Algebraic General Topology. Volume 1

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Todd Trimble, Andreas Blass, Robert Martin Solovay, Niels Diepeveen, and others (mentioned below) have proved some theorems which are now in this book.

Abstract. In this work I introduce and study in details the concepts of funcoids which generalize proximity spaces and reloids which generalize uniform spaces, and generalizations thereof. The concept of funcoid is generalized concept of proximity, the concept of reloid is cleared from superfluous details (generalized) concept of uniformity.

Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets). Also funcoids and reloids can be considered as a generalization of (oriented) graphs, this provides us with a common generalization of calculus and discrete mathematics.

The concept of continuity is defined by an algebraic formula (instead of old messy epsilon-delta notation) for arbitrary morphisms (including funcoids and reloids) of a partially ordered category. In one formula continuity, proximity continuity, and uniform continuity are generalized.

Also I define connectedness for funcoids and reloids.

Then I consider generalizations of funcoids: pointfree funcoids and generalization of pointfree funcoids: staroids and multifuncoids. Also I define several kinds of products of funcoids and other morphisms.

Before going to topology, this book studies properties of co-brouwerian lattices and filters.

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Part 1

Introductory chapters

CHAPTER 1

Introduction

The main purpose of this book is to record the current state of my research. The book is however written in such a way that it can be used as a textbook for studying my research.

For the latest version of this file, related materials, articles, research questions, and erratum consult the Web page of the author of the book:
http://www.mathematics21.org/algebraic-general-topology.html

1.1. License and editing

You can create your own copy of LaTeX source of the book and edit it (to correct errors, add new results, generalize existing results, enhance readability). The editable source of the book is presented at
https://bitbucket.org/portonv/algebraic-general-topology

Please consider reviewing this book at
http://www.euro-math-soc.eu/node/add/book-review

If you find any error (or some improvement idea), please report in our bug tracker:
https://bitbucket.org/portonv/algebraic-general-topology/issues

1.2. Intended audience

This book is suitable for any math student as well as for researchers.

To make this book be understandable even for first grade students, I made a chapter about basic concepts (posets, lattices, topological spaces, etc.), which an already knowledgeable person may skip reading. It is assumed that the reader knows basic set theory.

But it is also valuable for mature researchers, as it contains much original research which you could not find in any other source except of my work.

Knowledge of the basic set theory is expected from the reader.

Despite that this book presents new research, it is well structured and is suitable to be used as a textbook for a college course.

Your comments about this book are welcome to the email porton@narod.ru.

1.3. Reading Order

If you know basic order and lattice theory (including Galois connections and brouwerian lattices) and basics of category theory, you may skip reading the chapter "Common knowledge, part 1".

You are recommended to read the rest of this book by the order.

1.4. Our topic and rationale

From [42]: *Point-set topology, also called set-theoretic topology or general topology, is the study of the general abstract nature of continuity or "closeness" on spaces. Basic point-set topological notions are ones like continuity, dimension, compactness, and connectedness.*

In this work we study a new approach to point-set topology (and pointfree topology).

Traditionally general topology is studied using topological spaces (defined below in the section "Topological spaces"). I however argue that the theory of topological spaces is not the best method of studying general topology and introduce an alternative theory, the theory of *funcoids*. Despite of popularity of the theory of topological spaces it has some drawbacks and is in my opinion not the most appropriate formalism to study most of general topology. Because topological spaces are tailored for study of special sets, so called open and closed sets, studying general topology with topological spaces is a little anti-natural and ugly. In my opinion the theory of funcoids is more elegant than the theory of topological spaces, and it is better to study funcoids than topological spaces. One of the main purposes of this work is to present an alternative General Topology based on funcoids instead of being based on topological spaces as it is customary. In order to study funcoids the prior knowledge of topological spaces is not necessary. Nevertheless in this work I will consider topological spaces and the topic of interrelation of funcoids with topological spaces.

In fact funcoids are a generalization of topological spaces, so the well known theory of topological spaces is a special case of the below presented theory of funcoids.

But probably the most important reason to study funcoids is that funcoids are a generalization of proximity spaces (see section "Proximity spaces" for the definition of proximity spaces). Before this work it was written that the theory of proximity spaces was an example of a stalled research, almost nothing interesting was discovered about this theory. It was so because the proper way to research proximity spaces is to research their generalization, funcoids. And so it was stalled until discovery of funcoids. That generalized theory of proximity spaces will bring us yet many interesting results.

In addition to *funcoids* I research *reloids*. Using below defined terminology it may be said that reloids are (basically) filters on Cartesian product of sets, and this is a special case of uniform spaces.

Afterward we study some generalizations.

Somebody might ask, why to study it? My approach relates to traditional general topology like complex numbers to real numbers theory. Be sure this will find applications.

This book has a deficiency: It does not properly relate my theory with previous research in general topology and does not consider deeper category theory properties. It is however OK for now, as I am going to do this study in later volumes (continuation of this book).

Many proofs in this book may seem too easy and thus this theory not sophisticated enough. But it is largely a result of a well structured digraph of proofs, where more difficult results are made easy by reducing them to easier lemmas and propositions.

1.5. Earlier works

Some mathematicians were researching generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization

which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Proximity structures were introduced by Smirnov in [11].

Some references to predecessors:

- In [15, 16, 25, 2, 36] generalized uniformities and proximities are studied.
- Proximities and uniformities are also studied in [22, 23, 35, 37, 38].
- [20, 21] contains recent progress in quasi-uniform spaces. [21] has a very long list of related literature.

Some works ([34]) about proximity spaces consider relationships of proximities and compact topological spaces. In this work the attempt to define or research their generalization, compactness of funcoids or reloids is not done. It seems potentially productive to attempt to borrow the definitions and procedures from the above mentioned works. I hope to do this study in a separate volume.

[10] studies mappings between proximity structures. (In this volume no attempt to research mappings between funcoids is done.) [26] researches relationships of quasi-uniform spaces and topological spaces. [1] studies how proximity structures can be treated as uniform structures and compactification regarding proximity and uniform spaces.

This book is based partially on my articles [30, 28, 29].

1.6. Kinds of continuity

A research result based on this book but not fully included in this book (and not yet published) is that the following kinds of continuity are described by the same algebraic (or rather categorical) formulas for different kinds of continuity and have common properties:

- discrete continuity (between digraphs);
- (pre)topological continuity;
- proximal continuity;
- uniform continuity;
- Cauchy continuity;
- (probably other kinds of continuity).

Thus my research justifies using the same word "continuity" for these diverse kinds of continuity.

See http://www.mathematics21.org/algebraic-general-topology.html

1.7. Responses to some accusations against style of my exposition

> The proofs are generally hard to follow and unpleasant to read as they are just a bunch of equations thrown at you, without motivation or underlying reasoning, etc.

I don't think this is essential. The proofs are not the most important thing in my book. The most essential thing are definitions. The proofs are just to fill the gaps. So I deem it not important whether proofs are motivated.

Also, note "algebraic" in the title of my book. The proofs are meant to be just sequences of formulas for as much as possible :-) It is to be thought algebraically. The meaning are the formulas themselves.

Maybe it makes sense to read my book skipping all the proofs? Some proofs contain important ideas, but most don't. The important thing are definitions.

1.8. Structure of this book

In the chapter "Common knowledge, part 1" some well known definitions and theories are considered. You may skip its reading if you already know it. That chapter contains info about:

- posets;
- lattices and complete lattices;
- Galois connections;
- co-brouwerian lattices;
- a very short intro into category theory;
- a very short introduction to group theory.

Afterward there are my little additions to poset/lattice and category theory.

Afterward there is the theory of filters and filtrators.

Then there is "Common knowledge, part 2 (topology)", which considers briefly:

- metric spaces;
- topological spaces;
- pretopological spaces;
- proximity spaces.

Despite of the name "Common knowledge" this second common knowledge chapter is recommended to be read completely even if you know topology well, because it contains some rare theorems not known to most mathematicians and hard to find in literature.

Then the most interesting thing in this book, the theory of funcoids, starts.

Afterwards there is the theory of reloids.

Then I show relationships between funcoids and reloids.

The last I research generalizations of funcoids, *pointfree funcoids*, *staroids*, and *multifuncoids* and some different kinds of products of morphisms.

1.9. Basic notation

I will denote a set definition like $\left\{ \frac{x \in A}{P(x)} \right\}$ instead of customary $\{x \in A \mid P(x)\}$. I do this because otherwise formulas don't fit horizontally into the available space.

1.9.1. Grothendieck universes. We will work in ZFC with an infinite and uncountable Grothendieck universe.

A Grothendieck universe is just a set big enough to make all usual set theory inside it. For example if $\mho$ is a Grothendieck universe, and sets $X, Y \in \mho$, then also $X \cup Y \in \mho$, $X \cap Y \in \mho$, $X \times Y \in \mho$, etc.

A set which is a member of a Grothendieck universe is called a *small set* (regarding this Grothendieck universe). We can restrict our consideration to small sets in order to get rid troubles with proper classes.

Definition 1. Grothendieck universe is a set $\mho$ such that:

1°. If $x \in \mho$ and $y \in x$ then $y \in \mho$.
2°. If $x, y \in \mho$ then $\{x, y\} \in \mho$.
3°. If $x \in \mho$ then $\mathscr{P}x \in \mho$.
4°. If $\left\{ \frac{x_i}{i \in I \in \mho} \right\}$ is a family of elements of $\mho$, then $\bigcup_{i \in I} x_i \in \mho$.

One can deduce from this also:

1°. If $x \in \mho$, then $\{x\} \in \mho$.
2°. If x is a subset of $y \in \mho$, then $x \in \mho$.
3°. If $x, y \in \mho$ then the ordered pair $(x, y) = \{\{x, y\}, x\} \in \mho$.
4°. If $x, y \in \mho$ then $x \cup y$ and $x \times y$ are in $\mho$.
5°. If $\left\{ \frac{x_i}{i \in I \in \mho} \right\}$ is a family of elements of $\mho$, then the product $\prod_{i \in I} x_i \in \mho$.

6°. If $x \in \mho$, then the cardinality of x is strictly less than the cardinality of $\mho$.

1.9.2. Misc. In this book quantifiers bind tightly. That is $\forall x \in A : P(x) \wedge Q$ and $\forall x \in A : P(x) \Rightarrow Q$ should be read $(\forall x \in A : P(x)) \wedge Q$ and $(\forall x \in A : P(x)) \Rightarrow Q$ not $\forall x \in A : (P(x) \wedge Q)$ and $\forall x \in A : (P(x) \Rightarrow Q)$.

The set of functions from a set A to a set B is denoted as B^A.

I will often skip parentheses and write fx instead of $f(x)$ to denote the result of a function f acting on the argument x.

I will denote $\langle f \rangle^* X = \left\{ \frac{\beta \in \operatorname{im} f}{\exists \alpha \in X : \alpha f \beta} \right\}$ (in other words $\langle f \rangle^* X$ is the image of a set X under a function or binary relation f) and $X \, [f]^* \, Y \Leftrightarrow \exists x \in X, y \in Y : x \; f \; y$ for sets X, Y and a binary relation f. (Note that functions are a special case of binary relations.)

By just $\langle f \rangle^*$ and $[f]^*$ I will denote the corresponding function and relation on small sets.

Obvious 2. For a function f we have $\langle f \rangle^* X = \left\{ \frac{f(x)}{x \in X} \right\}$.

Definition 3. $\langle f^{-1} \rangle^* X$ is called the *preimage* of a set X by a function (or, more generally, a binary relation) f.

Obvious 4. $\{\alpha\} \, [f]^* \, \{\beta\} \Leftrightarrow \alpha \; f \; \beta$ for every α and β.

$\lambda x \in D : f(x) = \left\{ \frac{(x, f(x))}{x \in D} \right\}$ for a set D and and a form f depending on the variable x. In other words, $\lambda x \in D : f(x)$ is the function which maps elements x of a set D into $f(x)$.

I will denote source and destination of a morphism f of any category (See chapter 2 chapter for a definition of a category.) as $\operatorname{Src} f$ and $\operatorname{Dst} f$ correspondingly. Note that below defined domain and image of a funcoid are not the same as its source and destination.

I will denote $\operatorname{GR}(A, B, f) = f$ for any morphism (A, B, f) of either **Set** or **Rel**. (See definitions of **Set** and **Rel** below.)

1.10. Implicit arguments

Some notation such that $\bot^{\mathfrak{A}}$, $\top^{\mathfrak{A}}$, $\sqcup^{\mathfrak{A}}$, $\sqcap^{\mathfrak{A}}$ have indexes (in these examples $\mathfrak{A}$).

We will omit these indexes when they can be restored from the context. For example, having a function $f : \mathfrak{A} \to \mathfrak{B}$ where $\mathfrak{A}$, $\mathfrak{B}$ are posets with least elements, we will concisely denote $f\bot = \bot$ for $f\bot^{\mathfrak{A}} = \bot^{\mathfrak{B}}$. (See below for definitions of these operations.)

Note 5. In the above formula $f\bot = \bot$ we have the first $\bot$ and the second $\bot$ denoting different objects.

We will assume (skipping this in actual proofs) that all omitted indexes can be restored from context. (Note that so called dependent type theory computer proof assistants do this like we implicitly.)

1.11. Unusual notation

In the chapter "Common knowledge, part 1" (which you may skip reading if you are already knowledgeable) some non-standard notation is defined. I summarize here this notation for the case if you choose to skip reading that chapter:

Partial order is denoted as $\sqsubseteq$.

Meets and joins are denoted as $\sqcap$, $\sqcup$, $\bigsqcap$, $\bigsqcup$.

I call element b *substractive* from an elements a (of a distributive lattice $\mathfrak{A}$) when the difference $a \setminus b$ exists. I call b *complementive* to a when there exists $c \in \mathfrak{A}$

such that $b \sqcap c = \bot$ and $b \sqcup c = a$. We will prove that b is complementive to a iff b is substractive from a and $b \sqsubseteq a$.

DEFINITION 6. Call a and b of a poset $\mathfrak{A}$ *intersecting*, denoted $a \not\asymp b$, when there exists a non-least element c such that $c \sqsubseteq a \wedge c \sqsubseteq b$.

DEFINITION 7. $a \asymp b \overset{\text{def}}{=} \neg(a \not\asymp b)$.

DEFINITION 8. I call elements a and b of a poset $\mathfrak{A}$ *joining* and denote $a \equiv b$ when there are no non-greatest element c such that $c \sqsupseteq a \wedge c \sqsupseteq b$.

DEFINITION 9. $a \not\equiv b \overset{\text{def}}{=} \neg(a \equiv b)$.

OBVIOUS 10. $a \not\asymp b$ iff $a \sqcap b$ is non-least, for every elements a, b of a meet-semilattice.

OBVIOUS 11. $a \equiv b$ iff $a \sqcup b$ is the greatest element, for every elements a, b of a join-semilattice.

I extend the definitions of pseudocomplement and dual pseudocomplement to arbitrary posets (not just lattices as it is customary):

DEFINITION 12. Let $\mathfrak{A}$ be a poset. *Pseudocomplement* of a is

$$\max\left\{\frac{c \in \mathfrak{A}}{c \asymp a}\right\}.$$

If z is the pseudocomplement of a we will denote $z = a^*$.

DEFINITION 13. Let $\mathfrak{A}$ be a poset. *Dual pseudocomplement* of a is

$$\min\left\{\frac{c \in \mathfrak{A}}{c \equiv a}\right\}.$$

If z is the dual pseudocomplement of a we will denote $z = a^+$.

CHAPTER 2

Common knowledge, part 1

In this chapter we will consider some well known mathematical theories. If you already know them you may skip reading this chapter (or its parts).

2.1. Order theory

2.1.1. Posets.

DEFINITION 14. The *identity relation* on a set A is $\mathrm{id}_A = \left\{ \frac{(a,a)}{a\in A} \right\}$.

DEFINITION 15. A *preorder* on a set A is a binary relation $\sqsubseteq$ on A which is:

- *reflexive* on A that is $(\sqsubseteq) \supseteq \mathrm{id}_A$ or what is the same $\forall x \in A : x \sqsubseteq x$;
- *transitive* that is $(\sqsubseteq) \circ (\sqsubseteq) \subseteq (\sqsubseteq)$ or what is the same

$$\forall x, y, z : (x \sqsubseteq y \land y \sqsubseteq z \Rightarrow x \sqsubseteq z).$$

DEFINITION 16. A *partial order* on a set A is a preorder on A which is *antisymmetric* that is $(\sqsubseteq) \cap (\sqsupseteq) \subseteq \mathrm{id}_A$ or what is the same

$$\forall x, y \in A : (x \sqsubseteq y \land y \sqsubseteq x \Rightarrow x = y).$$

The reverse relation is denoted $\sqsupseteq$.

DEFINITION 17. a is a subelement of b (or what is the same a is *contained* in b or b *contains* a) iff $a \sqsubseteq b$.

OBVIOUS 18. The reverse of a partial order is also a partial order.

DEFINITION 19. A set A together with a partial order on it is called a *partially ordered set* (*poset* for short).

An example of a poset is the set $\mathbb{R}$ of real numbers with $\sqsubseteq = \leq$.

Another example is the set $\mathscr{P}A$ of all subsets of an arbitrary fixed set A with $\sqsubseteq = \subseteq$. Note that this poset is (in general) not linear (see definition of *linear* poset below.)

DEFINITION 20. Strict partial order $\sqsubset$ corresponding to the partial order $\sqsubseteq$ on a set A is defined by the formula $(\sqsubset) = (\sqsubseteq) \setminus \mathrm{id}_A$. In other words,

$$a \sqsubset b \Leftrightarrow a \sqsubseteq b \land a \neq b.$$

An example of strict partial order is $<$ on the set $\mathbb{R}$ of real numbers.

DEFINITION 21. A partial order on a set A *restricted* to a set $B \subseteq A$ is $(\sqsubseteq) \cap (B \times B)$.

OBVIOUS 22. A partial order on a set A restricted to a set $B \subseteq A$ is a partial order on B.

DEFINITION 23.

- The *least* element $\bot$ of a poset $\mathfrak{A}$ is defined by the formula $\forall a \in \mathfrak{A} : \bot \sqsubseteq a$.
- The *greatest* element $\top$ of a poset $\mathfrak{A}$ is defined by the formula $\forall a \in \mathfrak{A} : \top \sqsupseteq a$.

PROPOSITION 24. There exist no more than one least element and no more than one greatest element (for a given poset).

PROOF. By antisymmetry. □

DEFINITION 25. The *dual* order for $\sqsubseteq$ is $\sqsupseteq$.

OBVIOUS 26. Dual of a partial order is a partial order.

DEFINITION 27. The *dual* poset for a poset $(A, \sqsubseteq)$ is the poset $(A, \sqsupseteq)$.

I will denote dual of a poset $\mathfrak{A}$ as $(\operatorname{dual}\mathfrak{A})$ and dual of an element $a \in \mathfrak{A}$ (that is the same element in the dual poset) as $(\operatorname{dual} a)$.

Below we will sometimes use *duality* that is replacement of the partial order and all related operations and relations with their duals. In other words, it is enough to prove a theorem for an order $\sqsubseteq$ and the similar theorem for $\sqsupseteq$ follows by duality.

DEFINITION 28. A subset P of a poset $\mathfrak{A}$ is called *bounded above* if there exists $t \in \mathfrak{A}$ such that $\forall x \in P : t \sqsupseteq x$. *Bounded below* is defined dually.

2.1.1.1. *Intersecting and joining elements.* Let $\mathfrak{A}$ be a poset.

DEFINITION 29. Call elements a and b of $\mathfrak{A}$ *intersecting*, denoted $a \not\asymp b$, when there exists a non-least element c such that $c \sqsubseteq a \wedge c \sqsubseteq b$.

DEFINITION 30. $a \asymp b \stackrel{\text{def}}{=} \neg(a \not\asymp b)$.

OBVIOUS 31. $a_0 \not\asymp b_0 \wedge a_1 \sqsupseteq a_0 \wedge b_1 \sqsupseteq b_0 \Rightarrow a_1 \not\asymp b_1$.

DEFINITION 32. I call elements a and b of $\mathfrak{A}$ *joining* and denote $a \equiv b$ when there is no a non-greatest element c such that $c \sqsupseteq a \wedge c \sqsupseteq b$.

DEFINITION 33. $a \not\equiv b \stackrel{\text{def}}{=} \neg(a \equiv b)$.

OBVIOUS 34. Intersecting is the dual of non-joining.

OBVIOUS 35. $a_0 \equiv b_0 \wedge a_1 \sqsupseteq a_0 \wedge b_1 \sqsupseteq b_0 \Rightarrow a_1 \equiv b_1$.

2.1.2. Linear order.

DEFINITION 36. A poset $\mathfrak{A}$ is called *linearly ordered set* (or what is the same, *totally ordered set*) if $a \sqsupseteq b \vee b \sqsupseteq a$ for every $a, b \in \mathfrak{A}$.

EXAMPLE 37. The set of real numbers with the customary order is a linearly ordered set.

DEFINITION 38. A set $X \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset is called *chain* if $\mathfrak{A}$ restricted to X is a total order.

2.1.3. Meets and joins. Let $\mathfrak{A}$ be a poset.

DEFINITION 39. Given a set $X \in \mathscr{P}\mathfrak{A}$ the *least element* (also called *minimum* and denoted $\min X$) of X us such $a \in X$ that $\forall x \in X : a \sqsubseteq x$.

Least element does not necessarily exists. But if it exists:

PROPOSITION 40. For a given $X \in \mathscr{P}\mathfrak{A}$ there exist no more than one least element.

PROOF. It follows from anti-symmetry. □

Greatest element is the dual of least element:

DEFINITION 41. Given a set $X \in \mathscr{P}\mathfrak{A}$ the *greatest element* (also called *maximum* and denoted $\max X$) of X us such $a \in X$ that $\forall x \in X : a \sqsupseteq x$.

REMARK 42. Least and greatest elements of a set X is a trivial generalization of the above defined least and greatest element for the entire poset.

DEFINITION 43.

- A *minimal* element of a set $X \in \mathscr{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X : a \sqsupset x$.
- A *maximal* element of a set $X \in \mathscr{P}\mathfrak{A}$ is such $a \in \mathfrak{A}$ that $\nexists x \in X : a \sqsubset x$.

REMARK 44. Minimal element is not the same as minimum, and maximal element is not the same as maximum.

OBVIOUS 45.

1°. The least element (if it exists) is a minimal element.
2°. The greatest element (if it exists) is a maximal element.

EXERCISE 46. Show that there may be more than one minimal and more than one maximal element for some poset.

DEFINITION 47. *Upper bounds* of a set X is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsupseteq x} \right\}$.

The dual notion:

DEFINITION 48. *Lower bounds* of a set X is the set $\left\{ \frac{y \in \mathfrak{A}}{\forall x \in X : y \sqsubseteq x} \right\}$.

DEFINITION 49. *Join* $\bigsqcup X$ (also called *supremum* and denoted "$\sup X$") of a set X is the least element of its upper bounds (if it exists).

DEFINITION 50. *Meet* $\bigsqcap X$ (also called *infimum* and denoted "$\inf X$") of a set X is the greatest element of its lower bounds (if it exists).

We will also denote $\bigsqcup_{i \in X} f(i) = \bigsqcup \left\{ \frac{f(i)}{x \in X} \right\}$ and $\bigsqcap_{i \in X} f(i) = \bigsqcap \left\{ \frac{f(i)}{x \in X} \right\}$.

We will write $b = \bigsqcup X$ when $b \in \mathfrak{A}$ is the join of X or say that $\bigsqcup X$ does not exist if there are no such $b \in \mathfrak{A}$. (And dually for meets.)

EXERCISE 51. Provide an example of $\bigsqcup X \notin X$ for some set X on some poset.

PROPOSITION 52.

1°. If b is the greatest element of X then $\bigsqcup X = b$.
2°. If b is the least element of X then $\bigsqcap X = b$.

PROOF. We will prove only the first as the second is dual.

Let b be the greatest element of X. Then upper bounds of X are $\left\{ \frac{y \in \mathfrak{A}}{y \sqsupseteq b} \right\}$. Obviously b is the least element of this set, that is the join. □

DEFINITION 53. *Binary joins and meets* are defined by the formulas

$$x \sqcup y = \bigsqcup \{x, y\} \quad \text{and} \quad x \sqcup y = \bigsqcap \{x, y\}.$$

OBVIOUS 54. $\sqcup$ and $\sqcap$ are symmetric operations (whenever these are defined for given x and y).

THEOREM 55.

1°. If $\bigsqcup X$ exists then $y \sqsupseteq \bigsqcup X \Leftrightarrow \forall x \in X : y \sqsupseteq x$.
2°. If $\bigsqcap X$ exists then $y \sqsubseteq \bigsqcap X \Leftrightarrow \forall x \in X : y \sqsubseteq x$.

PROOF. I will prove only the first as the second follows by duality.
$y \sqsupseteq \bigsqcup X \Leftrightarrow y$ is an upper bound for $X \Leftrightarrow \forall x \in X : y \sqsupseteq x$. □

COROLLARY 56.

1°. If $a \sqcup b$ exists then $y \sqsupseteq a \sqcup b \Leftrightarrow y \sqsupseteq a \wedge y \sqsupseteq b$.
2°. If $a \sqcap b$ exists then $y \sqsubseteq a \sqcap b \Leftrightarrow y \sqsubseteq a \wedge y \sqsubseteq b$.

I will denote meets and joins for a specific poset $\mathfrak{A}$ as $\bigsqcap^{\mathfrak{A}}$, $\bigsqcup^{\mathfrak{A}}$, $\sqcap^{\mathfrak{A}}$, $\sqcup^{\mathfrak{A}}$.

2.1.4. Semilattices.

DEFINITION 57.

1°. A *join-semilattice* is a poset $\mathfrak{A}$ such that $a \sqcup b$ is defined for every $a, b \in \mathfrak{A}$.
2°. A *meet-semilattice* is a poset $\mathfrak{A}$ such that $a \sqcap b$ is defined for every $a, b \in \mathfrak{A}$.

THEOREM 58.

1°. The operation $\sqcup$ is associative for any join-semilattice.
2°. The operation $\sqcap$ is associative for any meet-semilattice.

PROOF. I will prove only the first as the second follows by duality.
We need to prove $(a \sqcup b) \sqcup c = a \sqcup (b \sqcup c)$ for every $a, b, c \in \mathfrak{A}$.
Taking into account the definition of join, it is enough to prove that

$$x \sqsupseteq (a \sqcup b) \sqcup c \Leftrightarrow x \sqsupseteq a \sqcup (b \sqcup c)$$

for every $x \in \mathfrak{A}$. Really, this follows from the chain of equivalences:

$$\begin{gathered} x \sqsupseteq (a \sqcup b) \sqcup c \Leftrightarrow \\ x \sqsupseteq a \sqcup b \wedge x \sqsupseteq c \Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \wedge x \sqsupseteq c \Leftrightarrow \\ x \sqsupseteq a \wedge x \sqsupseteq b \sqcup c \Leftrightarrow \\ x \sqsupseteq a \sqcup (b \sqcup c). \end{gathered}$$

□

OBVIOUS 59. $a \not\asymp b$ iff $a \sqcap b$ is non-least, for every elements a, b of a meet-semilattice.

OBVIOUS 60. $a \equiv b$ iff $a \sqcup b$ is the greatest element, for every elements a, b of a join-semilattice.

2.1.5. Lattices and complete lattices.

DEFINITION 61. A *bounded* poset is a poset having both least and greatest elements.

DEFINITION 62. *Lattice* is a poset which is both join-semilattice and meet-semilattice.

DEFINITION 63. A *complete lattice* is a poset $\mathfrak{A}$ such that for every $X \in \mathscr{P}\mathfrak{A}$ both $\bigsqcup X$ and $\bigsqcap X$ exist.

OBVIOUS 64. Every complete lattice is a lattice.

PROPOSITION 65. Every complete lattice is a bounded poset.

PROOF. $\bigsqcup \emptyset$ is the least and $\bigsqcap \emptyset$ is the greatest element. □

THEOREM 66. Let $\mathfrak{A}$ be a poset.

1°. If $\bigsqcup X$ is defined for every $X \in \mathscr{P}\mathfrak{A}$, then $\mathfrak{A}$ is a complete lattice.
2°. If $\bigsqcap X$ is defined for every $X \in \mathscr{P}\mathfrak{A}$, then $\mathfrak{A}$ is a complete lattice.

PROOF. See [27] or any lattice theory reference. □

OBVIOUS 67. If $X \subseteq Y$ for some $X, Y \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a complete lattice, then

1°. $\bigsqcup X \sqsubseteq \bigsqcup Y$;
2°. $\bigsqcap X \sqsupseteq \bigsqcap Y$.

PROPOSITION 68. If $S \in \mathscr{P}\mathscr{P}\mathfrak{A}$ then for every complete lattice $\mathfrak{A}$

1°. $\bigsqcup \bigcup S = \bigsqcup_{X \in S} \bigsqcup X$;
2°. $\bigsqcap \bigcup S = \bigsqcap_{X \in S} \bigsqcap X$.

PROOF. We will prove only the first as the second is dual.
By definition of joins, it is enough to prove $y \sqsupseteq \bigsqcup \bigcup S \Leftrightarrow y \sqsupseteq \bigsqcup_{X \in S} \bigsqcup X$.
Really,

$$\begin{aligned} y \sqsupseteq \bigsqcup \bigcup S &\Leftrightarrow \\ \forall x \in \bigcup S : y \sqsupseteq x &\Leftrightarrow \\ \forall X \in S \forall x \in X : y \sqsupseteq x &\Leftrightarrow \\ \forall X \in S : y \sqsupseteq \bigsqcup X &\Leftrightarrow \\ y \sqsupseteq \bigsqcup_{X \in S} \bigsqcup X&. \end{aligned}$$

□

DEFINITION 69. A *sublattice* of a lattice is it subset closed regarding $\sqcup$ and $\sqcap$.

OBVIOUS 70. Sublattice with induced order is also a lattice.

2.1.6. Distributivity of lattices.

DEFINITION 71. A *distributive* lattice is such lattice $\mathfrak{A}$ that for every $x, y, z \in \mathfrak{A}$

1°. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2°. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

THEOREM 72. For a lattice to be distributive it is enough just one of the conditions:

1°. $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$;
2°. $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

PROOF.

$$\begin{aligned} (x \sqcup y) \sqcap (x \sqcup z) &= \\ ((x \sqcup y) \sqcap x) \sqcup ((x \sqcup y) \sqcap z) &= \\ x \sqcup ((x \sqcap z) \sqcup (y \sqcap z)) &= \\ (x \sqcup (x \sqcap z)) \sqcup (y \sqcap z) &= \\ x \sqcup (y \sqcap z)& \end{aligned}$$

(applied $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$ twice). □

2.1.7. Difference and complement.

DEFINITION 73. Let $\mathfrak{A}$ be a distributive lattice with least element $\bot$. The *difference* (denoted $a \setminus b$) of elements a and b is such $c \in \mathfrak{A}$ that $b \sqcap c = \bot$ and $a \sqcup b = b \sqcup c$. I will call b *substractive* from a when $a \setminus b$ exists.

THEOREM 74. If $\mathfrak{A}$ is a distributive lattice with least element $\bot$, there exists no more than one difference of elements a, b.

PROOF. Let c and d be both differences $a \setminus b$. Then $b \sqcap c = b \sqcap d = \bot$ and $a \sqcup b = b \sqcup c = b \sqcup d$. So

$$c = c \sqcap (b \sqcup c) = c \sqcap (b \sqcup d) = (c \sqcap b) \sqcup (c \sqcap d) = \bot \sqcup (c \sqcap d) = c \sqcap d.$$

Similarly $d = d \sqcap c$. Consequently $c = c \sqcap d = d \sqcap c = d$. □

DEFINITION 75. I will call b *complementive* to a iff there exists $c \in \mathfrak{A}$ such that $b \sqcap c = \bot$ and $b \sqcup c = a$.

PROPOSITION 76. b is complementive to a iff b is substractive from a and $b \sqsubseteq a$.

PROOF.

$\Leftarrow$. Obvious.
$\Rightarrow$. We deduce $b \sqsubseteq a$ from $b \sqcup c = a$. Thus $a \sqcup b = a = b \sqcup c$.

□

PROPOSITION 77. If b is complementive to a then $(a \setminus b) \sqcup b = a$.

PROOF. Because $b \sqsubseteq a$ by the previous proposition. □

DEFINITION 78. Let $\mathfrak{A}$ be a bounded distributive lattice. The *complement* (denoted $\bar{a}$) of an element $a \in \mathfrak{A}$ is such $b \in \mathfrak{A}$ that $a \sqcap b = \bot$ and $a \sqcup b = \top$.

PROPOSITION 79. If $\mathfrak{A}$ is a bounded distributive lattice then $\bar{a} = \top \setminus a$.

PROOF. $b = \bar{a} \Leftrightarrow b \sqcap a = \bot \wedge b \sqcup a = \top \Leftrightarrow b \sqcap a = \bot \wedge \top \sqcup a = a \sqcup b \Leftrightarrow b = \top \setminus a$. □

COROLLARY 80. If $\mathfrak{A}$ is a bounded distributive lattice then exists no more than one complement of an element $a \in \mathfrak{A}$.

DEFINITION 81. An element of bounded distributive lattice is called *complemented* when its complement exists.

DEFINITION 82. A distributive lattice is a *complemented lattice* iff every its element is complemented.

PROPOSITION 83. For a distributive lattice $(a \setminus b) \setminus c = a \setminus (b \sqcup c)$ if $a \setminus b$ and $(a \setminus b) \setminus c$ are defined.

PROOF. $((a \setminus b) \setminus c) \sqcap c = \bot$; $((a \setminus b) \setminus c) \sqcup c = (a \setminus b) \sqcup c$; $(a \setminus b) \sqcap b = \bot$; $(a \setminus b) \sqcup b = a \sqcup b$.

We need to prove $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = \bot$ and $((a \setminus b) \setminus c) \sqcup (b \sqcup c) = a \sqcup (b \sqcup c)$.

In fact,

$$\begin{aligned} ((a \setminus b) \setminus c) \sqcap (b \sqcup c) &= \\ (((a \setminus b) \setminus c) \sqcap b) \sqcup (((a \setminus b) \setminus c) \sqcap c) &= \\ (((a \setminus b) \setminus c) \sqcap b) \sqcup \bot &= \\ ((a \setminus b) \setminus c) \sqcap b &\sqsubseteq \\ (a \setminus b) \sqcap b &= \bot, \end{aligned}$$

so $((a \setminus b) \setminus c) \sqcap (b \sqcup c) = \bot$;

$$\begin{aligned} ((a \setminus b) \setminus c) \sqcup (b \sqcup c) &= \\ (((a \setminus b) \setminus c) \sqcup c) \sqcup b &= \\ (a \setminus b) \sqcup c \sqcup b &= \\ ((a \setminus b) \sqcup b) \sqcup c &= \\ a \sqcup b \sqcup c. \end{aligned}$$

□

2.1.8. Boolean lattices.

DEFINITION 84. A *boolean lattice* is a complemented distributive lattice.

The most important example of a boolean lattice is $\mathscr{P}A$ where A is a set, ordered by set inclusion.

THEOREM 85. (DE MORGAN's laws) For every elements a, b of a boolean lattice
1°. $\overline{a \sqcup b} = \bar{a} \sqcap \bar{b}$;
2°. $\overline{a \sqcap b} = \bar{a} \sqcup b$.

PROOF. We will prove only the first as the second is dual.
It is enough to prove that $a \sqcup b$ is a complement of $\bar{a} \sqcap \bar{b}$. Really:

$$(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) \sqsubseteq a \sqcap (\bar{a} \sqcap \bar{b}) = (a \sqcap \bar{a}) \sqcap \bar{b} = \bot \sqcap \bar{b} = \bot;$$

$$(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = ((a \sqcup b) \sqcup \bar{a}) \sqcap ((a \sqcup b) \sqcup \bar{b}) \sqsupseteq (a \sqcup \bar{a}) \sqcap (b \sqcup \bar{b}) = \top \sqcap \top = \top.$$

Thus $(a \sqcup b) \sqcap (\bar{a} \sqcap \bar{b}) = \bot$ and $(a \sqcup b) \sqcup (\bar{a} \sqcap \bar{b}) = \top$. □

DEFINITION 86. A complete lattice $\mathfrak{A}$ is *join infinite distributive* when $x \sqcap \bigsqcup S = \bigsqcup \langle x\sqcap\rangle^* S$; a complete lattice $\mathfrak{A}$ is *meet infinite distributive* when $x \sqcup \bigsqcap S = \bigsqcap \langle x\sqcup\rangle^* S$ for all $x \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$.

DEFINITION 87. *Infinite distributive complete lattice* is a complete lattice which is both join infinite distributive and meet infinite distributive.

THEOREM 88. For every boolean lattice $\mathfrak{A}$, $x \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$ we have:
1°. $\bigsqcup \langle x\sqcap\rangle^* S$ is defined and $x \sqcap \bigsqcup S = \bigsqcup \langle x\sqcap\rangle^* S$ whenever $\bigsqcup S$ is defined.
2°. $\bigsqcap \langle x\sqcup\rangle^* S$ is defined and $x \sqcup \bigsqcap S = \bigsqcap \langle x\sqcup\rangle^* S$ whenever $\bigsqcap S$ is defined.

PROOF. We will prove only the first, as the other is dual.
We need to prove that $x \sqcap \bigsqcup S$ is the least upper bound of $\langle x\sqcap\rangle^* S$.
That $x \sqcap \bigsqcup S$ is an upper bound of $\langle x\sqcap\rangle^* S$ is obvious.
Now let u be any upper bound of $\langle x\sqcap\rangle^* S$, that is $x \sqcap y \sqsubseteq u$ for all $y \in S$. Then

$$y = y \sqcap (x \sqcup \bar{x}) = (y \sqcap x) \sqcup (y \sqcap \bar{x}) \sqsubseteq u \sqcup \bar{x},$$

and so $\bigsqcup S \sqsubseteq u \sqcup \bar{x}$. Thus

$$x \sqcap \bigsqcup S \sqsubseteq x \sqcap (u \sqcup \bar{x}) = (x \sqcap u) \sqcup (x \sqcap \bar{x}) = (x \sqcap u) \sqcup \bot = x \sqcap u \sqsubseteq u,$$

that is $x \sqcap \bigsqcup S$ is the least upper bound of $\langle x\sqcap\rangle^* S$. □

COROLLARY 89. Every complete boolean lattice is both join infinite distributive and meet infinite distributive.

THEOREM 90. (infinite DE MORGAN's laws) For every subset S of a complete boolean lattice
1°. $\overline{\bigsqcup S} = \bigsqcap_{x \in S} \bar{x}$;
2°. $\overline{\bigsqcap S} = \bigsqcup_{x \in S} \bar{x}$.

PROOF. It's enough to prove that $\bigsqcup S$ is a complement of $\bigsqcap_{x \in S} \bar{x}$ (the second follows from duality). Really, using the previous theorem:

$$\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = \bigsqcap_{x \in S} \left\langle \bigsqcup S \sqcup \right\rangle^* \bar{x} = \bigsqcap \left\{ \frac{\bigsqcup S \sqcup \bar{x}}{x \in S} \right\} \sqsupseteq \bigsqcap \left\{ \frac{x \sqcup \bar{x}}{x \in S} \right\} = \top;$$

$$\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = \bigsqcup_{y \in S} \left\langle \bigsqcap_{x \in S} \bar{x} \sqcap \right\rangle^* y = \bigsqcup \left\{ \frac{\bigsqcap_{x \in S} \bar{x} \sqcap y}{y \in S} \right\} \sqsubseteq \bigsqcup \left\{ \frac{\bar{y} \sqcap y}{y \in S} \right\} = \bot.$$

So $\bigsqcup S \sqcup \bigsqcap_{x \in S} \bar{x} = \top$ and $\bigsqcup S \sqcap \bigsqcap_{x \in S} \bar{x} = \bot$. □

2.1.9. Center of a lattice.

Definition 91. The *center* $Z(\mathfrak{A})$ of a bounded distributive lattice $\mathfrak{A}$ is the set of its complemented elements.

Remark 92. For a definition of center of non-distributive lattices see [5].

Remark 93. In [24] the word center and the notation $Z(\mathfrak{A})$ are used in a different sense.

Definition 94. A sublattice K of a complete lattice L is a *closed sublattice* of L if K contains the meet and the join of any its nonempty subset.

Theorem 95. Center of an infinitely distributive lattice is its closed sublattice.

Proof. See [17]. □

Remark 96. See [18] for a more strong result.

Theorem 97. The center of a bounded distributive lattice constitutes its sublattice.

Proof. Let $\mathfrak{A}$ be a bounded distributive lattice and $Z(\mathfrak{A})$ be its center. Let $a, b \in Z(\mathfrak{A})$. Consequently $\bar{a}, \bar{b} \in Z(\mathfrak{A})$. Then $\bar{a} \sqcup \bar{b}$ is the complement of $a \sqcap b$ because

$$(a \sqcap b) \sqcap (\bar{a} \sqcup \bar{b}) = (a \sqcap b \sqcap \bar{a}) \sqcup (a \sqcap b \sqcap \bar{b}) = \bot \sqcup \bot = \bot \quad \text{and}$$
$$(a \sqcap b) \sqcup (\bar{a} \sqcup \bar{b}) = (a \sqcup \bar{a} \sqcup \bar{b}) \sqcap (b \sqcup \bar{a} \sqcup \bar{b}) = \top \sqcap \top = \top.$$

So $a \sqcap b$ is complemented. Similarly $a \sqcup b$ is complemented. □

Theorem 98. The center of a bounded distributive lattice constitutes a boolean lattice.

Proof. Because it is a distributive complemented lattice. □

2.1.10. Atoms of posets.

Definition 99. An atom of a poset is an element a such that (for every its element x) $x \sqsubset a$ if and only if x is the least element.

Remark 100. This definition is valid even for posets without least element.

Proposition 101. Element a is an atom iff both:

1°. $x \sqsubset a$ implies x is the least element;
2°. a is non-least.

Proof.

⇒. Let a be an atom. 1° is obvious. If a is least then $a \sqsubset a$ what is impossible, so 2°.

⇐. Let 1° and 2° hold. We need to prove only that x is least implies that $x \sqsubset a$ but this follows from a being non-least.

□

Example 102. Atoms of the boolean algebra $\mathscr{P}A$ (ordered by set inclusion) are one-element sets.

I will denote $\operatorname{atoms}^{\mathfrak{A}} a$ or just $(\operatorname{atoms} a)$ the set of atoms contained in an element a of a poset $\mathfrak{A}$. I will denote $\operatorname{atoms}^{\mathfrak{A}}$ the set of all atoms of a poset $\mathfrak{A}$.

Definition 103. A poset $\mathfrak{A}$ is called *atomic* iff $\operatorname{atoms} a \neq \emptyset$ for every non-least element a of the poset $\mathfrak{A}$.

DEFINITION 104. *Atomistic poset* is such a poset that $a = \bigsqcup \operatorname{atoms} a$ for every element a of this poset.

OBVIOUS 105. Every atomistic poset is atomic.

PROPOSITION 106. Let $\mathfrak{A}$ be a poset. If a is an atom of $\mathfrak{A}$ and $B \in \mathfrak{A}$ then

$$a \in \operatorname{atoms} B \Leftrightarrow a \sqsubseteq B \Leftrightarrow a \not\asymp B.$$

PROOF.

$a \in \operatorname{atoms} B \Leftrightarrow a \sqsubseteq B$. Obvious.

$a \sqsubseteq B \Rightarrow a \not\asymp B$. $a \sqsubseteq B \Rightarrow a \sqsubseteq a \wedge a \sqsubseteq B$, thus $a \not\asymp B$ because a is not least.

$a \sqsubseteq B \Leftarrow a \not\asymp B$. $a \not\asymp B$ implies existence of non-least element x such that $x \sqsubseteq B$ and $x \sqsubseteq a$. Because a is an atom, we have $x = a$. So $a \sqsubseteq B$.

