mathcal{P}\text{Base}(a))^n$.

**Conjecture 18.89.** If $a$ is an ultrafilter, then $\uparrow id^\text{Strd}_a$ is an atom of the lattice of:

1. anchored relations of the form $\mathfrak{f}(\text{Base}(a))^n$;
2. staroids of the form $\mathfrak{f}(\text{Base}(a))^n$;
3. completary staroids of the form $\mathfrak{f}(\text{Base}(a))^n$.

Informal problem: Formulate and prove associativity of staroidal product.
Chapter 19
Postface

See this Web page for my research plans: http://www.mathematics21.org/agt-plans.html

I deem that now the most important research topics in Algebraic General Topology are:

- to solve the open problems mentioned in this work;
- define and research compactness of funcoids;
- research categories related with funcoids and reloids;
- research multifuncoids and staroids in more details;
- research generalized limit of compositions of functions.

We should also research relationships between complete funcoids and complete reloids.
All my research of funcoids and reloids is presented at http://www.mathematics21.org/algebraic-general-topology.html
Bibliography


Index

\begin{tabular}{ll}
\textbf{\textit{a-thick}} & 153 \\
\textbf{\textit{\beta-thick}} & 153 \\
\textbf{\textit{adjoint}} & 25 \\
\textbf{\textit{lower}} & 25 \\
\textbf{\textit{upper}} & 25 \\
\textbf{\textit{associative}} & 48 \\
\textbf{\textit{infinite}} & 23 \\
\textbf{\textit{atom}} & 24 \\
\textbf{\textit{atomic}} & 24 \\
\textbf{\textit{ball}} & 89 \\
\textbf{\textit{closed}} & 89 \\
\textbf{\textit{big staroid}} & 89 \\
\textbf{\textit{generated}} & 209 \\
\textbf{\textit{bounds}} & 19 \\
\textbf{\textit{lower}} & 19 \\
\textbf{\textit{upper}} & 19 \\
\textbf{\textit{category}} & 30 \\
\textbf{\textit{abrupt}} & 218 \\
\textbf{\textit{dagger}} & 40 \\
\textbf{\textit{of funcoid triples}} & 104 \\
\textbf{\textit{of funcoids}} & 104 \\
\textbf{\textit{of pointfree funcoid triples}} & 184 \\
\textbf{\textit{of pointfree funcoids}} & 184 \\
\textbf{\textit{of reloid triples}} & 126 \\
\textbf{\textit{of reloids}} & 126 \\
\textbf{\textit{partially ordered}} & 40 \\
\textbf{\textit{quasi-invertible}} & 201 \\
\textbf{\textit{Rel}} & 31 \\
\textbf{\textit{Set}} & 30 \\
\textbf{\textit{with star morphisms}} & 219 \\
\textbf{\textit{induced by dagger category}} & 219 \\
\textbf{\textit{with star-morphisms}} & 217 \\
\textbf{\textit{quasi-invertible}} & 218 \\
\textbf{\textit{category theory}} & 5, 30 \\
\textbf{\textit{chain}} & 18 \\
\textbf{\textit{closed}} & 18 \\
\textbf{\textit{regarding pointfree funcoid}} & 196 \\
\textbf{\textit{closure}} & 89 \\
\textbf{\textit{in metric space}} & 93 \\
\textbf{\textit{Kuratowski}} & 93 \\
\textbf{\textit{co-completion}} & 144 \\
\textbf{\textit{funcoid}} & 144 \\
\textbf{\textit{pointfree}} & 194 \\
\textbf{\textit{of funcoid}} & 114 \\
\textbf{\textit{of reloid}} & 128 \\
\textbf{\textit{co-metacomplete}} & 42 \\
\textbf{\textit{complement}} & 21 \\
\textbf{\textit{complemented}} & 21 \\
\textbf{\textit{element}} & 22 \\
\textbf{\textit{lattice}} & 22 \\
\textbf{\textit{compleventive}} & 21 \\
\textbf{\textit{complete}} & 17 \\
\textbf{\textit{multifuncoid}} & 14 \\
\textbf{\textit{stanoid}} & 14 \\
\textbf{\textit{complete lattice}} & 25 \\
\textbf{\textit{homomorphism}} & 13 \\
\textbf{\textit{completely starrish}} & 114 \\
\textbf{\textit{completion}} & 128 \\
\textbf{\textit{of funcoid}} & 97 \\
\textbf{\textit{of reloid}} & 121 \\
\textbf{\textit{composable}} & 121 \\
\textbf{\textit{funcoids}} & 97 \\
\textbf{\textit{reloids}} & 121 \\
\textbf{\textit{concatenation}} & 45 \\
\textbf{\textit{connected}} & 150 \\
\textbf{\textit{regarding endofuncoid}} & 149 \\
\textbf{\textit{regarding endoreloid}} & 149 \\
\textbf{\textit{regarding pointfree funcoid}} & 186 \\
\textbf{\textit{connected component}} & 149 \\
\textbf{\textit{connectedness}} & 148 \\
\textbf{\textit{regarding binary relation}} & 149 \\
\textbf{\textit{connectivity reloid}} & 149 \\
\textbf{\textit{contained}} & 17 \\
\textbf{\textit{contains}} & 17 \\
\textbf{\textit{continuity}} & 10, 11 \\
\textbf{\textit{coordinate-wise}} & 8, 144 \\
\textbf{\textit{generalized}} & 8, 144 \\
\textbf{\textit{in metric space}} & 90 \\
\textbf{\textit{of restricted morphism}} & 8, 145 \\
\textbf{\textit{pre-topology}} & 8, 143 \\
\textbf{\textit{proximity}} & 8, 143 \\
\textbf{\textit{uniformity}} & 8, 143 \\
\textbf{\textit{converges}} & 197, 197, 197 \\
\textbf{\textit{regarding funcoid}} & 52 \\
\textbf{\textit{core}} & 53 \\
\textbf{\textit{of filtrator}} & 53 \\
\textbf{\textit{core part}} & 53 \\
\textbf{\textit{dual}} & 53 \\
\textbf{\textit{core star}} & 55 \\
\textbf{\textit{currying}} & 43, 44 \\
\textbf{\textit{De Morgan's laws}} & 22 \\
\textbf{\textit{finite}} & 23 \\
\textbf{\textit{decomposition of composition}} & 8, 139 \\
\textbf{\textit{of reloids}} & 8, 139 \\
\textbf{\textit{destination}} & 30 \\
\textbf{\textit{difference}} & 21 \\
\textbf{\textit{directly isomorphic}} & 159 \\
\textbf{\textit{disjunction property of Wallman}} & 34 \\
\textbf{\textit{distributive}} & 22 \\
\textbf{\textit{domain}} & 103 \\
\textbf{\textit{of funcoid}} & 22
\end{tabular}