□

THEOREM 107. A poset is atomistic iff every its element can be represented as join of atoms.

PROOF.

$\Rightarrow$. Obvious.

$\Leftarrow$. Let $a = \bigsqcup S$ where S is a set of atoms. We will prove that a is the least upper bound of $\operatorname{atoms} a$.

That a is an upper bound of $\operatorname{atoms} a$ is obvious. Let x is an upper bound of $\operatorname{atoms} a$. Then $x \sqsupseteq \bigsqcup S$ because $S \subseteq \operatorname{atoms} a$. Thus $x \sqsupseteq a$.

□

THEOREM 108. $\operatorname{atoms} \bigsqcap S = \bigcap \langle \operatorname{atoms} \rangle^* S$ whenever $\bigsqcap S$ is defined for every $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset.

PROOF. For any atom

$$\begin{aligned} c \in \operatorname{atoms} \bigsqcap S &\Leftrightarrow \\ c \sqsubseteq \bigsqcap S &\Leftrightarrow \\ \forall a \in S : c \sqsubseteq a &\Leftrightarrow \\ \forall a \in S : c \in \operatorname{atoms} a &\Leftrightarrow \\ c \in \bigcap \langle \operatorname{atoms} \rangle^* S&. \end{aligned}$$

□

COROLLARY 109. $\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ for an arbitrary meet-semilattice.

THEOREM 110. A complete boolean lattice is atomic iff it is atomistic.

PROOF.

$\Leftarrow$. Obvious.

$\Rightarrow$. Let $\mathfrak{A}$ be an atomic boolean lattice. Let $a \in \mathfrak{A}$. Suppose $b = \bigsqcup \operatorname{atoms} a \sqsubset a$. If $x \in \operatorname{atoms}(a \setminus b)$ then $x \sqsubseteq a \setminus b$ and so $x \sqsubseteq a$ and hence $x \sqsubseteq b$. But we have $x = x \sqcap b \sqsubseteq (a \setminus b) \sqcap b = \bot$ what contradicts to our supposition.

□

2.1.11. Kuratowski's lemma.

THEOREM 111. (KURATOWSKI's lemma) Any chain in a poset is contained in a maximal chain (if we order chains by inclusion).

I will skip the proof of KURATOWSKI's lemma as this proof can be found in any set theory or order theory reference.

2.1.12. Homomorphisms of posets and lattices.

Definition 112. A *monotone* function (also called *order homomorphism*) from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ is such a function f that $x \sqsubseteq y \Rightarrow fx \sqsubseteq fy$ for every $x, y \in \mathfrak{A}$.

Definition 113. A *antitone* function (also called *antitone order homomorphism*) from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ is such a function f that $x \sqsubseteq y \Rightarrow fx \sqsupseteq fy$ for every $x, y \in \mathfrak{A}$.

Definition 114. *Order embedding* is a function f from poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ such that $x \sqsubseteq y \Leftrightarrow fx \sqsubseteq fy$ for every $x, y \in \mathfrak{A}$.

Proposition 115. Every order embedding is injective.

Proof. $fx = fy$ implies $x \sqsubseteq y$ and $y \sqsubseteq x$. □

Obvious 116. Every order embedding is an order homomorphism.

Definition 117. *Antitone order embedding* is a function f from poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ such that $x \sqsubseteq y \Leftrightarrow fx \sqsupseteq fy$ for every $x, y \in \mathfrak{A}$.

Obvious 118. Antitone order embedding is an order embedding between a poset and a dual of (another) poset.

Definition 119. *Order isomorphism* is a surjective order embedding.

Order isomorphism preserves properties of posets, such as order, joins and meets, etc.

Definition 120. *Antitone order isomorphism* is a surjective antitone order embedding.

Definition 121.

1°. *Join semilattice homomorphism* is a function f from a join semilattice $\mathfrak{A}$ to a join semilattice $\mathfrak{B}$, such that $f(x \sqcup y) = fx \sqcup fy$ for every $x, y \in \mathfrak{A}$.
2°. *Meet semilattice homomorphism* is a function f from a meet semilattice $\mathfrak{A}$ to a meet semilattice $\mathfrak{B}$, such that $f(x \sqcap y) = fx \sqcap fy$ for every $x, y \in \mathfrak{A}$.

Obvious 122.

1°. Join semilattice homomorphisms are monotone.
2°. Meet semilattice homomorphisms are monotone.

Definition 123. A *lattice homomorphism* is a function from a lattice to a lattice, which is both join semilattice homomorphism and meet semilattice homomorphism.

Definition 124. *Complete lattice homomorphism* from a complete lattice $\mathfrak{A}$ to a complete lattice $\mathfrak{B}$ is a function f from $\mathfrak{A}$ to $\mathfrak{B}$ which preserves all meets and joins, that is $f \bigsqcup S = \bigsqcup \langle f \rangle^* S$ and $f \bigsqcap S = \bigsqcap \langle f \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A}$.

2.1.13. Galois connections.

See [3, 12] for more detailed treatment of Galois connections.

Definition 125. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two posets. A *Galois connection* between $\mathfrak{A}$ and $\mathfrak{B}$ is a pair of functions $f = (f^*, f_*)$ with $f^* : \mathfrak{A} \to \mathfrak{B}$ and $f_* : \mathfrak{B} \to \mathfrak{A}$ such that:

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (f^* x \sqsubseteq y \Leftrightarrow x \sqsubseteq f_* y).$$

f_* is called *the upper adjoint* of f^* and f^* is called *the lower adjoint* of f_*.

Theorem 126. A pair (f^*, f_*) of functions $f^* : \mathfrak{A} \to \mathfrak{B}$ and $f_* : \mathfrak{B} \to \mathfrak{A}$ is a Galois connection iff both of the following:

1°. f^* and f_* are monotone.
2°. $x \sqsubseteq f_* f^* x$ and $f^* f_* y \sqsubseteq y$ for every $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$.

PROOF.

$\Rightarrow$.

2°. $x \sqsubseteq f_* f^* x$ since $f^* x \sqsubseteq f^* x$; $f^* f_* y \sqsubseteq y$ since $f_* y \sqsubseteq f_* y$.
1°. Let $a, b \in \mathfrak{A}$ and $a \sqsubseteq b$. Then $a \sqsubseteq b \sqsubseteq f_* f^* b$. So by definition $f^* a \sqsubseteq f^* b$ that is f^* is monotone. Analogously f_* is monotone.

$\Leftarrow$. $f^* x \sqsubseteq y \Rightarrow f_* f^* x \sqsubseteq f_* y \Rightarrow x \sqsubseteq f_* y$. The other direction is analogous.

□

THEOREM 127.
1°. $f^* \circ f_* \circ f^* = f^*$.
2°. $f_* \circ f^* \circ f_* = f_*$.

PROOF.
1°. Let $x \in \mathfrak{A}$. We have $x \sqsubseteq f_* f^* x$; consequently $f^* x \sqsubseteq f^* f_* f^* x$. On the other hand, $f^* f_* f^* x \sqsubseteq f^* x$. So $f^* f_* f^* x = f^* x$.
2°. Similar.

□

DEFINITION 128. A function f is called *idempotent* iff $f(f(X)) = f(X)$ for every argument X.

PROPOSITION 129. $f^* \circ f_*$ and $f_* \circ f^*$ are idempotent.

PROOF. $f^* \circ f_*$ is idempotent because $f^* f_* f^* f_* y = f^* f_* y$. $f_* \circ f^*$ is similar. □

THEOREM 130. Each of two adjoints is uniquely determined by the other.

PROOF. Let p and q be both upper adjoints of f. We have for all $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$:

$$x \sqsubseteq p(y) \Leftrightarrow f(x) \sqsubseteq y \Leftrightarrow x \sqsubseteq q(y).$$

For $x = p(y)$ we obtain $p(y) \sqsubseteq q(y)$ and for $x = q(y)$ we obtain $q(y) \sqsubseteq p(y)$. So $q(y) = p(y)$. □

THEOREM 131. Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.
1°. Both:
(a) If f is monotone and $g(b) = \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq b}\right\}$ is defined for every $b \in \mathfrak{B}$ then g is the upper adjoint of f.
(b) If $g : \mathfrak{B} \to \mathfrak{A}$ is the upper adjoint of f then $g(b) = \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq b}\right\}$ for every $b \in \mathfrak{B}$.
2°. Both:
(a) If f is monotone and $g(b) = \min\left\{\frac{x \in \mathfrak{A}}{fx \sqsupseteq b}\right\}$ is defined for every $b \in \mathfrak{B}$ then g is the lower adjoint of f.
(b) If $g : \mathfrak{B} \to \mathfrak{A}$ is the lower adjoint of f then $g(b) = \min\left\{\frac{x \in \mathfrak{A}}{fx \sqsupseteq b}\right\}$ for every $b \in \mathfrak{B}$.

PROOF. We will prove only the first as the second is its dual.

1°a. Let $g(b) = \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq b}\right\}$ for every $b \in \mathfrak{B}$. Then

$$x \sqsubseteq gy \Leftrightarrow x \sqsubseteq \max\left\{\frac{x \in \mathfrak{A}}{fx \sqsubseteq y}\right\} \Rightarrow fx \sqsubseteq y$$

(because f is monotone) and

$$x \sqsubseteq gy \Leftrightarrow x \sqsubseteq \max\left\{ \frac{x \in \mathfrak{A}}{fx \sqsubseteq y} \right\} \Leftarrow fx \sqsubseteq y.$$

So $fx \sqsubseteq y \Leftrightarrow x \sqsubseteq gy$ that is f is the lower adjoint of g.

1°b. We have

$$g(b) = \max\left\{ \frac{x \in \mathfrak{A}}{fx \sqsubseteq b} \right\} \Leftrightarrow fgb \sqsubseteq b \wedge \forall x \in \mathfrak{A} : (fx \sqsubseteq b \Rightarrow x \sqsubseteq gb).$$

what is true by properties of adjoints.

□

THEOREM 132. Let f be a function from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

1°. If f is an upper adjoint, f preserves all existing infima in $\mathfrak{A}$.
2°. If $\mathfrak{A}$ is a complete lattice and f preserves all infima, then f is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$.
3°. If f is a lower adjoint, f preserves all existing suprema in $\mathfrak{A}$.
4°. If $\mathfrak{A}$ is a complete lattice and f preserves all suprema, then f is a lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$.

PROOF. We will prove only first two items because the rest items are similar.

1°. Let $S \in \mathscr{P}\mathfrak{A}$ and $\bigsqcap S$ exists. $f \bigsqcap S$ is a lower bound for $\langle f\rangle^* S$ because f is order-preserving. If a is a lower bound for $\langle f\rangle^* S$ then $\forall x \in S : a \sqsubseteq fx$ that is $\forall x \in S : ga \sqsubseteq x$ where g is the lower adjoint of f. Thus $ga \sqsubseteq \bigsqcap S$ and hence $f \bigsqcap S \sqsupseteq a$. So $f \bigsqcap S$ is the greatest lower bound for $\langle f\rangle^* S$.

2°. Let $\mathfrak{A}$ be a complete lattice and f preserves all infima. Let

$$g(a) = \bigsqcap \left\{ \frac{x \in \mathfrak{A}}{fx \sqsupseteq a} \right\}.$$

Since f preserves infima, we have

$$f(g(a)) = \bigsqcap \left\{ \frac{f(x)}{x \in \mathfrak{A}, fx \sqsupseteq a} \right\} \sqsupseteq a.$$

$g(f(b)) = \bigsqcap \left\{ \frac{x \in \mathfrak{A}}{fx \sqsupseteq fb} \right\} \sqsubseteq b.$

Obviously f is monotone and thus g is also monotone.

So f is the upper adjoint of g.

□

COROLLARY 133. Let f be a function from a complete lattice $\mathfrak{A}$ to a poset $\mathfrak{B}$. Then:

1°. f is an upper adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all infima in $\mathfrak{A}$.
2°. f is a lower adjoint of a function $\mathfrak{B} \to \mathfrak{A}$ iff f preserves all suprema in $\mathfrak{A}$.

2.1.13.1. *Order and composition of Galois connections.* Following [32] we will denote the set of Galois connection between posets $\mathfrak{A}$ and $\mathfrak{B}$ as $\mathfrak{A} \otimes \mathfrak{B}$.

DEFINITION 134. I will order Galois connections by the formula: $f \sqsubseteq g \Leftrightarrow f^* \sqsubseteq g^*$ (where $f^* \sqsubseteq g^* \Leftrightarrow \forall x \in \mathfrak{A} : f^* x \sqsubseteq g^* x$).

OBVIOUS 135. Galois connections $\mathfrak{A} \otimes \mathfrak{B}$ between two given posets form a poset.

PROPOSITION 136. $f \sqsubseteq g \Leftrightarrow f_* \sqsupseteq g_*$.

Proof. It is enough to prove $f \sqsubseteq g \Rightarrow f_* \sqsupseteq g_*$ (the rest follows from the fact that a Galois connection is determined by one adjoint).

Really, let $f \sqsubseteq g$. Then $f_0^* \sqsubseteq f_1^*$ and thus:
$f_{0*}(b) = \max\left\{\frac{x \in \mathfrak{A}}{f_0^* x \sqsubseteq b}\right\}$, $f_{1*}(b) = \max\left\{\frac{x \in \mathfrak{A}}{f_1^* x \sqsubseteq b}\right\}$.
Thus $f_{0*}(b) \sqsupseteq f_{1*}(b)$ for every $b \in \mathfrak{B}$ and so $f_{0*} \sqsupseteq f_{1*}$. □

Definition 137. Composition of Galois connections is defined by the formula: $g \circ f = (g^* \circ f^*, f_* \circ g_*)$.

Proposition 138. Composition of Galois connections is a Galois connection.

Proof. $g^* \circ f^*$ and $f_* \circ g_*$ are monotone as composition of monotone functions;

$$(g^* \circ f^*)x \sqsubseteq z \Leftrightarrow g^* f^* x \sqsubseteq z \Leftrightarrow f^* x \sqsubseteq g_* z \Leftrightarrow x \sqsubseteq f_* g_* z \Leftrightarrow x \sqsubseteq (f_* \circ g_*)z.$$

□

Obvious 139. Composition of Galois connections preserves order.

2.1.13.2. *Antitone Galois connections.*

Definition 140. An *antitone Galois connection* between posets $\mathfrak{A}$ and $\mathfrak{B}$ is a Galois connection between $\mathfrak{A}$ and $\operatorname{dual}\mathfrak{B}$.

Obvious 141. An antitone Galois connection is a pair of antitone functions $f : \mathfrak{A} \to \mathfrak{B}$, $g : \mathfrak{B} \to \mathfrak{A}$ such that $b \sqsubseteq fa \Leftrightarrow a \sqsubseteq gb$ for every $a \in \mathfrak{A}$, $b \in \mathfrak{B}$.

Such f and g are called *polarities* (between $\mathfrak{A}$ and $\mathfrak{B}$).

Obvious 142. $f \bigsqcup S = \bigsqcap \langle f \rangle^* S$ if f is a polarity between $\mathfrak{A}$ and $\mathfrak{B}$ and $S \in \mathscr{P}\mathfrak{A}$.

Galois connections (particularly between boolean lattices) are studied in [32] and [33].

2.1.14. Co-Brouwerian lattices.

Definition 143. Let $\mathfrak{A}$ be a poset. *Pseudocomplement* of $a \in \mathfrak{A}$ is

$$\max\left\{\frac{c \in \mathfrak{A}}{c \asymp a}\right\}.$$

If z is the pseudocomplement of a we will denote $z = a^*$.

Definition 144. Let $\mathfrak{A}$ be a poset. *Dual pseudocomplement* of $a \in \mathfrak{A}$ is

$$\min\left\{\frac{c \in \mathfrak{A}}{c \equiv a}\right\}.$$

If z is the dual pseudocomplement of a we will denote $z = a^+$.

Proposition 145. If a is a complemented element of a bounded distributive lattice, then $\bar{a}$ is both pseudocomplement and dual pseudocomplement of a.

Proof. Because of duality it is enough to prove that $\bar{a}$ is pseudocomplement of a.

We need to prove $c \asymp a \Rightarrow c \sqsubseteq \bar{a}$ for every element c of our poset, and $\bar{a} \asymp a$. The second is obvious. Let's prove $c \asymp a \Rightarrow c \sqsubseteq \bar{a}$.

Really, let $c \asymp a$. Then $c \sqcap a = \bot$; $\bar{a} \sqcup (c \sqcap a) = \bar{a}$; $(\bar{a} \sqcup c) \sqcap (\bar{a} \sqcup a) = \bar{a}$; $\bar{a} \sqcup c = \bar{a}$; $c \sqsubseteq \bar{a}$. □

Definition 146. Let $\mathfrak{A}$ be a join-semilattice. Let $a, b \in \mathfrak{A}$. *Pseudodifference* of a and b is

$$\min\left\{\frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z}\right\}.$$

If z is a pseudodifference of a and b we will denote $z = a \setminus^* b$.

REMARK 147. I do not require that a^* is undefined if there are no pseudocomplement of a and likewise for dual pseudocomplement and pseudodifference. In fact below I will define quasicomplement, dual quasicomplement, and quasidifference which generalize pseudo-* counterparts. I will denote a^* the more general case of quasicomplement than of pseudocomplement, and likewise for other notation.

OBVIOUS 148. Dual pseudocomplement is the dual of pseudocomplement.

THEOREM 149. Let $\mathfrak{A}$ be a distributive lattice with least element. Let $a, b \in \mathfrak{A}$. If $a \setminus b$ exists, then $a \setminus^* b$ also exists and $a \setminus^* b = a \setminus b$.

PROOF. Because $\mathfrak{A}$ be a distributive lattice with least element, the definition of $a \setminus b$ is correct.

Let $x = a \setminus b$ and let $S = \left\{ \frac{y \in \mathfrak{A}}{a \sqsubseteq b \sqcup y} \right\}$.

We need to show

1°. $x \in S$;

2°. $y \in S \Rightarrow x \sqsubseteq y$ (for every $y \in \mathfrak{A}$).

Really,

1°. Because $b \sqcup x = a \sqcup b$.

2°.

$$\begin{aligned}
& y \in S \\
\Rightarrow\quad & a \sqsubseteq b \sqcup y && \text{(by definition of } S\text{)} \\
\Rightarrow\quad & a \sqcup b \sqsubseteq b \sqcup y \\
\Rightarrow\quad & x \sqcup b \sqsubseteq b \sqcup y && \text{(since } x \sqcup b = a \sqcup b\text{)} \\
\Rightarrow\quad & x \sqcap (x \sqcup b) \sqsubseteq x \sqcap (b \sqcup y) \\
\Rightarrow\quad & (x \sqcap x) \sqcup (x \sqcap b) \sqsubseteq (x \sqcap b) \sqcup (x \sqcap y) && \text{(by distributive law)} \\
\Rightarrow\quad & x \sqcup \bot \sqsubseteq \bot \sqcup (x \sqcap y) && \text{(since } x \sqcap b = \bot\text{)} \\
\Rightarrow\quad & x \sqsubseteq x \sqcap y \\
\Rightarrow\quad & x \sqsubseteq y.
\end{aligned}$$

□

DEFINITION 150. *Co-brouwerian lattice* is a lattice for which pseudodifference of any two its elements is defined.

PROPOSITION 151. Every non-empty co-brouwerian lattice $\mathfrak{A}$ has least element.

PROOF. Let a be an arbitrary lattice element. Then

$$a \setminus^* a = \min \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq a \sqcup z} \right\} = \min \mathfrak{A}.$$

So $\min \mathfrak{A}$ exists. □

DEFINITION 152. *Co-Heyting lattice* is co-brouwerian lattice with greatest element.

DEFINITION 153. A *co-frame* is the same as a complete co-brouwerian lattice.

THEOREM 154. For a co-brouwerian lattice $a \sqcup -$ is an upper adjoint of $- \setminus^* a$ for every $a \in \mathfrak{A}$.

PROOF. $g(b) = \min \left\{ \frac{x \in \mathfrak{A}}{a \sqcup x \sqsupseteq b} \right\} = b \setminus^* a$ exists for every $b \in \mathfrak{A}$ and thus is the lower adjoint of $a \sqcup -$. □

COROLLARY 155. $\forall a, x, y \in \mathfrak{A} : (x \setminus^* a \sqsubseteq y \Leftrightarrow x \sqsubseteq a \sqcup y)$ for a co-brouwerian lattice.

COROLLARY 156. For a co-brouwerian lattice $a \sqcup \bigsqcap S = \bigsqcap \langle a \sqcup \rangle^* S$ whenever $\bigsqcap S$ exists (for a being a lattice element and S being a set of lattice elements).

DEFINITION 157. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a complete lattice. *Quasidifference* $a \setminus^* b$ is defined by the formula:

$$a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}.$$

REMARK 158. A more detailed theory of quasidifference (as well as quasicomplement and dual quasicomplement) will be considered below.

LEMMA 159. $(a \setminus^* b) \sqcup b = a \sqcup b$ for elements a, b of a meet infinite distributive complete lattice.

PROOF.

$$\begin{aligned} &(a \setminus^* b) \sqcup b = \\ &\bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\} \sqcup b = \\ &\bigsqcap \left\{ \frac{z \sqcup b}{z \in \mathfrak{A}, a \sqsubseteq b \sqcup z} \right\} = \\ &\bigsqcap \left\{ \frac{t \in \mathfrak{A}}{t \sqsupseteq b, a \sqsubseteq t} \right\} = \\ &a \sqcup b. \end{aligned}$$

□

THEOREM 160. The following are equivalent for a complete lattice $\mathfrak{A}$:

1°. $\mathfrak{A}$ is a co-frame.
2°. $\mathfrak{A}$ is meet infinite distributive.
3°. $\mathfrak{A}$ is a co-brouwerian lattice.
4°. $\mathfrak{A}$ is a co-Heyting lattice.
5°. $a \sqcup -$ has lower adjoint for every $a \in \mathfrak{A}$.

PROOF. □

1°⇔3°. Because it is complete.
3°⇔4°. Obvious (taking into account completeness of $\mathfrak{A}$).
5°⇒2°. Let $- \setminus^* a$ be the lower adjoint of $a \sqcup -$. Let $S \in \mathscr{P}\mathfrak{A}$. For every $y \in S$ we have $y \sqsupseteq (a \sqcup y) \setminus^* a$ by properties of Galois connections; consequently $y \sqsupseteq \left(\bigsqcap \langle a \sqcup \rangle^* S\right) \setminus^* a$; $\bigsqcap S \sqsupseteq \left(\bigsqcap \langle a \sqcup \rangle^* S\right) \setminus^* a$. So

$$a \sqcup \bigsqcap S \sqsupseteq \left(\left(\bigsqcap \langle a \sqcup \rangle^* S\right) \setminus^* a\right) \sqcup a \sqsupseteq \bigsqcap \langle a \sqcup \rangle^* S.$$

But $a \sqcup \bigsqcap S \sqsubseteq \bigsqcap \langle a \sqcup \rangle^* S$ is obvious.
2°⇒3°. Let $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. To prove that $\mathfrak{A}$ is a co-brouwerian lattice it is enough to prove $a \sqsubseteq b \sqcup (a \setminus^* b)$. But it follows from the lemma.
3°⇒5°. $a \setminus^* b = \min \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. So $a \sqcup -$ is the upper adjoint of $- \setminus^* a$.
2°⇒5°. Because $a \sqcup -$ preserves all meets.

COROLLARY 161. Co-brouwerian lattices are distributive.

The following theorem is essentially borrowed from [19]:

THEOREM 162. A lattice $\mathfrak{A}$ with least element $\bot$ is co-brouwerian with pseudodifference $\setminus^*$ iff $\setminus^*$ is a binary operation on $\mathfrak{A}$ satisfying the following identities:

1°. $a \setminus^* a = \bot$;

2°. $a \sqcup (b \setminus^* a) = a \sqcup b$;
3°. $b \sqcup (b \setminus^* a) = b$;
4°. $(b \sqcup c) \setminus^* a = (b \setminus^* a) \sqcup (c \setminus^* a)$.

PROOF.

$\Leftarrow$. We have

$$c \sqsupseteq b \setminus^* a \Rightarrow c \sqcup a \sqsupseteq a \sqcup (b \setminus^* a) = a \sqcup b \sqsupseteq b;$$

$$c \sqcup a \sqsupseteq b \Rightarrow c = c \sqcup (c \setminus^* a) \sqsupseteq (a \setminus^* a) \sqcup (c \setminus^* a) = (a \sqcup c) \setminus^* a \sqsupseteq b \setminus^* a.$$

So $c \sqsupseteq b \setminus^* a \Leftrightarrow c \sqcup a \sqsupseteq b$ that is $a \sqcup -$ is an upper adjoint of $- \setminus^* a$. By a theorem above our lattice is co-brouwerian. By another theorem above $\setminus^*$ is a pseudodifference.

$\Rightarrow$.

1°. Obvious.

2°.

$$a \sqcup (b \setminus^* a) = a \sqcup \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{b \sqsubseteq a \sqcup z} \right\} = \bigsqcap \left\{ \frac{a \sqcup z}{z \in \mathfrak{A}, b \sqsubseteq a \sqcup z} \right\} = a \sqcup b.$$

3°. $b \sqcup (b \setminus^* a) = b \sqcup \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{b \sqsubseteq a \sqcup z} \right\} = \bigsqcap \left\{ \frac{b \sqcup z}{z \in \mathfrak{A}, b \sqsubseteq a \sqcup z} \right\} = b$.

4°. Obviously $(b \sqcup c) \setminus^* a \sqsupseteq b \setminus^* a$ and $(b \sqcup c) \setminus^* a \sqsupseteq c \setminus^* a$. Thus $(b \sqcup c) \setminus^* a \sqsupseteq (b \setminus^* a) \sqcup (c \setminus^* a)$. We have

$$(b \setminus^* a) \sqcup (c \setminus^* a) \sqcup a = ((b \setminus^* a) \sqcup a) \sqcup ((c \setminus^* a) \sqcup a) = (b \sqcup a) \sqcup (c \sqcup a) = a \sqcup b \sqcup c \sqsupseteq b \sqcup c.$$

From this by definition of adjoints: $(b \setminus^* a) \sqcup (c \setminus^* a) \sqsupseteq (b \sqcup c) \setminus^* a$. □

THEOREM 163. $(\bigsqcup S) \setminus^* a = \bigsqcup_{x \in S} (x \setminus^* a)$ for all $a \in \mathfrak{A}$ and $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a co-brouwerian lattice and $\bigsqcup S$ is defined.

PROOF. Because lower adjoint preserves all suprema. □

THEOREM 164. $(a \setminus^* b) \setminus^* c = a \setminus^* (b \sqcup c)$ for elements a, b, c of a co-frame.

PROOF. $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$.
$(a \setminus^* b) \setminus^* c = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \setminus^* b \sqsubseteq c \sqcup z} \right\}$.
$a \setminus^* (b \sqcup c) = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup c \sqcup z} \right\}$.
It is left to prove $a \setminus^* b \sqsubseteq c \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup c \sqcup z$. But this follows from corollary 155. □

COROLLARY 165. $(((a_0 \setminus^* a_1) \setminus^* \dots) \setminus^* a_n) = a_0 \setminus^* (a_1 \sqcup \dots \sqcup a_n)$.

PROOF. By math induction. □

2.1.15. Dual pseudocomplement on co-Heyting lattices.

THEOREM 166. For co-Heyting algebras $\top \setminus^* b = b^+$.

PROOF.
$$\top \setminus^* b = \min\left\{\frac{z \in \mathfrak{A}}{\top \sqsubseteq b \sqcup z}\right\} = \min\left\{\frac{z \in \mathfrak{A}}{\top = b \sqcup z}\right\} = \min\left\{\frac{z \in \mathfrak{A}}{b \equiv z}\right\} = b^+.$$
□

THEOREM 167. $(a \sqcap b)^+ = a^+ \sqcup b^+$ for every elements a, b of a co-Heyting algebra.

PROOF. $a \sqcup (a \sqcap b)^+ \sqsupseteq (a \sqcap b) \sqcup (a \sqcap b)^+ \sqsupseteq \top$. So $a \sqcup (a \sqcap b)^+ \sqsupseteq \top$; $(a \sqcap b)^+ \sqsupseteq \top \setminus^* a = a^+$.

We have $(a \sqcap b)^+ \sqsupseteq a^+$. Similarly $(a \sqcap b)^+ \sqsupseteq b^+$. Thus $(a \sqcap b)^+ \sqsupseteq a^+ \sqcup b^+$.

On the other hand, $a^+ \sqcup b^+ \sqcup (a \sqcap b) = (a^+ \sqcup b^+ \sqcup a) \sqcap (a^+ \sqcup b^+ \sqcup b)$. Obviously $a^+ \sqcup b^+ \sqcup a = a^+ \sqcup b^+ \sqcup b = \top$. So $a^+ \sqcup b^+ \sqcup (a \sqcap b) \sqsupseteq \top$ and thus $a^+ \sqcup b^+ \sqsupseteq \top \setminus^* (a \sqcap b) = (a \sqcap b)^+$.

So $(a \sqcap b)^+ = a^+ \sqcup b^+$. □

2.2. Intro to category theory

This is a *very* basic introduction to category theory.

DEFINITION 168. A *directed multigraph* (also known as *quiver*) is:

1°. a set $\mathcal{O}$ (*vertices*);
2°. a set $\mathcal{M}$ (*edges*);
3°. functions Src and Dst (*source* and *destination*) from $\mathcal{M}$ to $\mathcal{O}$.

Note that in category theory vertices are called *objects* and edges are called *morphisms*.

DEFINITION 169. A *precategory* is a directed multigraph together with a partial binary operation $\circ$ on the set $\mathcal{M}$ such that $g \circ f$ is defined iff $\operatorname{Dst} f = \operatorname{Src} g$ (for every morphisms f and g) such that

1°. $\operatorname{Src}(g \circ f) = \operatorname{Src} f$ and $\operatorname{Dst}(g \circ f) = \operatorname{Dst} g$ whenever the composition $g \circ f$ of morphisms f and g is defined.
2°. $(h \circ g) \circ f = h \circ (g \circ f)$ whenever compositions in this equation are defined.

DEFINITION 170. The set $\operatorname{Hom}(A, B)$ (also denoted as $\operatorname{Hom}_C(A, B)$ or just $C(A, B)$, where C is our category) (morphisms from an object A to an object B) is exactly morphisms which have A as the source and B as the destination.

DEFINITION 171. *Identity morphism* is such a morphism e that $e \circ f = f$ and $g \circ e = g$ whenever compositions in these formulas are defined.

DEFINITION 172. A *category* is a precategory with additional requirement that for every object X there exists identity morphism 1_X.

PROPOSITION 173. For every object X there exist no more than one identity morphism.

PROOF. Let p and q be both identity morphisms for a object X. Then $p = p \circ q = q$. □

DEFINITION 174. An *isomorphism* is such a morphism f of a category that there exists a morphism f^{-1} (*inverse* of f) such that $f \circ f^{-1} = 1_{\operatorname{Dst} f}$ and $f^{-1} \circ f = 1_{\operatorname{Src} f}$.

PROPOSITION 175. An isomorphism has exactly one inverse.

PROOF. Let g and h be both inverses of f. Then $h = h \circ 1_{\operatorname{Dst} f} = h \circ f \circ g = 1_{\operatorname{Src} f} \circ g = g$. □

DEFINITION 176. A *groupoid* is a category all of whose morphisms are isomorphisms.

DEFINITION 177. A morphism whose source is the same as destination is called *endomorphism*.

DEFINITION 178. An *involution* or *involutive morphism* is an endomorphism f that $f \circ f = 1_{\operatorname{Ob} f}$. In other words, an involution is such a self-inverse (that is conforming to the formula $f = f^{-1}$) isomorphism.

DEFINITION 179. *Functor* from category C to category D is a mapping F which associates every object X of C with an object $F(X)$ of D and every morphism $f : X \to Y$ of C with morphism $F(f) : F(X) \to F(Y)$ of D, such that:

1°. $F(g \circ f) = F(g) \circ F(f)$ for every composable morphisms f, g of C;
2°. $F(1^C_X) = 1^D_{FX}$ for every object X of C.

DEFINITION 180. *Opposite category* C^{op} of category C is the category where "all arrows are reversed" that is every morphism f is replaced with so called *opposite morphism* f^{op} such that $\operatorname{Src} f^{\mathrm{op}} = \operatorname{Dst} f$, $\operatorname{Dst} f^{\mathrm{op}} = \operatorname{Src} f$ and $g^{\mathrm{op}} \circ f^{\mathrm{op}} = (f \circ g)^{\mathrm{op}}$ (composition taken in the "opposite order").

DEFINITION 181. A *monomorphism* (also called a *monic* morphism or a *mono*) is a left-cancellative morphism. That is, an arrow $f \colon X \to Y$ such that for all objects Z and all morphisms $g_1, g_2 \colon Z \to X$,

$$f \circ g_1 = f \circ g_2 \Rightarrow g_1 = g_2.$$

Monomorphisms are a categorical generalization of injective functions (also called "one-to-one functions"); in some categories the notions coincide, but monomorphisms are more general.

DEFINITION 182. The categorical dual of a monomorphism is an *epimorphism*, i.e. a monomorphism in a category C is an epimorphism in the dual category C^{op} that is it conforms to the formula

$$g_1 \circ f = g_2 \circ f \Rightarrow g_1 = g_2.$$

2.2.1. Some important examples of categories.

EXERCISE 183. Prove that the below examples of categories are really categories.

DEFINITION 184. The category **Set** is:

- Objects are small sets.
- Morphisms from an object A to an object B are triples (A, B, f) where f is a function from A to B.
- Composition of morphisms is defined by the formula: $(B, C, g) \circ (A, B, f) = (A, C, g \circ f)$ where $g \circ f$ is function composition.

DEFINITION 185. The category **Rel** is:

- Objects are small sets.
- Morphisms from an object A to an object B are triples (A, B, f) where f is a binary relation between A and B.
- Composition of morphisms is defined by the formula: $(B, C, g) \circ (A, B, f) = (A, C, g \circ f)$ where $g \circ f$ is relation composition.

I will denote $\operatorname{GR}(A, B, f) = f$ for any morphism (A, B, f) of either **Set** or **Rel**.

DEFINITION 186. A *subcategory* of a category C is a category whose set of objects is a subset of the set of objects of C and whose set of morphisms is a subset of the set of morphisms of C.

DEFINITION 187. *Wide subcategory* of a category $(\mathcal{O}, \mathcal{M})$ is a category $(\mathcal{O}, \mathcal{M}')$ where $\mathcal{M} \subseteq \mathcal{M}'$ and the composition on $(\mathcal{O}, \mathcal{M}')$ is a restriction of composition of $(\mathcal{O}, \mathcal{M})$. (Similarly *wide sub-precategory* can be defined.)

2.2.2. Commutative diagrams.

DEFINITION 188. A *finite path in directed multigraph* is a tuple $[\![e_0, \ldots, e_n]\!]$ of edges (where $i \in \mathbb{N}$) such that $\operatorname{Dst} e_i = \operatorname{Src} e_{i+1}$ for every $i = 0, \ldots, n-1$.

DEFINITION 189. The vertices of a finite path are $\operatorname{Src} e_0$, $\operatorname{Dst} e_0 = \operatorname{Src} e_1$, $\operatorname{Dst} e_1 = \operatorname{Src} e_2$, $\ldots$, $\operatorname{Dst} e_n$.

DEFINITION 190. Composition of finite paths $[\![e_0, \ldots, e_n]\!]$ and $[\![e_k, \ldots, e_m]\!]$ (where $\operatorname{Dst} e_n = \operatorname{Src} e_k$) is the path $[\![e_0, \ldots, e_n, e_k, \ldots e_m]\!]$. (It is a path because $\operatorname{Dst} e_n = \operatorname{Src} e_k$.)

DEFINITION 191. A *cycle* is a finite path whose first vertex is the same as the last vertex (in other words $\operatorname{Dst} e_n = \operatorname{Src} e_0$).

DEFINITION 192. A *diagram* in C is a directed multigraph, whose vertices are labeled with objects of C and whose edges are labeled with morphisms of C.

I will denote the morphism corresponding to a edge e as $D(e)$.

DEFINITION 193. A diagram in C is *commutative* when the composition of morphisms corresponding to a finite path is always the same for finite paths from a fixed vertex A to a fixed vertex B independently of the path choice.

We will say "*commutative diagram*" when commutativity of a diagram is implied by the context.

REMARK 194. See Wikipedia for more on definition and examples of commutative diagrams.

The following is an example of a commutative diagram in **Set** (because $x+5-3 = x+4-2$):

$$\begin{tikzcd} \mathbb{N} \arrow[r, "+5"] \arrow[d, "+4"] & \mathbb{N} \arrow[d, "-3"] \\ \mathbb{N} \arrow[r, "-2"] & \mathbb{N} \end{tikzcd}$$

We are especially interested in the special case of commutative diagrams every morphism of which is an isomorphism. So, the below theorem.

THEOREM 195. If morphisms corresponding to every edge e_i of a cycle $[\![e_0, \ldots, e_n]\!]$ are isomorphisms then the following are equivalent:

- The morphism induced by $[\![e_0, \ldots, e_n]\!]$ is identity.
- The morphism induced by $[\![e_n, e_0, \ldots, e_{n-1}]\!]$ is identity.
- The morphism induced by $[\![e_{n-1}, e_n, e_0, \ldots, e_{n-2}]\!]$ is identity.
- ...
- The morphism induced by $[\![e_1, e_2, \ldots, e_n, e_0]\!]$ is identity.

In other words, the cycle being an identity does not depend on the choice of the start edge in the cycle.

Proof. Each step in the proof is like:

$$D(n) \circ \cdots \circ D(e_0) = 1_{\operatorname{Src} D(e_0)} \Leftrightarrow$$
$$D(n)^{-1} \circ D(n) \circ \cdots \circ D(e_0) \circ D(n) = D(n)^{-1} \circ 1_{\operatorname{Src} D(e_0)} \circ D(n) \Leftrightarrow$$
$$D(n-1) \circ \cdots \circ D(e_0) \circ D(n) = 1_{\operatorname{Src} D(e_n)}.$$

□

Lemma 196. Let f, g, h be isomorphisms. Let $g \circ f = h^{-1}$. The diagram at the figure 1 is commutative, every cycle in the diagram is an identity.

$$\begin{tikzcd}
& \bullet \arrow[dl, shift left, "h^{-1}"] \arrow[dr, shift left, "f"] & \\
\bullet \arrow[ur, shift left, "h"] \arrow[rr, shift left, "g^{-1}"] & & \bullet \arrow[ul, shift left, "f^{-1}"] \arrow[ll, shift left, "g"]
\end{tikzcd}$$

Figure 1

Proof. We will prove by induction that every cycle of the length N in the diagram is an identity.

For cycles of length 2 it holds by definition of isomorphism.

For cycles of length 3 it holds by theorem 195.

Consider a cycle of length above 3. It is easy to show that this cycle contains a sub-cycle of length 3 or below. (Consider three first edges $a \xrightarrow{e_0} b \xrightarrow{e_1} c \xrightarrow{e_2} d$ of the path, by pigeonhole principle we have that there are equal elements among a, b, c, d.) We can exclude the sub-cycle because it is identity. Thus we reduce to cycles of lesser length. Applying math induction, we get that every cycle in the diagram is an identity.

That the diagram is commutative follows from it (because for paths σ, τ we have the paths $\sigma \circ \tau^{-1}$ and $\tau \circ \sigma^{-1}$ being identities). □

Lemma 197. Let f, g, h, t be isomorphisms. Let $t \circ h \circ g \circ f = 1_{\operatorname{Src} f}$. The diagram at the figure 2 is commutative, every cycle in the diagram is an identity.

$$\begin{tikzcd}
(0,0) \arrow[r, shift left, "f"] \arrow[d, shift left, "t^{-1}"] & (0,1) \arrow[l, shift left, "f^{-1}"] \arrow[d, shift left, "g"] \\
(1,0) \arrow[u, shift left, "t"] \arrow[r, shift left, "h^{-1}"] & (1,1) \arrow[l, shift left, "h"] \arrow[u, shift left, "g^{-1}"]
\end{tikzcd}$$

Figure 2

TABLE 1

i	j	$W(i,j)$
0	0	$1_{\mathrm{Src}\, f}$
0	1	f
1	0	t^{-1}
1	1	$g \circ f$

PROOF. Assign to every vertex (i, j) of the diagram morphism $W(i, j)$ defined by the table 1.

It is easy to verify by induction that the morphism corresponding every cycle in the diagram starting at the vertex $(0, 0)$ and ending with a vertex (x, y) is $W(x, y)$.

Thus the morphism corresponding to every cycle starting at the vertex $(0, 0)$ is identity.

By symmetry, the morphism corresponding to every cycle is identity.

That the diagram is commutative follows from it (because for paths σ, τ we have the paths $\sigma \circ \tau^{-1}$ and $\tau \circ \sigma^{-1}$ being identities). □

2.3. Intro to group theory

DEFINITION 198. A *semigroup* is a pair of a set G and an associative binary operation on G.

DEFINITION 199. A *group* is a pair of a set G and a binary operation $\cdot$ on G such that:

1°. $(h \cdot g) \cdot f = h \cdot (g \cdot f)$ for every $f, g, h \in G$.
2°. There exists an element e (*identity*) of G such that $f \cdot e = e \cdot f = f$ for every $f \in G$.
3°. For every element f there exists an element f^{-1} (*inverse* of f) such that $f \cdot f^{-1} = f^{-1} \cdot f = e$.

OBVIOUS 200. Every group is a semigroup.

PROPOSITION 201. In every group there exists exactly one identity element.

PROOF. If p and q are both identities, then $p = p \cdot q = q$. □

PROPOSITION 202. Every group element has exactly one inverse.

PROOF. Let p and q be both inverses of $f \in G$. Then $f \cdot p = p \cdot f = e$ and $f \cdot q = q \cdot f = e$. Then $p = p \cdot e = p \cdot f \cdot q = e \cdot q = q$. □

PROPOSITION 203. $(g \cdot f)^{-1} = f^{-1} \cdot g^{-1}$ for every group elements f and g.

PROOF. $(f^{-1} \cdot g^{-1}) \cdot (g \cdot f) = f^{-1} \cdot g^{-1} \cdot g \cdot f = f^{-1} \cdot e \cdot f = f^{-1} \cdot f = e$. Similarly $(g \cdot f) \cdot (f^{-1} \cdot g^{-1}) = e$. So $f^{-1} \cdot g^{-1}$ is the inverse of $g \cdot f$. □

DEFINITION 204. A *permutation group* on a set D is a group whose elements are functions on D and whose composition is function composition.

OBVIOUS 205. Elements of a permutation group are bijections.

DEFINITION 206. A *transitive* permutation group on a set D is such a permutation group G on D that for every $x, y \in D$ there exists $r \in G$ such that $y = r(x)$.

A groupoid with single (arbitrarily chosen) object corresponds to every group. The morphisms of this category are elements of the group and the composition of morphisms is the group operation.

CHAPTER 3

More on order theory

3.1. Straight maps and separation subsets

3.1.1. Straight maps.

Definition 207. An *order reflecting* map from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ is such a function f that (for every $x, y \in \mathfrak{A}$)

$$fx \sqsubseteq fy \Rightarrow x \sqsubseteq y.$$

Obvious 208. Order embeddings are exactly the same as monotone and order reflecting maps.

Definition 209. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. I call f a *straight* map when

$$\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa = f(a \sqcap b)).$$

Proposition 210. The following statements are equivalent for a monotone map f:

1°. f is a straight map.
2°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa \sqsubseteq f(a \sqcap b))$.
3°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow fa \not\sqsupset f(a \sqcap b))$.
4°. $\forall a, b \in \mathfrak{A} : (fa \sqsupset f(a \sqcap b) \Rightarrow fa \not\sqsubseteq fb)$.

Proof.
1°⇔2°⇔3°. Due $fa \sqsupseteq f(a \sqcap b)$.
3°⇔4°. Obvious.

□

Remark 211. The definition of straight map can be generalized for any poset $\mathfrak{A}$ by the formula

$$\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow \exists c \in \mathfrak{A} : (c \sqsubseteq a \wedge c \sqsubseteq b \wedge fa = fc)).$$

This generalization is not yet researched however.

Proposition 212. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to a meet-semilattice $\mathfrak{B}$. If

$$\forall a, b \in \mathfrak{A} : f(a \sqcap b) = fa \sqcap fb$$

then f is a straight map.

Proof. Let $fa \sqsubseteq fb$. Then $f(a \sqcap b) = fa \sqcap fb = fa$. □

Proposition 213. Let f be a monotone map from a meet-semilattice $\mathfrak{A}$ to some poset $\mathfrak{B}$. If f is order reflecting, then f is a straight map.

Proof. $fa \sqsubseteq fb \Rightarrow a \sqsubseteq b \Rightarrow a = a \sqcap b \Rightarrow fa = f(a \sqcap b)$. □

The following theorem is the main reason of why we are interested in straight maps:

THEOREM 214. If f is a straight monotone map from a meet-semilattice $\mathfrak{A}$ then the following statements are equivalent:

1°. f is an injection.
2°. f is order reflecting.
3°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \sqsubset fb)$.
4°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \neq fb)$.
5°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \not\sqsupseteq fb)$.
6°. $\forall a, b \in \mathfrak{A} : (fa \sqsubseteq fb \Rightarrow a \not\sqsupset b)$.

PROOF.

1°⇒3°. Let $a, b \in \mathfrak{A}$. Let $fa = fb \Rightarrow a = b$. Let $a \sqsubset b$. $fa \neq fb$ because $a \neq b$. $fa \sqsubseteq fb$ because $a \sqsubseteq b$. So $fa \sqsubset fb$.

2°⇒1°. Let $a, b \in \mathfrak{A}$. Let $fa \sqsubseteq fb \Rightarrow a \sqsubseteq b$. Let $fa = fb$. Then $a \sqsubseteq b$ and $b \sqsubseteq a$ and consequently $a = b$.

3°⇒2°. Let $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow fa \sqsubset fb)$. Let $a \not\sqsubseteq b$. Then $a \sqsupset a \sqcap b$. So $fa \sqsupset f(a \sqcap b)$. If $fa \sqsubseteq fb$ then $fa \sqsubseteq f(a \sqcap b)$ what is a contradiction.

3°⇒5°⇒4°. Obvious.

4°⇒3°. Because $a \sqsubset b \Rightarrow a \sqsubseteq b \Rightarrow fa \sqsubseteq fb$.

5°⇔6°. Obvious.

□

3.1.2. Separation subsets and full stars.

DEFINITION 215. $\partial_Y a = \left\{ \frac{x \in Y}{x \not\asymp a} \right\}$ for an element a of a poset $\mathfrak{A}$ and $Y \in \mathscr{P}\mathfrak{A}$.

DEFINITION 216. *Full star* of $a \in \mathfrak{A}$ is $\star a = \partial_{\mathfrak{A}} a$.

PROPOSITION 217. If $\mathfrak{A}$ is a meet-semilattice, then $\star$ is a straight monotone map.

PROOF. Monotonicity is obvious. Let $\star a \not\sqsubseteq \star(a \sqcap b)$. Then it exists $x \in \star a$ such that $x \notin \star(a \sqcap b)$. So $x \sqcap a \notin \star b$ but $x \sqcap a \in \star a$ and consequently $\star a \not\sqsubseteq \star b$. □

DEFINITION 218. A *separation subset* of a poset $\mathfrak{A}$ is such its subset Y that

$$\forall a, b \in \mathfrak{A} : (\partial_Y a = \partial_Y b \Rightarrow a = b).$$

DEFINITION 219. I call *separable* such poset that $\star$ is an injection.

DEFINITION 220. I call *strongly separable* such poset that $\star$ is order reflecting.

OBVIOUS 221. A poset is separable iff it has a separation subset.

OBVIOUS 222. A poset is strongly separable iff $\star$ is order embedding.

OBVIOUS 223. Strong separability implies separability.

DEFINITION 224. A poset $\mathfrak{A}$ has *disjunction property of Wallman* iff for any $a, b \in \mathfrak{A}$ either $b \sqsubseteq a$ or there exists a non-least element $c \sqsubseteq b$ such that $a \asymp c$.

THEOREM 225. For a meet-semilattice with least element the following statements are equivalent:

1°. $\mathfrak{A}$ is separable.
2°. $\mathfrak{A}$ is strongly separable.
3°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \star a \sqsubset \star b)$.
4°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \star a \neq \star b)$.
5°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \star a \not\sqsupseteq \star b)$.
6°. $\forall a, b \in \mathfrak{A} : (\star a \sqsubseteq \star b \Rightarrow a \not\sqsupset b)$.
7°. $\mathfrak{A}$ conforms to Wallman's disjunction property.

8°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a \wedge c \sqsubseteq b))$.

PROOF.

1°⇔2°⇔3°⇔4°⇔5°⇔6°. By the above theorem.

8°⇒4°. Let property 8° hold. Let $a \sqsubset b$. Then it exists element $c \sqsubseteq b$ such that $c \neq \bot$ and $c \sqcap a = \bot$. But $c \sqcap b \neq \bot$. So $\star a \neq \star b$.

2°⇒7°. Let property 2° hold. Let $a \not\sqsubseteq b$. Then $\star a \not\sqsubseteq \star b$ that is it there exists $c \in \star a$ such that $c \notin \star b$, in other words $c \sqcap a \neq \bot$ and $c \sqcap b = \bot$. Let $d = c \sqcap a$. Then $d \sqsubseteq a$ and $d \neq \bot$ and $d \sqcap b = \bot$. So disjunction property of Wallman holds.

7°⇒8°. Obvious.

8°⇒7°. Let $b \not\sqsubseteq a$. Then $a \sqcap b \sqsubset b$ that is $a' \sqsubset b$ where $a' = a \sqcap b$. Consequently $\exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a' \wedge c \sqsubseteq b)$. We have $c \sqcap a = c \sqcap b \sqcap a = c \sqcap a' = \bot$. So $c \sqsubseteq b$ and $c \sqcap a = \bot$. Thus Wallman's disjunction property holds.

□

PROPOSITION 226. Every boolean lattice is strongly separable.

PROOF. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a boolean lattice an $a \neq b$. Then $a \sqcap \bar{b} \neq \bot$ or $\bar{a} \sqcap b \neq \bot$ because otherwise $a \sqcap \bar{b} = \bot$ and $a \sqcup \bar{b} = \top$ and thus $a = b$. Without loss of generality assume $a \sqcap \bar{b} \neq \bot$. Then $a \sqcap c \neq \bot$ and $b \sqcap c = \bot$ for $c = a \sqcap \bar{b} \neq \bot$, that is our lattice is separable.

It is strongly separable by theorem 225. □

3.1.3. Atomically Separable Lattices.

PROPOSITION 227. "atoms" is a straight monotone map (for any meet-semilattice).

PROOF. Monotonicity is obvious. The rest follows from the formula

$$\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$$

(corollary 109). □

DEFINITION 228. I will call *atomically separable* such a poset that "atoms" is an injection.

PROPOSITION 229. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \operatorname{atoms} a \subset \operatorname{atoms} b)$ iff $\mathfrak{A}$ is atomically separable for a poset $\mathfrak{A}$.

PROOF.

⇐. Obvious.

⇒. Let $a \neq b$ for example $a \not\sqsubseteq b$. Then $a \sqcap b \sqsubset a$; $\operatorname{atoms} a \supset \operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ and thus $\operatorname{atoms} a \neq \operatorname{atoms} b$.

□

PROPOSITION 230. Any atomistic poset is atomically separable.

PROOF. We need to prove that $\operatorname{atoms} a = \operatorname{atoms} b \Rightarrow a = b$. But it is obvious because

$$a = \bigsqcup \operatorname{atoms} a \quad \text{and} \quad b = \bigsqcup \operatorname{atoms} b.$$

□

THEOREM 231. A complete lattice is atomistic iff it is atomically separable.

PROOF. Direct implication is the above proposition. Let's prove the reverse implication.

Let "atoms" be injective. Consider an element a of our poset. Let $b = \bigsqcup \operatorname{atoms} a$. Obviously $b \sqsubseteq a$ and thus $\operatorname{atoms} b \subseteq \operatorname{atoms} a$. But if $x \in \operatorname{atoms} a$ then $x \sqsubseteq b$ and thus $x \in \operatorname{atoms} b$. So $\operatorname{atoms} a = \operatorname{atoms} b$. By injectivity $a = b$ that is $a = \bigsqcup \operatorname{atoms} a$. □

THEOREM 232. If a lattice with least element is atomic and separable then it is atomistic.

PROOF. Suppose the contrary that is $a \sqsupset \bigsqcup \operatorname{atoms} a$. Then, because our lattice is separable, there exists $c \in \mathfrak{A}$ such that $c \sqcap a \neq \bot$ and $c \sqcap \bigsqcup \operatorname{atoms} a = \bot$. There exists atom $d \sqsubseteq c$ such that $d \sqsubseteq c \sqcap a$. $d \sqcap \bigsqcup \operatorname{atoms} a \sqsubseteq c \sqcap \bigsqcup \operatorname{atoms} a = \bot$. But $d \in \operatorname{atoms} a$. Contradiction. □

THEOREM 233. Let $\mathfrak{A}$ be an atomic meet-semilattice with least element. Then the following statements are equivalent:

1°. $\mathfrak{A}$ is separable.
2°. $\mathfrak{A}$ is strongly separable.
3°. $\mathfrak{A}$ is atomically separable.
4°. $\mathfrak{A}$ conforms to Wallman's disjunction property.
5°. $\forall a, b \in \mathfrak{A} : (a \sqsubset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{\bot\} : (c \asymp a \wedge c \sqsubseteq b))$.

PROOF.

1°⇔2°⇔4°⇔5°. Proved above.

3°⇒5°. Let our semilattice be atomically separable. Let $a \sqsubset b$. Then $\operatorname{atoms} a \subset \operatorname{atoms} b$ and there exists $c \in \operatorname{atoms} b$ such that $c \notin \operatorname{atoms} a$. $c \neq \bot$ and $c \sqsubseteq b$, from which (taking into account that c is an atom) $c \sqsubseteq b$ and $c \sqcap a = \bot$. So our semilattice conforms to the formula 5°.

5°⇒3°. Let formula 5° hold. Then for any elements $a \sqsubset b$ there exists $c \neq \bot$ such that $c \sqsubseteq b$ and $c \sqcap a = \bot$. Because $\mathfrak{A}$ is atomic there exists atom $d \sqsubseteq c$. $d \in \operatorname{atoms} b$ and $d \notin \operatorname{atoms} a$. So $\operatorname{atoms} a \neq \operatorname{atoms} b$ and $\operatorname{atoms} a \subseteq \operatorname{atoms} b$. Consequently $\operatorname{atoms} a \subset \operatorname{atoms} b$.

□

THEOREM 234. Any atomistic poset is strongly separable.

PROOF. $\star x \sqsubseteq \star y \Rightarrow \operatorname{atoms} x \sqsubseteq \operatorname{atoms} y \Rightarrow x \sqsubseteq y$ because $\operatorname{atoms} x = \star x \cap \operatorname{atoms}^{\mathfrak{A}}$. □

3.2. Quasidifference and Quasicomplement

I've got quasidifference and quasicomplement (and dual quasicomplement) replacing max and min in the definition of pseudodifference and pseudocomplement (and dual pseudocomplement) with $\bigsqcup$ and $\bigsqcap$. Thus quasidifference and (dual) quasicomplement are generalizations of their pseudo- counterparts.

REMARK 235. *Pseudocomplements* and *pseudodifferences* are standard terminology. *Quasi-* counterparts are my neologisms.

DEFINITION 236. Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$. *Quasicomplement* of a is

$$a^* = \bigsqcup \left\{ \frac{c \in \mathfrak{A}}{c \asymp a} \right\}.$$

DEFINITION 237. Let $\mathfrak{A}$ be a poset, $a \in \mathfrak{A}$. *Dual quasicomplement* of a is

$$a^+ = \bigsqcap \left\{ \frac{c \in \mathfrak{A}}{c \equiv a} \right\}.$$

I will denote quasicomplement and dual quasicomplement for a specific poset $\mathfrak{A}$ as $a^{*(\mathfrak{A})}$ and $a^{+(\mathfrak{A})}$.

DEFINITION 238. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice. *Quasidifference* of a and b is

$$a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}.$$

DEFINITION 239. Let $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice. *Second quasidifference* of a and b is

$$a \# b = \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \asymp b} \right\}.$$

THEOREM 240. $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\}$ where $\mathfrak{A}$ is a distributive lattice and $a, b \in \mathfrak{A}$.

PROOF. Obviously $\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\} \subseteq \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. Thus $\bigsqcap \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\} \sqsupseteq a \setminus^* b$.

Let $z \in \mathfrak{A}$ and $z' = z \sqcap a$.

$a \sqsubseteq b \sqcup z \Rightarrow a \sqsubseteq (b \sqcup z) \sqcap a \Leftrightarrow a \sqsubseteq (b \sqcap a) \sqcup (z \sqcap a) \Leftrightarrow a \sqsubseteq (b \sqcap a) \sqcup z' \Rightarrow a \sqsubseteq b \sqcup z'$ and $a \sqsubseteq b \sqcup z \Leftarrow a \sqsubseteq b \sqcup z'$. Thus $a \sqsubseteq b \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup z'$.

If $z \in \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ then $a \sqsubseteq b \sqcup z$ and thus

$$z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\}.$$

But $z' \sqsubseteq z$ thus having $\bigsqcap \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge a \sqsubseteq b \sqcup z} \right\} \sqsubseteq \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$. □

REMARK 241. If we drop the requirement that $\mathfrak{A}$ is distributive, two formulas for quasidifference (the definition and the last theorem) fork.

OBVIOUS 242. Dual quasicomplement is the dual of quasicomplement.

OBVIOUS 243.

- Every pseudocomplement is quasicomplement.
- Every dual pseudocomplement is dual quasicomplement.
- Every pseudodifference is quasidifference.

Below we will stick to the more general quasies than pseudos. If needed, one can check that a quasicomplement a^* is a pseudocomplement by the equation $a^* \asymp a$ (and analogously with other quasies).

Next we will express quasidifference through quasicomplement.

PROPOSITION 244.

1°. $a \setminus^* b = a \setminus^* (a \sqcap b)$ for any distributive lattice;
2°. $a \# b = a \# (a \sqcap b)$ for any distributive lattice with least element.

PROOF.

1°. $a \sqsubseteq (a \sqcap b) \sqcup z \Leftrightarrow a \sqsubseteq (a \sqcup z) \sqcap (b \sqcup z) \Leftrightarrow a \sqsubseteq a \sqcup z \wedge a \sqsubseteq b \sqcup z \Leftrightarrow a \sqsubseteq b \sqcup z$. Thus $a \setminus^* (a \sqcap b) = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq (a \sqcap b) \sqcup z} \right\} = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\} = a \setminus^* b$.

2°.

$$a \#(a \sqcap b) = \bigsqcup\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap a \sqcap b = \bot} \right\} = \bigsqcup\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge (z \sqcap a) \sqcap a \sqcap b = \bot} \right\} = \bigsqcup\left\{ \frac{z \sqcap a}{z \in \mathfrak{A}, z \sqcap a \sqcap b = \bot} \right\} = \bigsqcup\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a, z \sqcap b = \bot} \right\} = a \# b.$$

□

I will denote Da the lattice $\left\{ \frac{x \in \mathfrak{A}}{x \sqsubseteq a} \right\}$.

THEOREM 245. For $a, b \in \mathfrak{A}$ where $\mathfrak{A}$ is a distributive lattice
1°. $a \setminus^* b = (a \sqcap b)^{+(Da)}$;
2°. $a \# b = (a \sqcap b)^{*(Da)}$ if $\mathfrak{A}$ has least element.

PROOF.
1°.

$$(a \sqcap b)^{+(Da)} = \bigsqcap\left\{ \frac{c \in Da}{c \sqcup (a \sqcap b) = a} \right\} = \bigsqcap\left\{ \frac{c \in Da}{c \sqcup (a \sqcap b) \sqsupseteq a} \right\} = \bigsqcap\left\{ \frac{c \in Da}{(c \sqcup a) \sqcap (c \sqcup b) \sqsupseteq a} \right\} = \bigsqcap\left\{ \frac{c \in \mathfrak{A}}{c \sqsubseteq a \wedge c \sqcup b \sqsupseteq a} \right\} = a \setminus^* b.$$

2°.

$$(a \sqcap b)^{*(Da)} = \bigsqcup\left\{ \frac{c \in Da}{c \sqcap a \sqcap b = \bot} \right\} = \bigsqcup\left\{ \frac{c \in \mathfrak{A}}{c \sqsubseteq a \wedge c \sqcap a \sqcap b = \bot} \right\} = \bigsqcup\left\{ \frac{c \in \mathfrak{A}}{c \sqsubseteq a \wedge c \sqcap b = \bot} \right\} = a \# b.$$

□

PROPOSITION 246. $(a \sqcup b) \setminus^* b \sqsubseteq a$ for an arbitrary complete lattice.

PROOF. $(a \sqcup b) \setminus^* b = \bigsqcap\left\{ \frac{z \in \mathfrak{A}}{a \sqcup b \sqsubseteq b \sqcup z} \right\}$.
But $a \sqsubseteq z \Rightarrow a \sqcup b \sqsubseteq b \sqcup z$. So $\left\{ \frac{z \in \mathfrak{A}}{a \sqcup b \sqsubseteq b \sqcup z} \right\} \supseteq \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq z} \right\}$.
Consequently, $(a \sqcup b) \setminus^* b \sqsubseteq \bigsqcap\left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq z} \right\} = a$. □

3.3. Several equal ways to express pseudodifference

THEOREM 247. For an atomistic co-brouwerian lattice $\mathfrak{A}$ and $a, b \in \mathfrak{A}$ the following expressions are always equal:

1°. $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ (quasidifference of a and b);

2°. $a \# b = \bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ (second quasidifference of a and b);

3°. $\bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b)$.

PROOF.

Proof of 1°=3°.

$$\begin{aligned}
a \setminus^* b &= \\
\left(\bigsqcup \operatorname{atoms} a \right) \setminus^* b &= \quad \text{(theorem 163)} \\
\bigsqcup_{A \in \operatorname{atoms} a} (A \setminus^* b) &= \\
\bigsqcup_{A \in \operatorname{atoms} a} \left(\begin{cases} A & \text{if } A \notin \operatorname{atoms} b \\ \bot & \text{if } A \in \operatorname{atoms} b \end{cases} \right) &= \\
\bigsqcup \left\{ \frac{A}{A \in \operatorname{atoms} a, A \notin \operatorname{atoms} b} \right\} &= \\
\bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b)&.
\end{aligned}$$

Proof of 2°=3°. $a \setminus^* b$ is defined because our lattice is co-brouwerian. Taking the above into account, we have

$$\begin{aligned}
a \setminus^* b &= \\
\bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b) &= \\
\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}&.
\end{aligned}$$

So $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$ is defined.

If $z \sqsubseteq a \wedge z \sqcap b = \bot$ then $z' = \bigsqcup \left\{ \frac{x \in \operatorname{atoms} z}{x \sqcap b = \bot} \right\}$ is defined because $z' = z \setminus^* b$ (atomisticity taken into account). z' is a lower bound for $\left\{ \frac{z \in \operatorname{atoms} a}{x \sqcap b = \bot} \right\}$.

Thus $z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ and so $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$ is an upper bound of $\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$.

If y is above every $z' \in \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ then y is above every $z \in \operatorname{atoms} a$ such that $z \sqcap b = \bot$ and thus y is above $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$.

Thus $\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\}$ is least upper bound of

$$\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\},$$

that is

$$\begin{aligned}
\bigsqcup \left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\} &= \\
\bigsqcup \left\{ \frac{z \in \operatorname{atoms} a}{z \sqcap b = \bot} \right\} &= \\
\bigsqcup (\operatorname{atoms} a \setminus \operatorname{atoms} b)&.
\end{aligned}$$

□

3.4. Partially ordered categories

3.4.1. Definition.

Definition 248. I will call a partially ordered (pre)category a (pre)category together with partial order $\sqsubseteq$ on each of its Mor-sets with the additional requirement that

$$f_1 \sqsubseteq f_2 \wedge g_1 \sqsubseteq g_2 \Rightarrow g_1 \circ f_1 \sqsubseteq g_2 \circ f_2$$

for every morphisms f_1, g_1, f_2, g_2 such that $\operatorname{Src} f_1 = \operatorname{Src} f_2$ and $\operatorname{Dst} f_1 = \operatorname{Dst} f_2 = \operatorname{Src} g_1 = \operatorname{Src} g_2$ and $\operatorname{Dst} g_1 = \operatorname{Dst} g_2$.

I will denote lattice operations on a Hom-set $C(A, B)$ of a category (or any directed multigraph) like $\sqcup^C$ instead of writing $\sqcup^{C(A,B)}$ explicitly.

3.4.2. Dagger categories.

Definition 249. I will call a *dagger precategory* a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^\dagger$.

In other words, a dagger precategory is a precategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g:

1°. $f^{\dagger\dagger} = f$;
2°. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

Definition 250. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^\dagger$.

In other words, a dagger category is a category equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A:

1°. $f^{\dagger\dagger} = f$;
2°. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
3°. $(1_A)^\dagger = 1_A$.

Theorem 251. If a category is a dagger precategory then it is a dagger category.

Proof. We need to prove only that $(1_A)^\dagger = 1_A$. Really,

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A.$$

□

For a partially ordered dagger (pre)category I will additionally require (for every morphisms f and g with the same source and destination)

$$f^\dagger \sqsubseteq g^\dagger \Leftrightarrow f \sqsubseteq g.$$

An example of dagger category is the category **Rel** whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^\dagger = f^{-1}$.

Definition 252. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^\dagger = f^{-1}$.

Definition 253. *Symmetric* (endo)morphism of a dagger precategory is such a morphism f that $f = f^\dagger$.

Definition 254. *Transitive* (endo)morphism of a precategory is such a morphism f that $f = f \circ f$.

THEOREM 255. The following conditions are equivalent for a morphism f of a dagger precategory:

1°. f is symmetric and transitive.
2°. $f = f^\dagger \circ f$.

PROOF.
1°⇒2°. If f is symmetric and transitive then $f^\dagger \circ f = f \circ f = f$.
2°⇒1°. $f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger \circ f = f \circ f$, so f is transitive.

□

3.4.2.1. *Some special classes of morphisms.*

DEFINITION 256. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^\dagger \sqsubseteq 1_{\operatorname{Dst} f}$.

DEFINITION 257. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^\dagger \circ f \sqsupseteq 1_{\operatorname{Src} f}$.

DEFINITION 258. For a partially ordered dagger category I will call *injective* morphism such a morphism f that $f^\dagger \circ f \sqsubseteq 1_{\operatorname{Src} f}$.

DEFINITION 259. For a partially ordered dagger category I will call *surjective* morphism such a morphism f that $f \circ f^\dagger \sqsupseteq 1_{\operatorname{Dst} f}$.

REMARK 260. It is easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective functions as morphisms of the category **Rel**.

OBVIOUS 261. "Injective morphism" is a dual of "monovalued morphism" and "surjective morphism" is a dual of "entirely defined morphism".

DEFINITION 262. For a given partially ordered dagger category C the *category of monovalued* (*entirely defined*, *injective*, *surjective*) morphisms of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined, injective, surjective) morphisms of C with the composition of morphisms the same as in C.

We need to prove that these are really categories, that is that composition of monovalued (entirely defined, injective, surjective) morphisms is monovalued (entirely defined, injective, surjective) and that identity morphisms are monovalued, entirely defined, injective, and surjective.

PROOF. We will prove only for monovalued morphisms and entirely defined morphisms, as injective and surjective morphisms are their duals.

Monovalued. Let f and g be monovalued morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. Then

$$\begin{aligned}(g \circ f) \circ (g \circ f)^\dagger &= \\ g \circ f \circ f^\dagger \circ g^\dagger &\sqsubseteq \\ g \circ 1_{\operatorname{Src} g} \circ g^\dagger &= \\ g \circ g^\dagger &\sqsubseteq \\ 1_{\operatorname{Dst} g} &= 1_{\operatorname{Dst}(g \circ f)}.\end{aligned}$$

So $g \circ f$ is monovalued.
That identity morphisms are monovalued follows from the following:

$$1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A = 1_{\operatorname{Dst} 1_A} \sqsubseteq 1_{\operatorname{Dst} 1_A}.$$

Entirely defined. Let f and g be entirely defined morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. Then

$$\begin{aligned}(g\circ f)^\dagger\circ(g\circ f) &= \\ f^\dagger\circ g^\dagger\circ g\circ f &\sqsupseteq \\ f^\dagger\circ 1_{\operatorname{Src} g}\circ f &= \\ f^\dagger\circ 1_{\operatorname{Dst} f}\circ f &= \\ f^\dagger\circ f &\sqsupseteq \\ 1_{\operatorname{Src} f} &= 1_{\operatorname{Src}(g\circ f)}.\end{aligned}$$

So $g\circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following:

$$(1_A)^\dagger\circ 1_A = 1_A\circ 1_A = 1_A = 1_{\operatorname{Src} 1_A} \sqsupseteq 1_{\operatorname{Src} 1_A}.$$

□

DEFINITION 263. I will call a *bijective* morphism a morphism which is entirely defined, monovalued, injective, and surjective.

PROPOSITION 264. If a morphism is bijective then it is an isomorphism.

PROOF. Let f be bijective. Then $f\circ f^\dagger \sqsubseteq 1_{\operatorname{Dst} f}$, $f^\dagger\circ f \sqsupseteq 1_{\operatorname{Src} f}$, $f^\dagger\circ f \sqsubseteq 1_{\operatorname{Src} f}$, $f\circ f^\dagger \sqsupseteq 1_{\operatorname{Dst} f}$. Thus $f\circ f^\dagger = 1_{\operatorname{Dst} f}$ and $f^\dagger\circ f = 1_{\operatorname{Src} f}$ that is $f^\dagger$ is an inverse of f. □

Let Hom-sets be complete lattices.

DEFINITION 265. A morphism f of a partially ordered category is *metamonovalued* when $(\bigsqcap G)\circ f = \bigsqcap_{g\in G}(g\circ f)$ whenever G is a set of morphisms with a suitable domain and image.

DEFINITION 266. A morphism f of a partially ordered category is *metainjective* when $f\circ(\bigsqcap G) = \bigsqcap_{g\in G}(f\circ g)$ whenever G is a set of morphisms with a suitable domain and image.

OBVIOUS 267. Metamonovaluedness and metainjectivity are dual to each other.

DEFINITION 268. A morphism f of a partially ordered category is *metacomplete* when $f\circ(\bigsqcup G) = \bigsqcup_{g\in G}(f\circ g)$ whenever G is a set of morphisms with a suitable domain and image.

DEFINITION 269. A morphism f of a partially ordered category is *co-metacomplete* when $(\bigsqcup G)\circ f = \bigsqcup_{g\in G}(g\circ f)$ whenever G is a set of morphisms with a suitable domain and image.

Let now Hom-sets be meet-semilattices.

DEFINITION 270. A morphism f of a partially ordered category is *weakly metamonovalued* when $(g\sqcap h)\circ f = (g\circ f)\sqcap(h\circ f)$ whenever g and h are morphisms with a suitable domain and image.

DEFINITION 271. A morphism f of a partially ordered category is *weakly metainjective* when $f\circ(g\sqcap h) = (f\circ g)\sqcap(f\circ h)$ whenever g and h are morphisms with a suitable domain and image.

Let now Hom-sets be join-semilattices.

DEFINITION 272. A morphism f of a partially ordered category is *weakly metacomplete* when $f\circ(g\sqcup h) = (f\circ g)\sqcup(f\circ h)$ whenever g and h are morphisms with a suitable domain and image.

Definition 273. A morphism f of a partially ordered category is *weakly co-metacomplete* when $(g \sqcup h) \circ f = (g \circ f) \sqcup (h \circ f)$ whenever g and h are morphisms with a suitable domain and image.

Obvious 274.

1°. Metamonovalued morphisms are weakly metamonovalued.
2°. Metainjective morphisms are weakly metainjective.
3°. Metacomplete morphisms are weakly metacomplete.
4°. Co-metacomplete morphisms are weakly co-metacomplete.

3.5. Partitioning

Definition 275. Let $\mathfrak{A}$ be a complete lattice. *Torning* of an element $a \in \mathfrak{A}$ is a set $S \in \mathscr{P}\mathfrak{A} \setminus \{\bot\}$ such that

$$\bigsqcup S = a \quad \text{and} \quad \forall x, y \in S : (x \neq y \Rightarrow x \asymp y).$$

Definition 276. Let $\mathfrak{A}$ be a complete lattice. *Weak partition* of an element $a \in \mathfrak{A}$ is a set $S \in \mathscr{P}\mathfrak{A} \setminus \{\bot\}$ such that

$$\bigsqcup S = a \quad \text{and} \quad \forall x \in S : x \asymp \bigsqcup (S \setminus \{x\}).$$

Definition 277. Let $\mathfrak{A}$ be a complete lattice. *Strong partition* of an element $a \in \mathfrak{A}$ is a set $S \in \mathscr{P}\mathfrak{A} \setminus \{\bot\}$ such that

$$\bigsqcup S = a \quad \text{and} \quad \forall A, B \in \mathscr{P}S : \left(A \asymp B \Rightarrow \bigsqcup A \asymp \bigsqcup B\right).$$

Obvious 278.

1°. Every strong partition is a weak partition.
2°. Every weak partition is a torning.

Definition 279. *Complete lattice generated by* a set P (on a complete lattice) is the set (obviously having the structure of complete lattice) $P_0 \cup P_1 \cup \dots$ where $P_0 = P$ and $P_{i+1} = \left\{ \frac{\bigsqcup K, \bigsqcap K}{K \in \mathscr{P} P_i} \right\}$.

Obvious 280. Complete lattice generated by a set is indeed a complete lattice.

Example 281. $[S] \neq \left\{ \frac{\bigsqcup^{\mathfrak{A}} X}{X \in \mathscr{P} S} \right\}$, where $[S]$ is the complete lattice generated by a strong partition S of a filter on a set.

Proof. Consider any infinite set U and its strong partition $S = \left\{ \frac{\uparrow^U \{x\}}{x \in U} \right\}$. The set S consists only of principal filters. But $[S]$ contains (exercise!) some nonprincipal filters. □

By the way:

Proposition 282. $\left\{ \frac{\bigsqcup^{\mathfrak{A}} X}{X \in \mathscr{P} S} \right\}$ is closed under binary meets, if S is a strong partition of an element of a complete lattice.

Proof. Let $R = \left\{ \frac{\bigsqcup^{\mathfrak{A}} X}{X \in \mathscr{P}S} \right\}$. Then for every $X, Y \in \mathscr{P}S$

$$\begin{aligned}
\bigsqcup^{\mathfrak{A}} X \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y &= \\
\bigsqcup^{\mathfrak{A}} ((X \cap Y) \cup (X \setminus Y)) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y &= \\
\left(\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcup^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} (X \setminus Y) \right) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y &= \\
\left(\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y \right) \sqcup^{\mathfrak{A}} \left(\bigsqcup^{\mathfrak{A}} (X \setminus Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y \right) &= \\
\left(\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y \right) \sqcup^{\mathfrak{A}} \bot^{\mathfrak{A}} &= \\
\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y&.
\end{aligned}$$

Applying the formula $\bigsqcup^{\mathfrak{A}} X \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y = \bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y$ twice we get

$$\begin{aligned}
\bigsqcup^{\mathfrak{A}} X \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} Y &= \\
\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} (Y \cap (X \cap Y)) &= \\
\bigsqcup^{\mathfrak{A}} (X \cap Y) \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} (X \cap Y) &= \\
\bigsqcup^{\mathfrak{A}} (X \cap Y)&.
\end{aligned}$$

But for any $A, B \in R$ there exist $X, Y \in \mathscr{P}S$ such that $A = \bigsqcup^{\mathfrak{A}} X$, $B = \bigsqcup^{\mathfrak{A}} Y$. So $A \sqcap^{\mathfrak{A}} B = \bigsqcup^{\mathfrak{A}} X \sqcap \bigsqcup^{\mathfrak{A}} Y = \bigsqcup^{\mathfrak{A}} (X \cap Y) \in R$. □

3.6. A proposition about binary relations

Proposition 283. Let f, g, h be binary relations. Then $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}$.

Proof.

$$\begin{aligned}
g \circ f \not\asymp h &\Leftrightarrow \\
\exists a, c : a\ ((g \circ f) \cap h)\ c &\Leftrightarrow \\
\exists a, c : (a\ (g \circ f)\ c \wedge a\ h\ c) &\Leftrightarrow \\
\exists a, b, c : (a\ f\ b \wedge b\ g\ c \wedge a\ h\ c) &\Leftrightarrow \\
\exists b, c : (b\ g\ c \wedge b\ (h \circ f^{-1})\ c) &\Leftrightarrow \\
\exists b, c : b\ (g \cap (h \circ f^{-1}))\ c &\Leftrightarrow \\
g \not\asymp h \circ f^{-1}&.
\end{aligned}$$

□

3.7. Infinite associativity and ordinated product

3.7.1. Introduction. We will consider some function f which takes an arbitrary ordinal number of arguments. That is f can be taken for arbitrary (small, if to be precise) ordinal number of arguments. More formally: Let $x = x_{i \in n}$ be a

family indexed by an ordinal n. Then $f(x)$ can be taken. The same function f can take different number of arguments. (See below for the exact definition.)

Some of such functions f are associative in the sense defined below. If a function is associative in the below defined sense, then the binary operation induced by this function is associative in the usual meaning of the word "associativity" as defined in basic algebra.

I also introduce and research an important example of infinitely associative function, which I call *ordinated product*.

Note that my searching about infinite associativity and ordinals in Internet has provided no useful results. As such there is a reason to assume that my research of generalized associativity in terms of ordinals is novel.

3.7.2. Used notation. We identify natural numbers with finite Von Neumann's ordinals (further just *ordinals* or *ordinal numbers*).

For simplicity we will deal with small sets (members of a Grothendieck universe). We will denote the Grothendieck universe (aka *universal set*) as $\mho$.

I will denote a tuple of n elements like $[\![a_0, \ldots, a_{n-1}]\!]$. By definition

$$[\![a_0, \ldots, a_{n-1}]\!] = \{(0, a_0), \ldots, (n-1, a_{n-1})\}.$$

Note that an ordered pair (a, b) is not the same as the tuple $[\![a, b]\!]$ of two elements. (However, we will use them interchangeably.)

DEFINITION 284. An *anchored relation* is a tuple $[\![n, r]\!]$ where n is an index set and r is an n-ary relation.

For an anchored relation $\operatorname{arity}[\![n, r]\!] = n$. The graph[1] of $[\![n, r]\!]$ is defined as follows: $\operatorname{GR}[\![n, r]\!] = r$.

DEFINITION 285. $\Pr_i f$ is a function defined by the formula

$$\Pr_i f = \left\{ \frac{x_i}{x \in f} \right\}$$

for every small n-ary relation f where n is an ordinal number and $i \in n$. Particularly for every n-ary relation f and $i \in n$ where $n \in \mathbb{N}$

$$\Pr_i f = \left\{ \frac{x_i}{[\![x_0, \ldots, x_{n-1}]\!] \in f} \right\}.$$

Recall that Cartesian product is defined as follows:

$$\prod a = \left\{ \frac{z \in (\bigcup \operatorname{im} a)^{\operatorname{dom} a}}{\forall i \in \operatorname{dom} a : z(i) \in a_i} \right\}.$$

OBVIOUS 286. If a is a small function, then $\prod a = \left\{ \frac{z \in \mho^{\operatorname{dom} a}}{\forall i \in \operatorname{dom} a : z(i) \in a_i} \right\}$.

3.7.2.1. *Currying and uncurrying.*

The customary definition. Let X, Y, Z be sets.

We will consider variables $x \in X$ and $y \in Y$.

Let a function $f \in Z^{X \times Y}$. Then $\operatorname{curry}(f) \in (Z^Y)^X$ is the function defined by the formula $(\operatorname{curry}(f)x)y = f(x, y)$.

Let now $f \in (Z^Y)^X$. Then $\operatorname{uncurry}(f) \in Z^{X \times Y}$ is the function defined by the formula $\operatorname{uncurry}(f)(x, y) = (fx)y$.

OBVIOUS 287.

1°. $\operatorname{uncurry}(\operatorname{curry}(f)) = f$ for every $f \in Z^{X \times Y}$.
2°. $\operatorname{curry}(\operatorname{uncurry}(f)) = f$ for every $f \in (Z^Y)^X$.

[1]It is unrelated with graph theory.

Currying and uncurrying with a dependent variable. Let X, Z be sets and Y be a function with the domain X. (Vaguely saying, Y is a variable dependent on X.)

The disjoint union $\coprod Y = \bigcup_{i\in\operatorname{dom}Y}(\{i\}\times Y_i) = \left\{\frac{(i,x)}{i\in\operatorname{dom}Y, x\in Y_i}\right\}$.

We will consider variables $x\in X$ and $y\in Y_x$.

Let a function $f\in Z^{\coprod_{i\in X}Y_i}$ (or equivalently $f\in Z^{\coprod Y}$). Then $\operatorname{curry}(f)\in\prod_{i\in X}Z^{Y_i}$ is the function defined by the formula $(\operatorname{curry}(f)x)y = f(x,y)$.

Let now $f\in\prod_{i\in X}Z^{Y_i}$. Then $\operatorname{uncurry}(f)\in Z^{\coprod_{i\in X}Y_i}$ is the function defined by the formula $\operatorname{uncurry}(f)(x,y) = (fx)y$.

Obvious 288.

1°. $\operatorname{uncurry}(\operatorname{curry}(f)) = f$ for every $f\in Z^{\coprod_{i\in X}Y_i}$.
2°. $\operatorname{curry}(\operatorname{uncurry}(f)) = f$ for every $f\in\prod_{i\in X}Z^{Y_i}$.

3.7.2.2. *Functions with ordinal numbers of arguments.* Let Ord be the set of small ordinal numbers.

If X and Y are sets and n is an ordinal number, the set of functions taking n arguments on the set X and returning a value in Y is Y^{X^n}.

The set of all small functions taking ordinal numbers of arguments is $Y^{\bigcup_{n\in\mathrm{Ord}}X^n}$.

I will denote $\operatorname{OrdVar}(X) = \mho^{\bigcup_{n\in\mathrm{Ord}}X^n}$ and call it *ordinal variadic*. ("Var" in this notation is taken from the word *variadic* in the collocation *variadic function* used in computer science.)

3.7.3. On sums of ordinals. Let a be an ordinal-indexed family of ordinals.

Proposition 289. $\coprod a$ with lexicographic order is a well-ordered set.

Proof. Let S be non-empty subset of $\coprod a$.

Take $i_0 = \min\Pr_0 S$ and $x_0 = \min\left\{\frac{\Pr_1 y}{y\in S, y(0)=i_0}\right\}$ (these exist by properties of ordinals). Then (i_0,x_0) is the least element of S. □

Definition 290. $\sum a$ is the unique ordinal order-isomorphic to $\coprod a$.

Exercise 291. Prove that for finite ordinals it is just a sum of natural numbers.

This ordinal exists and is unique because our set is well-ordered.

Remark 292. An infinite sum of ordinals is not customary defined.

The *structured sum* $\bigoplus a$ of a is an order isomorphism from lexicographically ordered set $\coprod a$ into $\sum a$.

There exists (for a given a) exactly one structured sum, by properties of well-ordered sets.

Obvious 293. $\sum a = \operatorname{im}\bigoplus a$.

Theorem 294. $(\bigoplus a)(n,x) = \sum_{i\in n}a_i + x$.

Proof. We need to prove that it is an order isomorphism. Let's prove it is an injection that is $m>n\Rightarrow\sum_{i\in m}a_i+x>\sum_{i\in n}a_i+x$ and $y>x\Rightarrow\sum_{i\in n}a_i+y>\sum_{i\in n}a_i+x$.

Really, if $m>n$ then $\sum_{i\in m}a_i+x\geq\sum_{i\in n+1}a_i+x>\sum_{i\in n}a_i+x$. The second formula is true by properties of ordinals.

Let's prove that it is a surjection. Let $r\in\sum a$. There exist $n\in\operatorname{dom}a$ and $x\in a_n$ such that $r=(\bigoplus a)(n,x)$. Thus $r=(\bigoplus a)(n,0)+x=\sum_{i\in n}a_i+x$ because $(\bigoplus a)(n,0)=\sum_{i\in n}a_i$ since $(n,0)$ has $\sum_{i\in n}a_i$ predecessors. □

3.7.4. Ordinated product.

3.7.4.1. *Introduction.* *Ordinated product* defined below is a variation of Cartesian product, but is associative unlike Cartesian product. However, ordinated product unlike Cartesian product is defined not for arbitrary sets, but only for relations having ordinal numbers of arguments.

Let F indexed by an ordinal number be a small family of anchored relations.

3.7.4.2. *Concatenation.*

DEFINITION 295. Let z be an indexed by an ordinal number family of functions each taking an ordinal number of arguments. The *concatenation* of z is

$$\operatorname{concat} z = \operatorname{uncurry}(z) \circ \left(\bigoplus(\operatorname{dom} \circ z)\right)^{-1}.$$

EXERCISE 296. Prove, that if z is a finite family of finitary tuples, it is concatenation of $\operatorname{dom} z$ tuples in the usual sense (as it is commonly used in computer science).

PROPOSITION 297. If $z \in \prod(\mathrm{GR} \circ F)$ then $\operatorname{concat} z = \operatorname{uncurry}(z) \circ (\bigoplus(\operatorname{arity} \circ F))^{-1}$.

PROOF. If $z \in \prod(\mathrm{GR} \circ F)$ then $\operatorname{dom} z(i) = \operatorname{dom}(\mathrm{GR} \circ F)_i = \operatorname{arity} F_i$ for every $i \in \operatorname{dom} F$. Thus $\operatorname{dom} \circ z = \operatorname{arity} \circ F$. □

PROPOSITION 298. $\operatorname{dom} \operatorname{concat} z = \sum_{i \in \operatorname{dom} z} \operatorname{dom} z_i$.

PROOF. Because $\operatorname{dom}(\bigoplus(\operatorname{dom} \circ z))^{-1} = \sum_{i \in \operatorname{dom} f}(\operatorname{dom} \circ z)$, it is enough to prove that

$$\operatorname{dom} \operatorname{uncurry}(z) = \operatorname{dom} \bigoplus(\operatorname{dom} \circ z).$$

Really,

$$\begin{gathered}\sum_{i \in \operatorname{dom} f}(\operatorname{dom} \circ z) = \\ \left\{\frac{(i,x)}{i \in \operatorname{dom}(\operatorname{dom} \circ z), x \in \operatorname{dom} z_i}\right\} = \\ \left\{\frac{(i,x)}{i \in \operatorname{dom} z, x \in \operatorname{dom} z_i}\right\} = \\ \coprod z\end{gathered}$$

and $\operatorname{dom} \operatorname{uncurry}(z) = \coprod_{i \in X} z_i = \coprod z$. □

3.7.4.3. *Finite example.* If F is a finite family (indexed by a natural number $\operatorname{dom} F$) of anchored finitary relations, then by definition

$$\mathrm{GR} \prod^{(\mathrm{ord})} = \left\{ \frac{\llbracket a_{0,0}, \ldots, a_{0,\operatorname{arity} F_0 - 1}, \ldots, a_{\operatorname{dom} F-1,0}, \ldots, a_{\operatorname{dom} F-1, \operatorname{arity} F_{\operatorname{dom} F-1}-1} \rrbracket}{\begin{matrix}\llbracket a_{0,0}, \ldots, a_{0,\operatorname{arity} F_0-1} \rrbracket \in \mathrm{GR}\, F_0 \wedge \ldots \wedge \\ \llbracket a_{\operatorname{dom} F-1, \operatorname{arity} F_{\operatorname{dom} F-1}-1} \rrbracket \in \mathrm{GR}\, F_{\operatorname{dom} F-1}\end{matrix}} \right\}$$

and

$$\operatorname{arity} \prod^{(\mathrm{ord})} F = \operatorname{arity} F_0 + \ldots + \operatorname{arity} F_{\operatorname{dom} F-1}.$$

The above formula can be shortened to

$$\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{concat} z}{z \in \prod(\mathrm{GR} \circ F)} \right\}.$$

3.7.4.4. *The definition.*

DEFINITION 299. The anchored relation (which I call *ordinated product*) $\prod^{(\mathrm{ord})} F$ is defined by the formulas:

$$\operatorname{arity} \prod^{(\mathrm{ord})} F = \sum (\operatorname{arity} \circ f);$$

$$\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{concat} z}{z \in \prod (\mathrm{GR} \circ F)} \right\}.$$

PROPOSITION 300. $\prod^{(\mathrm{ord})} F$ is a properly defined anchored relation.