255
<table>
<thead>
<tr>
<th>Term</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>downgrading</td>
<td>208</td>
</tr>
<tr>
<td>staroid</td>
<td>9, 208</td>
</tr>
<tr>
<td>dual</td>
<td>17</td>
</tr>
<tr>
<td>order</td>
<td>18</td>
</tr>
<tr>
<td>poset</td>
<td>59</td>
</tr>
<tr>
<td>duality</td>
<td>18</td>
</tr>
<tr>
<td>partial order</td>
<td>18</td>
</tr>
<tr>
<td>edge</td>
<td>30</td>
</tr>
<tr>
<td>edge part</td>
<td>59</td>
</tr>
<tr>
<td>element</td>
<td>52</td>
</tr>
<tr>
<td>of filtrator</td>
<td>52</td>
</tr>
<tr>
<td>embedding</td>
<td>52</td>
</tr>
<tr>
<td>reloids into funcoids</td>
<td>8, 141</td>
</tr>
<tr>
<td>endo-funcoid</td>
<td>147</td>
</tr>
<tr>
<td>endomorphism</td>
<td>31</td>
</tr>
<tr>
<td>symmetric</td>
<td>41</td>
</tr>
<tr>
<td>transitive</td>
<td>41</td>
</tr>
<tr>
<td>endomorphism series</td>
<td>147</td>
</tr>
<tr>
<td>endo-reloid</td>
<td>147</td>
</tr>
<tr>
<td>entirely defined morphism</td>
<td>41</td>
</tr>
<tr>
<td>equivalent</td>
<td></td>
</tr>
<tr>
<td>filters</td>
<td>157</td>
</tr>
<tr>
<td>filter</td>
<td></td>
</tr>
<tr>
<td>closed</td>
<td>119</td>
</tr>
<tr>
<td>cofinite</td>
<td>81</td>
</tr>
<tr>
<td>directly isomorphic</td>
<td>159</td>
</tr>
<tr>
<td>Fréchet</td>
<td>81</td>
</tr>
<tr>
<td>on a meet-semilattice</td>
<td>52</td>
</tr>
<tr>
<td>on a poset</td>
<td>52</td>
</tr>
<tr>
<td>on a set</td>
<td>51</td>
</tr>
<tr>
<td>on meet-semilattice</td>
<td>62</td>
</tr>
<tr>
<td>on poset</td>
<td>62</td>
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<tr>
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