PROOF. $\operatorname{dom} \operatorname{concat} z = \sum_{i \in \operatorname{dom} F} \operatorname{dom} z_i = \sum_{i \in \operatorname{dom} F} \operatorname{arity} f_i = \sum (\operatorname{arity} \circ F)$. □

3.7.4.5. *Definition with composition for every multiplier.*

$$q(F)_i \stackrel{\mathrm{def}}{=} \left(\operatorname{curry} \left(\bigoplus (\operatorname{arity} \circ F) \right) \right) i.$$

PROPOSITION 301. $\prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \mho^{\sum (\operatorname{arity} \circ F)}}{\forall i \in \operatorname{dom} F : L \circ q(F)_i \in \mathrm{GR}\, F_i} \right\}$.

PROOF. $\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{concat} z}{z \in \prod (\mathrm{GR} \circ F)} \right\}$;
$\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{uncurry}(z) \circ \left(\bigoplus (\operatorname{arity} \circ f) \right)^{-1}}{z \in \prod_{i \in \operatorname{dom} F} \mho^{\operatorname{arity} F_i}, \forall i \in \operatorname{dom} F : z(i) \in \mathrm{GR}\, F_i} \right\}$.
Let $L = \operatorname{uncurry}(z)$. Then $z = \operatorname{curry}(L)$.
$\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{L \circ \left(\bigoplus (\operatorname{arity} \circ f) \right)^{-1}}{\operatorname{curry}(L) \in \prod_{i \in \operatorname{dom} F} \mho^{\operatorname{arity} F_i}, \forall i \in \operatorname{dom} F : \operatorname{curry}(L) i \in \mathrm{GR}\, F_i} \right\}$;
$\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{L \circ \left(\bigoplus (\operatorname{arity} \circ f) \right)^{-1}}{L \in \mho^{\coprod_{i \in \operatorname{dom} F} \operatorname{arity} F_i}, \forall i \in \operatorname{dom} F : \operatorname{curry}(L) i \in \mathrm{GR}\, F_i} \right\}$;
$\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \mho^{\sum (\operatorname{arity} \circ f)}}{\forall i \in \operatorname{dom} F : \operatorname{curry} \left(L \circ \bigoplus (\operatorname{arity} \circ F) \right) i \in \mathrm{GR}\, F_i} \right\}$;
$(\operatorname{curry}(L \circ \bigoplus (\operatorname{arity} \circ F)) i) x = L((\operatorname{curry}(\bigoplus (\operatorname{arity} \circ F)) i) x) = L(q(F)_i x) = (L \circ q(F)_i) x$;
$\operatorname{curry}(L \circ \bigoplus (\operatorname{arity} \circ F)) i = L \circ q(F)_i$;
$\prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \mho^{\sum (\operatorname{arity} \circ F)}}{\forall i \in \operatorname{dom} F : L \circ q(F)_i \in \mathrm{GR}\, F_i} \right\}$. □

COROLLARY 302. $\prod^{(\mathrm{ord})} F = \left\{ \frac{L \in \left(\bigcup \operatorname{im} (\mathrm{GR} \circ F) \right)^{\sum (\operatorname{arity} \circ F)}}{\forall i \in \operatorname{dom} F : L \circ q(F)_i \in \mathrm{GR}\, F_i} \right\}$.

COROLLARY 303. $\prod^{(\mathrm{ord})} F$ is small if F is small.

3.7.4.6. *Definition with shifting arguments.* Let $F'_i = \left\{ \frac{L \circ \Pr_1 |_{\{i\} \times \operatorname{arity} F_i}}{L \in \mathrm{GR}\, F_i} \right\}$.

PROPOSITION 304. $F'_i = \left\{ \frac{L \circ \Pr_1 |_{\{i\} \times \mho}}{L \in \mathrm{GR}\, F_i} \right\}$.

PROOF. If $L \in \mathrm{GR}\, F_i$ then $\operatorname{dom} L = \operatorname{arity} F_i$. Thus

$$L \circ \Pr_1 |_{\{i\} \times \operatorname{arity} F_i} = L \circ \Pr_1 |_{\{i\} \times \operatorname{dom} L} = L \circ \Pr_1 |_{\{i\} \times \mho}.$$

□

PROPOSITION 305. F'_i is an $(\{i\} \times \operatorname{arity} F_i)$-ary relation.

PROOF. We need to prove that $\operatorname{dom}\left(L \circ \Pr_1|_{\{i\}\times\operatorname{arity} F_i}\right) = \{i\} \times \operatorname{arity} F_i$ for $L \in \operatorname{GR} F_i$, but that's obvious. □

OBVIOUS 306. $\coprod(\operatorname{arity} \circ F) = \bigcup_{i\in\operatorname{dom} F}(\{i\} \times \operatorname{arity} F_i) = \bigcup_{i\in\operatorname{dom} F} \operatorname{dom} F'_i$.

LEMMA 307. $P \in \prod_{i\in\operatorname{dom} F} F'_i \Leftrightarrow \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod(\operatorname{GR} \circ F)$ for a $(\operatorname{dom} F)$-indexed family P where $P_i \in \mho^{\{i\}\times\operatorname{arity} F_i}$ for every $i \in \operatorname{dom} F$, that is for $P \in \coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i}$.

PROOF. For every $P \in \coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i}$ we have:

$$P \in \prod_{i\in\operatorname{dom} F} F'_i \Leftrightarrow$$

$$P \in \left\{ \frac{z \in \mho^{\operatorname{dom} F}}{\forall i \in \operatorname{dom} F : z(i) \in F'_i} \right\} \Leftrightarrow$$

$$P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F : P(i) \in F'_i \Leftrightarrow$$

$$P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F \exists L \in \operatorname{GR} F_i : P_i = L \circ (\Pr_1|_{\{i\}\times\mho}) \Leftrightarrow$$

$$P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F \exists L \in \operatorname{GR} F_i : (P_i \in \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \forall x \in \operatorname{arity} F_i : P_i(i,x) = Lx) \Leftrightarrow$$

$$P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F \exists L \in \operatorname{GR} F_i : (P_i \in \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \operatorname{curry}(P_i)i = L) \Leftrightarrow$$

$$P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F : (P_i \in \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \operatorname{curry}(P_i)i \in \operatorname{GR} F_i) \Leftrightarrow$$

$$\forall i \in \operatorname{dom} F \exists Q_i \in (\mho^{\operatorname{arity} F_i})^{\{i\}} : (P_i = \operatorname{uncurry}(Q_i) \wedge (Q_i)i \in \mho^{\operatorname{arity} F_i} \wedge Q_i i \in \operatorname{GR} F_i) \Leftrightarrow$$

$$\forall i \in \operatorname{dom} F \exists Q_i \in (\mho^{\operatorname{arity} F_i})^{\{i\}} : \left(P_i = \operatorname{uncurry}(Q_i) \wedge \left(\bigcup_{i\in\operatorname{dom} F} Q_i\right) i \in \operatorname{GR} F_i\right) \Leftrightarrow$$

$$\forall i \in \operatorname{dom} F \exists Q_i \in (\mho^{\operatorname{arity} F_i})^{\{i\}} : \left(P_i = \operatorname{uncurry}(Q_i) \wedge \bigcup_{i\in\operatorname{dom} F} Q_i \in \prod(\operatorname{GR} \circ F)\right) \Leftrightarrow$$

$$\forall i \in \operatorname{dom} F : \bigcup_{i\in\operatorname{dom} F} \operatorname{curry}(P_i) \in \prod(\operatorname{GR} \circ F) \Leftrightarrow$$

$$\operatorname{curry}\left(\bigcup_{i\in\operatorname{dom} F} P_i\right) \in \prod(\operatorname{GR} \circ F) \Leftrightarrow$$

$$\operatorname{curry}\left(\bigcup \operatorname{im} P\right) \in \prod(\operatorname{GR} \circ F).$$

□

LEMMA 308. $\left\{ \frac{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity} \circ F)}{f\in\operatorname{GR}\prod^{(\mathrm{ord})} F} \right\} = \prod(\operatorname{GR} \circ F)$.

PROOF. First $\operatorname{GR}\prod^{(\mathrm{ord})} F = \left\{ \frac{\operatorname{uncurry}(z)\circ\left(\bigoplus(\operatorname{dom} \circ z)\right)^{-1}}{z\in\prod(\operatorname{GR} \circ F)} \right\}$, that is $\left\{ \frac{f}{f\in\operatorname{GR}\prod^{(\mathrm{ord})} F} \right\} = \left\{ \frac{\operatorname{uncurry}(z)\circ\left(\bigoplus(\operatorname{arity} \circ F)\right)^{-1}}{z\in\prod(\operatorname{GR} \circ F)} \right\}$.
Since $\bigoplus(\operatorname{arity} \circ F)$ is a bijection, we have $\left\{ \frac{f\circ\bigoplus(\operatorname{arity} \circ F)}{f\in\operatorname{GR}\prod^{(\mathrm{ord})} F} \right\} = \left\{ \frac{\operatorname{uncurry}(z)}{z\in\prod(\operatorname{GR} \circ F)} \right\}$ what is equivalent to $\left\{ \frac{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity} \circ F)}{f\in\operatorname{GR}\prod^{(\mathrm{ord})} F} \right\} = \left\{ \frac{z}{z\in\prod(\operatorname{GR} \circ F)} \right\}$ that is $\left\{ \frac{\operatorname{curry}(f)\circ\bigoplus(\operatorname{arity} \circ F)}{f\in\operatorname{GR}\prod^{(\mathrm{ord})} F} \right\} = \prod(\operatorname{GR} \circ F)$. □

LEMMA 309. $\left\{ \frac{\bigcup \operatorname{im} P}{P\in\coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i}\wedge\operatorname{curry}(\bigcup \operatorname{im} P)\in\prod(\operatorname{GR} \circ F)} \right\} = \left\{ \frac{L\in\mho^{\coprod_{i\in\operatorname{dom} F} \operatorname{arity} F_i}}{\operatorname{curry}(L)\in\prod(\operatorname{GR} \circ F)} \right\}$.

Proof. Let $L' \in \left\{ \frac{L \in \mho^{\coprod_{i\in\operatorname{dom} F} \operatorname{arity} F_i}}{\operatorname{curry}(L) \in \prod(\mathrm{GR} \circ F)} \right\}$. Then $L' \in \mho^{\coprod_{i\in\operatorname{dom} F} \operatorname{arity} F_i}$ and $\operatorname{curry}(L') \in \prod(\mathrm{GR} \circ F)$.

Let $P = \lambda i \in \operatorname{dom} F : L'|_{\{i\}\times\operatorname{arity} F_i}$. Then $P \in \coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i}$ and $\bigcup \operatorname{im} P = L'$. So $L' \in \left\{ \frac{\bigcup \operatorname{im} P}{P \in \coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod(\mathrm{GR} \circ F)} \right\}$.

Let now $L' \in \left\{ \frac{\bigcup \operatorname{im} P}{P \in \coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod(\mathrm{GR} \circ F)} \right\}$. Then there exists $P \in \coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i}$ such that $L' = \bigcup \operatorname{im} P$ and $\operatorname{curry}(L') \in \prod(\mathrm{GR} \circ F)$. Evidently $L' \in \mho^{\coprod_{i\in\operatorname{dom} F} \operatorname{arity} F_i}$. So $L' \in \left\{ \frac{L \in \mho^{\coprod_{i\in\operatorname{dom} F} \operatorname{arity} F_i}}{\operatorname{curry}(L) \in \prod(\mathrm{GR} \circ F)} \right\}$. □

Lemma 310. $\left\{ \frac{f \circ \bigoplus(\operatorname{arity} \circ F)}{f \in \mathrm{GR} \prod^{(\mathrm{ord})} F} \right\} = \left\{ \frac{\bigcup \operatorname{im} P}{P \in \prod_{i\in\operatorname{dom} F} F'_i} \right\}$.

Proof.

$$L \in \left\{ \frac{\bigcup \operatorname{im} P}{P \in \prod_{i\in\operatorname{dom} F} F'_i} \right\} \Leftrightarrow$$

$$L \in \left\{ \frac{\bigcup \operatorname{im} P}{P \in \coprod_{i\in\operatorname{dom} F} \mho^{\{i\}\times\operatorname{arity} F_i} \wedge \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod(\mathrm{GR} \circ F)} \right\} \Leftrightarrow$$

$$L \in \mho^{\coprod_{i\in\operatorname{dom} F} \operatorname{arity} F_i} \wedge \operatorname{curry}(L) \in \prod(\mathrm{GR} \circ F) \Leftrightarrow$$

$$L \in \mho^{\coprod_{i\in\operatorname{dom} F} \operatorname{arity} F_i} \wedge \operatorname{curry}(L) \in \left\{ \frac{\operatorname{curry}(f) \circ \bigoplus(\operatorname{arity} \circ F)}{f \in \mathrm{GR} \prod^{(\mathrm{ord})} F} \right\} \Leftrightarrow$$

(because $\bigoplus(\operatorname{arity} \circ F)$ is a bijection)

$$\operatorname{curry}(L) \circ \left(\bigoplus(\operatorname{arity} \circ F) \right)^{-1} \in \left\{ \frac{\operatorname{curry}(f)}{f \in \mathrm{GR} \prod^{(\mathrm{ord})} F} \right\} \Leftrightarrow$$

$$L \circ \left(\bigoplus(\operatorname{arity} \circ F) \right)^{-1} \in \left\{ \frac{f}{f \in \mathrm{GR} \prod^{(\mathrm{ord})} F} \right\} \Leftrightarrow$$

(because $\bigoplus(\operatorname{arity} \circ F)$ is a bijection)

$$L \in \left\{ \frac{f \circ \bigoplus(\operatorname{arity} \circ F)}{f \in \mathrm{GR} \prod^{(\mathrm{ord})} F} \right\}.$$

□

Theorem 311. $\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{(\bigcup \operatorname{im} P) \circ \left(\bigoplus(\operatorname{arity} \circ F) \right)^{-1}}{P \in \prod_{i\in\operatorname{dom} F} F'_i} \right\}$.

Proof. From the lemma, because $\bigoplus(\operatorname{arity} \circ F)$ is a bijection. □

Theorem 312. $\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\bigcup_{i\in\operatorname{dom} F} \left(P_i \circ \left(\bigoplus(\operatorname{arity} \circ f) \right)^{-1} \right)}{P \in \prod_{i\in\operatorname{dom} F} F'_i} \right\}$.

Proof. From the previous theorem. □

Theorem 313. $\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{\bigcup \operatorname{im} P}{P \in \prod_{i\in\operatorname{dom} F} \left\{ \frac{f \circ \left(\bigoplus(\operatorname{arity} \circ f) \right)^{-1}}{f \in F'_i} \right\}} \right\}$.

Proof. From the previous. □

REMARK 314. Note that the above formulas contain both $\bigcup_{i\in\operatorname{dom}F}\operatorname{dom}F'_i$ and $\bigcup_{i\in\operatorname{dom}F}F'_i$. These forms are similar but different.

3.7.4.7. *Associativity of ordinated product.* Let f be an ordinal variadic function.

Let S be an ordinal indexed family of functions of ordinal indexed families of functions each taking an ordinal number of arguments in a set X.

I call f *infinite associative* when

1°. $f(f\circ S)=f(\operatorname{concat}S)$ for every S;
2°. $f(\llbracket x\rrbracket)=x$ for $x\in X$.

Infinite associativity implies associativity.

PROPOSITION 315. Let f be an infinitely associative function taking an ordinal number of arguments in a set X. Define $x\star y=f\llbracket x,y\rrbracket$ for $x,y\in X$. Then the binary operation $\star$ is associative.

PROOF. Let $x,y,z\in X$. Then $(x\star y)\star z=f\llbracket f\llbracket x,y\rrbracket,z\rrbracket=f(f\llbracket x,y\rrbracket,f\llbracket z\rrbracket)=f\llbracket x,y,z\rrbracket$. Similarly $x\star(y\star z)=f\llbracket x,y,z\rrbracket$. So $(x\star y)\star z=x\star(y\star z)$. □

Concatenation is associative. First we will prove some lemmas.

Let a and b be functions on a poset. Let $a\sim b$ iff there exist an order isomorphism f such that $a=b\circ f$. Evidently $\sim$ is an equivalence relation.

OBVIOUS 316. $\operatorname{concat}a=\operatorname{concat}b\Leftrightarrow\operatorname{uncurry}(a)\sim\operatorname{uncurry}(b)$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

Thank to the above, we can reduce properties of concat to properties of uncurry.

LEMMA 317. $a\sim b\Rightarrow\operatorname{uncurry}a\sim\operatorname{uncurry}b$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

PROOF. There exist an order isomorphism f such that $a=b\circ f$.

$\operatorname{uncurry}(a)(x,y)=(ax)y=(bfx)y=\operatorname{uncurry}(b)(fx,y)=\operatorname{uncurry}(b)g(x,y)$ where $g(x,y)=(fx,y)$.

g is an order isomorphism because $g(x_0,y_0)\geq g(x_1,y_1)\Leftrightarrow(x_0,y_0)\geq(x_1,y_1)$. (Injectivity and surjectivity are obvious.) □

LEMMA 318. Let $a_i\sim b_i$ for every i. Then $\operatorname{uncurry}a\sim\operatorname{uncurry}b$ for every ordinal indexed families a and b of ordinal indexed families of functions taking an ordinal number of arguments.

PROOF. Let $a_i=b_i\circ f_i$ where f_i is an order isomorphism for every i.

$\operatorname{uncurry}(a)(i,y)=a_iy=b_if_iy=\operatorname{uncurry}(b)(i,f_iy)=\operatorname{uncurry}(b)g(i,y)=(\operatorname{uncurry}(b)\circ g)(i,y)$ where $g(i,y)=(i,f_iy)$.

g is an order isomorphism because $g(i,y_0)\geq g(i,y_1)\Leftrightarrow f_iy_0\geq f_iy_1\Leftrightarrow y_0\geq y_1$ and $i_0>i_1\Rightarrow g(i,y_0)>g(i,y_1)$. (Injectivity and surjectivity are obvious.) □

Let now S be an ordinal indexed family of ordinal indexed families of functions taking an ordinal number of arguments.

LEMMA 319. $\operatorname{uncurry}(\operatorname{uncurry}\circ S)\sim\operatorname{uncurry}(\operatorname{uncurry}S)$.

PROOF. $\operatorname{uncurry}\circ S=\lambda i\in S:\operatorname{uncurry}(S_i)$;
$(\operatorname{uncurry}(\operatorname{uncurry}\circ S))((i,x),y)=(\operatorname{uncurry}S_i)(x,y)=(S_ix)y$;
$(\operatorname{uncurry}(\operatorname{uncurry}S))((i,x),y)=((\operatorname{uncurry}S)(i,x))y=(S_ix)y$.

Thus $(\operatorname{uncurry}(\operatorname{uncurry}\circ S))((i,x),y)=(\operatorname{uncurry}(\operatorname{uncurry}S))((i,x),y)$ and thus evidently $\operatorname{uncurry}(\operatorname{uncurry}\circ S)\sim\operatorname{uncurry}(\operatorname{uncurry}S)$. □

THEOREM 320. concat is an infinitely associative function.

PROOF. $\operatorname{concat}(\llbracket x \rrbracket) = x$ for a function x taking an ordinal number of argument is obvious. It is remained to prove

$$\operatorname{concat}(\operatorname{concat} \circ S) = \operatorname{concat}(\operatorname{concat} S);$$

We have, using the lemmas,

$$\begin{gathered}\operatorname{concat}(\operatorname{concat} \circ S) \sim \\ \operatorname{uncurry}(\operatorname{concat} \circ S) \sim \\ \text{(by lemma 318)} \\ \operatorname{uncurry}(\operatorname{uncurry} \circ S) \sim \\ \operatorname{uncurry}(\operatorname{uncurry} S) \sim \\ \operatorname{uncurry}(\operatorname{concat} S) \sim \\ \operatorname{concat}(\operatorname{concat} S).\end{gathered}$$

Consequently $\operatorname{concat}(\operatorname{concat} \circ S) = \operatorname{concat}(\operatorname{concat} S)$. □

COROLLARY 321. Ordinated product is an infinitely associative function.

3.8. Galois surjections

DEFINITION 322. *Galois surjection* is the special case of Galois connection such that $f^* \circ f_*$ is identity.

PROPOSITION 323. For Galois surjection $\mathfrak{A} \to \mathfrak{B}$ such that $\mathfrak{A}$ is a join-semilattice we have (for every $y \in \mathfrak{B}$)

$$f_* y = \max \left\{ \frac{x \in \mathfrak{A}}{f^* x = y} \right\}.$$

PROOF. We need to prove (theorem 131)

$$\max \left\{ \frac{x \in \mathfrak{A}}{f^* x = y} \right\} = \max \left\{ \frac{x \in \mathfrak{A}}{f^* x \sqsubseteq y} \right\}.$$

To prove it, it's enough to show that for each $f^* x \sqsubseteq y$ there exists an $x' \sqsupseteq x$ such that $f^* x' = y$.

Really, $y = f^* f_* y$. It's enough to prove $f^*(x \sqcup f_* y) = y$.

Indeed (because lower adjoints preserve joins), $f^*(x \sqcup f_* y) = f^* x \sqcup f^* f_* y = f^* x \sqcup y = y$. □

3.9. Some properties of frames

This section is based on a TODD TRIMBLE's proof. A shorter but less elementary proof (also by TODD TRIMBLE) is available at
http://ncatlab.org/toddtrimble/published/topogeny

I will abbreviate *join-semilattice with least element* as JSWLE.

OBVIOUS 324. JSWLEs are the same as finitely join-closed posets (with nullary joins included).

DEFINITION 325. It is said that a function f from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ *preserves finite joins*, when for every finite set $S \in \mathscr{P}\mathfrak{A}$ such that $\bigsqcup^{\mathfrak{A}} S$ exists we have $\bigsqcup^{\mathfrak{B}} \langle f \rangle^* S = f \bigsqcup^{\mathfrak{A}} S$.

OBVIOUS 326. A function between JSWLEs preserves finite joins iff it preserves binary joins ($f(x \sqcup y) = fx \sqcup fy$) and nullary joins ($f(\bot^{\mathfrak{A}}) = \bot^{\mathfrak{B}}$).

DEFINITION 327. A *fixed point* of a function F is such x that $F(x) = x$. We will denote $\operatorname{Fix}(F)$ the set of all fixed points of a function F.

DEFINITION 328. Let $\mathfrak{A}$ be a JSWLE. A *co-nucleus* is a function $F : \mathfrak{A} \to \mathfrak{A}$ such that for every $p, q \in \mathfrak{A}$ we have:

1°. $F(p) \sqsubseteq p$;
2°. $F(F(p)) = F(p)$;
3°. $F(p \sqcup q) = F(p) \sqcup F(q)$.

PROPOSITION 329. Every co-nucleus is a monotone function.

PROOF. It follows from $F(p \sqcup q) = F(p) \sqcup F(q)$. □

LEMMA 330. $\bigsqcup^{\operatorname{Fix}(F)} S = \bigsqcup S$ for every $S \in \mathscr{P} \operatorname{Fix}(F)$ for every co-nucleus F on a complete lattice.

PROOF. Obviously $\bigsqcup S \sqsupseteq x$ for every $x \in S$.

Suppose $z \sqsupseteq x$ for every $x \in S$ for a $z \in \operatorname{Fix}(F)$. Then $z \sqsupseteq \bigsqcup S$.

$F(\bigsqcup S) \sqsupseteq F(x)$ for every $x \in S$. Thus $F(\bigsqcup S) \sqsupseteq \bigsqcup_{x \in S} F(x) = \bigsqcup S$. But $F(\bigsqcup S) \sqsubseteq \bigsqcup S$. Thus $F(\bigsqcup S) = \bigsqcup S$ that is $\bigsqcup S \in \operatorname{Fix}(F)$.

So $\bigsqcup^{\operatorname{Fix}(F)} S = \bigsqcup S$ by the definition of join. □

COROLLARY 331. $\bigsqcup^{\operatorname{Fix}(F)} S$ is defined for every $S \in \mathscr{P} \operatorname{Fix}(F)$.

LEMMA 332. $\bigsqcap^{\operatorname{Fix}(F)} S = F(\bigsqcap S)$ for every $S \in \mathscr{P} \operatorname{Fix}(F)$ for every co-nucleus F on a complete lattice.

PROOF. Obviously $F(\bigsqcap S) \sqsubseteq x$ for every $x \in S$.

Suppose $z \sqsubseteq x$ for every $x \in S$ for a $z \in \operatorname{Fix}(F)$. Then $z \sqsubseteq \bigsqcap S$ and thus $z \sqsubseteq F(\bigsqcap S)$.

So $\bigsqcap^{\operatorname{Fix}(F)} S = F(\bigsqcap S)$ by the definition of meet. □

COROLLARY 333. $\bigsqcap^{\operatorname{Fix}(F)} S$ is defined for every $S \in \mathscr{P} \operatorname{Fix}(F)$.

OBVIOUS 334. $\operatorname{Fix}(F)$ with induced order is a complete lattice.

LEMMA 335. If F is a co-nucleus on a co-frame $\mathfrak{A}$, then the poset $\operatorname{Fix}(F)$ of fixed points of F, with order inherited from $\mathfrak{A}$, is also a co-frame.

PROOF. Let $b \in \operatorname{Fix}(F)$, $S \in \mathscr{P} \operatorname{Fix}(F)$. Then

$$\begin{aligned} b \sqcup^{\operatorname{Fix}(F)} \bigsqcap^{\operatorname{Fix}(F)} S &= \\ b \sqcup^{\operatorname{Fix}(F)} F\left(\bigsqcap S\right) &= \\ F(b) \sqcup F\left(\bigsqcap S\right) &= \\ F\left(b \sqcup \bigsqcap S\right) &= \\ F\left(\bigsqcap \langle b \sqcup \rangle^* S\right) &= \\ \bigsqcap^{\operatorname{Fix}(F)} \langle b \sqcup \rangle^* S &= \\ \bigsqcap^{\operatorname{Fix}(F)} \langle b \sqcup^{\operatorname{Fix}(F)} \rangle^* S. & \end{aligned}$$

□

DEFINITION 336. Denote $\operatorname{Up}(\mathfrak{A})$ the set of upper sets on $\mathfrak{A}$ ordered *reverse* to set theoretic inclusion.

DEFINITION 337. Denote $\uparrow a = \left\{ \frac{x \in \mathfrak{A}}{x \sqsupseteq a} \right\} \in \operatorname{Up}(\mathfrak{A})$.

LEMMA 338. The set $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary meets and joins.

PROOF. Let $S \in \mathscr{P}\operatorname{Up}(\mathfrak{A})$.

Let $X \in \bigcup S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then there is $P \in S$ such that $X \in P$ and thus $Y \in P$ and so $Y \in \bigcup S$. So $\bigcup S \in \operatorname{Up}(\mathfrak{A})$.

Let now $X \in \bigcap S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then $\forall T \in S : X \in T$ and so $\forall T \in S : Y \in T$, thus $Y \in \bigcap S$. So $\bigcap S \in \operatorname{Up}(\mathfrak{A})$. □

THEOREM 339. A poset $\mathfrak{A}$ is a complete lattice iff there is a antitone map $s : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ such that

1°. $s(\uparrow p) = p$ for every $p \in \mathfrak{A}$;
2°. $D \subseteq \uparrow s(D)$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

Moreover, in this case $s(D) = \bigsqcap D$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

PROOF.

⇒. Take $s(D) = \bigsqcap D$.

⇐. $\forall x \in D : x \sqsupseteq s(D)$ from the second formula.

Let $\forall x \in D : y \sqsubseteq x$. Then $x \in \uparrow y$, $D \subseteq \uparrow y$; because s is an antitone map, thus follows $s(D) \sqsupseteq s(\uparrow y) = y$. So $\forall x \in D : y \sqsubseteq s(D)$.

That s is the meet follows from the definition of meets.

It remains to prove that $\mathfrak{A}$ is a complete lattice.

Take any subset S of $\mathfrak{A}$. Let D be the smallest upper set containing S. (It exists because $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary joins.) This is

$$D = \left\{ \frac{x \in \mathfrak{A}}{\exists s \in S : x \sqsupseteq s} \right\}.$$

Any lower bound of D is clearly a lower bound of S since $D \supseteq S$. Conversely any lower bound of S is a lower bound of D. Thus S and D have the same set of lower bounds, hence have the same greatest lower bound. □

PROPOSITION 340. For any poset $\mathfrak{A}$ the following are mutually reverse order isomorphisms between upper sets F (ordered reverse to set-theoretic inclusion) on $\mathfrak{A}$ and order homomorphisms $\varphi : \mathfrak{A}^{\mathrm{op}} \to 2$ (here 2 is the partially ordered set of two elements: 0 and 1 where $0 \sqsubseteq 1$), defined by the formulas

1°. $\varphi(a) = \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$ for every $a \in \mathfrak{A}$;
2°. $F = \varphi^{-1}(1)$.

PROOF. Let $X \in \varphi^{-1}(1)$ and $Y \sqsupseteq X$. Then $\varphi(X) = 1$ and thus $\varphi(Y) = 1$. Thus $\varphi^{-1}(1)$ is a upper set.

It is easy to show that φ defined by the formula 1° is an order homomorphism $\mathfrak{A}^{\mathrm{op}} \to 2$ whenever F is a upper set.

Finally we need to prove that they are mutually inverse. Really: Let φ be defined by the formula 1°. Then take $F' = \varphi^{-1}(1)$ and define $\varphi'(a)$ by the formula 1°. We have

$$\varphi'(a) = \begin{cases} 1 & \text{if } a \in \varphi^{-1}(1) \\ 0 & \text{if } a \notin \varphi^{-1}(1) \end{cases} = \begin{cases} 1 & \text{if } \varphi(a) = 1 \\ 0 & \text{if } \varphi(a) \neq 1 \end{cases} = \varphi(a).$$

Let now F be defined by the formula 2°. Then take $\varphi'(a) = \begin{cases} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{cases}$ as defined by the formula 1° and define $F' = \varphi'^{-1}(1)$. Then

$$F' = \varphi'^{-1}(1) = F.$$

□

LEMMA 341. For a complete lattice $\mathfrak{A}$, the map $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves arbitrary meets.

PROOF. Let $S \in \mathscr{P}\operatorname{Up}(\mathfrak{A})$. We have $\bigsqcap S \in \operatorname{Up}(\mathfrak{A})$.
$\bigsqcap\bigsqcap S = \bigsqcap\bigsqcap_{X\in S} X = \bigsqcap_{X\in S}\bigsqcap X$ is what we needed to prove. □

LEMMA 342. A complete lattice $\mathfrak{A}$ is a co-frame iff $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins.

PROOF.

$\Rightarrow$. Let $\mathfrak{A}$ be a co-frame. Let $D, D' \in \operatorname{Up}(\mathfrak{A})$. Obviously $\bigsqcap(D \sqcup D') \sqsupseteq \bigsqcap D$ and $\bigsqcap(D \sqcup D') \sqsupseteq \bigsqcap D'$, so $\bigsqcap(D \sqcup D') \sqsupseteq \bigsqcap D \sqcup \bigsqcap D'$.

Also

$$\bigsqcap D \sqcup \bigsqcap D' = \bigcup D \sqcup \bigcup D' = \text{(because } \mathfrak{A} \text{ is a co-frame)} = \bigcup\left\{\frac{d \sqcup d'}{d \in D, d' \in D'}\right\}.$$

Obviously $d \sqcup d' \in D \cap D'$, thus $\bigsqcap D \sqcup \bigsqcap D' \subseteq \bigcup(D \cap D') = \bigsqcap(D \cap D')$ that is $\bigsqcap D \sqcup \bigsqcap D' \sqsupseteq \bigsqcap(D \cap D')$. So $\bigsqcap(D \sqcup D') = \bigsqcap D \sqcup \bigsqcap D'$ that is $\bigsqcap : \operatorname{Up}(A) \to A$ preserves binary joins.

It preserves nullary joins since $\bigsqcap^{\operatorname{Up}(\mathfrak{A})} \bot^{\operatorname{Up}(\mathfrak{A})} = \bigsqcap^{\operatorname{Up}(\mathfrak{A})} \mathfrak{A} = \bot^{\mathfrak{A}}$.

$\Leftarrow$. Suppose $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins. Let $b \in \mathfrak{A}$, $S \in \mathscr{P}\mathfrak{A}$. Let D be the smallest upper set containing S (so $D = \bigcup\langle\uparrow\rangle^* S$). Then

$$\begin{gathered}
b \sqcup \bigsqcap S = \\
\bigsqcap \uparrow b \sqcup \bigcup\bigsqcap\langle\uparrow\rangle^* S = \\
\bigsqcap \uparrow b \sqcup \bigsqcap\bigcup\langle\uparrow\rangle^* S = \text{(since } \textstyle\bigsqcap \text{ preserves finite joins)} \\
\bigsqcap\left(\uparrow b \sqcup \bigcup\langle\uparrow\rangle^* S\right) = \\
\bigcup\left(\uparrow b \cap \bigcup\langle\uparrow\rangle^* S\right) = \\
\bigsqcap\bigcup_{a\in S}(\uparrow b \cap \uparrow a) = \\
\bigsqcap\bigcup_{a\in S} \uparrow (b \sqcup a) = \text{(since } \textstyle\bigsqcap \text{ preserves all meets)} \\
\bigcup_{a\in S}\bigsqcap \uparrow (b \sqcup a) = \\
\bigcup_{a\in S}(b \sqcup a) = \\
\bigsqcap_{a\in S}(b \sqcup a).
\end{gathered}$$

□

COROLLARY 343. If $\mathfrak{A}$ is a co-frame, then the composition $F = \uparrow \circ \bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \operatorname{Up}(\mathfrak{A})$ is a co-nucleus. The embedding $\uparrow : \mathfrak{A} \to \operatorname{Up}(\mathfrak{A})$ is an isomorphism of $\mathfrak{A}$ onto the co-frame $\operatorname{Fix}(F)$.

PROOF. $D \sqsupseteq F(D)$ follows from theorem 339.

We have $F(F(D)) = F(D)$ for all $D \in \operatorname{Up}(\mathfrak{A})$ since $F(F(D)) = \uparrow\bigsqcap\uparrow\bigsqcap D =$ (because $\bigsqcap\uparrow s = s$ for any s) $= \uparrow\bigsqcap D = F(D)$.

And since both $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ and $\uparrow$ preserve finite joins, F preserves finite joins. Thus F is a co-nucleus.

Finally, we have $a \sqsupseteq a'$ if and only if $\uparrow a \subseteq \uparrow a'$, so that $\uparrow : \mathfrak{A} \to \mathrm{Up}(\mathfrak{A})$ maps $\mathfrak{A}$ isomorphically onto its image $\langle \uparrow \rangle^* \mathfrak{A}$. This image is $\mathrm{Fix}(F)$ because if D is any fixed point (i.e. if $D = \uparrow \bigsqcap D$), then D clearly belongs to $\langle \uparrow \rangle^* \mathfrak{A}$; and conversely $\uparrow a$ is always a fixed point of $F = \uparrow \circ \bigsqcap$ since $F(\uparrow a) = \uparrow \bigsqcap \uparrow a = \uparrow a$. □

Definition 344. If $\mathfrak{A}$, $\mathfrak{B}$ are two JSWLEs, then $\mathrm{Join}(\mathfrak{A}, \mathfrak{B})$ is the (ordered pointwise) set of finite joins preserving maps $\mathfrak{A} \to \mathfrak{B}$.

Obvious 345. $\mathrm{Join}(\mathfrak{A}, \mathfrak{B})$ is a JSWLE, where $f \sqcup g$ is given by the formula $(f \sqcup g)(p) = f(p) \sqcup g(p)$, $\bot^{\mathrm{Join}(\mathfrak{A}, \mathfrak{B})}$ is given by the formula $\bot^{\mathrm{Join}(\mathfrak{A}, \mathfrak{B})}(p) = \bot^{\mathfrak{B}}$.

Definition 346. Let $h : Q \to R$ be a finite joins preserving map. Then by definition $\mathrm{Join}(P, h) : \mathrm{Join}(P, Q) \to \mathrm{Join}(P, R)$ takes $f \in \mathrm{Join}(P, Q)$ into the composition $h \circ f \in \mathrm{Join}(P, R)$.

Lemma 347. Above defined $\mathrm{Join}(P, h)$ is a finite joins preserving map.

Proof.

$$(h \circ (f \sqcup f'))x = h(f \sqcup f')x = h(fx \sqcup f'x) = \\ hfx \sqcup hf'x = (h \circ f)x \sqcup (h \circ f')x = ((h \circ f) \sqcup (h \circ f'))x.$$

Thus $h \circ (f \sqcup f') = (h \circ f) \sqcup (h \circ f')$.

$(h \circ \bot^{\mathrm{Join}(P,Q)})x = h \bot^{\mathrm{Join}(P,Q)} x = h \bot^Q = \bot^R$. □

Proposition 348. If $h, h' : Q \to R$ are finite join preserving maps and $h \sqsupseteq h'$, then $\mathrm{Join}(P, h) \sqsupseteq \mathrm{Join}(P, h')$.

Proof. $\mathrm{Join}(P, h)(f)(x) = (h \circ f)(x) = hfx \sqsupseteq h'fx = (h' \circ f)(x) = \mathrm{Join}(P, h')(f)(x)$. □

Lemma 349. If $g : Q \to R$ and $h : R \to S$ are finite joins preserving, then the composition $\mathrm{Join}(P, h) \circ \mathrm{Join}(P, g)$ is equal to $\mathrm{Join}(P, h \circ g)$. Also $\mathrm{Join}(P, \mathrm{id}_Q)$ for identity map id_Q on Q is the identity map $\mathrm{id}_{\mathrm{Join}(P,Q)}$ on $\mathrm{Join}(P, Q)$.

Proof. $\mathrm{Join}(P, h) \mathrm{Join}(P, g) f = \mathrm{Join}(P, h)(g \circ f) = h \circ g \circ f = \mathrm{Join}(P, h \circ g) f$.

$\mathrm{Join}(P, \mathrm{id}_Q) f = \mathrm{id}_Q \circ f = f$. □

Corollary 350. If Q is a JSWLE and $F : Q \to Q$ is a co-nucleus, then for any JSWLE P we have that

$$\mathrm{Join}(P, F) : \mathrm{Join}(P, Q) \to \mathrm{Join}(P, Q)$$

is also a co-nucleus.

Proof. From $\mathrm{id}_Q \sqsupseteq F$ (co-nucleus axiom 1°) we have $\mathrm{Join}(P, \mathrm{id}_Q) \sqsupseteq \mathrm{Join}(P, F)$ and since by the last lemma the left side is the identity on $\mathrm{Join}(P, Q)$, we see that $\mathrm{Join}(P, F)$ also satisfies co-nucleus axiom 1°.

$\mathrm{Join}(P, F) \circ \mathrm{Join}(P, F) = \mathrm{Join}(P, F \circ F)$ by the same lemma and thus $\mathrm{Join}(P, F) \circ \mathrm{Join}(P, F) = \mathrm{Join}(P, F)$ by the second co-nucleus axiom for F, showing that $\mathrm{Join}(P, F)$ satisfies the second co-nucleus axiom.

By an other lemma, we have that $\mathrm{Join}(P, F)$ preserves binary joins, given that F preserves binary joins, which is the third co-nucleus axiom. □

Lemma 351. $\mathrm{Fix}(\mathrm{Join}(P, F)) = \mathrm{Join}(P, \mathrm{Fix}(F))$ for every JSWLEs P, Q and a join preserving function $F : Q \to Q$.

PROOF. $a \in \operatorname{Fix}(\operatorname{Join}(P, F)) \Leftrightarrow a \in F^P \wedge F \circ a = a \Leftrightarrow a \in F^P \wedge \forall x \in P : F(a(x)) = a(x)$.

$a \in \operatorname{Join}(P, \operatorname{Fix}(F)) \Leftrightarrow a \in \operatorname{Fix}(F)^P \Leftrightarrow a \in F^P \wedge \forall x \in P : F(a(x)) = a(x)$.

Thus $\operatorname{Fix}(\operatorname{Join}(P, F)) = \operatorname{Join}(P, \operatorname{Fix}(F))$. That the order of the left and right sides of the equality agrees is obvious. $\square$

DEFINITION 352. $\mathbf{Pos}(\mathfrak{A}, \mathfrak{B})$ is the pointwise ordered poset of monotone maps from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

LEMMA 353. If Q, R are JSWLEs and P is a poset, then $\mathbf{Pos}(P, R)$ is a JSWLE and $\mathbf{Pos}(P, \operatorname{Join}(Q, R))$ is isomorphic to $\operatorname{Join}(Q, \mathbf{Pos}(P, R))$. If R is a co-frame, then also $\mathbf{Pos}(P, R)$ is a co-frame.

PROOF. Let $f, g \in \mathbf{Pos}(P, R)$. Then $\lambda x \in P : (fx \sqcup gx)$ is obviously monotone and then it is evident that $f \sqcup^{\mathbf{Pos}(P,R)} g = \lambda x \in P : (fx \sqcup gx)$. $\lambda x \in P : \bot^R$ is also obviously monotone and it is evident that $\bot^{\mathbf{Pos}(P,R)} = \lambda x \in P : \bot^R$.

Obviously both $\mathbf{Pos}(P, \operatorname{Join}(Q, R))$ and $\operatorname{Join}(Q, \mathbf{Pos}(P, R))$ are sets of order preserving maps.

Let f be a monotone map.

$f \in \mathbf{Pos}(P, \operatorname{Join}(Q, R))$ iff $f \in \operatorname{Join}(Q, R)^P$ iff $f \in \left\{ \frac{g \in R^Q}{g \text{ preserves finite joins}} \right\}^P$ iff $f \in (R^Q)^P$ and every $g = f(x)$ (for $x \in P$) preserving finite joins. This is bijectively equivalent ($f \mapsto f'$) to $f' \in (R^P)^Q$ preserving finite joins.

$f' \in \operatorname{Join}(Q, \mathbf{Pos}(P, R))$ iff f' preserves finite joins and $f' \in \mathbf{Pos}(P, R)^Q$ iff f' preserves finite joins and $f' \in \left\{ \frac{g \in (R^P)^Q}{g(x) \text{ is monotone}} \right\}$ iff f' preserves finite joins and $f' \in (R^P)^Q$.

So we have proved that $f \mapsto f'$ is a bijection between $\mathbf{Pos}(P, \operatorname{Join}(Q, R))$ and $\operatorname{Join}(Q, \mathbf{Pos}(P, R))$. That it preserves order is obvious.

It remains to prove that if R is a co-frame, then also $\mathbf{Pos}(P, R)$ is a co-frame.

First, we need to prove that $\mathbf{Pos}(P, R)$ is a complete lattice. But it is easy to prove that for every set $S \in \mathscr{P}\mathbf{Pos}(P, R)$ we have $\lambda x \in P : \bigsqcup_{f \in S} f(x)$ and $\lambda x \in P : \bigsqcap_{f \in S} f(x)$ are monotone and thus are the joins and meets on $\mathbf{Pos}(P, R)$.

Next we need to prove that

$$b \sqcup^{\mathbf{Pos}(P,R)} \bigsqcap^{\mathbf{Pos}(P,R)} S = \bigsqcap^{\mathbf{Pos}(P,R)} \left\langle b \sqcup^{\mathbf{Pos}(P,R)} \right\rangle^* S.$$

Really (for every $x \in P$),

$$\left(b \sqcup^{\mathbf{Pos}(P,R)} \bigsqcap^{\mathbf{Pos}(P,R)} S \right) x = b(x) \sqcup \left(\bigsqcap^{\mathbf{Pos}(P,R)} S \right) x = b(x) \sqcup \bigsqcap_{f \in S} f(x) = \bigsqcap_{f \in S} (b(x) \sqcup f(x)) = \bigsqcap_{f \in S} \left(b \sqcup^{\mathbf{Pos}(P,R)} f \right) x = \left(\bigsqcap_{f \in S}^{\mathbf{Pos}(P,R)} \left(b \sqcup^{\mathbf{Pos}(P,R)} f \right) \right) x.$$

Thus $b \sqcup^{\mathbf{Pos}(P,R)} \bigsqcap^{\mathbf{Pos}(P,R)} S = \bigsqcap_{f \in S}^{\mathbf{Pos}(P,R)} \left(b \sqcup^{\mathbf{Pos}(P,R)} f \right) = \bigsqcap^{\mathbf{Pos}(P,R)} \left\langle b \sqcup^{\mathbf{Pos}(P,R)} \right\rangle^* S$. $\square$

DEFINITION 354. $P \cong Q$ means that posets P and Q are isomorphic.

CHAPTER 4

Typed sets and category Rel

4.1. Relational structures

Definition 355. A *relational structure* is a pair consisting of a set and a tuple of relations on this set.

A poset $(\mathfrak{A}, \sqsubseteq)$ can be considered as a relational structure: $(\mathfrak{A}, \llbracket \sqsubseteq \rrbracket)$.

A set can X be considered as a relational structure with zero relations: $(X, \llbracket\rrbracket)$.

This book is not about relational structures. So I will not introduce more examples.

Think about relational structures as a common place for sets or posets, as far as they are considered in this book.

We will denote $x \in (\mathfrak{A}, R)$ iff $x \in \mathfrak{A}$ for a relational structure $(\mathfrak{A}, R)$.

4.2. Typed elements and typed sets

We sometimes want to differentiate between the same element of two different sets. For example, we may want to consider different the natural number 3 and the rational number 3. In order to describe this in a formal way we consider elements of sets together with sets themselves. For example, we can consider the pairs $(\mathbb{N}, 3)$ and $(\mathbb{Q}, 3)$.

Definition 356. A *typed element* is a pair $(\mathfrak{A}, a)$ where $\mathfrak{A}$ is a relational structure and $a \in \mathfrak{A}$.

I denote $\operatorname{type}(\mathfrak{A}, a) = \mathfrak{A}$ and $\operatorname{GR}(\mathfrak{A}, a) = a$.

Definition 357. I will denote typed element $(\mathfrak{A}, a)$ as $@^{\mathfrak{A}}a$ or just $@a$ when $\mathfrak{A}$ is clear from context.

Definition 358. A *typed set* is a typed element equal to $(\mathscr{P}U, A)$ where U is a set and A is its subset.

Remark 359. *Typed sets* is an awkward formalization of type theory sets in ZFC (U is meant to express the *type* of the set). This book could be better written using type theory instead of ZFC, but I want my book to be understandable for everyone knowing ZFC. $(\mathscr{P}U, A)$ should be understood as a set A of type U. For an example, consider $(\mathscr{P}\mathbb{R}, [0; 10])$; it is the closed interval $[0; 10]$ whose elements are considered as real numbers.

Definition 360. $\mathfrak{T}\mathfrak{A} = \left\{ \frac{(\mathfrak{A}, a)}{a \in \mathfrak{A}} \right\} = \{\mathfrak{A}\} \times \mathfrak{A}$ for every relational structure $\mathfrak{A}$.

Remark 361. $\mathfrak{T}\mathfrak{A}$ is the set of typed elements of $\mathfrak{A}$.

Definition 362. If $\mathfrak{A}$ is a poset, we introduce order on its typed elements isomorphic to the order of the original poset: $(\mathfrak{A}, a) \sqsubseteq (\mathfrak{A}, b) \Leftrightarrow a \sqsubseteq b$.

Definition 363. I denote $\operatorname{GR}(\mathfrak{A}, a) = a$ for a typed element $(\mathfrak{A}, a)$.

Definition 364. I will denote *typed subsets* of a typed poset $(\mathscr{P}U, A)$ as $\mathscr{P}(\mathscr{P}U, A) = \left\{ \frac{(\mathscr{P}U, X)}{X \in \mathscr{P}A} \right\} = \{\mathscr{P}U\} \times \mathscr{P}A$.

Obvious 365. $\mathscr{P}(\mathscr{P}U, A)$ is also a set of typed sets.

Definition 366. I will denote $\mathscr{T}U = \mathfrak{T}\mathscr{P}U$.

Remark 367. This means that $\mathscr{T}U$ is the set of typed subsets of a set U.

Obvious 368. $\mathscr{T}U = \left\{ \frac{(\mathscr{P}U, X)}{X \in \mathscr{P}U} \right\} = \{\mathscr{P}U\} \times \mathscr{P}U = \mathscr{P}(\mathscr{P}U, U)$.

Obvious 369. $\mathscr{T}U$ is a complete atomistic boolean lattice. Particularly:

1°. $\bot^{\mathscr{T}U} = (\mathscr{P}U, \emptyset)$;
2°. $\top^{\mathscr{T}U} = (\mathscr{P}U, U)$;
3°. $(\mathscr{P}U, A) \sqcup (\mathscr{P}U, B) = (\mathscr{P}U, A \cup B)$;
4°. $(\mathscr{P}U, A) \sqcap (\mathscr{P}U, B) = (\mathscr{P}U, A \cap B)$;
5°. $\bigsqcup_{A \in S}(\mathscr{P}U, A) = (\mathscr{P}U, \bigcup_{A \in S} A)$;
6°. $\bigsqcap_{A \in S}(\mathscr{P}U, A) = \left(\mathscr{P}U, \begin{cases} \bigcap_{A \in S} A & \text{if } A \neq \emptyset \\ U & \text{if } A = \emptyset \end{cases} \right)$;
7°. $\overline{(\mathscr{P}U, A)} = (\mathscr{P}U, U \setminus A)$;
8°. atomic elements are $(\mathscr{P}U, \{x\})$ where $x \in U$.

Typed sets are "better" than regular sets as (for example) for a set U and a typed set X the following are defined by regular order theory:

- $\operatorname{atoms} X$;
- $\overline{X}$;
- $\bigsqcap^{\mathscr{T}U} \emptyset$.

For regular ("non-typed") sets these are not defined (except of $\operatorname{atoms} X$ which however needs a special definition instead of using the standard order-theory definition of atoms).

Typed sets are convenient to be used together with filters on sets (see below), because both typed sets and filters have a set $\mathscr{P}U$ as their type.

Another advantage of typed sets is that their binary product (as defined below) is a **Rel**-morphism. This is especially convenient because below defined products of filters are also morphisms of related categories.

Well, typed sets are also quite awkward, but the proper way of doing modern mathematics is *type theory* not ZFC, what is however outside of the topic of this book.

4.3. Category Rel

I remind that **Rel** is the category of (small) binary relations between sets, and **Set** is its subcategory where only monovalued entirely defined morphisms (functions) are considered.

Definition 370. Order on $\mathbf{Rel}(A, B)$ is defined by the formula $f \sqsubseteq g \Leftrightarrow \operatorname{GR} f \subseteq \operatorname{GR} g$.

Obvious 371. This order is isomorphic to the natural order of subsets of the set $A \times B$.

Definition 372. $X \; [f]^* \; Y \Leftrightarrow \operatorname{GR} X \; [\operatorname{GR} f]^* \; \operatorname{GR} Y$ and $\langle f \rangle^* X = (\operatorname{Dst} f, \langle \operatorname{GR} f \rangle^* \operatorname{GR} X)$ for a **Rel**-morphism f and typed sets $X \in \mathscr{T} \operatorname{Src} f$, $Y \in \mathscr{T} \operatorname{Dst} f$.

Definition 373. For category **Rel** there is defined reverse morphism: $(A, B, F)^{-1} = (B, A, F^{-1})$.

Obvious 374. $(f^{-1})^{-1} = f$ for every **Rel**-morphism f.

OBVIOUS 375. $\left[f^{-1}\right]^* = [f]^{*-1}$ for every **Rel**-morphism f.

OBVIOUS 376. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every composable **Rel**-morphisms f and g.

PROPOSITION 377. $\langle g \circ f \rangle^* = \langle g \rangle^* \circ \langle f \rangle^*$ for every composable **Rel**-morphisms f and g.

PROOF. Exercise. □

PROPOSITION 378. The above definitions of monovalued morphisms of **Rel** and of injective morphisms of **Set** coincide with how mathematicians usually define monovalued functions (that is morphisms of **Set**) and injective functions.

PROOF. Let f be a **Rel**-morphism $A \to B$.

The following are equivalent:

- f is a monovalued relation;
- $\forall x \in A, y_0, y_1 \in B : (x\ f\ y_0 \wedge x\ f\ y_1 \Rightarrow y_0 = y_1)$;
- $\forall x \in A, y_0, y_1 \in B : (y_0 \neq y_1 \Rightarrow \neg(x\ f\ y_0) \vee \neg(x\ f\ y_1))$;
- $\forall y_0, y_1 \in B \forall x \in A : (y_0 \neq y_1 \Rightarrow \neg(x\ f\ y_0) \vee \neg(x\ f\ y_1))$;
- $\forall y_0, y_1 \in B : (y_0 \neq y_1 \Rightarrow \forall x \in A : (\neg(x\ f\ y_0) \vee \neg(x\ f\ y_1)))$;
- $\forall y_0, y_1 \in B : (\exists x \in A : (x\ f\ y_0 \wedge x\ f\ y_1) \Rightarrow y_0 = y_1)$;
- $\forall y_0, y_1 \in B : y_0\ (f \circ f^{-1})\ y_1 \Rightarrow y_0 = y_1$;
- $f \circ f^{-1} \sqsubseteq 1_B$.

Let now f be a **Set**-morphism $A \to B$.

The following are equivalent:

- f is an injective function;
- $\forall y \in B, a, b \in A : (a\ f\ y \wedge b\ f\ y \Rightarrow a = b)$;
- $\forall y \in B, a, b \in A : (a \neq b \Rightarrow \neg(a\ f\ y) \vee \neg(b\ f\ y))$;
- $\forall y \in B : (a \neq b \Rightarrow \forall a, b \in A : (\neg(a\ f\ y) \vee \neg(b\ f\ y)))$;
- $\forall y \in B : (\exists a, b \in A : (a\ f\ y \wedge b\ f\ y) \Rightarrow a = b)$;
- $f^{-1} \circ f \sqsubseteq 1_A$.

□

PROPOSITION 379. For a binary relation f we have:

1°. $\langle f \rangle^* \bigcup S = \bigcup \langle \langle f \rangle^* \rangle^* S$ for a set of sets S;
2°. $\bigcup S\ [f]^*\ Y \Leftrightarrow \exists X \in S : X\ [f]^*\ Y$ for a set of sets S;
3°. $X\ [f]^* \bigcup T \Leftrightarrow \exists Y \in T : X\ [f]^*\ Y$ for a set of sets T;
4°. $\bigcup S\ [f]^* \bigcup T \Leftrightarrow \exists X \in S, Y \in T : X\ [f]^*\ Y$ for sets of sets S and T;
5°. $X\ [f]^*\ Y \Leftrightarrow \exists \alpha \in X, \beta \in Y : \{\alpha\}\ [f]^*\ \{\beta\}$ for sets X and Y;
6°. $\langle f \rangle^* X = \bigcup \langle \langle f \rangle^* \rangle^* \operatorname{atoms} X$ for a set X (where $\operatorname{atoms} X = \left\{ \frac{\{x\}}{x \in X} \right\}$).

PROOF.

1°.

$$y \in \langle f \rangle^* \bigcup S \Leftrightarrow \exists x \in \bigcup S : x\ f\ y \Leftrightarrow \exists P \in S, x \in P : x\ f\ y \Leftrightarrow \\ \exists P \in S : y \in \langle f \rangle^* P \Leftrightarrow \exists Q \in \langle \langle f \rangle^* \rangle^* S : y \in Q \Leftrightarrow y \in \bigcup \langle \langle f \rangle^* \rangle^* S.$$

2°.

$$\bigcup S\ [f]^*\ Y \Leftrightarrow \exists x \in \bigcup S, y \in Y : x\ f\ y \Leftrightarrow \\ \exists X \in S, x \in X, y \in Y : x\ f\ y \Leftrightarrow \exists X \in S : X\ [f]^*\ Y.$$

3°. By symmetry.

4°. From two previous formulas.
5°. $X\;[f]^*\;Y \Leftrightarrow \exists \alpha \in X, \beta \in Y : \alpha\; f\; \beta \Leftrightarrow \exists \alpha \in X, \beta \in Y : \{\alpha\}\;[f]^*\;\{\beta\}$.
6°. Obvious.

□

Corollary 380. For a **Rel**-morphism f we have:

1°. $\langle f\rangle^* \bigsqcup S = \bigsqcup \langle \langle f\rangle^*\rangle^* S$ for $S \in \mathscr{P}\mathscr{T}\operatorname{Src} f$;
2°. $\bigsqcup S\;[f]^*\;Y \Leftrightarrow \exists X \in S : X\;[f]^*\;Y$ for $S \in \mathscr{P}\mathscr{T}\operatorname{Src} f$;
3°. $X\;[f]^* \bigsqcup T \Leftrightarrow \exists Y \in T : X\;[f]^*\;Y$ for $T \in \mathscr{P}\mathscr{T}\operatorname{Dst} f$;
4°. $\bigsqcup S\;[f]^*\;\bigsqcup T \Leftrightarrow \exists X \in S, Y \in T : X\;[f]^*\;Y$ for $S \in \mathscr{P}\mathscr{T}\operatorname{Src} f$, $T \in \mathscr{P}\mathscr{T}\operatorname{Dst} f$;
5°. $X\;[f]^*\;Y \Leftrightarrow \exists x \in \operatorname{atoms} X, y \in \operatorname{atoms} Y : x\;[f]^*\;y$ for $X \in \mathscr{T}\operatorname{Src} f$, $Y \in \mathscr{T}\operatorname{Dst} f$;
6°. $\langle f\rangle^* X = \bigsqcup \langle \langle f\rangle^*\rangle^* \operatorname{atoms} X$ for $X \in \mathscr{T}\operatorname{Src} f$.

Corollary 381. A **Rel**-morphism f can be restored knowing either $\langle f\rangle^* x$ for atoms $x \in \mathscr{T}\operatorname{Src} f$ or $x\;[f]^*\;y$ for atoms $x \in \mathscr{T}\operatorname{Src} f$, $y \in \mathscr{T}\operatorname{Dst} f$.

Proposition 382. Let A, B be sets, R be a set of binary relations.

1°. $\langle \bigcup R\rangle^* X = \bigcup_{f\in R} \langle f\rangle^* X$ for every set X;
2°. $\langle \bigcap R\rangle^* \{\alpha\} = \bigcap_{f\in R} \langle f\rangle^* \{\alpha\}$ for every α, if R is nonempty;
3°. $X\;[\bigcup R]^*\;Y \Leftrightarrow \exists f \in R : X\;[f]^*\;Y$ for every sets X, Y;
4°. $\alpha\;(\bigcap R)\;\beta \Leftrightarrow \forall f \in R : \alpha\; f\; \beta$ for every α and β, if R is nonempty.

Proof.

1°.

$$y \in \left\langle \bigcup R \right\rangle^* X \Leftrightarrow \exists x \in X : x \left(\bigcup R\right) y \Leftrightarrow \exists x \in X, f \in R : x\; f\; y \Leftrightarrow \exists f \in R : y \in \langle f\rangle^* X \Leftrightarrow y \in \bigcup_{f\in R} \langle f\rangle^* X.$$

2°.

$$y \in \left\langle \bigcap R \right\rangle^* \{\alpha\} \Leftrightarrow \forall f \in R : \alpha\; f\; y \Leftrightarrow \forall f \in R : y \in \langle f\rangle^* \{\alpha\} \Leftrightarrow y \in \bigcap_{f\in R} \langle f\rangle^* \{\alpha\}.$$

3°.

$$X \left[\bigcup R\right]^* Y \Leftrightarrow \exists x \in X, y \in Y : x \left(\bigcup R\right) y \Leftrightarrow \exists x \in X, y \in Y, f \in R : x\; f\; y \Leftrightarrow \exists f \in R : X\;[f]^*\;Y.$$

4°. Obvious.

□

Corollary 383. Let A, B be sets, $R \in \mathscr{P}\mathbf{Rel}(A, B)$.

1°. $\langle \bigsqcup R\rangle^* X = \bigsqcup_{f\in R} \langle f\rangle^* X$ for $X \in \mathscr{T} A$;
2°. $\langle \bigsqcap R\rangle^* x = \bigsqcap_{f\in R} \langle f\rangle^* x$ for atomic $x \in \mathscr{T} A$;
3°. $X\;[\bigsqcup R]^*\;Y \Leftrightarrow \exists f \in R : X\;[f]^*\;Y$ for $X \in \mathscr{T} A$, $Y \in \mathscr{T} B$;
4°. $x\;[\bigsqcap R]^*\;y \Leftrightarrow \forall f \in R : x\;[f]^*\;y$ for every atomic $x \in \mathscr{T} A$, $y \in \mathscr{T} B$.

Proposition 384. $X\;[g \circ f]^*\;Z \Leftrightarrow \exists \beta : (X\;[f]^*\;\{\beta\} \wedge \{\beta\}\;[g]^*\;Z)$ for every binary relation f and sets X and Y.

Proof.

$$X\ [g\circ f]^*\ Z \Leftrightarrow \exists x\in X, z\in Z : x\ (g\circ f)\ z \Leftrightarrow \\ \exists x\in X, z\in Z, \beta : (x\ f\ \beta\wedge\beta\ g\ z) \Leftrightarrow \\ \exists\beta : (\exists x\in X : x\ f\ \beta\wedge\exists y\in Y : \beta\ g\ z) \Leftrightarrow \exists\beta : (X\ [f]^*\ \{\beta\}\wedge\{\beta\}\ [g]^*\ Z).$$

□

Corollary 385. $X\ [g\circ f]^*\ Z \Leftrightarrow \exists y\in\operatorname{atoms}^{\mathscr{T}B} : (X\ [f]^*\ y\wedge y\ [g]^*\ Z)$ for $f\in\mathbf{Rel}(A,B)$, $g\in\mathbf{Rel}(B,C)$ (for sets A, B, C).

Proposition 386. $f\circ\bigcup G = \bigcup_{g\in G}(f\circ g)$ and $\bigcup G\circ f = \bigcup_{g\in G}(g\circ f)$ for every binary relation f and set G of binary relations.

Proof. We will prove only $\bigcup G\circ f = \bigcup_{g\in G}(g\circ f)$ as the other formula follows from duality. Really

$$(x,z)\in\bigcup G\circ f \Leftrightarrow \exists y : ((x,y)\in f\wedge(y,z)\in\bigcup G) \Leftrightarrow \\ \exists y, g\in G : ((x,y)\in f\wedge(y,z)\in g) \Leftrightarrow \exists g\in G : (x,z)\in g\circ f \Leftrightarrow (x,z)\in\bigcup_{g\in G}(g\circ f).$$

□

Corollary 387. Every **Rel**-morphism is metacomplete and co-metacomplete.

Proposition 388. The following are equivalent for a **Rel**-morphism f:

1°. f is monovalued.
2°. f is metamonovalued.
3°. f is weakly metamonovalued.
4°. $\langle f\rangle^* a$ is either atomic or least whenever $a\in\operatorname{atoms}^{\mathscr{T}\operatorname{Src} f}$.
5°. $\left\langle f^{-1}\right\rangle^*(I\sqcap J) = \left\langle f^{-1}\right\rangle^* I\sqcap\left\langle f^{-1}\right\rangle^* J$ for every $I,J\in\mathscr{T}\operatorname{Src} f$.
6°. $\left\langle f^{-1}\right\rangle^*\bigsqcap S = \bigsqcap_{Y\in S}\left\langle f^{-1}\right\rangle^* Y$ for every $S\in\mathscr{P}\mathscr{T}\operatorname{Src} f$.

Proof.

2°⇒3°. Obvious.
1°⇒2°. Take $x\in\operatorname{atoms}^{\mathscr{T}\operatorname{Src} f}$; then $fx\in\operatorname{atoms}^{\mathscr{T}\operatorname{Dst} f}\cup\{\bot^{\mathscr{T}\operatorname{Dst} f}\}$ and thus

$$\left\langle\left(\bigsqcap G\right)\circ f\right\rangle^* x = \left\langle\bigsqcap G\right\rangle^*\langle f\rangle^* x = \bigsqcap_{g\in G}\langle g\rangle^*\langle f\rangle^* x = \\ \bigsqcap_{g\in G}\langle g\circ f\rangle^* x = \left\langle\bigsqcap_{g\in G}(g\circ f)\right\rangle^* x;$$

so $(\bigsqcap G)\circ f = \bigsqcap_{g\in G}(g\circ f)$.
3°⇒1°. Take $g=\{(a,y)\}$ and $h=\{(b,y)\}$ for arbitrary $a\neq b$ and arbitrary y. We have $g\cap h=\emptyset$; thus $(g\circ f)\cap(h\circ f) = (g\cap h)\circ f = \bot$ and thus impossible $x\ f\ a\wedge x\ f\ b$ as otherwise $(x,y)\in(g\circ f)\cap(h\circ f)$. Thus f is monovalued.

4°⇒6°. Let $a \in \operatorname{atoms}^{\mathscr{T} \operatorname{Src} f}$, $\langle f\rangle^* a = b$. Then because $b \in \operatorname{atoms}^{\mathscr{T} \operatorname{Dst} f} \cup \{\bot^{\mathscr{T} \operatorname{Dst} f}\}$

$$\bigsqcap S \sqcap b \neq \bot \Leftrightarrow \forall Y \in S : Y \sqcap b \neq \bot;$$

$$a \, [f]^* \bigsqcap S \Leftrightarrow \forall Y \in S : a \, [f]^* \, Y;$$

$$\bigsqcap S \, \left[f^{-1}\right]^* a \Leftrightarrow \forall Y \in S : Y \, \left[f^{-1}\right]^* a;$$

$$a \not\asymp \left\langle f^{-1}\right\rangle^* \bigsqcap S \Leftrightarrow \forall Y \in S : a \not\asymp \left\langle f^{-1}\right\rangle^* Y;$$

$$a \not\asymp \left\langle f^{-1}\right\rangle^* \bigsqcap S \Leftrightarrow a \not\asymp \bigsqcap_{Y \in S} \left\langle f^{-1}\right\rangle^* Y;$$

$$\left\langle f^{-1}\right\rangle^* \bigsqcap S = \bigsqcap_{X \in S} \left\langle f^{-1}\right\rangle^* X.$$

6°⇒5°. Obvious.

5°⇒1°. $\left\langle f^{-1}\right\rangle^* a \sqcap \left\langle f^{-1}\right\rangle^* b = \left\langle f^{-1}\right\rangle^* (a \sqcap b) = \left\langle f^{-1}\right\rangle^* \bot = \bot$ for every two distinct atoms $a = \{\alpha\}, b = \{\beta\} \in \mathscr{T} \operatorname{Dst} f$. From this

$$\alpha \left(f \circ f^{-1}\right) \beta \Leftrightarrow \exists y \in \operatorname{Dst} f : \left(\alpha \; f^{-1} \; y \wedge y \; f \; \beta\right) \Leftrightarrow$$
$$\exists y \in \operatorname{Dst} f : \left(y \in \left\langle f^{-1}\right\rangle^* a \wedge y \in \left\langle f^{-1}\right\rangle^* b\right)$$

is impossible. Thus $f \circ f^{-1} \sqsubseteq 1^{\mathbf{Rel}}_{\operatorname{Dst} f}$.

¬4°⇒ ¬1°. Suppose $\langle f\rangle^* a \notin \operatorname{atoms}^{\mathscr{T} \operatorname{Dst} f} \cup \{\bot^{\mathscr{T} \operatorname{Dst} f}\}$ for some $a \in \operatorname{atoms}^{\mathscr{T} \operatorname{Src} f}$. Then there exist distinct points p, q such that $p, q \in \langle f\rangle^* a$. Thus $p \left(f \circ f^{-1}\right) q$ and so $f \circ f^{-1} \not\sqsubseteq 1^{\mathbf{Rel}}_{\operatorname{Dst} f}$.

□

4.4. Product of typed sets

Definition 389. Product of typed sets is defined by the formula

$$(\mathscr{P} U, A) \times (\mathscr{P} W, B) = (U, W, A \times B).$$

Proposition 390. Product of typed sets is a **Rel**-morphism.

Proof. We need to prove $A \times B \subseteq U \times W$, but this is obvious. □

Obvious 391. Atoms of $\mathbf{Rel}(A, B)$ are exactly products $a \times b$ where a and b are atoms correspondingly of $\mathscr{T} A$ and $\mathscr{T} B$. $\mathbf{Rel}(A, B)$ is an atomistic poset.

Proposition 392. $f \not\asymp A \times B \Leftrightarrow A \, [f]^* \, B$ for every **Rel**-morphism f and $A \in \mathscr{T} \operatorname{Src} f$, $B \in \mathscr{T} \operatorname{Dst} f$.

Proof.

$$A \, [f]^* \, B \Leftrightarrow \exists x \in \operatorname{atoms} A, y \in \operatorname{atoms} B : x \, [f]^* \, y \Leftrightarrow$$
$$\exists x \in \operatorname{atoms}^{\mathscr{T} \operatorname{Src} f}, y \in \operatorname{atoms}^{\mathscr{T} \operatorname{Dst} f} : (x \times y \sqsubseteq f \wedge x \times y \sqsubseteq A \times B) \Leftrightarrow f \not\asymp A \times B.$$

□

Definition 393. *Image* and *domain* of a **Rel**-morphism f are typed sets defined by the formulas

$$\operatorname{dom}(U, W, f) = (\mathscr{P} U, \operatorname{dom} f) \quad \text{and} \quad \operatorname{im}(U, W, f) = (\mathscr{P} W, \operatorname{im} f).$$

Obvious 394. Image and domain of a **Rel**-morphism are really typed sets.

Definition 395. *Restriction* of a **Rel**-morphism to a typed set is defined by the formula $(U, W, f)|_{(\mathscr{P} U, X)} = (U, W, f|_X)$.

OBVIOUS 396. Restriction of a **Rel**-morphism is **Rel**-morphism.

OBVIOUS 397. $f|_A = f \sqcap (A \times \top^{\mathscr{T} \operatorname{Dst} f})$ for every **Rel**-morphism f and $A \in \mathscr{T} \operatorname{Src} f$.

OBVIOUS 398. $\langle f \rangle^* X = \langle f \rangle^* (X \sqcap \operatorname{dom} f) = \operatorname{im}(f|_X)$ for every **Rel**-morphism f and $X \in \mathscr{T} \operatorname{Src} f$.

OBVIOUS 399. $f \sqsubseteq A \times B \Leftrightarrow \operatorname{dom} f \sqsubseteq A \wedge \operatorname{im} f \sqsubseteq B$ for every **Rel**-morphism f and $A \in \mathscr{T} \operatorname{Src} f$, $B \in \mathscr{T} \operatorname{Dst} f$.

THEOREM 400. Let A, B be sets. If $S \in \mathscr{P}(\mathscr{T} A \times \mathscr{T} B)$ then

$$\bigsqcap_{(A,B) \in S} (A \times B) = \bigsqcap \operatorname{dom} S \times \bigsqcap \operatorname{im} S.$$

PROOF. For every atomic $x \in \mathscr{T} A$, $y \in \mathscr{T} B$ we have

$$x \times y \sqsubseteq \bigsqcap_{(A,B) \in S} (A \times B) \Leftrightarrow \forall (A,B) \in S : x \times y \sqsubseteq A \times B \Leftrightarrow$$
$$\forall (A,B) \in S : (x \sqsubseteq A \wedge y \sqsubseteq B) \Leftrightarrow \forall A \in \operatorname{dom} S : x \sqsubseteq A \wedge \forall B \in \operatorname{im} S : y \sqsubseteq B \Leftrightarrow$$
$$x \sqsubseteq \bigsqcap \operatorname{dom} S \wedge y \sqsubseteq \bigsqcap \operatorname{im} S \Leftrightarrow x \times y \sqsubseteq \bigsqcap \operatorname{dom} S \times \bigsqcap \operatorname{im} S.$$

□

OBVIOUS 401. If U, W are sets and $A \in \mathscr{T}(U)$ then $A \times$ is a complete homomorphism from the lattice $\mathscr{T}(W)$ to the lattice $\mathbf{Rel}(U, W)$, if also $A \neq \bot$ then it is an order embedding.

CHAPTER 5

Filters and filtrators

This chapter is based on my article [30].

This chapter is grouped in the following way:

- First it goes a short introduction in pedagogical order (first less general stuff and examples, last the most general stuff):
  - filters on a set;
  - filters on a meet-semilattice;
  - filters on a poset.
- Then it goes the formal part.

5.1. Implication tuples

DEFINITION 402. An *implications tuple* is a tuple $(P_1, \ldots, P_n)$ such that $P_1 \Rightarrow \ldots \Rightarrow P_n$.

OBVIOUS 403. $(P_1, \ldots, P_n)$ is an implications tuple iff $P_i \Rightarrow P_j$ for every $i < j$ (where $i, j \in \{1, \ldots, n\}$).

The following is an example of a theorem using an implication tuple:

EXAMPLE 404. The following is an implications tuple:

1°. A.
2°. B.
3°. C.

This example means just that $A \Rightarrow B \Rightarrow C$.

I prefer here a verbal description instead of symbolic implications $A \Rightarrow B \Rightarrow C$, because A, B, C may be long English phrases and they may not fit into the formula layout.

The main (intuitive) idea of the theorem is expressed by the implication $P_1 \Rightarrow P_n$, the rest implications ($P_2 \Rightarrow P_n$, $P_3 \Rightarrow P_n$, ...) are purely technical, as they express generalizations of the main idea.

For uniformity theorems in the section about filters and filtrators start with the same P_1: "$(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator." (defined below) That means that the main idea of the theorem is about powerset filtrators, the rest implications (like $P_2 \Rightarrow P_n$, $P_3 \Rightarrow P_n$, ...) are just technical generalizations.

5.2. Introduction to filters and filtrators

5.2.1. Filters on a set. We sometimes want to define something resembling an infinitely small (or infinitely big) set, for example the infinitely small interval near 0 on the real line. Of course there is no such set, just like as there is no natural number which is the difference $2 - 3$. To overcome this shortcoming we introduce whole numbers, and $2-3$ becomes well defined. In the same way to consider things which are like infinitely small (or infinitely big) sets we introduce *filters*.

An example of a filter is the infinitely small interval near 0 on the real line. To come to infinitely small, we consider all intervals $]-\epsilon; \epsilon[$ for all $\epsilon > 0$. This filter

consists of all intervals $]-\epsilon;\epsilon[$ for all $\epsilon > 0$ and also all subsets of $\mathbb{R}$ containing such intervals as subsets. Informally speaking, this is the greatest filter contained in every interval $]-\epsilon;\epsilon[$ for all $\epsilon > 0$.

Definition 405. A filter on a set $\mho$ is a $\mathcal{F} \in \mathscr{P}\mathscr{P}\mho$ such that:

1°. $\forall A, B \in \mathcal{F} : A \cap B \in \mathcal{F}$;
2°. $\forall A, B \in \mathscr{P}\mho : (A \in \mathcal{F} \wedge B \supseteq A \Rightarrow B \in \mathcal{F})$.

Exercise 406. Verify that the above introduced infinitely small interval near 0 on the real line is a filter on $\mathbb{R}$.

Exercise 407. Describe "the neighborhood of positive infinity" filter on $\mathbb{R}$.

Definition 408. A filter not containing empty set is called a *proper filter*.

Obvious 409. The non-proper filter is $\mathscr{P}\mho$.

Remark 410. Some other authors require that all filters are proper. This is a stupid idea and we allow non-proper filters, in the same way as we allow to use the number 0.

5.2.2. Intro to filters on a meet-semilattice. A trivial generalization of the above:

Definition 411. A filter on a meet-semilattice $\mathfrak{Z}$ is a $\mathcal{F} \in \mathscr{P}\mathfrak{Z}$ such that:

1°. $\forall A, B \in \mathcal{F} : A \sqcap B \in \mathcal{F}$;
2°. $\forall A, B \in \mathfrak{Z} : (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

5.2.3. Intro to filters on a poset.

Definition 412. A filter on a poset $\mathfrak{Z}$ is a $\mathcal{F} \in \mathscr{P}\mathfrak{Z}$ such that:

1°. $\forall A, B \in \mathcal{F} \exists C \in \mathcal{F} : C \sqsubseteq A, B$;
2°. $\forall A, B \in \mathfrak{Z} : (A \in \mathcal{F} \wedge B \sqsupseteq A \Rightarrow B \in \mathcal{F})$.

It is easy to show (and there is a proof of it somewhere below) that this coincides with the above definition in the case if $\mathfrak{Z}$ is a meet-semilattice.

5.3. Filters on a poset

5.3.1. Filters on posets. Let $\mathfrak{Z}$ be a poset.

Definition 413. *Filter base* is a nonempty subset F of $\mathfrak{Z}$ such that

$$\forall X, Y \in F \exists Z \in F : (Z \sqsubseteq X \wedge Z \sqsubseteq Y).$$

Definition 414. *Ideal base* is a nonempty subset F of $\mathfrak{Z}$ such that

$$\forall X, Y \in F \exists Z \in F : (Z \sqsupseteq X \wedge Z \sqsupseteq Y).$$

Obvious 415. Ideal base is the dual of filter base.

Obvious 416.

1°. A poset with a lowest element is a filter base.
2°. A poset with a greatest element is an ideal base.

Obvious 417.

1°. A meet-semilattice is a filter base.
2°. A join-semilattice is an ideal base.

Obvious 418. A nonempty chain is a filter base and an ideal base.

Definition 419. *Filter* is a subset of $\mathfrak{Z}$ which is both a filter base and an upper set.

I will denote the set of filters (for a given or implied poset $\mathfrak{Z}$) as $\mathfrak{F}$ and call $\mathfrak{F}$ the set of filters over the poset $\mathfrak{Z}$.

PROPOSITION 420. If $\top$ is the maximal element of $\mathfrak{Z}$ then $\top \in F$ for every filter F.

PROOF. If $\top \notin F$ then $\forall K \in \mathfrak{Z} : K \notin F$ and so F is empty what is impossible. □

PROPOSITION 421. Let S be a filter base on a poset. If $A_0, \ldots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S : (C \sqsubseteq A_0 \wedge \ldots \wedge C \sqsubseteq A_n).$$

PROOF. It can be easily proved by induction. □

DEFINITION 422. A function f from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ *preserves filtered meets* iff whenever $\bigsqcap S$ is defined for a filter base S on $\mathfrak{A}$ we have $f \bigsqcap S = \bigsqcap \langle f \rangle^* S$.

5.3.2. Filters on meet-semilattices.

THEOREM 423. If $\mathfrak{Z}$ is a meet-semilattice and F is a nonempty subset of $\mathfrak{Z}$ then the following conditions are equivalent:

1°. F is a filter.
2°. $\forall X, Y \in F : X \sqcap Y \in F$ and F is an upper set.
3°. $\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Leftrightarrow X \sqcap Y \in F)$.

PROOF.

1°⇒2°. Let F be a filter. Then F is an upper set. If $X, Y \in F$ then $Z \sqsubseteq X \wedge Z \sqsubseteq Y$ for some $Z \in F$. Because F is an upper set and $Z \sqsubseteq X \sqcap Y$ then $X \sqcap Y \in F$.

2°⇒1°. Let $\forall X, Y \in F : X \sqcap Y \in F$ and F be an upper set. We need to prove that F is a filter base. But it is obvious taking $Z = X \sqcap Y$ (we have also taken into account that $F \neq \emptyset$).

2°⇒3°. Let $\forall X, Y \in F : X \sqcap Y \in F$ and F be an upper set. Then

$$\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Rightarrow X \sqcap Y \in F).$$

Let $X \sqcap Y \in F$; then $X, Y \in F$ because F is an upper set.

3°⇒2°. Let

$$\forall X, Y \in \mathfrak{Z} : (X, Y \in F \Leftrightarrow X \sqcap Y \in F).$$

Then $\forall X, Y \in F : X \sqcap Y \in F$. Let $X \in F$ and $X \sqsubseteq Y \in \mathfrak{Z}$. Then $X \sqcap Y = X \in F$. Consequently $X, Y \in F$. So F is an upper set.

□

PROPOSITION 424. Let S be a filter base on a meet-semilattice. If $A_0, \ldots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S : C \sqsubseteq A_0 \sqcap \cdots \sqcap A_n.$$

PROOF. It can be easily proved by induction. □

PROPOSITION 425. If $\mathfrak{Z}$ is a meet-semilattice and S is a filter base on it, $A \in \mathfrak{Z}$, then $\langle A \sqcap \rangle^* S$ is also a filter base.

PROOF. $\langle A \sqcap \rangle^* S \neq \emptyset$ because $S \neq \emptyset$.

Let $X, Y \in \langle A \sqcap \rangle^* S$. Then $X = A \sqcap X'$ and $Y = A \sqcap Y'$ where $X', Y' \in S$. There exists $Z' \in S$ such that $Z' \sqsubseteq X' \sqcap Y'$. So $X \sqcap Y = A \sqcap X' \sqcap Y' \sqsupseteq A \sqcap Z' \in \langle A \sqcap \rangle^* S$. □

5.3.3. Order of filters. Principal filters. I will make the set of filters $\mathfrak{F}$ into a poset by the order defined by the formula: $a \sqsubseteq b \Leftrightarrow a \supseteq b$.

DEFINITION 426. The principal filter corresponding to an element $a \in \mathfrak{Z}$ is

$$\uparrow a = \left\{ \frac{x \in \mathfrak{Z}}{x \sqsupseteq a} \right\}.$$

Elements of $\mathfrak{P} = \langle \uparrow \rangle^* \mathfrak{Z}$ are called *principal filters*.

OBVIOUS 427. Principal filters are filters.

OBVIOUS 428. $\uparrow$ is an order embedding from $\mathfrak{Z}$ to $\mathfrak{F}$.

COROLLARY 429. $\uparrow$ is an order isomorphism between $\mathfrak{Z}$ and $\mathfrak{P}$.

We will equate principal filters with corresponding elements of the base poset (in the same way as we equate for example nonnegative whole numbers and natural numbers).

PROPOSITION 430. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$.

PROOF. $\uparrow K \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow K \subseteq \mathcal{A} \Leftrightarrow K \in \mathcal{A}$. □

5.4. Filters on a Set

Consider filters on the poset $\mathfrak{Z} = \mathscr{P}\mho$ (where $\mho$ is some fixed set) with the order $A \sqsubseteq B \Leftrightarrow A \subseteq B$ (for $A, B \in \mathscr{P}\mathfrak{A}$).

In fact, it is a complete atomistic boolean lattice with $\bigsqcap S = \bigcap S$, $\bigsqcup S = \bigcup S$, $\overline{A} = \mho \setminus A$ for every $S \in \mathscr{P}\mathscr{P}\mho$ and $A \in \mathscr{P}\mho$, atoms being one-element sets.

DEFINITION 431. I will call a filter on the lattice of all subsets of a given set $\mho$ as a *filter on set*.

DEFINITION 432. I will denote the set on which a filter $\mathcal{F}$ is defined as $\operatorname{Base}(\mathcal{F})$.

OBVIOUS 433. $\operatorname{Base}(\mathcal{F}) = \bigcup \mathcal{F}$.

PROPOSITION 434. The following are equivalent for a non-empty set $F \in \mathscr{P}\mathscr{P}\mho$:

1°. F is a filter.
2°. $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set.
3°. $\forall X, Y \in \mathscr{P}\mho : (X, Y \in F \Leftrightarrow X \cap Y \in F)$.

PROOF. By theorem 423. □

OBVIOUS 435. The minimal filter on $\mathscr{P}\mho$ is $\mathscr{P}\mho$.

OBVIOUS 436. The maximal filter on $\mathscr{P}\mho$ is $\{\mho\}$.

I will denote $\uparrow A = \uparrow^{\mho} A = \uparrow^{\mathscr{P}\mho} A$. (The distinction between conflicting notations $\uparrow^{\mho} A$ and $\uparrow^{\mathscr{P}\mho} A$ will be clear from the context.)

PROPOSITION 437. Every filter on a finite set is principal.

PROOF. Let $\mathcal{F}$ be a filter on a finite set. Then obviously $\mathcal{F} = \bigsqcap^{\mathfrak{Z}} \operatorname{up} \mathcal{F}$ and thus $\mathcal{F}$ is principal. □

5.5. Filtrators

$(\mathfrak{F}, \mathfrak{P})$ is a poset and its subset (with induced order on the subset). I call pairs of a poset and its subset like this *filtrators*.

Definition 438. I will call a *filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a poset $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$. I call $\mathfrak{A}$ the *base* of the filtrator and $\mathfrak{Z}$ the *core* of the filtrator. I will also say that $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator *over* poset $\mathfrak{Z}$.

I will denote $\operatorname{base}(\mathfrak{A}, \mathfrak{Z}) = \mathfrak{A}$, $\operatorname{core}(\mathfrak{A}, \mathfrak{Z}) = \mathfrak{Z}$ for a filtrator $(\mathfrak{A}, \mathfrak{Z})$.

While *filters* are customary and well known mathematical objects, the concept of *filtrators* is probably first researched by me.

When speaking about filters, we will imply that we consider the filtrator $(\mathfrak{F}, \mathfrak{P})$ or what is the same (as we equate principal filters with base elements) the filtrator $(\mathfrak{F}, \mathfrak{Z})$.

Definition 439. I will call a *lattice filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a lattice $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 440. I will call a *complete lattice filtrator* a pair $(\mathfrak{A}, \mathfrak{Z})$ of a complete lattice $\mathfrak{A}$ and its subset $\mathfrak{Z} \subseteq \mathfrak{A}$.

Definition 441. I will call a *central filtrator* a filtrator $(\mathfrak{A}, Z(\mathfrak{A}))$ where $Z(\mathfrak{A})$ is the center of a bounded lattice $\mathfrak{A}$.

Definition 442. I will call *element* of a filtrator an element of its base.

Definition 443. $\operatorname{up}^{\mathfrak{Z}} a = \operatorname{up} a = \left\{ \frac{c \in \mathfrak{Z}}{c \sqsupseteq a} \right\}$ for an element a of a filtrator.

Definition 444. $\operatorname{down}^{\mathfrak{Z}} a = \operatorname{down} a = \left\{ \frac{c \in \mathfrak{Z}}{c \sqsubseteq a} \right\}$ for an element a of a filtrator.

Obvious 445. “up” and “down” are dual.

Our main purpose here is knowing properties of the core of a filtrator to infer properties of the base of the filtrator, specifically properties of $\operatorname{up} a$ for every element a.

Definition 446. I call a filtrator *with join-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\bigsqcup^{\mathfrak{Z}} S = \bigsqcup^{\mathfrak{A}} S$ whenever $\bigsqcup^{\mathfrak{Z}} S$ exists for $S \in \mathscr{P}\mathfrak{Z}$.

Definition 447. I call a filtrator *with meet-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\bigsqcap^{\mathfrak{Z}} S = \bigsqcap^{\mathfrak{A}} S$ whenever $\bigsqcap^{\mathfrak{Z}} S$ exists for $S \in \mathscr{P}\mathfrak{Z}$.

Definition 448. I call a filtrator with *binarily join-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $a \sqcup^{\mathfrak{Z}} b = a \sqcup^{\mathfrak{A}} b$ whenever $a \sqcup^{\mathfrak{Z}} b$ exists for $a, b \in \mathfrak{Z}$.

Definition 449. I call a filtrator with *binarily meet-closed core* such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $a \sqcap^{\mathfrak{Z}} b = a \sqcap^{\mathfrak{A}} b$ whenever $a \sqcap^{\mathfrak{Z}} b$ exists for $a, b \in \mathfrak{Z}$.

Definition 450. *Prefiltered filtrator* is a filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that “up” is injective.

Definition 451. *Filtered filtrator* is a filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that

$$\forall a, b \in \mathfrak{A} : (\operatorname{up} a \supseteq \operatorname{up} b \Rightarrow a \sqsubseteq b).$$

Theorem 452. A filtrator $(\mathfrak{A}, \mathfrak{Z})$ is filtered iff $\forall a \in \mathfrak{A} : a = \bigsqcap^{\mathfrak{A}} \operatorname{up} a$.

Proof.

$\Leftarrow$. $\operatorname{up} a \supseteq \operatorname{up} b \Rightarrow \bigsqcap^{\mathfrak{A}} \operatorname{up} a \sqsubseteq \bigsqcap^{\mathfrak{A}} \operatorname{up} b \Rightarrow a \sqsubseteq b$.

⇒. $a = \bigsqcap^{\mathfrak{A}} \operatorname{up} a$ is equivalent to a is a greatest lower bound of $\operatorname{up} a$. That is the implication that b is lower bound of $\operatorname{up} a$ implies $a \sqsupseteq b$.

b is lower bound of $\operatorname{up} a$ implies $\operatorname{up} b \supseteq \operatorname{up} a$. So as it is filtered $a \sqsupseteq b$.

□

OBVIOUS 453. Every filtered filtrator is prefiltered.

OBVIOUS 454. "up" is a straight map from $\mathfrak{A}$ to the dual of the poset $\mathscr{P}\mathfrak{Z}$ if $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.

DEFINITION 455. An *isomorphism* between filtrators $(\mathfrak{A}_0, \mathfrak{Z}_0)$ and $(\mathfrak{A}_1, \mathfrak{Z}_1)$ is an isomorphism between posets $\mathfrak{A}_0$ and $\mathfrak{A}_1$ such that it maps $\mathfrak{Z}_0$ into $\mathfrak{Z}_1$.

OBVIOUS 456. Isomorphism isomorphically maps the order on $\mathfrak{Z}_0$ into order on $\mathfrak{Z}_1$.

DEFINITION 457. Two filtrators are *isomorphic* when there exists an isomorphism between them.

DEFINITION 458. I will call *primary filtrator* a filtrator isomorphic to the filtrator consisting of the set of filters on a poset and the set of principal filters on this poset.

OBVIOUS 459. The order on a primary filtrator is defined by the formula $a \sqsubseteq b \Leftrightarrow \operatorname{up} a \supseteq \operatorname{up} b$.

DEFINITION 460. I will call a primary filtrator over a poset isomorphic to a powerset as *powerset filtrator*.

OBVIOUS 461. $\operatorname{up} \mathcal{F}$ is a filter for every element $\mathcal{F}$ of a primary filtrator. Reversely, there exists a filter $\mathcal{F}$ if $\operatorname{up} \mathcal{F}$ is a filter.

THEOREM 462. For every poset $\mathfrak{Z}$ there exists a poset $\mathfrak{A} \supseteq \mathfrak{Z}$ such that $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.

PROOF. See appendix A. □

5.5.1. Filtrators with Separable Core.

DEFINITION 463. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with separable core* when

$$\forall x, y \in \mathfrak{A} : (x \asymp^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{up} x : X \asymp^{\mathfrak{A}} y).$$

PROPOSITION 464. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with separable core* iff

$$\forall x, y \in \mathfrak{A} : (x \asymp^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{up} x, Y \in \operatorname{up} y : X \asymp^{\mathfrak{A}} Y).$$

PROOF.

⇒. Apply the definition twice.

⇐. Obvious.

□

DEFINITION 465. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with co-separable core* when

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{down} x : X \equiv^{\mathfrak{A}} y).$$

OBVIOUS 466. Co-separability is the dual of separability.

DEFINITION 467. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. It is a *filtrator with co-separable core* when

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \operatorname{down} x, Y \in \operatorname{down} y : X \equiv^{\mathfrak{A}} Y).$$

PROOF. By duality. □

5.6. Alternative primary filtrators

5.6.1. Lemmas.

LEMMA 468. A set F is a lower set iff $\overline{F}$ is an upper set.

PROOF. $X \in \overline{F} \wedge Z \sqsupseteq X \Rightarrow Z \in \overline{F}$ is equivalent to $Z \in F \Rightarrow X \in F \vee Z \not\sqsupseteq X$ is equivalent $Z \in F \Rightarrow (Z \sqsupseteq X \Rightarrow X \in F)$ is equivalent $Z \in F \wedge X \sqsubseteq Z \Rightarrow X \in F$. □

PROPOSITION 469. Let $\mathfrak{Z}$ be a poset with least element $\bot$. Then for upper set F we have $F \neq \mathscr{P}\mathfrak{Z} \Leftrightarrow \bot \notin F$.

PROOF.
$\Rightarrow$. If $\bot \in F$ then $F = \mathscr{P}\mathfrak{Z}$ because F is an upper set.
$\Leftarrow$. Obvious.

□

5.6.2. Informal introduction. We have already defined filters on a poset. Now we will define three other sets which are order-isomorphic to the set of filters on a poset: ideals ($\mathfrak{I}$), free stars ($\mathfrak{S}$), and mixers ($\mathfrak{M}$).

These four kinds of objects are related through commutative diagrams. First we will paint an informal commutative diagram (it makes no formal sense because it is not pointed the poset for which the filters are defined):

$$\begin{tikzcd} \mathfrak{F} \arrow[r, leftrightarrow, "\langle \mathrm{dual} \rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathfrak{I} \arrow[d, leftrightarrow, "\neg"] \\ \mathfrak{M} \arrow[r, leftrightarrow, "\langle \mathrm{dual} \rangle^*"] & \mathfrak{S} \end{tikzcd}$$

Then we can define ideals, free stars, and mixers as sets following certain formulas. You can check that the intuition behind these formulas follows the above commutative diagram. (That is transforming these formulas by the course of the above diagram, you get formulas of the other objects in this list.)

After this, we will paint some formal commutative diagrams similar to the above diagram but with particular posets at which filters, ideals, free stars, and mixers are defined.

5.6.3. Definitions of ideals, free stars, and mixers. *Filters* and *ideals* are well known concepts. The terms *free stars* and *mixers* are my new terminology.

Recall that *filters* are nonempty sets F with $A, B \in F \Leftrightarrow \exists Z \in F : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 470. *Ideals* are nonempty sets F with $A, B \in F \Leftrightarrow \exists Z \in F : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 471. *Free stars* are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Leftrightarrow \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

DEFINITION 472. *Mixers* are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Leftrightarrow \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

By duality and and an above theorem about filters, we have:

PROPOSITION 473.

- Filters are nonempty upper sets F with $A, B \in F \Rightarrow \exists Z \in F : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Ideals are nonempty lower sets F with $A, B \in F \Rightarrow \exists Z \in F : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

- Free stars are upper sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Rightarrow \exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are lower sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A, B \in \overline{F} \Rightarrow \exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

Proposition 474. The following are equivalent:

1°. F is a free star.
2°. $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and $F \neq \mathscr{P}\mathfrak{Z}$.
3°. $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Rightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and F is an upper set and $F \neq \mathscr{P}\mathfrak{Z}$.

Proof.

1°⇔2°. The following is a chain of equivalencies:

$$\exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B) \Leftrightarrow A \notin F \wedge B \notin F;$$
$$\forall Z \in \overline{F} : \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B) \Leftrightarrow A \in F \vee B \in F;$$
$$\forall Z \in \mathfrak{Z} : (Z \notin F \Rightarrow \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B)) \Leftrightarrow A \in F \vee B \in F;$$
$$\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F.$$

2°⇒3°. Let $A = B \in F$. Then $A \in F \vee B \in F$. So $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F)$ that is $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \Rightarrow Z \in F)$ that is F is an upper set.
3°⇒2°. We need to prove that F is an upper set. let $A \in F$ and $A \sqsubseteq B \in \mathfrak{Z}$. Then $A \in F \vee B \in F$ and thus $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F)$ that is $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq B \Rightarrow Z \in F)$ and so $B \in F$.

□

Corollary 475. The following are equivalent:

1°. F is a mixer.
2°. $\forall Z \in \mathfrak{Z} : (Z \sqsubseteq A \wedge Z \sqsubseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and $F \neq \mathscr{P}\mathfrak{Z}$.
3°. $\forall Z \in \mathfrak{Z} : (Z \sqsubseteq A \wedge Z \sqsubseteq B \Rightarrow Z \in F) \Rightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{Z}$ and F is an lower set and $F \neq \mathscr{P}\mathfrak{Z}$.

Obvious 476.

1°. A free star cannot contain the least element of the poset.
2°. A mixer cannot contain the greatest element of the poset.

5.6.4. Filters, ideals, free stars, and mixers on semilattices.

Proposition 477.

- Free stars are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg\exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Free stars are upper sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftarrow \neg\exists Z \in \overline{F} : (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftrightarrow \neg\exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).
- Mixers are lower sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \in F \vee B \in F \Leftarrow \neg\exists Z \in \overline{F} : (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

Proof. By duality. □

By duality and and an above theorem about filters, we have:

Proposition 478.

- Filters are nonempty sets F with $A \sqcap B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a meet-semilattice.
- Ideals are nonempty sets F with $A \sqcup B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a join-semilattice.
- Free stars are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \sqcup B \in F \Leftrightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a join-semilattice.
- Mixers are sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \sqcap B \in F \Leftrightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a meet-semilattice.

By duality and and an above theorem about filters, we have:

PROPOSITION 479.

- Filters are nonempty upper sets F with $A \sqcap B \in F \Leftarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a meet-semilattice.
- Ideals are nonempty lower sets F with $A \sqcup B \in F \Leftarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a join-semilattice.
- Free stars are upper sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \sqcup B \in F \Rightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a join-semilattice.
- Mixers are lower sets F not equal to $\mathscr{P}\mathfrak{Z}$ with $A \sqcap B \in F \Rightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$), whenever $\mathfrak{Z}$ is a meet-semilattice.

5.6.5. The general diagram. Let $\mathfrak{A}$ and $\mathfrak{B}$ be two posets connected by an order reversing isomorphism $\theta : \mathfrak{A} \to \mathfrak{B}$. We have commutative diagram on the figure 1 in the category **Set**:

$$\begin{tikzcd}
\mathscr{P}\mathfrak{A} \arrow[r, shift left, "\langle\theta\rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathscr{P}\mathfrak{B} \arrow[l, shift left, "\langle\theta^{-1}\rangle^*"] \arrow[d, leftrightarrow, "\neg"] \\
\mathscr{P}\mathfrak{A} \arrow[r, shift left, "\langle\theta\rangle^*"] & \mathscr{P}\mathfrak{B} \arrow[l, shift left, "\langle\theta^{-1}\rangle^*"]
\end{tikzcd}$$

FIGURE 3

THEOREM 480. This diagram is commutative, every arrow of this diagram is an isomorphism, every cycle in this diagrams is an identity (therefore "parallel" arrows are mutually inverse).

PROOF. That every arrow is an isomorphism is obvious.
Show that $\langle\theta\rangle^*\neg X = \neg\langle\theta\rangle^* X$ for every set $X \in \mathscr{P}\mathfrak{A}$.
Really,

$$p \in \langle\theta\rangle^*\neg X \Leftrightarrow \exists q \in \neg X : p = \theta q \Leftrightarrow \exists q \in \neg X : \theta^{-1}p = q \Leftrightarrow \theta^{-1}p \in \neg X \Leftrightarrow \\ \nexists q \in X : q = \theta^{-1}p \Leftrightarrow \nexists q \in X : \theta q = p \Leftrightarrow p \notin \langle\theta\rangle^* X \Leftrightarrow p \in \neg\langle\theta\rangle^* X.$$

Thus the theorem follows from lemma 197. □

This diagram can be restricted to filters, ideals, free stars, and mixers, see figure 2:

THEOREM 481. It is a restriction of the above diagram. Every arrow of this diagram is an isomorphism, every cycle in these diagrams is an identity. (To prove that, is an easy application of duality and the above lemma.)

$$\begin{tikzcd}
\mathfrak{F}(\mathfrak{A}) \arrow[r, shift left, "\langle\theta\rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathfrak{I}(\mathfrak{B}) \arrow[l, shift left, "\langle\theta^{-1}\rangle^*"] \arrow[d, leftrightarrow, "\neg"] \\
\mathfrak{M}(\mathfrak{A}) \arrow[r, shift left, "\langle\theta\rangle^*"] & \mathfrak{S}(\mathfrak{B}) \arrow[l, shift left, "\langle\theta^{-1}\rangle^*"]
\end{tikzcd}$$

FIGURE 4

5.6.6. Special diagrams. Here are two important special cases of the above diagram:

$$\begin{tikzcd}
\mathfrak{F}(\mathfrak{A}) \arrow[r, leftrightarrow, "\langle\mathrm{dual}\rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathfrak{I}(\mathrm{dual}\,\mathfrak{A}) \arrow[d, leftrightarrow, "\neg"] \\
\mathfrak{M}(\mathfrak{A}) \arrow[r, leftrightarrow, "\langle\mathrm{dual}\rangle^*"] & \mathfrak{S}(\mathrm{dual}\,\mathfrak{A})
\end{tikzcd}
\quad\text{and}\quad
\begin{tikzcd}
\mathfrak{F}(\mathfrak{A}) \arrow[r, leftrightarrow, "\langle\neg\rangle^*"] \arrow[d, leftrightarrow, "\neg"] & \mathfrak{I}(\mathfrak{A}) \arrow[d, leftrightarrow, "\neg"] \\
\mathfrak{M}(\mathfrak{A}) \arrow[r, leftrightarrow, "\langle\neg\rangle^*"] & \mathfrak{S}(\mathfrak{A})
\end{tikzcd} \tag{1}$$

(the second diagram is defined for a boolean lattice $\mathfrak{A}$).

5.6.7. Order of ideals, free stars, mixers. Define order of ideals, free stars, mixers in such a way that the above diagrams isomorphically preserve order of filters:

- $A \sqsubseteq B \Leftrightarrow A \supseteq B$ for filters and ideals;
- $A \sqsubseteq B \Leftrightarrow A \subseteq B$ for free stars and mixers.

5.6.8. Principal ideals, free stars, mixers.

DEFINITION 482. *Principal* ideal generated by an element a of poset $\mathfrak{A}$ is $\downarrow a = \left\{ \frac{x\in\mathfrak{A}}{x\sqsubseteq a} \right\}$.

DEFINITION 483. An ideal is *principal* iff it is generated by some poset element.

DEFINITION 484. The *filtrator of ideals* on a given poset is the pair consisting of the set of ideals and the set of principal ideals.

The above poset isomorphism maps principal filters into principal ideals and thus is an isomorphism between the filtrator of filters on a poset and the filtrator of ideals on the dual poset.

EXERCISE 485. Define principal free stars and mixers, filtrators of free stars and mixers and isomorphisms of these with the filtrator of filters (these isomorphisms exist because the posets of free stars and mixers are isomorphic to the poset of filters).

OBVIOUS 486. The following filtrators are primary:

- filtrators of filters;
- filtrators of ideals;
- filtrators of free stars;
- filtrators of mixers.

5.6.8.1. *Principal free stars.*

PROPOSITION 487. An upper set $F \in \mathscr{P}\mathfrak{Z}$ is a principal filter iff $\exists Z \in F \forall P \in F : Z \sqsubseteq P$.

PROOF.

$\Rightarrow$. Obvious.

$\Leftarrow$. Let $Z \in F$ and $\forall P \in F : Z \sqsubseteq P$. F is nonempty because $Z \in F$. It remains to prove that $Z \sqsubseteq P \Leftrightarrow P \in F$. The reverse implication follows from $\forall P \in F : Z \sqsubseteq P$. The direct implication follows from that F is an upper set.

□

LEMMA 488. If $S \in \mathscr{P}\mathfrak{Z}$ is not the complement of empty set and for every $T \in \mathscr{P}\mathfrak{Z}$

$$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset,$$

then S is a free star.

PROOF. Take $T = \{A, B\}$. Then $\forall Z \in \mathfrak{Z} : (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in S) \Leftrightarrow A \in S \vee B \in S$. So S is a free star. □

PROPOSITION 489. A set $S \in \mathscr{P}\mathfrak{Z}$ is a principal free star iff S is not the complement of empty set and for every $T \in \mathscr{P}\mathfrak{Z}$

$$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset.$$

PROOF. Let $S = \overline{\langle \operatorname{dual} \rangle^* F}$. We need to prove that F is a principal filter iff the above formula holds. Really, we have the following chain of equivalencies:

$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset$;

$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \notin \langle \operatorname{dual} \rangle^* F) \Leftrightarrow T \cap \overline{\langle \operatorname{dual} \rangle^* F} \neq \emptyset$;

$\forall Z \in \operatorname{dual} \mathfrak{Z} : (\forall X \in T : Z \sqsubseteq X \Rightarrow Z \notin F) \Leftrightarrow T \cap \overline{F} \neq \emptyset$;

$\forall Z \in \operatorname{dual} \mathfrak{Z} : (\forall X \in T : Z \sqsubseteq X \Rightarrow Z \notin F) \Leftrightarrow T \nsubseteq F$;

$T \subseteq F \Leftrightarrow \neg \forall Z \in \operatorname{dual} \mathfrak{Z} : (Z \in F \Rightarrow \neg \forall X \in T : Z \sqsubseteq X)$;

$T \subseteq F \Leftrightarrow \neg \forall Z \in \operatorname{dual} \mathfrak{Z} : (Z \notin F \vee \neg \forall X \in T : Z \sqsubseteq X)$;

$T \subseteq F \Leftrightarrow \exists Z \in \operatorname{dual} \mathfrak{Z} : (Z \in F \wedge \forall X \in T : Z \sqsubseteq X)$;

$T \subseteq F \Leftrightarrow \exists Z \in F \forall X \in T : Z \sqsubseteq X$;

$\exists Z \in F \forall X \in F : Z \sqsubseteq X$ that is F is a principal filter (S is an upper set because by the lemma it is a free star; thus F is also an upper set). □

PROPOSITION 490. $S \in \mathscr{P}\mathfrak{Z}$ where $\mathfrak{Z}$ is a poset is a principal free star iff all the following:

1°. The least element (if it exists) is not in S.
2°. $\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$ for every $T \in \mathscr{P}\mathfrak{Z}$.
3°. S is an upper set.

PROOF.

$\Rightarrow$. 1° and 2° are obvious. S is an upper set because S is a free star.

$\Leftarrow$. We need to prove that

$$\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Leftarrow T \cap S \neq \emptyset.$$

Let $X' \in T \cap S$. Then $\forall X \in T : Z \sqsupseteq X \Rightarrow Z \sqsupseteq X' \Rightarrow Z \in S$ because S is an upper set.

□

PROPOSITION 491. Let $\mathfrak{Z}$ be a complete lattice. $S \in \mathscr{P}\mathfrak{Z}$ is a principal free star iff all the following:

1°. The least element is not in S.
2°. $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$ for every $T \in \mathscr{P}\mathfrak{Z}$.
3°. S is an upper set.

PROOF.

$\Rightarrow$. We need to prove only $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$. Let $\bigsqcup T \in S$. Because S is an upper set, we have $\forall X \in T : Z \sqsupseteq X \Rightarrow Z \sqsupseteq \bigsqcup T \Rightarrow Z \in S$ for every $Z \in \mathfrak{Z}$; from which we conclude $T \cap S \neq \emptyset$.

$\Leftarrow$. We need to prove only $\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$.

Really, if $\forall Z \in \mathfrak{Z} : (\forall X \in T : Z \sqsupseteq X \Rightarrow Z \in S)$ then $\bigsqcup T \in S$ and thus $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$.

□

PROPOSITION 492. Let $\mathfrak{Z}$ be a complete lattice. $S \in \mathscr{P}\mathfrak{Z}$ is a principal free star iff the least element is not in S and for every $T \in \mathscr{P}\mathfrak{Z}$

$$\bigsqcup T \in S \Leftrightarrow T \cap S \neq \emptyset.$$

PROOF.

$\Rightarrow$. We need to prove only $\bigsqcup T \in S \Leftarrow T \cap S \neq \emptyset$ what follows from that S is an upper set.

$\Leftarrow$. We need to prove only that S is an upper set. To prove this we can use the fact that S is a free star.

□

EXERCISE 493. Write down similar formulas for mixers.

5.6.9. Starrish posets.

DEFINITION 494. I will call a poset *starrish* when the full star $\star a$ is a free star for every element a of this poset.

PROPOSITION 495. Every distributive lattice is starrish.

PROOF. Let $\mathfrak{A}$ be a distributive lattice, $a \in \mathfrak{A}$. Obviously $\bot \notin \star a$ (if $\bot$ exists); obviously $\star a$ is an upper set. If $x \sqcup y \in \star a$, then $(x \sqcup y) \sqcap a$ is non-least that is $(x \sqcap a) \sqcup (y \sqcap a)$ is non-least what is equivalent to $x \sqcap a$ or $y \sqcap a$ being non-least that is $x \in \star a \vee y \in \star a$. □

THEOREM 496. If $\mathfrak{A}$ is a starrish join-semilattice lattice then

$$\operatorname{atoms}(a \sqcup b) = \operatorname{atoms} a \cup \operatorname{atoms} b$$

for every $a, b \in \mathfrak{A}$.

PROOF. For every atom c we have:

$$\begin{aligned} c \in \operatorname{atoms}(a \sqcup b) &\Leftrightarrow \\ c \not\asymp a \sqcup b &\Leftrightarrow \\ a \sqcup b \in \star c &\Leftrightarrow \\ a \in \star c \vee b \in \star c &\Leftrightarrow \\ c \not\asymp a \vee c \not\asymp b &\Leftrightarrow \\ c \in \operatorname{atoms} a \vee c \in \operatorname{atoms} b&. \end{aligned}$$

□

5.6.9.1. *Completely starrish posets.*

DEFINITION 497. I will call a poset *completely starrish* when the full star $\star a$ is a principal free star for every element a of this poset.

OBVIOUS 498. Every completely starrish poset is starrish.

PROPOSITION 499. Every complete join infinite distributive lattice is completely starrish.

PROOF. Let $\mathfrak{A}$ be a join infinite distributive lattice, $a \in \mathfrak{A}$. Obviously $\bot \notin \star a$ (if $\bot$ exists); obviously $\star a$ is an upper set. If $\bigsqcup T \in \star a$, then $(\bigsqcup T) \sqcap a$ is non-least that is $\bigsqcup \langle a \sqcap \rangle^* T$ is non-least what is equivalent to $a \sqcap x$ being non-least for some $x \in T$ that is $x \in \star a$. □

THEOREM 500. If $\mathfrak{A}$ is a completely starrish complete lattice lattice then

$$\operatorname{atoms} \bigsqcup T = \bigcup \langle \operatorname{atoms} \rangle^* T.$$

for every $T \in \mathscr{P}\mathfrak{A}$.

PROOF. For every atom c we have:

$$c \in \operatorname{atoms} \bigsqcup T \Leftrightarrow c \not\asymp \bigsqcup T \Leftrightarrow \bigsqcup T \in \star c \Leftrightarrow \exists X \in T : X \in \star c \Leftrightarrow \\ \exists X \in T : X \not\asymp c \Leftrightarrow \exists X \in T : c \in \operatorname{atoms} X \Leftrightarrow c \in \bigcup \langle \operatorname{atoms} \rangle^* T.$$

□

5.7. Basic properties of filters

PROPOSITION 501. $\operatorname{up} \mathcal{A} = \mathcal{A}$ for every filter $\mathcal{A}$ (provided that we equate elements of the base poset $\mathfrak{Z}$ with corresponding principal filters.

PROOF. $A \in \operatorname{up} \mathcal{A} \Leftrightarrow A \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow A \sqsupseteq \mathcal{A} \Leftrightarrow \uparrow A \subseteq \mathcal{A} \Leftrightarrow A \in \mathcal{A}$. □

5.7.1. Minimal and maximal filters.

OBVIOUS 502. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $\bot^{\mathfrak{A}}$ (equal to the principal filter for the least element of $\mathfrak{Z}$ if it exists) defined by the formula $\operatorname{up} \bot^{\mathfrak{A}} = \mathfrak{Z}$ is the least element of $\mathfrak{A}$.

PROPOSITION 503. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator with greatest element.
3°. $\top^{\mathfrak{A}}$ defined by the formula $\operatorname{up} \top^{\mathfrak{A}} = \{\top^{\mathfrak{Z}}\}$ is the greatest element of $\mathfrak{A}$.

PROOF. Take into account that filters are nonempty. □

5.7.2. Alignment.

DEFINITION 504. I call *down-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\mathfrak{A}$ and $\mathfrak{Z}$ have common least element. (Let's denote it $\bot$.)

DEFINITION 505. I call *up-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that $\mathfrak{A}$ and $\mathfrak{Z}$ have common greatest element. (Let's denote it $\top$.)

OBVIOUS 506.

1°. If $\mathfrak{Z}$ has least element, the primary filtrator is down-aligned.
2°. If $\mathfrak{Z}$ has greatest element, the primary filtrator is up-aligned.

COROLLARY 507. Every powerset filtrator is both up and down-aligned.

We can also define (without requirement of having least and greatest elements, but coinciding with the above definitions if least/greatest elements are present):

DEFINITION 508. I call *weakly down-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that whenever $\bot^{\mathfrak{Z}}$ exists, $\bot^{\mathfrak{A}}$ also exists and $\bot^{\mathfrak{Z}} = \bot^{\mathfrak{A}}$.

DEFINITION 509. I call *weakly up-aligned* filtrator such a filtrator $(\mathfrak{A}, \mathfrak{Z})$ that whenever $\top^{\mathfrak{Z}}$ exists, $\top^{\mathfrak{A}}$ also exists and $\top^{\mathfrak{Z}} = \top^{\mathfrak{A}}$.

Obvious 510.

1°. Every up-aligned filtrator is weakly up-aligned.
2°. Every down-aligned filtrator is weakly down-aligned.

Obvious 511.

1°. Every primary filtrator is weakly down-aligned.
2°. Every primary filtrator is weakly up-aligned.

5.8. More advanced properties of filters

5.8.1. Formulas for Meets and Joins of Filters.

Lemma 512. If f is an order embedding from a poset $\mathfrak{A}$ to a complete lattice $\mathfrak{B}$ and $S \in \mathscr{P}\mathfrak{A}$ and there exists such $\mathcal{F} \in \mathfrak{A}$ that $f\mathcal{F} = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$.

Proof. f is an order isomorphism from $\mathfrak{A}$ to $\mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}$. $f\mathcal{F} \in \mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}$.
Consequently, $\bigsqcup^{\mathfrak{B}} \langle f \rangle^* S \in \mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}$ and $\bigsqcup^{\mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}} \langle f \rangle^* S = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$.
$f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}|_{\langle f \rangle^* \mathfrak{A}}} \langle f \rangle^* S$ because f is an order isomorphism.
Combining, $f \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle f \rangle^* S$. □

Corollary 513. If $\mathfrak{B}$ is a complete lattice and $\mathfrak{A}$ is its subset and $S \in \mathscr{P}\mathfrak{A}$ and $\bigsqcup^{\mathfrak{B}} S \in \mathfrak{A}$, then $\bigsqcup^{\mathfrak{A}} S$ exists and $\bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} S$.

Exercise 514. The below theorem does not work for $S = \emptyset$. Formulate the general case.

Theorem 515.

1°. If $\mathfrak{Z}$ is a meet-semilattice, then $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S$ exists and $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S = \bigcap S$ for every bounded above set $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$.
2°. If $\mathfrak{Z}$ is a join-semilattice, then $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S$ exists and $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S = \bigcap S$ for for every bounded below set $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$.

Proof.

1°. Taking into account the lemma, it is enough to prove that $\bigcap S$ is a filter. Let's prove that $\bigcap S$ is nonempty. There is an upper bound $\mathcal{T}$ of S. Take arbitrary $T \in \mathcal{T}$. We have $T \in \mathcal{X}$ for every $\mathcal{X} \in S$. Thus S is nonempty.

For every $A, B \in \mathfrak{Z}$ we have:

$$A, B \in \bigcap S \Leftrightarrow \forall P \in S : A, B \in P \Leftrightarrow \forall P \in S : A \sqcap B \in P \Leftrightarrow A \sqcap B \in \bigcap S.$$

So $\bigcap S$ is a filter.

2°. By duality.

□

Theorem 516.

1°. If $\mathfrak{Z}$ is a meet-semilattice with greatest element, then $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S$ exists and $\bigsqcup^{\mathfrak{F}(\mathfrak{Z})} S = \bigcap S$ for every $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$.
2°. If $\mathfrak{Z}$ is a join-semilattice with least element, then $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S$ exists and $\bigsqcap^{\mathfrak{I}(\mathfrak{Z})} S = \bigcap S$ for every $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$.
3°. If $\mathfrak{Z}$ is a join-semilattice with least element, then $\bigsqcup^{\mathfrak{S}(\mathfrak{Z})} S$ exists and $\bigsqcup^{\mathfrak{S}(\mathfrak{Z})} S = \bigcup S$ for every $S \in \mathscr{P}\mathfrak{S}(\mathfrak{Z})$.
4°. If $\mathfrak{Z}$ is a meet-semilattice with greatest element, then $\bigsqcap^{\mathfrak{M}(\mathfrak{Z})} S$ exists and $\bigsqcap^{\mathfrak{M}(\mathfrak{Z})} S = \bigcup S$ for every $S \in \mathscr{P}\mathfrak{M}(\mathfrak{Z})$.

Proof.

1°. From the previous theorem.
2°. By duality.
3°. Taking into account the lemma, it is enough to prove that $\bigcup S$ is a free star. $\bigcup S$ is not the complement of empty set because $\bot \notin \bigcup S$. For every $A, B \in \mathfrak{Z}$ we have:

$$A \in \bigcup S \vee B \in \bigcup S \Leftrightarrow \exists P \in S : (A \in P \vee B \in P) \Leftrightarrow \\ \exists P \in S : A \sqcup B \in P \Leftrightarrow A \sqcup B \in \bigcup S.$$

4°. By duality.

□

Corollary 517. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with greatest element $\top$.
3°. $\bigsqcup^{\mathfrak{A}} S$ exists and $\operatorname{up} \bigsqcup^{\mathfrak{A}} S = \bigcap \langle \operatorname{up} \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A} \setminus \{\emptyset\}$.

Proof.
1°⇒2°. Obvious.
2°⇒3°. By the theorem.

□

Corollary 518. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with greatest element $\top$.
3°. $\mathfrak{A}$ is a complete lattice.

We will denote meets and joins on the lattice of filters just as $\sqcap$ and $\sqcup$.

Proposition 519. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over an ideal base.
3°. $\mathfrak{A}$ is a join-semilattice and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$

$$\operatorname{up}(\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{B}) = \operatorname{up} \mathcal{A} \cap \operatorname{up} \mathcal{B}.$$

Proof.
1°⇒2°. Obvious.
2°⇒3°. Taking in account the lemma it is enough to prove that $R = \operatorname{up} \mathcal{A} \cap \operatorname{up} \mathcal{B}$ is a filter.

R is nonempty because we can take $X \in \operatorname{up} \mathcal{A}$ and $Y \in \operatorname{up} \mathcal{B}$ and $Z \sqsupseteq X \wedge Z \sqsupseteq Y$ and then $R \ni Z$.

Let $A, B \in R$. Then $A, B \in \operatorname{up} \mathcal{A}$; so exists $C \in \operatorname{up} \mathcal{A}$ such that $C \sqsubseteq A \wedge C \sqsubseteq B$. Analogously exists $D \in \operatorname{up} \mathcal{B}$ such that $D \sqsubseteq A \wedge D \sqsubseteq B$. Take $E \sqsupseteq C \wedge E \sqsupseteq D$. Then $E \in \operatorname{up} \mathcal{A}$ and $E \in \operatorname{up} \mathcal{B}$; $E \in R$ and $E \sqsubseteq A \wedge E \sqsubseteq B$. So R is a filter base.

That R is an upper set is obvious.

□

Theorem 520. Let $\mathfrak{Z}$ be a distributive lattice. Then
1°. $\bigsqcap^{\mathfrak{F}(\mathfrak{Z})} S = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i=0,\ldots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$;
2°. $\bigsqcup^{\mathfrak{I}(\mathfrak{Z})} S = \left\{ \frac{K_0 \sqcup^{\mathfrak{Z}} \cdots \sqcup^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i=0,\ldots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$.

Proof. We will prove only the first, as the second is dual.

Let's denote the right part of the equality to be proven as R. First we will prove that R is a filter. R is nonempty because S is nonempty.

Let $A, B \in R$. Then $A = X_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} X_k$, $B = Y_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} Y_l$ where $X_i, Y_j \in \bigcup S$. So

$$A \sqcap^{\mathfrak{Z}} B = X_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} X_k \sqcap^{\mathfrak{Z}} Y_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} Y_l \in R.$$

Let element $C \sqsupseteq A \in R$. Consequently (distributivity used)

$$C = C \sqcup^{\mathfrak{Z}} A = (C \sqcup^{\mathfrak{Z}} X_0) \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} (C \sqcup^{\mathfrak{Z}} X_k).$$

$X_i \in P_i$ for some $P_i \in S$; $C \sqcup^{\mathfrak{Z}} X_i \in P_i$; $C \sqcup^{\mathfrak{Z}} X_i \in \bigcup S$; consequently $C \in R$.

We have proved that that R is a filter base and an upper set. So R is a filter.

Let $\mathcal{A} \in S$. Then $\mathcal{A} \subseteq \bigcup S$;

$$R \supseteq \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \mathcal{A} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\} = \mathcal{A}.$$

Consequently $\mathcal{A} \sqsupseteq R$.

Let now $\mathcal{B} \in \mathfrak{A}$ and $\forall \mathcal{A} \in S : \mathcal{A} \sqsupseteq \mathcal{B}$. Then $\forall \mathcal{A} \in S : \mathcal{A} \subseteq \mathcal{B}$; $\mathcal{B} \supseteq \bigcup S$. Thus $\mathcal{B} \supseteq T$ for every finite set $T \subseteq \bigcup S$. Consequently $\operatorname{up} \mathcal{B} \ni \bigsqcap^{\mathfrak{Z}} T$. Thus $\mathcal{B} \supseteq R$; $\mathcal{B} \sqsubseteq R$.

Comparing we get $\bigsqcap^{\mathfrak{F}(\mathfrak{Z})} S = R$.

□

Corollary 521. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.

3°. $\operatorname{up} \bigsqcap^{\mathfrak{A}} S = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{A} \setminus \{\emptyset\}$.

Proof.

1°⇒2°. Obvious.

2°⇒3°. By the theorem.

□

Theorem 522. Let $\mathfrak{Z}$ be a distributive lattice. Then:

1°. $\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \cdots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i = 0, \ldots, m} \right\}$ for any $\mathcal{F}_0, \ldots, \mathcal{F}_m \in \mathfrak{F}(\mathfrak{Z})$;

2°. $\mathcal{F}_0 \sqcup^{\mathfrak{I}(\mathfrak{Z})} \cdots \sqcup^{\mathfrak{I}(\mathfrak{Z})} \mathcal{F}_m = \left\{ \frac{K_0 \sqcup^{\mathfrak{Z}} \cdots \sqcup^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i = 0, \ldots, m} \right\}$ for any $\mathcal{F}_0, \ldots, \mathcal{F}_m \in \mathfrak{I}(\mathfrak{Z})$.

Proof. We will prove only the first as the second is dual.

Let's denote the right part of the equality to be proven as R. First we will prove that R is a filter. Obviously R is nonempty.

Let $A, B \in R$. Then $A = X_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} X_m$, $B = Y_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} Y_m$ where $X_i, Y_i \in \mathcal{F}_i$.

$$A \sqcap^{\mathfrak{Z}} B = (X_0 \sqcap^{\mathfrak{Z}} Y_0) \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} (X_m \sqcap^{\mathfrak{Z}} Y_m),$$

consequently $A \sqcap^{\mathfrak{Z}} B \in R$.

Let filter $C \sqsupseteq A \in R$

$$C = A \sqcup^{\mathfrak{Z}} C = (X_0 \sqcup^{\mathfrak{Z}} C) \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} (X_m \sqcup^{\mathfrak{Z}} C) \in R.$$

So R is a filter.

Let $P_i \in \mathcal{F}_i$. Then $P_i \in R$ because $P_i = (P_i \sqcup^{\mathfrak{Z}} P_0) \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} (P_i \sqcup^{\mathfrak{Z}} P_m)$. So $\mathcal{F}_i \subseteq R$; $\mathcal{F}_i \sqsupseteq R$.

Let now $\mathcal{B} \in \mathfrak{A}$ and $\forall i \in \{0, \ldots, m\} : \mathcal{F}_i \sqsupseteq \mathcal{B}$. Then $\forall i \in \{0, \ldots, m\} : \mathcal{F}_i \subseteq \mathcal{B}$.

Let $L_i \in \mathcal{B}$ for every $L_i \in \mathcal{F}_i$. $L_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} L_m \in \mathcal{B}$. So $\mathcal{B} \supseteq R$; $\mathcal{B} \sqsubseteq R$.

So $\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \cdots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m = R$. □

COROLLARY 523. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.
3°. $\operatorname{up}(\mathcal{F}_0 \sqcap^{\mathfrak{A}} \cdots \sqcap^{\mathfrak{A}} \mathcal{F}_m) = \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \operatorname{up} \mathcal{F}_i \text{ where } i=0,\ldots,m} \right\}$ for any $\mathcal{F}_0, \ldots, \mathcal{F}_m \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. By the theorem.

□

More general case of semilattices follows:

THEOREM 524.

1°. $\prod^{\mathfrak{F}(\mathfrak{Z})} S = \bigcup \left\{ \frac{\uparrow (K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n)}{K_i \in \bigcup S \text{ where } i=0,\ldots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$ if $\mathfrak{Z}$ is a meet-semilattice;
2°. $\coprod^{\mathfrak{I}(\mathfrak{Z})} S = \bigcup \left\{ \frac{\uparrow (K_0 \sqcup^{\mathfrak{Z}} \cdots \sqcup^{\mathfrak{Z}} K_n)}{K_i \in \bigcup S \text{ where } i=0,\ldots,n \text{ for } n \in \mathbb{N}} \right\}$ for $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$ if $\mathfrak{Z}$ is a join-semilattice.

PROOF. We will prove only the first as the second is dual.
It follows from the fact that
$$\prod^{\mathfrak{F}(\mathfrak{Z})} S = \prod^{\mathfrak{F}(\mathfrak{Z})} \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\}$$
and that $\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup S \text{ where } i=0,\ldots,n \text{ for } n \in \mathbb{N}} \right\}$ is a filter base. □

COROLLARY 525. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. $\operatorname{up} \prod S = \bigcup \left\{ \frac{\operatorname{up}(K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n)}{K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i=0,\ldots,n \text{ for } n \in \mathbb{N}} \right\}$ for every $S \in \mathscr{P}\mathfrak{A} \setminus \{\emptyset\}$.

THEOREM 526.

1°. $\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \cdots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m = \bigcup \left\{ \frac{\uparrow (K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_m)}{K_i \in \mathcal{F}_i \text{ where } i=0,\ldots,m} \right\}$ for $S \in \mathscr{P}\mathfrak{F}(\mathfrak{Z}) \setminus \{\emptyset\}$ if $\mathfrak{Z}$ is a meet-semilattice;
2°. $\mathcal{F}_0 \sqcup^{\mathfrak{I}(\mathfrak{Z})} \cdots \sqcup^{\mathfrak{I}(\mathfrak{Z})} \mathcal{F}_m = \bigcup \left\{ \frac{\uparrow (K_0 \sqcup^{\mathfrak{Z}} \cdots \sqcup^{\mathfrak{Z}} K_m)}{K_i \in \mathcal{F}_i \text{ where } i=0,\ldots,m} \right\}$ for $S \in \mathscr{P}\mathfrak{I}(\mathfrak{Z}) \setminus \{\emptyset\}$ if $\mathfrak{Z}$ is a join-semilattice.

PROOF. We will prove only the first as the second is dual.
It follows from the fact that
$$\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \cdots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m = \prod^{\mathfrak{F}(\mathfrak{Z})} \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i = 0, \ldots, m} \right\}$$
and that $\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_m}{K_i \in \mathcal{F}_i \text{ where } i=0,\ldots,m} \right\}$ is a filter base. □

COROLLARY 527. $\operatorname{up}(\mathcal{F}_0 \sqcap^{\mathfrak{F}(\mathfrak{Z})} \cdots \sqcap^{\mathfrak{F}(\mathfrak{Z})} \mathcal{F}_m) = \bigcup \left\{ \frac{\operatorname{up}(K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_m)}{K_i \in \mathcal{F}_i \text{ where } i=0,\ldots,m} \right\}$ if $\mathfrak{Z}$ is a meet-semilattice.

LEMMA 528. If $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator and $\mathfrak{Z}$ is a meet-semilattice and an ideal base, then $\mathfrak{A}$ is a lattice.

PROOF. It is a join-semilattice by proposition 519. It is a meet-semilattice by theorem 524. □

COROLLARY 529. If $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator and $\mathfrak{Z}$ is a lattice, then $\mathfrak{A}$ is a lattice.

5.8.2. Distributivity of the Lattice of Filters.

THEOREM 530. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.
3°. $\mathcal{A} \sqcup^{\mathfrak{A}} \prod^{\mathfrak{A}} S = \prod^{\mathfrak{A}} \langle \mathcal{A} \sqcup^{\mathfrak{A}} \rangle^* S$ for $S \in \mathscr{P}\mathfrak{A}$ and $\mathcal{A} \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. Taking into account the previous section, we have:

$$\operatorname{up}\left(\mathcal{A} \sqcup^{\mathfrak{A}} \prod^{\mathfrak{A}} S\right) =$$

$$\operatorname{up}\mathcal{A} \cap \operatorname{up}\prod^{\mathfrak{A}} S =$$

$$\operatorname{up}\mathcal{A} \cap \left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\} =$$

$$\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n \in \operatorname{up}\mathcal{A}, K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\} =$$

$$\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \operatorname{up}\mathcal{A}, K_i \in \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\} =$$

$$\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \operatorname{up}\mathcal{A} \cap \bigcup \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\} =$$

$$\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \langle \operatorname{up}\mathcal{A} \cap \rangle^* \langle \operatorname{up} \rangle^* S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\} =$$

$$\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \left\{ \frac{\operatorname{up}\mathcal{A} \cap \operatorname{up}\mathcal{X}}{\mathcal{X} \in S} \right\} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\} =$$

$$\left\{ \frac{K_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} K_n}{K_i \in \bigcup \left\{ \frac{\operatorname{up}(\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{X})}{\mathcal{X} \in S} \right\} \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}} \right\} =$$

$$\operatorname{up}\prod^{\mathfrak{A}} \left\{ \frac{\mathcal{A} \sqcup^{\mathfrak{A}} \mathcal{X}}{\mathcal{X} \in S} \right\} =$$

$$\operatorname{up}\prod^{\mathfrak{A}} \langle \mathcal{A} \sqcup^{\mathfrak{A}} \rangle^* S.$$

□

COROLLARY 531. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is an ideal base.
3°. $\mathfrak{A}$ is a distributive and co-brouwerian lattice.

COROLLARY 532. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice with greatest element.

3°. $\mathfrak{A}$ is a co-frame.

The below theorem uses the notation and results from section 3.9.

THEOREM 533. If $\mathfrak{A}$ is a co-frame and L is a bounded distributive lattice which, then $\mathrm{Join}(L, \mathfrak{A})$ is also a co-frame.

PROOF. Let $F = \uparrow \circ \bigsqcap : \mathrm{Up}(\mathfrak{A}) \to \mathrm{Up}(\mathfrak{A})$; F is a co-nucleus by above.

Since $\mathrm{Up}(\mathfrak{A}) \cong \mathbf{Pos}(\mathfrak{A}, 2)$ by proposition 340, we may regard F as a co-nucleus on $\mathbf{Pos}(\mathfrak{A}, 2)$.

$\mathrm{Join}(L, \mathfrak{A}) \cong \mathrm{Join}(L, \mathrm{Fix}(F))$ by corollary 343.

$\mathrm{Join}(L, \mathrm{Fix}(F)) \cong \mathrm{Fix}(\mathrm{Join}(L, F))$ by lemma 351.

By corollary 350 the function $\mathrm{Join}(L, F)$ is a co-nucleus on $\mathrm{Join}(L, \mathbf{Pos}(\mathfrak{A}, 2))$.

$$\begin{aligned} \mathrm{Join}(L, \mathbf{Pos}(\mathfrak{A}, 2)) &\cong \quad \text{(by lemma 353)} \\ \mathbf{Pos}(\mathfrak{A}, \mathrm{Join}(L, 2)) &\cong \\ \mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X)). \end{aligned}$$

$\mathfrak{F}(X)$ is a co-frame by corollary 532. Thus $\mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X))$ is a co-frame by lemma 353.

Thus $\mathrm{Join}(L, \mathfrak{A})$ is isomorphic to a poset of fixed points of a co-nucleus on the co-frame $\mathbf{Pos}(\mathfrak{A}, \mathfrak{F}(X))$. By lemma 335 $\mathrm{Join}(L, \mathfrak{A})$ is also a co-frame. □

5.9. Misc filtrator properties

THEOREM 534. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with join-closed core.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. The formula $\forall a, b \in \mathfrak{A} : (\mathrm{up}\, a \supseteq \mathrm{up}\, b \Rightarrow a \sqsubseteq b)$ is obvious for primary filtrators.

3°⇒4°. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtered filtrator. Let $S \in \mathscr{P}\mathfrak{Z}$ and $\bigsqcup^{\mathfrak{Z}} S$ be defined. We need to prove $\bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{Z}} S$. That $\bigsqcup^{\mathfrak{Z}} S$ is an upper bound for S is obvious. Let $a \in \mathfrak{A}$ be an upper bound for S. It's enough to prove that $\bigsqcup^{\mathfrak{Z}} S \sqsubseteq a$. Really,

$$c \in \mathrm{up}\, a \Rightarrow c \sqsupseteq a \Rightarrow \forall x \in S : c \sqsupseteq x \Rightarrow c \sqsupseteq \bigsqcup^{\mathfrak{Z}} S \Rightarrow c \in \mathrm{up} \bigsqcup^{\mathfrak{Z}} S;$$

so $\mathrm{up}\, a \subseteq \mathrm{up} \bigsqcup^{\mathfrak{Z}} S$ and thus $a \sqsupseteq \bigsqcup^{\mathfrak{Z}} S$ because it is filtered.

□

5.10. Characterization of Binarily Meet-Closed Filtrators

THEOREM 535. The following are equivalent for a filtrator $(\mathfrak{A}, \mathfrak{Z})$ whose core is a meet semilattice such that $\forall a \in \mathfrak{A} : \mathrm{up}\, a \neq \emptyset$:

1°. The filtrator is with binarily meet-closed core.
2°. $\mathrm{up}\, a$ is a filter for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Let $X, Y \in \mathrm{up}\, a$. Then $X \sqcap^{\mathfrak{Z}} Y = X \sqcap^{\mathfrak{A}} Y \sqsupseteq a$. That $\mathrm{up}\, a$ is an upper set is obvious. So taking into account that $\mathrm{up}\, a \neq \emptyset$, $\mathrm{up}\, a$ is a filter.

2°⇒1°. It is enough to prove that $a \sqsubseteq A, B \Rightarrow a \sqsubseteq A \sqcap^{\mathfrak{Z}} B$ for every $A, B \in \mathfrak{A}$. Really:

$$a \sqsubseteq A, B \Rightarrow A, B \in \operatorname{up} a \Rightarrow A \sqcap^{\mathfrak{Z}} B \in \operatorname{up} a \Rightarrow a \sqsubseteq A \sqcap^{\mathfrak{Z}} B.$$

□

COROLLARY 536. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily meet-closed core.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. From the theorem.

□

5.10.1. Separability of Core for Primary Filtrators.

THEOREM 537. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet semilattice with least element.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is with separable core.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Let $\mathcal{A} \asymp^{\mathfrak{A}} \mathcal{B}$ where $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$.

$$\operatorname{up}(\mathcal{A} \sqcap^{\mathfrak{A}} \mathcal{B}) = \bigcup \left\{ \frac{\operatorname{up}(A \sqcap^{\mathfrak{Z}} B)}{A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}} \right\}.$$

So

$$\begin{aligned} \bot \in \operatorname{up}(\mathcal{A} \sqcap^{\mathfrak{A}} \mathcal{B}) \Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} : \bot \in \operatorname{up}(A \sqcap^{\mathfrak{Z}} B) \Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} : A \sqcap^{\mathfrak{Z}} B = \bot \Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} : A \sqcap^{\mathfrak{A}} B = \bot^{\mathfrak{A}} \end{aligned}$$

(used proposition 536).

□

5.11. Core Part

Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator.

DEFINITION 538. The *core part* of an element $a \in \mathfrak{A}$ is $\operatorname{Cor} a = \bigsqcap^{\mathfrak{Z}} \operatorname{up} a$.

DEFINITION 539. The *dual core part* of an element $a \in \mathfrak{A}$ is $\operatorname{Cor}' a = \bigsqcup^{\mathfrak{Z}} \operatorname{down} a$.

OBVIOUS 540. Cor' is dual of Cor.

OBVIOUS 541. $\operatorname{Cor} a = \operatorname{Cor}' a = a$ for every element a of the core of a filtrator.

THEOREM 542. The following is an implications tuple:

1°. a is a filter on a set.
2°. a is a filter on a complete lattice.
3°. a is an element of a filtered filtrator and $\operatorname{Cor} a$ exists.
4°. $\operatorname{Cor} a \sqsubseteq a$ and $\operatorname{Cor} a \in \operatorname{down} a$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Theorem 534.
3°⇒4°. $\operatorname{Cor} a = \bigsqcap^{\mathfrak{Z}} \operatorname{up} a \sqsubseteq \bigsqcap^{\mathfrak{A}} \operatorname{up} a = a$. Then obviously $\operatorname{Cor} a \in \operatorname{down} a$.

□

THEOREM 543. The following is an implications tuple:

1°. a is a filter on a set.
2°. a is a filter on a complete lattice.
3°. a is an element a of a filtrator with join-closed core and $\operatorname{Cor}' a$ exists.
4°. $\operatorname{Cor}' a \sqsubseteq a$ and $\operatorname{Cor}' a \in \operatorname{down} a$ and $\operatorname{Cor}' a = \max \operatorname{down} a$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. It is join closed by 534. $\operatorname{Cor}' a$ exists because our filtrator is join-closed.
3°⇒4°. $\operatorname{Cor}' a = \bigsqcup^{\mathfrak{Z}} \operatorname{down} a = \bigsqcup^{\mathfrak{A}} \operatorname{down} a \sqsubseteq a$. Now $\operatorname{Cor}' a \in \operatorname{down} a$ is obvious. Thus $\operatorname{Cor}' a = \max \operatorname{down} a$.

□

PROPOSITION 544. $\operatorname{Cor}' a \sqsubseteq \operatorname{Cor} a$ whenever both $\operatorname{Cor} a$ and $\operatorname{Cor}' a$ exist for any element a of a filtrator with join-closed core.

PROOF. $\operatorname{Cor} a = \bigsqcap^{\mathfrak{Z}} \operatorname{up} a \sqsupseteq \operatorname{Cor}' a$ because $\forall A \in \operatorname{up} a : \operatorname{Cor}' a \sqsubseteq A$. □

THEOREM 545. The following is an implications tuple:

1°. a is a filter on a set.
2°. a is a filter on a complete lattice.
3°. a is an element of a filtered filtrator and both $\operatorname{Cor} a$ and $\operatorname{Cor}' a$ exist.
4°. $\operatorname{Cor}' a = \operatorname{Cor} a$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. By theorem 534.
3°⇒4°. It is with join-closed core because it is filtered. So $\operatorname{Cor}' a \sqsubseteq \operatorname{Cor} a$. $\operatorname{Cor} a \in \operatorname{down} a$. So $\operatorname{Cor} a \sqsubseteq \bigsqcup^{\mathfrak{Z}} \operatorname{down} a = \operatorname{Cor}' a$.

□

COROLLARY 546. $\operatorname{Cor}' a = \operatorname{Cor} a = \bigcap a$ for every filter a on a set.

5.12. Intersection and Joining with an Element of the Core

DEFINITION 547. A filtrator $(\mathfrak{A}; \mathfrak{Z})$ is with *correct intersection* iff $\forall a, b \in \mathfrak{Z} : (a \not\asymp^{\mathfrak{Z}} b \Leftrightarrow a \not\asymp^{\mathfrak{A}} b)$.

DEFINITION 548. A filtrator $(\mathfrak{A}; \mathfrak{Z})$ is with *correct joining* iff $\forall a, b \in \mathfrak{Z} : (a \equiv^{\mathfrak{Z}} b \Leftrightarrow a \equiv^{\mathfrak{A}} b)$.

PROPOSITION 549. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily meet-closed core, weakly down-aligned filtrator, and $\mathfrak{Z}$ is a meet-semilattice.
4°. $(\mathfrak{A}, \mathfrak{Z})$ is with correct intersection.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Corollary 536.

3°⇒4°. $a \not\asymp^{\mathfrak{Z}} b \Rightarrow a \not\asymp^{\mathfrak{A}} b$ is obvious. Let $a \asymp^{\mathfrak{Z}} b$. Then $a \sqcap^{\mathfrak{Z}} b$ exists; so $\bot^{\mathfrak{Z}}$ exists and $a \sqcap^{\mathfrak{Z}} b = \bot^{\mathfrak{Z}}$ (as otherwise $a \sqcap^{\mathfrak{Z}} b$ is non-least). So $\bot^{\mathfrak{Z}} = \bot^{\mathfrak{A}}$. We have $a \sqcap^{\mathfrak{A}} b = \bot^{\mathfrak{A}}$. Thus $a \asymp^{\mathfrak{A}} b$.

□

PROPOSITION 550. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a join-semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily join-closed core, weakly up-aligned filtrator, and $\mathfrak{Z}$ is a join-semilattice.
4°. $(\mathfrak{A}, \mathfrak{Z})$ is with correct joining.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Corollary 534.
3°⇒4°. Dual of the previous proposition.

□

LEMMA 551. For a filtrator $(\mathfrak{A}, \mathfrak{Z})$ where $\mathfrak{Z}$ is a boolean lattice, for every $B \in \mathfrak{Z}$, $\mathcal{A} \in \mathfrak{A}$:

1°. $B \asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsupseteq \mathcal{A}$ if it is with separable core and with correct intersection;
2°. $B \equiv^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsubseteq \mathcal{A}$ if it is with co-separable core and with correct joining.

PROOF. We will prove only the first as the second is dual.

$$\begin{gathered} B \asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : B \asymp^{\mathfrak{A}} A \Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : B \asymp^{\mathfrak{Z}} A \Leftrightarrow \\ \exists A \in \operatorname{up} \mathcal{A} : \overline{B} \sqsupseteq A \Leftrightarrow \\ \overline{B} \in \operatorname{up} \mathcal{A} \Leftrightarrow \\ \overline{B} \sqsupseteq \mathcal{A}. \end{gathered}$$

□

5.13. Stars of Elements of Filtrators

DEFINITION 552. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. *Core star* of an element a of the filtrator is

$$\partial a = \left\{ \frac{x \in \mathfrak{Z}}{x \not\asymp^{\mathfrak{A}} a} \right\}.$$

PROPOSITION 553. $\operatorname{up} a \subseteq \partial a$ for any non-least element a of a filtrator.

PROOF. For any element $X \in \mathfrak{Z}$

$$X \in \operatorname{up} a \Rightarrow a \sqsubseteq X \wedge a \sqsubseteq a \Rightarrow X \not\asymp^{\mathfrak{A}} a \Rightarrow X \in \partial a.$$

□

THEOREM 554. Let $(\mathfrak{A}, \mathfrak{Z})$ be a distributive lattice filtrator with least element and binarily join-closed core which is a join-semilattice. Then ∂a is a free star for each $a \in \mathfrak{A}$.

Proof. For every $A, B \in \mathfrak{Z}$

$$\begin{aligned}A \sqcup^{\mathfrak{Z}} B \in \partial a \Leftrightarrow\\ A \sqcup^{\mathfrak{A}} B \in \partial a \Leftrightarrow\\ (A \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} a \neq \bot^{\mathfrak{A}} \Leftrightarrow\\ (A \sqcap^{\mathfrak{A}} a) \sqcup^{\mathfrak{A}} (B \sqcap^{\mathfrak{A}} a) \neq \bot^{\mathfrak{A}} \Leftrightarrow\\ A \sqcap^{\mathfrak{A}} a \neq \bot^{\mathfrak{A}} \vee B \sqcap^{\mathfrak{A}} a \neq \bot^{\mathfrak{A}} \Leftrightarrow\\ A \in \partial a \vee B \in \partial a.\end{aligned}$$

That ∂a doesn't contain $\bot^{\mathfrak{A}}$ is obvious. □

Definition 555. I call a filtrator *star-separable* when its core is a separation subset of its base.

5.14. Atomic Elements of a Filtrator

See [4, 9] for more detailed treatment of ultrafilters and prime filters.

Proposition 556. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with greatest element.
3°. $\mathfrak{A}$ is a complete lattice.
4°. $\operatorname{atoms} \bigsqcap S = \bigcap \langle \operatorname{atoms} \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A}$.
5°. $\operatorname{atoms}(a \sqcap b) = \operatorname{atoms} a \cap \operatorname{atoms} b$ for $a, b \in \mathfrak{A}$.

Proof.

1°⇒2°. Obvious.
2°⇒3°. Corollary 518.
3°⇒4°. Theorem 108.
4°⇒5°. Obvious.

□

Proposition 557. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is and ideal base.
3°. $\mathfrak{A}$ is a starrish join-semilattice.
4°. $\operatorname{atoms}(a \sqcup b) = \operatorname{atoms} a \cup \operatorname{atoms} b$ for $a, b \in \mathfrak{A}$.

Proof.

1°⇒2°. Obvious.
2°⇒3°. Corollary 531.
3°⇒4°. Corollary 496.

□

Theorem 558. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered weakly down-aligned filtrator with binarily meet-closed core $\mathfrak{Z}$ which is a meet-semilattice.
4°. a is an atom of $\mathfrak{Z}$ iff $a \in \mathfrak{Z}$ and a is an atom of $\mathfrak{A}$.

Proof.

1°⇒2°. Obvious.
2°⇒3°. It is filtered by the theorem 534, binarily meet-closed by corollary 536.

3°⇒4°.

⇐. Let a be an atom of $\mathfrak{A}$ and $a \in \mathfrak{Z}$. Then either a is an atom of $\mathfrak{Z}$ or a is the least element of $\mathfrak{Z}$. But if a is the least element of $\mathfrak{Z}$ then a is also least element of $\mathfrak{A}$ and thus is not an atom of $\mathfrak{A}$. So the only possible outcome is that a is an atom of $\mathfrak{Z}$.

⇒. We need to prove that if a is an atom of $\mathfrak{Z}$ then a is an atom of $\mathfrak{A}$. Suppose the contrary that a is not an atom of $\mathfrak{A}$. Then there exists $x \in \mathfrak{A}$ such that $x \sqsubset a$ and x is not least element of $\mathfrak{A}$. Because "up" is a straight monotone map to the dual of the poset $\mathscr{P}\mathfrak{Z}$ (obvious 454), $\operatorname{up} a \subset \operatorname{up} x$. So there exists $K \in \operatorname{up} x$ such that $K \notin \operatorname{up} a$. Also $a \in \operatorname{up} x$. We have $K \sqcap^{\mathfrak{Z}} a = K \sqcap^{\mathfrak{A}} a \in \operatorname{up} x$; $K \sqcap^{\mathfrak{Z}} a$ is not least of $\mathfrak{Z}$ (Suppose for the contrary that $K \sqcap^{\mathfrak{Z}} a = \bot^{\mathfrak{Z}}$, then $K \sqcap^{\mathfrak{Z}} a = \bot^{\mathfrak{A}} \notin \operatorname{up} x$.) and $K \sqcap^{\mathfrak{Z}} a \sqsubset a$. So a is not an atom of $\mathfrak{Z}$.

□

THEOREM 559. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. $a \in \mathfrak{A}$ is an atom of $\mathfrak{A}$ iff $\operatorname{up} a = \partial a$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. By the theorem 534.
3°⇒4°.

⇒. For any $K \in \mathfrak{A}$

$$K \in \operatorname{up} a \Leftrightarrow K \sqsupseteq a \Leftrightarrow K \not\asymp^{\mathfrak{A}} a \Leftrightarrow K \in \partial a.$$

⇐. Let $\operatorname{up} a = \partial a$. Then a is not least element of $\mathfrak{A}$. Consequently for every $x \in \mathfrak{A}$ if x is not the least element of $\mathfrak{A}$ we have

$$\begin{aligned} x \sqsubseteq a &\Rightarrow \\ x \not\asymp^{\mathfrak{A}} a &\Rightarrow \\ \forall K \in \operatorname{up} x : K \in \partial a &\Rightarrow \\ \forall K \in \operatorname{up} x : K \in \operatorname{up} a &\Rightarrow \\ \operatorname{up} x \subseteq \operatorname{up} a &\Rightarrow \\ x \sqsupseteq a&. \end{aligned}$$

So a is an atom of $\mathfrak{A}$.

□

PROPOSITION 560. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. Coatoms of $\mathfrak{A}$ are exactly coatoms of $\mathfrak{Z}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Suppose a is a coatom of $\mathfrak{Z}$. Then a is the only non-greatest element in $\operatorname{up} a$. Suppose $b \sqsupset a$ for some $b \in \mathfrak{A}$. Then a cannot be in $\operatorname{up} b$ and thus the only possible element of $\operatorname{up} b$ is the greatest element of $\mathfrak{Z}$ (if it exists) from what follows $b = \top^{\mathfrak{A}}$. So a is a coatom of $\mathfrak{A}$.

Suppose now that a is a coatom of $\mathfrak{A}$. To finish the proof it is enough to show that a is principal. (Then a is non-greatest and thus is a coatom of $\mathfrak{Z}$.)

Suppose a is non-principal. Then obviously exist two distinct elements x and y of the core such that $x, y \in \operatorname{up} a$. Thus a is not an atom of $\mathfrak{A}$.

□

COROLLARY 561. Coatoms of the set of filters on a set U are exactly sets $U \setminus \{x\}$ where $x \in U$.

PROPOSITION 562. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a coatomic poset.
3°. $\mathfrak{A}$ is coatomic.

PROOF.

1°⇒2°. Obvious.

2°⇒2°. Suppose $\mathcal{A} \in \mathfrak{A}$ and $\mathcal{A} \neq \top^{\mathfrak{A}}$. Then there exists $A \in \operatorname{up} \mathcal{A}$ such that A is not greatest element of $\mathfrak{Z}$. Consequently there exists a coatom $a \in \mathfrak{Z}$ such that $a \sqsupseteq A$. Thus $a \in \operatorname{up} \mathcal{A}$ and a is not greatest.

□

5.15. Prime Filtrator Elements

DEFINITION 563. Let $(\mathfrak{A}, \mathfrak{Z})$ be a filtrator. *Prime* filtrator elements are such $a \in \mathfrak{A}$ that $\operatorname{up} a$ is a free star (in lattice $\mathfrak{Z}$).

PROPOSITION 564. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is an ideal base.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with binarily join-closed core, where $\mathfrak{A}$ is a starrish join-semilattice and $\mathfrak{Z}$ is a join-semilattice.
4°. Atomic elements of this filtrator are prime.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is with binarily join-closed core by the theorem 534, $\mathfrak{A}$ is a distributive lattice by theorem 531.

3°⇒4°. Let a be an atom of the lattice $\mathfrak{A}$. We have for every $X, Y \in \mathfrak{Z}$

$$\begin{aligned} X \sqcup^{\mathfrak{Z}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \in \operatorname{up} a &\Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \sqsupseteq a &\Leftrightarrow \\ X \sqcup^{\mathfrak{A}} Y \not\asymp^{\mathfrak{A}} a &\Leftrightarrow \\ X \not\asymp^{\mathfrak{A}} a \vee Y \not\asymp^{\mathfrak{A}} a &\Leftrightarrow \\ X \sqsupseteq a \vee Y \sqsupseteq a &\Leftrightarrow \\ X \in \operatorname{up} a \vee Y \in \operatorname{up} a. \end{aligned}$$

□

The following theorem is essentially borrowed from [19]:

THEOREM 565. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. Let $a \in \mathfrak{A}$. Then the following are equivalent:
(a) a is prime.
(b) For every $A \in \mathfrak{Z}$ exactly one of $\{A, \overline{A}\}$ is in $\operatorname{up} a$.
(c) a is an atom of $\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°.

3°a⇒3°b. Let a be prime. Then $A \sqcup^{\mathfrak{Z}} \overline{A} = \top^{\mathfrak{A}} \in \operatorname{up} a$. Therefore $A \in \operatorname{up} a \vee \overline{A} \in \operatorname{up} a$. But since $A \sqcap^{\mathfrak{Z}} \overline{A} = \bot^{\mathfrak{Z}}$ it is impossible $A \in \operatorname{up} a \wedge \overline{A} \in \operatorname{up} a$.

3°b⇒3°c. Obviously $a \neq \bot^{\mathfrak{A}}$.
Let a filter $b \sqsubset a$. Take $X \in \operatorname{up} b$ such that $X \notin \operatorname{up} a$. Then $\overline{X} \in \operatorname{up} a$ because a is prime and thus $\overline{X} \in \operatorname{up} b$. So $\bot^{\mathfrak{Z}} = X \sqcap^{\mathfrak{Z}} \overline{X} \in \operatorname{up} b$ and thus $b = \bot^{\mathfrak{A}}$. So a is atomic.

3°c⇒3°a. By the previous proposition.

□

5.16. Stars for filters

THEOREM 566. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice which is an ideal base and has least element.
3°. ∂a is a free star for each $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $\mathfrak{A}$ is a distributive lattice by the corollary 531. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is binarily join-closed by corollary 534. So we can apply the theorem 554.

□

5.16.1. Stars of Filters on Boolean Lattices.

In this section we will consider the set of filters $\mathfrak{A}$ on a boolean lattice $\mathfrak{Z}$.

THEOREM 567. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\partial \mathcal{A} = \neg \langle \neg \rangle^* \operatorname{up} \mathcal{A} = \langle \neg \rangle^* \neg \operatorname{up} \mathcal{A}$ and $\operatorname{up} \mathcal{A} = \neg \langle \neg \rangle^* \partial \mathcal{A} = \langle \neg \rangle^* \neg \partial \mathcal{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. Because of properties of diagram (1), it is enough to prove just $\partial \mathcal{A} = \neg \langle \neg \rangle^* \operatorname{up} \mathcal{A}$. Really, $X \in \operatorname{up} \mathcal{A} \Leftrightarrow X \sqsupseteq \mathcal{A} \Leftrightarrow \overline{X} \asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{X} \notin \partial \mathcal{A}$ for any $X \in \mathfrak{Z}$ (taking into account theorems 535, 537, and lemma 551).

□

COROLLARY 568. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. ∂ is an order isomorphism from $\mathfrak{A}$ to $\mathfrak{S}(\mathfrak{Z})$.

PROOF. By properties of the diagram (1). □

COROLLARY 569. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\partial \bigsqcup^{\mathfrak{A}} S = \bigcup \langle \partial \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. $\partial \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{S}(\mathfrak{Z})} \langle \partial \rangle^* S = \bigcup \langle \partial \rangle^* S$.

□

5.17. Generalized Filter Base

DEFINITION 570. *Generalized filter base* is a filter base on the set $\mathfrak{A}$ where $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.

DEFINITION 571. If S is a generalized filter base and $\mathcal{A} = \bigsqcap^{\mathfrak{A}} S$ for some $\mathcal{A} \in \mathfrak{A}$, then we call S a generalized filter base of $\mathcal{A}$.

THEOREM 572. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. For a generalized filter base S of $\mathcal{F} \in \mathfrak{A}$ and $K \in \mathfrak{Z}$ we have

$$K \in \operatorname{up} \mathcal{F} \Leftrightarrow \exists \mathcal{L} \in S : K \in \operatorname{up} \mathcal{L}.$$

PROOF.

1°⇒2°. Obvious.
2°⇒3°.

⇐. Because $\mathcal{F} = \bigsqcap^{\mathfrak{A}} S$.

⇒. Let $K \in \operatorname{up} \mathcal{F}$. Then (taken into account corollary 525 and that S is nonempty) there exist $X_1, \ldots, X_n \in \bigcup \langle \operatorname{up} \rangle^* S$ such that $K \in \operatorname{up}(X_1 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} X_n)$ that is $K \in \operatorname{up}(\uparrow X_1 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} \uparrow X_n)$. Consequently (by theorem 535) $K \in \operatorname{up}(\uparrow X_1 \sqcap^{\mathfrak{A}} \cdots \sqcap^{\mathfrak{A}} \uparrow X_n)$. Replacing every $\uparrow X_i$ with such $\mathcal{X}_i \in S$ that $X_i \in \operatorname{up} \mathcal{X}_i$ (this is obviously possible to do), we get a finite set $T_0 \subseteq S$ such that $K \in \operatorname{up} \bigsqcap^{\mathfrak{A}} T_0$. From this there exists $\mathcal{C} \in S$ such that $\mathcal{C} \sqsubseteq \bigsqcap^{\mathfrak{A}} T_0$ and so $K \in \operatorname{up} \mathcal{C}$.

□

COROLLARY 573. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with least element.
3°. For a generalized filter base S of a $\mathcal{F} \in \mathfrak{A}$ we have

$$\bot^{\mathfrak{A}} \in S \Leftrightarrow \mathcal{F} = \bot^{\mathfrak{A}}.$$

PROOF. Substitute $\bot^{\mathfrak{A}}$ as K. □

THEOREM 574. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with least element.
3°. Let $\mathcal{F}_0 \sqcap^{\mathfrak{A}} \cdots \sqcap^{\mathfrak{A}} \mathcal{F}_n \neq \bot^{\mathfrak{A}}$ for every $\mathcal{F}_0, \ldots, \mathcal{F}_n \in S$, where S is a nonempty set of elements of $\mathfrak{A}$. Then $\bigsqcap^{\mathfrak{A}} S \neq \bot^{\mathfrak{A}}$.

PROOF. Consider the set

$$S' = \left\{ \frac{\mathcal{F}_0 \sqcap^{\mathfrak{A}} \cdots \sqcap^{\mathfrak{A}} \mathcal{F}_n}{\mathcal{F}_0, \ldots, \mathcal{F}_n \in S} \right\}.$$

Obviously S' is nonempty and binarily meet-closed. So S' is a generalized filter base. Obviously $\bot^{\mathfrak{A}} \notin S$. So by properties of generalized filter bases $\bigsqcap^{\mathfrak{A}} S' \neq \bot^{\mathfrak{A}}$. But obviously $\bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{A}} S'$. So $\bigsqcap^{\mathfrak{A}} S \neq \bot^{\mathfrak{A}}$. □

Corollary 575. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with least element.
3°. Let $S \in \mathscr{P}\mathfrak{Z}$ such that $S \neq \emptyset$ and $A_0 \sqcap^{\mathfrak{Z}} \cdots \sqcap^{\mathfrak{Z}} A_n \neq \bot^{\mathfrak{Z}}$ for every $A_0, \ldots, A_n \in S$. Then $\bigsqcap^{\mathfrak{A}} S \neq \bot^{\mathfrak{A}}$.

Proof.

1°⇒2°. Obvious.
2°⇒3°. Because $(\mathfrak{A}, \mathfrak{Z})$ is binarily meet-closed (by the theorem 535).

□

Theorem 576. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a bounded meet-semilattice.
3°. $\mathfrak{A}$ is an atomic lattice.

Proof.

1°⇒2°. Obvious.
2°⇒3°. Let $\mathcal{F} \in \mathfrak{A}$. Let choose (by Kuratowski's lemma) a maximal chain S from $\bot^{\mathfrak{A}}$ to $\mathcal{F}$. Let $S' = S \setminus \{\bot^{\mathfrak{A}}\}$. $a = \bigsqcap^{\mathfrak{A}} S' \neq \bot^{\mathfrak{A}}$ by properties of generalized filter bases (the corollary 573 which uses the fact that $\mathfrak{Z}$ is a meet-semilattice with least element). If $a \notin S$ then the chain S can be extended adding there element a because $\bot^{\mathfrak{A}} \sqsubset a \sqsubseteq \mathcal{X}$ for any $\mathcal{X} \in S'$ what contradicts to maximality of the chain. So $a \in S$ and consequently $a \in S'$. Obviously a is the minimal element of S'. Consequently (taking into account maximality of the chain) there is no $\mathcal{Y} \in \mathfrak{A}$ such that $\bot^{\mathfrak{A}} \sqsubset \mathcal{Y} \sqsubset a$. So a is an atomic filter. Obviously $a \sqsubseteq \mathcal{F}$.

□

Definition 577. A complete lattice is *co-compact* iff $\bigsqcap S = \bot$ for a set S of elements of this lattice implies that there is its finite subset $T \subseteq S$ such that $\bigsqcap T = \bot$.

Theorem 578. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a bounded meet-semilattice.
3°. $\mathfrak{A}$ is co-compact.

Proof.

1°⇒2°. Obvious.
2°⇒3°. Poset $\mathfrak{A}$ is complete by corollary 518.

If $\bot \in \operatorname{up} \bigsqcap^{\mathfrak{A}} S$ then there are $K_i \in \operatorname{up} \bigcup S$ such that $\bot \in \operatorname{up}(K_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} K_n)$ that is $K_0 \sqcap^{\mathfrak{Z}} \ldots \sqcap^{\mathfrak{Z}} K_n = \bot$ from which easily follows $\mathcal{F}_0 \sqcap^{\mathfrak{A}} \ldots \sqcap^{\mathfrak{A}} \mathcal{F}_n = \bot$ for some $\mathcal{F}_i \in S$.

□

5.18. Separability of filters

Proposition 579. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.

3°. $\mathfrak{A}$ is strongly separable.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. By properties of stars of filters.

□

REMARK 580. [14] seems to show that the above theorem cannot be generalized for a wider class of lattices.

THEOREM 581. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\mathfrak{A}$ is an atomistic poset.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Because (used theorem 232) $\mathfrak{A}$ is atomic (theorem 576) and separable.

□

COROLLARY 582. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\mathfrak{A}$ is atomically separable.

PROOF. By theorem 230. □

5.19. Some Criteria

THEOREM 583. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a down-aligned, with join-closed, binarily meet-closed and separable core which is a complete boolean lattice.
4°. The following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:
   (a) $\mathcal{F} \in \mathfrak{Z}$;
   (b) $\forall S \in \mathscr{P}\mathfrak{A} : \left(\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Rightarrow \exists \mathcal{K} \in S : \mathcal{F} \sqcap^{\mathfrak{A}} \mathcal{K} \neq \bot\right)$;
   (c) $\forall S \in \mathscr{P}\mathfrak{Z} : \left(\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Rightarrow \exists K \in S : \mathcal{F} \sqcap^{\mathfrak{A}} K \neq \bot\right)$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is with with join-closed core by theorem 534, binarily meet-closed core by corollary 536, with separable core by theorem 537.
3°⇒4°.
4°a⇒4°b. Let $\mathcal{F} \in \mathfrak{Z}$. Then (taking into account the lemma 551)

$$\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Leftrightarrow \overline{\mathcal{F}} \not\sqsupseteq \bigsqcup^{\mathfrak{A}} S \Rightarrow \exists \mathcal{K} \in S : \overline{\mathcal{F}} \not\sqsupseteq \mathcal{K} \Leftrightarrow \exists \mathcal{K} \in S : \mathcal{F} \sqcap^{\mathfrak{A}} \mathcal{K} \neq \bot.$$

4°b⇒4°c. Obvious.

4°c⇒4°a.

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bot \Rightarrow \exists K \in S : \mathcal{F} \sqcap^{\mathfrak{A}} K \neq \bot \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\mathcal{F} \not\asymp^{\mathfrak{A}} \bigsqcup^{\mathfrak{Z}} S \Rightarrow \exists K \in S : \mathcal{F} \not\asymp^{\mathfrak{A}} K \right) \Leftrightarrow \text{(lemma 551)}$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\overline{\bigsqcup^{\mathfrak{Z}} S} \not\sqsupseteq \mathcal{F} \Rightarrow \exists K \in S : \overline{K} \not\sqsupseteq \mathcal{F} \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\forall K \in S : \overline{K} \sqsupseteq \mathcal{F} \Rightarrow \overline{\bigsqcup^{\mathfrak{Z}} S} \sqsupseteq \mathcal{F} \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\forall K \in S : \overline{K} \sqsupseteq \mathcal{F} \Rightarrow \bigsqcap^{\mathfrak{Z}} \langle \neg \rangle^* S \sqsupseteq \mathcal{F} \right) \Leftrightarrow$$

$$\forall S \in \mathscr{P}\mathfrak{Z} : \left(\forall K \in S : K \sqsupseteq \mathcal{F} \Rightarrow \bigsqcap^{\mathfrak{Z}} S \sqsupseteq \mathcal{F} \right) \Rightarrow$$

$$\bigsqcap^{\mathfrak{Z}} \operatorname{up} \mathcal{F} \sqsupseteq \mathcal{F} \Leftrightarrow$$

$$\bigsqcap^{\mathfrak{Z}} \operatorname{up} \mathcal{F} \in \operatorname{up} \mathcal{F} \Rightarrow$$

$$\mathcal{F} \in \mathfrak{Z}.$$

□

Remark 584. The above theorem strengthens theorem 53 in [30]. Both the formulation of the theorem and the proof are considerably simplified.

Definition 585. Let S be a subset of a meet-semilattice. The *filter base generated by* S is the set

$$[S]_{\sqcap} = \left\{ \frac{a_0 \sqcap \cdots \sqcap a_n}{a_i \in S, n = 0, 1, \ldots} \right\}.$$

Lemma 586. The set of all finite subsets of an infinite set A has the same cardinality as A.

Proof. Let denote the number of n-element subsets of A as s_n. Obviously $s_n \leq \operatorname{card} A^n = \operatorname{card} A$. Then the number S of all finite subsets of A is equal to

$$s_0 + s_1 + \cdots \leq \operatorname{card} A + \operatorname{card} A + \cdots = \operatorname{card} A.$$

That $S \geq \operatorname{card} A$ is obvious. So $S = \operatorname{card} A$. □

Lemma 587. A filter base generated by an infinite set has the same cardinality as that set.

Proof. From the previous lemma. □

Definition 588. Let $\mathfrak{A}$ be a complete lattice. A set $S \in \mathscr{P}\mathfrak{A}$ is *filter-closed* when for every filter base $T \in \mathscr{P}S$ we have $\bigsqcap T \in S$.

Theorem 589. A subset S of a complete lattice is filter-closed iff for every nonempty chain $T \in \mathscr{P}S$ we have $\bigsqcap T \in S$.

Proof. (proof sketch by Joel David Hamkins)

⇒. Because every nonempty chain is a filter base.

$\Leftarrow$. We will assume that cardinality of a set is an ordinal defined by von Neumann cardinal assignment (what is a standard practice in ZFC). Recall that $\alpha < \beta \Leftrightarrow \alpha \in \beta$ for ordinals α, β.

We will take it as given that for every nonempty chain $T \in \mathscr{P}S$ we have $\bigsqcap T \in S$.

We will prove the following statement: If $\operatorname{card} S = n$ then S is filter closed, for any cardinal n.

Instead we will prove it not only for cardinals but for wider class of ordinals: If $\operatorname{card} S = n$ then S is filter-closed, for any ordinal n.

We will prove it using transfinite induction by n.

For finite n we have $\bigsqcap T \in S$ because $T \subseteq S$ has minimal element.

Let $\operatorname{card} T = n$ be an infinite ordinal.

Let the assumption hold for every $m \in \operatorname{card} T$.

We can assign $T = \left\{ \frac{a_\alpha}{\alpha \in \operatorname{card} T} \right\}$ for some a_α because $\operatorname{card} \operatorname{card} T = \operatorname{card} T$.

Consider $\beta \in \operatorname{card} T$.

Let $P_\beta = \left\{ \frac{a_\alpha}{\alpha \in \beta} \right\}$. Let $b_\beta = \bigsqcap P_\beta$. Obviously $b_\beta = \bigsqcap [P_\beta]_\sqcap$. We have

$$\operatorname{card}[P_\beta]_\sqcap = \operatorname{card} P_\beta = \operatorname{card} \beta < \operatorname{card} T$$

(used the lemma and von Neumann cardinal assignment). By the assumption of induction $b_\beta \in S$.

$\forall \beta \in \operatorname{card} T : P_\beta \subseteq T$ and thus $b_\beta \sqsupseteq \bigsqcap T$.

It is easy to see that the set $\left\{ \frac{P_\beta}{\beta \in \operatorname{card} T} \right\}$ is a chain. Consequently $\left\{ \frac{b_\beta}{\beta \in \operatorname{card} T} \right\}$ is a chain.

By the theorem conditions $b = \bigsqcap_{\beta \in \operatorname{card} T} b_\beta \in S$ (taken into account that $b_\beta \in S$ by the assumption of induction).

Obviously $b \sqsupseteq \bigsqcap T$.

$b \sqsubseteq b_\beta$ and so $\forall \beta \in \operatorname{card} T, \alpha \in \beta : b \sqsubseteq a_\alpha$. Let $\alpha \in \operatorname{card} T$. Then (because $\operatorname{card} T$ is a limit ordinal, see [44]) there exists $\beta \in \operatorname{card} T$ such that $\alpha \in \beta \in \operatorname{card} T$. So $b \sqsubseteq a_\alpha$ for every $\alpha \in \operatorname{card} T$. Thus $b \sqsubseteq \bigsqcap T$.

Finally $\bigsqcap T = b \in S$.

□

5.20. Co-Separability of Core

THEOREM 590. The following is an implications tuple.

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet infinite distributive complete lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned filtered filtrator whose core is a meet infinite distributive complete lattice.
4°. This filtrator is with co-separable core.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. It is obviously up-aligned, and filtered by theorem 534.
3°⇒4°. Our filtrator is with join-closed core (theorem 534).

Let $a, b \in \mathfrak{A}$. $\operatorname{Cor} a$ and $\operatorname{Cor} b$ exist since $\mathfrak{Z}$ is a complete lattice.

$\operatorname{Cor} a \in \operatorname{down} a$ and $\operatorname{Cor} b \in \operatorname{down} b$ by the theorem 542 since our filtrator is filtered. So we have

$$\begin{aligned}
\exists x \in \operatorname{down} a, y \in \operatorname{down} b : x \sqcup^{\mathfrak{A}} y = \top &\Leftarrow \\
\operatorname{Cor} a \sqcup^{\mathfrak{A}} \operatorname{Cor} b = \top &\Leftrightarrow \text{(by finite join-closedness of the core)} \\
\operatorname{Cor} a \sqcup^{\mathfrak{Z}} \operatorname{Cor} b = \top &\Leftrightarrow \\
\prod^{\mathfrak{Z}} \operatorname{up} a \sqcup^{\mathfrak{Z}} \prod^{\mathfrak{Z}} \operatorname{up} b = \top &\Leftrightarrow \text{(by infinite distributivity)} \\
\prod^{\mathfrak{Z}} \left\{ \frac{x \sqcup^{\mathfrak{Z}} y}{x \in \operatorname{up} a, y \in \operatorname{up} b} \right\} = \top &\Leftarrow \\
\forall x \in \operatorname{up} a, y \in \operatorname{up} b : x \sqcup^{\mathfrak{Z}} y = \top &\Leftrightarrow \text{(by binary join-closedness of the core)} \\
\forall x \in \operatorname{up} a, y \in \operatorname{up} b : x \sqcup^{\mathfrak{A}} y = \top &\Leftarrow \\
a \sqcup^{\mathfrak{A}} b = \top&.
\end{aligned}$$

□

5.21. Complements and Core Parts

LEMMA 591. If $(\mathfrak{A}, \mathfrak{Z})$ is a filtered, up-aligned filtrator with co-separable core which is a complete lattice, then for any $a, c \in \mathfrak{A}$

$$c \equiv^{\mathfrak{A}} a \Leftrightarrow c \equiv^{\mathfrak{A}} \operatorname{Cor} a.$$

PROOF.

$\Rightarrow$. If $c \equiv^{\mathfrak{A}} a$ then by co-separability of the core exists $K \in \operatorname{down} a$ such that $c \equiv^{\mathfrak{A}} K$. To finish the proof we will show that $K \sqsubseteq \operatorname{Cor} a$. To show this is enough to show that $\forall X \in \operatorname{up} a : K \sqsubseteq X$ what is obvious.

$\Leftarrow$. $\operatorname{Cor} a \sqsubseteq a$ (by theorem 542 using that our filtrator is filtered).

□

THEOREM 592. If $(\mathfrak{A}, \mathfrak{Z})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+ = \overline{\operatorname{Cor} a}$ for every $a \in \mathfrak{A}$.

PROOF. Our filtrator is with join-closed core (theorem 534).

$$\begin{aligned}
a^+ &= \\
\prod^{\mathfrak{A}} \left\{ \frac{c \in \mathfrak{A}}{c \sqcup^{\mathfrak{A}} a = \top^{\mathfrak{A}}} \right\} &= \\
\prod^{\mathfrak{A}} \left\{ \frac{c \in \mathfrak{A}}{c \sqcup^{\mathfrak{A}} \operatorname{Cor} a = \top^{\mathfrak{A}}} \right\} &= \\
\prod^{\mathfrak{A}} \left\{ \frac{c \in \mathfrak{A}}{c \sqsupseteq \overline{\operatorname{Cor} a}} \right\} &= \\
&\overline{\operatorname{Cor} a}
\end{aligned}$$

(used the lemma above and lemma 551). □

COROLLARY 593. If $(\mathfrak{A}, \mathfrak{Z})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice, then $a^+ \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.

THEOREM 594.

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.

3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered complete lattice filtrator with down-aligned, binarily meet-closed, separable core which is a complete boolean lattice.

4°. $a^* = \overline{\operatorname{Cor} a} = \overline{\operatorname{Cor}' a}$ for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2° Obvious.

2°⇒3° It is filtered by theorem 534. It is complete lattice filtrator by 518. It is with binarily meet-closed core (proposition 536), with separable core (theorem 537).

3°⇒4° Our filtrator is with join-closed core (theorem 534).

$$a^* = \bigsqcup^{\mathfrak{A}} \left\{ \frac{c \in \mathfrak{A}}{c \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}.$$

But $c \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}} \Rightarrow \exists C \in \operatorname{up} c : C \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}$. So

$$\begin{aligned} a^* &= \\ \bigsqcup^{\mathfrak{A}} \left\{ \frac{C \in \mathfrak{Z}}{C \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\} &= \\ \bigsqcup^{\mathfrak{A}} \left\{ \frac{C \in \mathfrak{Z}}{a \sqsubseteq \overline{C}} \right\} &= \\ \bigsqcup^{\mathfrak{A}} \left\{ \frac{\overline{C}}{C \in \mathfrak{Z}, a \sqsubseteq C} \right\} &= \\ \bigsqcup^{\mathfrak{A}} \left\{ \frac{\overline{C}}{C \in \operatorname{up} a} \right\} &= \\ \bigsqcup^{\mathfrak{Z}} \left\{ \frac{\overline{C}}{C \in \operatorname{up} a} \right\} &= \\ \overline{\bigsqcap^{\mathfrak{Z}} \left\{ \frac{C}{C \in \operatorname{up} a} \right\}} &= \\ \overline{\bigsqcap^{\mathfrak{Z}} \operatorname{up} a} &= \\ \overline{\operatorname{Cor} a} \end{aligned}$$

(used lemma 551).

$\operatorname{Cor} a = \operatorname{Cor}' a$ by theorem 545. □

THEOREM 595. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.

3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned and up-aligned complete lattice filtrator with binarily meet-closed, separable and co-separable core which is a complete boolean lattice.

4°. $a^* = a^+ = \overline{\operatorname{Cor} a} = \overline{\operatorname{Cor}' a} \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is filtered by the theorem 534. $\mathfrak{A}$ is a complete lattice by corollary 518. $(\mathfrak{A}, \mathfrak{Z})$ is with co-separable core by theorem 590. $(\mathfrak{A}, \mathfrak{Z})$ is binarily meet-closed by proposition 536, with separable core by theorem 537.

3°⇒4°. Comparing two last theorems.

□

THEOREM 596. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete lattice.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a complete lattice filtrator with join-closed separable core which is a complete lattice.
3°. $a^* \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. $\mathfrak{A}$ is a complete lattice by corollary 518. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with join-closed core by theorem 534. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with separable core by theorem 537.

2°⇒3°. $\left\{ \frac{c \in \mathfrak{A}}{c \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\} \supseteq \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$; consequently $a^* \sqsupseteq \bigsqcup^{\mathfrak{A}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$.
But if $c \in \left\{ \frac{c \in \mathfrak{A}}{c \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$ then there exists $A \in \mathfrak{Z}$ such that $A \sqsupseteq c$ and $A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}$ that is $A \in \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$. Consequently $a^* \sqsubseteq \bigsqcup^{\mathfrak{A}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\}$.
We have $a^* = \bigsqcup^{\mathfrak{A}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\} = \bigsqcup^{\mathfrak{Z}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqcap^{\mathfrak{A}} a = \bot^{\mathfrak{A}}} \right\} \in \mathfrak{Z}$.

□

THEOREM 597. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned filtered complete lattice filtrator with co-separable core which is a complete boolean lattice.
4°. a^+ is dual pseudocomplement of a, that is
$$a^+ = \min \left\{ \frac{c \in \mathfrak{A}}{c \sqcup^{\mathfrak{A}} a = \top^{\mathfrak{A}}} \right\}$$
for every $a \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is filtered by the theorem 534. It is with co-separable core by theorem 590. $\mathfrak{A}$ is a complete lattice by corollary 518.

3°⇒4°. Our filtrator is with join-closed core (theorem 534). It's enough to prove that $a^+ \sqcup^{\mathfrak{A}} a = \top^{\mathfrak{A}}$. But $a^+ \sqcup^{\mathfrak{A}} a = \overline{\operatorname{Cor} a} \sqcup^{\mathfrak{A}} a \sqsupseteq \overline{\operatorname{Cor} a} \sqcup^{\mathfrak{A}} \operatorname{Cor} a = \overline{\operatorname{Cor} a} \sqcup^{\mathfrak{Z}} \operatorname{Cor} a = \top^{\mathfrak{A}}$ (used the theorem 542 and the fact that our filtrator is filtered).

□

DEFINITION 598. The *edge part* of an element $a \in \mathfrak{A}$ is $\operatorname{Edg} a = a \setminus \operatorname{Cor} a$, the *dual edge part* is $\operatorname{Edg}' a = a \setminus \operatorname{Cor}' a$.

Knowing core part and edge part or dual core part and dual edge part of an element of a filtrator, the filter can be restored by the formulas:
$$a = \operatorname{Cor} a \sqcup^{\mathfrak{A}} \operatorname{Edg} a \quad \text{and} \quad a = \operatorname{Cor}' a \sqcup^{\mathfrak{A}} \operatorname{Edg}' a.$$

5.22. Core Part and Atomic Elements

PROPOSITION 599. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over an atomistic lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with join-closed core and $\mathfrak{Z}$ be an atomistic lattice.
4°. $\operatorname{Cor}' a = \bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a} \right\}$ for every $a \in \mathfrak{A}$ such that $\operatorname{Cor}' a$ exists.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is with join-closed core by corollary 534.

3°⇒4°.

$$\begin{aligned}\operatorname{Cor}' a &= \\ \bigsqcup^{\mathfrak{Z}} \left\{ \frac{A \in \mathfrak{Z}}{A \sqsubseteq a} \right\} &= \\ \bigsqcup^{\mathfrak{Z}} \left\{ \frac{\bigsqcup^{\mathfrak{Z}} \operatorname{atoms}^{\mathfrak{Z}} A}{A \in \mathfrak{Z}, A \sqsubseteq a} \right\} &= \\ \bigsqcup^{\mathfrak{Z}} \bigcup \left\{ \frac{\operatorname{atoms}^{\mathfrak{Z}} A}{A \in \mathfrak{Z}, A \sqsubseteq a} \right\} &= \\ \bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a} \right\}&.\end{aligned}$$

□

COROLLARY 600. $\operatorname{Cor} a = \uparrow \left\{ \frac{p \in \mathfrak{U}}{\uparrow\{p\} \sqsubseteq a} \right\}$ and $\bigcap a = \left\{ \frac{p \in \mathfrak{U}}{\uparrow\{p\} \sqsubseteq a} \right\}$ for every filter a on a set $\mathfrak{U}$.

PROOF. By proposition 546. □

5.23. Distributivity of Core Part over Lattice Operations

THEOREM 601. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a join-closed filtrator and $\mathfrak{A}$ is a meet-semilattice and $\mathfrak{Z}$ is a meet-semilattice.
4°. $\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b) = \operatorname{Cor}' a \sqcap^{\mathfrak{Z}} \operatorname{Cor}' b$ for every $a, b \in \mathfrak{A}$. whenever $\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b)$, $\operatorname{Cor}' a$, and $\operatorname{Cor}' b$ exist

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is with join-closed core by corollary 534. $\mathfrak{A}$ is a meet-semilattice by corollary 518.

3°⇒4°. We have $\operatorname{Cor}' p \sqsubseteq p$ for every $p \in \mathfrak{A}$ whenever $\operatorname{Cor}' p$ exists, because our filtrator is with join-closed core (theorem 543).

Obviously $\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b) \sqsubseteq \operatorname{Cor}' a$ and $\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b) \sqsubseteq \operatorname{Cor}' b$.

If $x \sqsubseteq \operatorname{Cor}' a$ and $x \sqsubseteq \operatorname{Cor}' b$ for some $x \in \mathfrak{Z}$ then $x \sqsubseteq a$ and $x \sqsubseteq b$, thus $x \sqsubseteq a \sqcap^{\mathfrak{A}} b$ and $x \sqsubseteq \operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b)$.

□

THEOREM 602. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a join-closed filtrator.
4°. $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{Z}} \langle \operatorname{Cor}' \rangle^* S$ for every $S \in \mathscr{P}\mathfrak{A}$ whenever both sides of the equality are defined. Also $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} T = \bigsqcap^{\mathfrak{Z}} T$ for every $T \in \mathscr{P}\mathfrak{Z}$ whenever both sides of the equality are defined.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is with join-closed core by theorem 534. $\mathfrak{A}$ is a complete lattice by corollary 518.

3°⇒4°. We have $\operatorname{Cor}' p \sqsubseteq p$ for every $p \in \mathfrak{A}$ because our filtrator is with join-closed core (theorem 543).

Obviously $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} S \sqsubseteq \operatorname{Cor}' a$ for every $a \in S$.

If $x \sqsubseteq \operatorname{Cor}' a$ for every $a \in S$ for some $x \in \mathfrak{Z}$ then $x \sqsubseteq a$, thus $x \sqsubseteq \bigsqcap^{\mathfrak{A}} S$ and $x \sqsubseteq \operatorname{Cor}' \bigsqcap^{\mathfrak{A}} S$.

So $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{Z}} \langle \operatorname{Cor}' \rangle^* S$. $\operatorname{Cor}' \bigsqcap^{\mathfrak{A}} T = \bigsqcap^{\mathfrak{Z}} T$ trivially follows from this.

□

THEOREM 603. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete atomistic distributive lattice.

3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned filtrator with binarily meet-closed core $\mathfrak{Z}$ which is a complete atomistic lattice and $\mathfrak{A}$ is a complete starrish lattice.

4°. $\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b) = \operatorname{Cor}' a \sqcup^{\mathfrak{Z}} \operatorname{Cor}' b$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(\mathfrak{A}, \mathfrak{Z})$ is filtered by theorem 534. It is with binarily meet-close core by corollary 536. $\mathfrak{A}$ is starrish by corollary 531. $\mathfrak{A}$ is complete by corollary 518.

3°⇒4°. From theorem conditions it follows that $\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b)$ exists.

$\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b) = \bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a \sqcup^{\mathfrak{A}} b} \right\}$ (used proposition 599).

By theorem 558 we have

$$\begin{aligned}
&\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b) = \\
&\bigsqcup^{\mathfrak{Z}} ((\operatorname{atoms}^{\mathfrak{A}}(a \sqcup^{\mathfrak{A}} b)) \cap \mathfrak{Z}) = \\
&\bigsqcup^{\mathfrak{Z}} ((\operatorname{atoms}^{\mathfrak{A}} a \cup \operatorname{atoms}^{\mathfrak{A}} b) \cap \mathfrak{Z}) = \\
&\bigsqcup^{\mathfrak{Z}} ((\operatorname{atoms}^{\mathfrak{A}} a \cap \mathfrak{Z}) \cup (\operatorname{atoms}^{\mathfrak{A}} b \cap \mathfrak{Z})) = \\
&\bigsqcup^{\mathfrak{Z}} (\operatorname{atoms}^{\mathfrak{A}} a \cap \mathfrak{Z}) \sqcup^{\mathfrak{Z}} \bigsqcup^{\mathfrak{Z}} (\operatorname{atoms}^{\mathfrak{A}} b \cap \mathfrak{Z})
\end{aligned}$$

(used the theorem 496). Again using theorem 558, we get

$$\begin{aligned}
&\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b) = \\
&\bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq a} \right\} \sqcup^{\mathfrak{Z}} \bigsqcup^{\mathfrak{Z}} \left\{ \frac{x}{x \text{ is an atom of } \mathfrak{Z}, x \sqsubseteq b} \right\} = \\
&\operatorname{Cor}' a \sqcup^{\mathfrak{Z}} \operatorname{Cor}' b
\end{aligned}$$

(again used proposition 599).

□

See also theorem 167 above.

5.24. Separability criteria

THEOREM 604. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with correct intersection, with binarily meet-closed and separable core.
4°. $B \asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsupseteq \mathcal{A}$ for every $B \in \mathfrak{Z}$, $\mathcal{A} \in \mathfrak{A}$.

Proof.

1°⇒2°. Obvious.
2°⇒3°. Using proposition 549, corollary 536, theorem 537.
3°⇒4°. By the lemma 551.

□

Theorem 605. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator over a boolean lattice with correct joining and co-separable core.
4°. $B \equiv^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \sqsubseteq \mathcal{A}$ for every $B \in \mathfrak{Z}$, $\mathcal{A} \in \mathfrak{A}$.

Proof.

1°⇒2°. Obvious.
2°⇒3°. Using obvious 550, theorem 590.
3°⇒4°. By the lemma 551.

□

5.25. Filtrators over Boolean Lattices

Proposition 606. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a down-aligned and up-aligned binarily meet-closed and binarily join-closed distributive lattice filtrator and $\mathfrak{Z}$ is a boolean lattice.
4°. $a \setminus^{\mathfrak{A}} B = a \sqcap^{\mathfrak{A}} \overline{B}$ for every $a \in \mathfrak{A}$, $B \in \mathfrak{Z}$.

Proof.

1°⇒2°. Obvious.
2°⇒3°. $\mathfrak{A}$ is a distributive lattice by corollary 531. Our filtrator is binarily meet-closed by the corollary 536 and with join-closed core by the theorem 534. It is also up and down aligned.
3°⇒4°.

$$(a \sqcap^{\mathfrak{A}} \overline{B}) \sqcup^{\mathfrak{A}} B = (a \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} (\overline{B} \sqcup^{\mathfrak{A}} B) =$$
$$(a \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} (\overline{B} \sqcup^{\mathfrak{Z}} B) = (a \sqcup^{\mathfrak{A}} B) \sqcap^{\mathfrak{A}} \top = a \sqcup^{\mathfrak{A}} B.$$
$$(a \sqcap^{\mathfrak{A}} \overline{B}) \sqcap^{\mathfrak{A}} B = a \sqcap^{\mathfrak{A}} (\overline{B} \sqcap^{\mathfrak{A}} B) = a \sqcap^{\mathfrak{A}} (\overline{B} \sqcap^{\mathfrak{Z}} B) = a \sqcap^{\mathfrak{A}} \bot = \bot.$$

So $a \sqcap^{\mathfrak{A}} \overline{B}$ is the difference of a and B.

□

Proposition 607. For a primary filtrator over a complete boolean lattice both edge part and dual edge part are always defined.

Proof. Core part and dual core part are defined because the core is a complete lattice. Using the theorem 606. □

Theorem 608. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a complete co-brouwerian atomistic down-aligned lattice filtrator with binarily meet-closed and separable boolean core.

3°. The three expressions of pseudodifference of a and b in theorem 247 are also equal to $\bigsqcup\left\{ \frac{a \sqcap \overline{B}}{B \in \operatorname{up} b} \right\}$.

PROOF.

1°⇒2°. The filtrator of filters on a boolean lattice is:

- complete by corollary 518;
- atomistic by theorem 581;
- co-brouwerian by corollary 531;
- with separable core by theorem 537;
- with binarily meet-closed core by corollary 536.

2°⇒3°. $\bigsqcup\left\{ \frac{z \in \mathscr{F}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\} \sqsubseteq \bigsqcup\left\{ \frac{a \sqcap \overline{B}}{B \in \operatorname{up} b} \right\}$ because

$$z \in \left\{ \frac{z \in \mathscr{F}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\} \Leftrightarrow z \sqsubseteq a \wedge z \sqcap b = \bot \Leftrightarrow \text{(separability)}$$

$$z \sqsubseteq a \wedge \exists B \in \operatorname{up} b : z \sqcap B = \bot \Leftrightarrow \text{(theorem 604)} \Leftrightarrow z \sqsubseteq a \wedge \exists B \in \operatorname{up} b : z \sqsubseteq \overline{B} \Leftrightarrow$$
$$\exists B \in \operatorname{up} b : \left(z \sqsubseteq a \wedge z \sqsubseteq \overline{B}\right) \Leftrightarrow \exists B \in \operatorname{up} b : z \sqsubseteq a \sqcap \overline{B} \Rightarrow$$
$$z \sqsubseteq \bigsqcup\left\{ \frac{a \sqcap \overline{B}}{B \in \operatorname{up} b} \right\}.$$

But $a \sqcap \overline{B} \in \left\{ \frac{z \in \mathscr{F}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ because

$$\left(a \sqcap \overline{B}\right) \sqcap b = a \sqcap \left(\overline{B} \sqcap b\right) \sqsubseteq a \sqcap \left(\overline{B} \sqcap^{\mathfrak{A}} B\right) = a \sqcap \left(\overline{B} \sqcap^{\mathfrak{Z}} B\right) = a \sqcap \bot = \bot$$

and thus

$$a \sqcap \overline{B} \sqsubseteq \bigsqcup\left\{ \frac{z \in \mathscr{F}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$$

so $\bigsqcup\left\{ \frac{z \in \mathscr{F}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\} \sqsupseteq \bigsqcup\left\{ \frac{a \sqcap \overline{B}}{B \in \operatorname{up} b} \right\}$.

□

5.26. Distributivity for an Element of Boolean Core

LEMMA 609. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.

3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned binarily join-closed and binarily meet-closed distributive lattice filtrator over a boolean lattice.

4°. $A \sqcap^{\mathfrak{A}}$ is a lower adjoint of $\overline{A} \sqcup^{\mathfrak{A}}$ for every $A \in \mathfrak{Z}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is binarily join closed by theorem 534. It is binarily meet-closed by corollary 536. It is distributive by corollary 531.

3°⇒4°. We will use the theorem 126.

That $A \sqcap^{\mathfrak{A}}$ and $\overline{A} \sqcup^{\mathfrak{A}}$ are monotone is obvious.

We need to prove (for every $x, y \in \mathfrak{A}$) that

$$x \sqsubseteq \overline{A} \sqcup^{\mathfrak{A}} \left(A \sqcap^{\mathfrak{A}} x\right) \quad \text{and} \quad A \sqcap^{\mathfrak{A}} \left(\overline{A} \sqcup^{\mathfrak{A}} y\right) \sqsubseteq y.$$

Really,

$$\overline{A} \sqcup^{\mathfrak{A}} \left(A \sqcap^{\mathfrak{A}} x\right) = \left(\overline{A} \sqcup^{\mathfrak{A}} A\right) \sqcap^{\mathfrak{A}} \left(\overline{A} \sqcup^{\mathfrak{A}} x\right) =$$
$$\left(\overline{A} \sqcup^{\mathfrak{Z}} A\right) \sqcap^{\mathfrak{A}} \left(\overline{A} \sqcup^{\mathfrak{A}} x\right) = \top \sqcap^{\mathfrak{A}} \left(\overline{A} \sqcup^{\mathfrak{A}} x\right) = \overline{A} \sqcup^{\mathfrak{A}} x \sqsupseteq x$$

and

$$A \sqcap^{\mathfrak{A}} (\overline{A} \sqcup^{\mathfrak{A}} y) = (A \sqcap^{\mathfrak{A}} \overline{A}) \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} y) = (A \sqcap^{\mathfrak{Z}} \overline{A}) \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} y) = \bot \sqcup^{\mathfrak{A}} (A \sqcap^{\mathfrak{A}} y) = A \sqcap^{\mathfrak{A}} y \sqsubseteq y.$$

□

THEOREM 610. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is an up-aligned binarily join-closed and binarily meet-closed distributive lattice filtrator over a boolean lattice.
4°. $A \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} \langle A \sqcap^{\mathfrak{A}} \rangle^* S$ for every $A \in \mathfrak{Z}$ and every set $S \in \mathscr{P}\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. It is binarily join-closed by theorem 534. It is binarily meet-closed by corollary 536. It is distributive by corollary 531.
3°⇒4°. Direct consequence of the lemma.

□

5.27. More about the Lattice of Filters

DEFINITION 611. Atoms of $\mathfrak{F}$ are called *ultrafilters*.

DEFINITION 612. Principal ultrafilters are also called *trivial ultrafilters*.

THEOREM 613. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is central.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. We can conclude that $\mathfrak{A}$ is atomically separable (the corollary 582), with separable core (the theorem 537), and with join-closed core (theorem 534), binarily meet-closed by corollary 536.

We need to prove $Z(\mathfrak{A}) = \mathfrak{Z}$.

Let $\mathcal{X} \in Z(\mathfrak{A})$. Then there exists $\mathcal{Y} \in Z(\mathfrak{A})$ such that $\mathcal{X} \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$ and $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = \top^{\mathfrak{A}}$. Consequently there is $X \in \operatorname{up} \mathcal{X}$ such that $X \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$; we also have $X \sqcup^{\mathfrak{A}} \mathcal{Y} = \top^{\mathfrak{A}}$. Suppose $X \sqsupset \mathcal{X}$. Then there exists $a \in \operatorname{atoms}^{\mathfrak{A}} X$ such that $a \notin \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$. We can conclude also $a \notin \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y}$ (otherwise $X \sqcap^{\mathfrak{A}} \mathcal{Y} \neq \bot^{\mathfrak{A}}$). Thus $a \notin \operatorname{atoms}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y})$ and consequently $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \neq \top^{\mathfrak{A}}$ what is a contradiction. We have $\mathcal{X} = X \in \mathfrak{Z}$.

Let now $X \in \mathfrak{Z}$. Let $Y = \overline{X}$. We have $X \sqcap^{\mathfrak{Z}} Y = \bot^{\mathfrak{A}}$ and $X \sqcup^{\mathfrak{Z}} Y = \top^{\mathfrak{A}}$. Thus $X \sqcap^{\mathfrak{A}} Y = \bigsqcap^{\mathfrak{A}} \{X \sqcap^{\mathfrak{Z}} Y\} = \bot^{\mathfrak{A}}$; $X \sqcap^{\mathfrak{A}} Y = X \sqcap^{\mathfrak{Z}} Y = \top^{\mathfrak{A}}$. We have shown that $X \in Z(\mathfrak{A})$.

□

5.28. More Criteria

THEOREM 614. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. For every $S \in \mathscr{P}\mathfrak{A}$ the condition $\exists \mathcal{F} \in \mathfrak{A} : S = \star\mathcal{F}$ is equivalent to conjunction of the following items:

(a) S is a free star on $\mathfrak{A}$;
(b) S is filter-closed.

PROOF.

1°⇒2°. Obvious.

2°⇒3°.

⇒.

3°a. That $\bot^{\mathfrak{A}} \notin \star\mathcal{F}$ is obvious. For every $a, b \in \mathfrak{A}$

$$\begin{aligned} a \sqcup^{\mathfrak{A}} b \in \star\mathcal{F} \Leftrightarrow \\ (a \sqcup^{\mathfrak{A}} b) \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ (a \sqcap^{\mathfrak{A}} \mathcal{F}) \sqcup^{\mathfrak{A}} (b \sqcap^{\mathfrak{A}} \mathcal{F}) \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ a \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}} \vee b \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ a \in \star\mathcal{F} \vee \in \star\mathcal{F} \end{aligned}$$

(taken into account corollary 531). So $\star\mathcal{F}$ is a free star on $\mathfrak{A}$.

3°b. We have a filter base $T \subseteq S$ and need to prove that $\bigsqcap^{\mathfrak{A}} T \sqcap \mathcal{F} \neq \bot^{\mathfrak{A}}$. Because $\langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* T$ is a generalized filter base, $\bot^{\mathfrak{A}} \in \langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* T \Leftrightarrow \bigsqcap^{\mathfrak{A}} \langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* T = \bot^{\mathfrak{A}} \Leftrightarrow \bigsqcap^{\mathfrak{A}} T \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}}$. So it is left to prove $\bot^{\mathfrak{A}} \notin \langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* T$ what follows from $T \subseteq S$.

⇐. Let S be a free star on $\mathfrak{A}$. Then for every $A, B \in \mathfrak{Z}$

$$\begin{aligned} A, B \in S \cap \mathfrak{Z} \Leftrightarrow \\ A, B \in S \Leftrightarrow \\ A \sqcup^{\mathfrak{A}} B \in S \Leftrightarrow \\ A \sqcup^{\mathfrak{Z}} B \in S \Leftrightarrow \\ A \sqcup^{\mathfrak{Z}} B \in S \cap \mathfrak{Z} \end{aligned}$$

(taken into account the theorem 534). So $S \cap \mathfrak{Z}$ is a free star on $\mathfrak{Z}$. Thus there exists $\mathcal{F} \in \mathfrak{A}$ such that $\partial\mathcal{F} = S \cap \mathfrak{Z}$. We have $\operatorname{up} \mathcal{X} \subseteq S \Leftrightarrow \mathcal{X} \in S$ (because S is filter-closed) for every $\mathcal{X} \in \mathfrak{A}$; then (taking into account properties of generalized filter bases)

$$\begin{aligned} \mathcal{X} \in S \Leftrightarrow \\ \operatorname{up} \mathcal{X} \subseteq S \Leftrightarrow \\ \operatorname{up} \mathcal{X} \subseteq \partial\mathcal{F} \Leftrightarrow \\ \forall X \in \operatorname{up} \mathcal{X} : X \sqcap^{\mathfrak{A}} \mathcal{F} \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ \bot^{\mathfrak{A}} \notin \langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* \operatorname{up} \mathcal{X} \Leftrightarrow \\ \bigsqcap^{\mathfrak{A}} \langle \mathcal{F} \sqcap^{\mathfrak{A}} \rangle^* \operatorname{up} \mathcal{X} \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ \mathcal{F} \sqcap^{\mathfrak{A}} \bigsqcap^{\mathfrak{A}} \operatorname{up} \mathcal{X} \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ \mathcal{F} \sqcap^{\mathfrak{A}} \mathcal{X} \neq \bot^{\mathfrak{A}} \Leftrightarrow \\ \mathcal{X} \in \star\mathcal{F}. \end{aligned}$$

□

5.29. Filters and a Special Sublattice

Remind that $Z(X)$ is the center of lattice X and Da is the lattice $\left\{ \frac{x \in \mathfrak{A}}{x \sqsubseteq a} \right\}$.

THEOREM 615. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. Let $\mathcal{A} \in \mathfrak{A}$. Then for each $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \in Z(D\mathcal{A}) \Leftrightarrow \exists X \in \mathfrak{Z} : \mathcal{X} = X \sqcap^{\mathfrak{A}} \mathcal{A}.$$

PROOF.

1°⇒2°. Obvious.

2°⇒3°.

⇐. Let $\mathcal{X} = X \sqcap^{\mathfrak{A}} \mathcal{A}$ where $X \in \mathfrak{Z}$. Let also $\mathcal{Y} = \overline{X} \sqcap^{\mathfrak{A}} \mathcal{A}$. Then $\mathcal{X} \sqcap^{\mathfrak{A}} \mathcal{Y} = X \sqcap^{\mathfrak{A}} \overline{X} \sqcap^{\mathfrak{A}} \mathcal{A} = (X \sqcap^{\mathfrak{Z}} \overline{X}) \sqcap^{\mathfrak{A}} \mathcal{A} = \bot^{\mathfrak{A}} \sqcap^{\mathfrak{A}} \mathcal{A} = \bot^{\mathfrak{A}}$ (used corollary 536) and $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = (X \sqcup^{\mathfrak{A}} \overline{X}) \sqcap^{\mathfrak{A}} \mathcal{A} = (X \sqcup^{\mathfrak{Z}} \overline{X}) \sqcap^{\mathfrak{A}} \mathcal{A} = \top^{\mathfrak{A}} \sqcap^{\mathfrak{A}} \mathcal{A} = \mathcal{A}$ (used theorem 534 and corollary 531). So $\mathcal{X} \in Z(D\mathcal{A})$.

⇒. Let $\mathcal{X} \in Z(D\mathcal{A})$. Then there exists $\mathcal{Y} \in Z(D\mathcal{A})$ such that $\mathcal{X} \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$ and $\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} = \mathcal{A}$. Then (used theorem 537) there exists $X \in \operatorname{up} \mathcal{X}$ such that $X \sqcap^{\mathfrak{A}} \mathcal{Y} = \bot^{\mathfrak{A}}$. We have

$$\mathcal{X} = \mathcal{X} \sqcup (X \sqcap^{\mathfrak{A}} \mathcal{Y}) = X \sqcap^{\mathfrak{A}} (\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y}) = X \sqcap^{\mathfrak{A}} \mathcal{A}.$$

□

THEOREM 616. The following is an implication tuple:

1°. $(\mathfrak{A}; \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}; \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. $\mathfrak{F}(Z(D\mathcal{A}))$ is order-isomorphic to $D\mathcal{A}$ by the formulas
- $\mathcal{Y} = \bigsqcap \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(Z(D\mathcal{A}))$;
- $\mathcal{X} = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \mathcal{Y}} \right\}$ for every $\mathcal{Y} \in D\mathcal{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. We need to prove that the above formulas define a bijection, then it becomes evident that it's an order isomorphism (take into account that the order of filters is *reverse* to set inclusion).

First prove that these formulas describe correspondences between $\mathfrak{F}(Z(D\mathcal{A}))$ and $D\mathcal{A}$.

Let $\mathcal{X} \in \mathfrak{F}(Z(D\mathcal{A}))$. Consider $\mathcal{Y} = \bigsqcap \mathcal{X}$. Every element of $\mathcal{X}$ is below $\mathcal{A}$, consequently $\mathcal{Y} \in D\mathcal{A}$.

Let now $\mathcal{Y} \in D\mathcal{A}$. Then $\left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \mathcal{Y}} \right\}$ is a filter.

It remains to prove that these correspondences are mutually inverse.

Let $\mathcal{X} = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \mathcal{Y}_0} \right\}$ and $\mathcal{Y}_1 = \bigsqcap \mathcal{X}$ for some $\mathcal{Y}_0 \in D\mathcal{A}$.

$\mathcal{Y}_1 \sqsupseteq \mathcal{Y}_0$ is obvious. By theorem 615 and the condition 2° we have $\mathcal{Y}_1 = \bigsqcap \mathcal{X} \sqsubseteq \bigsqcap^{\mathfrak{A}} \left\{ \frac{F \sqcap \mathcal{A}}{F \in \operatorname{up} \mathcal{Y}_0} \right\} = \bigsqcap^{\mathfrak{A}} \left\{ \frac{F}{F \in \operatorname{up} \mathcal{Y}_0} \right\} \sqcap \mathcal{A} = \mathcal{Y}_0 \sqcap \mathcal{A} = \mathcal{Y}_0$. So $\mathcal{Y}_1 = \mathcal{Y}_0$.

Let now $\mathcal{Y} = \bigsqcap \mathcal{X}_0$ and $\mathcal{X}_1 = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \mathcal{Y}} \right\}$ for some $\mathcal{X}_0 \in \mathfrak{F}(Z(D\mathcal{A}))$.

$\mathcal{X}_1 = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \sqsupseteq \bigsqcap \mathcal{X}_0} \right\} =$ (by generalized filter bases) $= \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\exists X \in \mathcal{X}_0 : \mathcal{F} \sqsupseteq X} \right\} = \left\{ \frac{\mathcal{F} \in Z(D\mathcal{A})}{\mathcal{F} \in \mathcal{X}_0} \right\} = \mathcal{X}_0$ because $\mathcal{F} \in \mathcal{X}_0 \Leftrightarrow \exists X \in \mathcal{X}_0 : \mathcal{F} \sqsupseteq X$ if $\mathcal{F} \in Z(D\mathcal{A})$.

□

5.30. Distributivity of quasicomplements

THEOREM 617. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned and up-aligned complete lattice filtrator with binarily meet-closed, separable and co-separable core which is a complete boolean lattice.
4°. $(a \sqcap^{\mathfrak{A}} b)^* = (a \sqcap^{\mathfrak{A}} b)^+ = a^* \sqcup^{\mathfrak{A}} b^* = a^+ \sqcup^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is filtered by the theorem 534. $\mathfrak{A}$ is a complete lattice by corollary 518. $(\mathfrak{A}, \mathfrak{Z})$ is with co-separable core by theorem 590. $(\mathfrak{A}, \mathfrak{Z})$ is binarily meet-closed by proposition 536, with separable core by theorem 537.

3°⇒4°. Theorem 595 apply. Also theorem 601 apply because every filtered filtrator is join-closed. So

$$(a \sqcap^{\mathfrak{A}} b)^* = (a \sqcap^{\mathfrak{A}} b)^+ = \overline{\operatorname{Cor}(a \sqcap^{\mathfrak{A}} b)} = \overline{\operatorname{Cor} a \sqcap^{\mathfrak{Z}} \operatorname{Cor} b} = \overline{\operatorname{Cor} a} \sqcup^{\mathfrak{A}} \overline{\operatorname{Cor} b} = a^+ \sqcup^{\mathfrak{A}} b^+ = a^* \sqcup^{\mathfrak{A}} b^*.$$

□

THEOREM 618. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered starrish down-aligned and up-aligned complete lattice filtrator with binarily meet-closed, separable and co-separable core which is a complete atomistic boolean lattice.
3°. $(a \sqcup^{\mathfrak{A}} b)^* = (a \sqcup^{\mathfrak{A}} b)^+ = a^* \sqcap^{\mathfrak{A}} b^* = a^+ \sqcap^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered (theorem 534), distributive (corollary 531) complete lattice filtrator (corollary 518), with binarily meet-closed core (corollary 536), with separable core (theorem 537), with co-separable core (theorem 590).

2°⇒3°. $(a \sqcup^{\mathfrak{A}} b)^+ = (a \sqcup^{\mathfrak{A}} b)^* = \overline{\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}' a \sqcup^{\mathfrak{Z}} \operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \sqcap^{\mathfrak{Z}} \overline{\operatorname{Cor}' b} = a^* \sqcap^{\mathfrak{Z}} b^* = a^* \sqcap^{\mathfrak{A}} b^* = a^+ \sqcap^{\mathfrak{A}} b^+$ (used theorems 594, 603, 595).

□

THEOREM 619. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered complete lattice filtrator with down-aligned, binarily meet-closed, separable core which is a complete boolean lattice.
4°. $(a \sqcap^{\mathfrak{A}} b)^* = a^* \sqcup^{\mathfrak{A}} b^*$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is filtered by theorem 534. It is complete lattice filtrator by 518. It is with binarily meet-closed core (corollary 536), with separable core (theorem 537).

3°⇒4°. It is join closed because it is filtered. $(a \sqcap^{\mathfrak{A}} b)^* = \overline{\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}' a \sqcap^{\mathfrak{Z}} \operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \sqcup^{\mathfrak{Z}} \overline{\operatorname{Cor}' b} = a^* \sqcup^{\mathfrak{Z}} b^* = a^* \sqcup^{\mathfrak{A}} b^*$ (theorems 601, 594).

□

THEOREM 620. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered starrish down-aligned complete lattice filtrator with binarily meet-closed, separable core which is a complete atomistic boolean lattice.
3°. $(a \sqcup^{\mathfrak{A}} b)^* = a^* \sqcap^{\mathfrak{A}} b^*$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered (theorem 534), distributive (corollary 531) complete lattice filtrator (corollary 518), with binarily meet-closed core (corollary 536), with separable core (theorem 537).

2°⇒3°. $(a \sqcup^{\mathfrak{A}} b)^* = \overline{\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}' a \sqcup^{\mathfrak{Z}} \operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \sqcap^{\mathfrak{Z}} \overline{\operatorname{Cor}' b} = a^* \sqcap^{\mathfrak{Z}} b^* = a^* \sqcap^{\mathfrak{A}} b^*$ (used theorems 594, 603).

□

THEOREM 621. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered up-aligned complete lattice filtrator with co-separable core which is a complete boolean lattice.
4°. $(a \sqcap^{\mathfrak{A}} b)^+ = a^+ \sqcup^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is filtered by theorem 534, is a complete lattice by corollary 518, is with co-separable core by theorem 590.

3°⇒4°. $(a \sqcap^{\mathfrak{A}} b)^+ = \overline{\operatorname{Cor}(a \sqcap^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}'(a \sqcap^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}' a \sqcap^{\mathfrak{Z}} \operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \sqcup^{\mathfrak{Z}} \overline{\operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \sqcup^{\mathfrak{A}} \overline{\operatorname{Cor}' b} = a^+ \sqcup^{\mathfrak{A}} b^+$ using theorems 592, 545, 601 and the fact that filtered filtrator is join-closed.

□

THEOREM 622. The following is an implications tuple:
1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned and up-aligned filtrator with binarily meet-closed core, with co-separable core $\mathfrak{Z}$ which is a complete atomistic boolean lattice and $\mathfrak{A}$ is a complete starrish lattice.
3°. $(a \sqcup^{\mathfrak{A}} b)^+ = a^+ \sqcap^{\mathfrak{A}} b^+$ for every $a, b \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. $(a \sqcup^{\mathfrak{A}} b)^+ = \overline{\operatorname{Cor}(a \sqcup^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}'(a \sqcup^{\mathfrak{A}} b)} = \overline{\operatorname{Cor}' a \sqcup^{\mathfrak{Z}} \operatorname{Cor}' b} = \overline{\operatorname{Cor}' a} \sqcap^{\mathfrak{Z}} \overline{\operatorname{Cor}' b} = \overline{\operatorname{Cor} a} \sqcap^{\mathfrak{A}} \overline{\operatorname{Cor} b} = a^+ \sqcap^{\mathfrak{A}} b^+$ using theorems 592, 545, 603.

□

5.31. Complementive Filters and Factoring by a Filter

DEFINITION 623. Let $\mathfrak{A}$ be a meet-semilattice and $\mathcal{A} \in \mathfrak{A}$. The relation $\sim$ on $\mathfrak{A}$ is defined by the formula

$$\forall X, Y \in \mathfrak{A} : (X \sim Y \Leftrightarrow X \sqcap^{\mathfrak{A}} \mathcal{A} = Y \sqcap^{\mathfrak{A}} \mathcal{A}).$$

PROPOSITION 624. The relation $\sim$ is an equivalence relation.

PROOF.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Obvious.

□

DEFINITION 625. When $X, Y \in \mathfrak{Z}$ and $\mathcal{A} \in \mathfrak{A}$ we define $X \sim Y \Leftrightarrow \uparrow X \sim \uparrow Y$.

THEOREM 626. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.
3°. For every $\mathcal{A} \in \mathfrak{A}$ and $X, Y \in \mathfrak{Z}$ we have

$$X \sim Y \Leftrightarrow \exists A \in \operatorname{up} \mathcal{A} : X \sqcap^{\mathfrak{Z}} A = Y \sqcap^{\mathfrak{Z}} A.$$

PROOF.

1°⇒2°. Obvious.

2°⇒3°.

$$\begin{aligned}
\exists A \in \operatorname{up} \mathcal{A} : X \sqcap^{\mathfrak{Z}} A = Y \sqcap^{\mathfrak{Z}} A \Leftrightarrow & \quad \text{(corollary 536)} \\
\exists A \in \operatorname{up} \mathcal{A} : \uparrow X \sqcap^{\mathfrak{A}} \uparrow A = \uparrow Y \sqcap^{\mathfrak{A}} \uparrow A \Rightarrow & \\
\exists A \in \operatorname{up} \mathcal{A} : \uparrow X \sqcap^{\mathfrak{A}} \uparrow A \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \uparrow A \sqcap^{\mathfrak{A}} \mathcal{A} \Leftrightarrow & \\
\exists A \in \operatorname{up} \mathcal{A} : \uparrow X \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \mathcal{A} \Leftrightarrow & \\
\uparrow X \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \mathcal{A} \Leftrightarrow & \\
\uparrow X \sim \uparrow Y \Leftrightarrow & \\
X \sim Y. &
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\uparrow X \sqcap^{\mathfrak{A}} \mathcal{A} = \uparrow Y \sqcap^{\mathfrak{A}} \mathcal{A} \Leftrightarrow & \\
\left\{ \frac{X \sqcap^{\mathfrak{Z}} A_0}{A_0 \in \mathcal{A}} \right\} = \left\{ \frac{Y \sqcap^{\mathfrak{Z}} A_1}{A_1 \in \mathcal{A}} \right\} \Rightarrow & \\
\exists A_0, A_1 \in \operatorname{up} \mathcal{A} : X \sqcap^{\mathfrak{Z}} A_0 = Y \sqcap^{\mathfrak{Z}} A_1 \Rightarrow & \\
\exists A_0, A_1 \in \operatorname{up} \mathcal{A} : X \sqcap^{\mathfrak{Z}} A_0 \sqcap^{\mathfrak{Z}} A_1 = Y \sqcap^{\mathfrak{Z}} A_0 \sqcap^{\mathfrak{Z}} A_1 \Rightarrow & \\
\exists A \in \operatorname{up} \mathcal{A} : Y \sqcap^{\mathfrak{Z}} A = X \sqcap^{\mathfrak{Z}} A. &
\end{aligned}$$

□

PROPOSITION 627. The relation $\sim$ is a congruence[1] for each of the following:

1°. a meet-semilattice $\mathfrak{A}$;
2°. a distributive lattice $\mathfrak{A}$.

PROOF. Let $a_0, a_1, b_0, b_1 \in \mathfrak{A}$ and $a_0 \sim a_1$ and $b_0 \sim b_1$.

1°. $a_0 \sqcap b_0 \sim a_1 \sqcap b_1$because $(a_0 \sqcap b_0) \sqcap \mathcal{A} = a_0 \sqcap (b_0 \sqcap \mathcal{A}) = a_0 \sqcap (b_1 \sqcap \mathcal{A}) = b_1 \sqcap (a_0 \sqcap \mathcal{A}) = b_1 \sqcap (a_1 \sqcap \mathcal{A}) = (a_1 \sqcap b_1) \sqcap \mathcal{A}$.

2°. Taking the above into account, we need to prove only $a_0 \sqcup b_0 \sim a_1 \sqcup b_1$. We have

$$(a_0 \sqcup b_0) \sqcap \mathcal{A} = (a_0 \sqcap \mathcal{A}) \sqcup (b_0 \sqcap \mathcal{A}) = (a_1 \sqcap \mathcal{A}) \sqcup (b_1 \sqcap \mathcal{A}) = (a_1 \sqcup b_1) \sqcap \mathcal{A}.$$

□

DEFINITION 628. We will denote $A/(\sim) = A/((\sim) \cap A \times A)$ for a set A and an equivalence relation $\sim$ on a set $B \supseteq A$. I will call $\sim$ a congruence on A when $(\sim) \cap (A \times A)$ is a congruence on A.

THEOREM 629. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

[1] See Wikipedia for a definition of congruence.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.

3°. Let $\mathcal{A} \in \mathfrak{A}$. Consider the function $\gamma : Z(D\mathcal{A}) \to \mathfrak{Z}/\sim$ defined by the formula (for every $p \in Z(D\mathcal{A})$)

$$\gamma p = \left\{ \frac{X \in \mathfrak{Z}}{X \sqcap^{\mathfrak{A}} \mathcal{A} = p} \right\}.$$

Then:

(a) γ is a lattice isomorphism.

(b) $\forall Q \in q : \gamma^{-1} q = Q \sqcap^{\mathfrak{A}} \mathcal{A}$ for every $q \in \mathfrak{Z}/\sim$.

Proof.

1°⇒2°. Obvious.

2°⇒3°. $\forall p \in Z(D\mathcal{A}) : \gamma p \neq \emptyset$ because of theorem 615. Thus it is easy to see that $\gamma p \in \mathfrak{Z}/\sim$ and that γ is an injection.

Let's prove that γ is a lattice homomorphism:

$\gamma(p_0 \sqcap^{\mathfrak{A}} p_1) = \left\{ \frac{X \in \mathfrak{Z}}{X \sqcap^{\mathfrak{A}} \mathcal{A} = p_0 \sqcap^{\mathfrak{A}} p_1} \right\}$;

$$\gamma p_0 \sqcap^{\mathfrak{Z}/\sim} \gamma p_1 = \left\{ \frac{X_0 \in \mathfrak{Z}}{X_0 \sqcap^{\mathfrak{A}} \mathcal{A} = p_0} \right\} \sqcap^{\mathfrak{Z}/\sim} \left\{ \frac{X_1 \in \mathfrak{Z}}{X_1 \sqcap^{\mathfrak{A}} \mathcal{A} = p_1} \right\} = \left\{ \frac{X_0 \sqcap^{\mathfrak{A}} X_1}{X_0, X_1 \in \mathfrak{Z}, X_0 \sqcap^{\mathfrak{A}} \mathcal{A} = p_0 \wedge X_1 \sqcap^{\mathfrak{A}} \mathcal{A} = p_1} \right\} \subseteq \left\{ \frac{X' \in \mathfrak{Z}}{X' \sqcap^{\mathfrak{A}} \mathcal{A} = p_0 \sqcap^{\mathfrak{A}} p_1} \right\} = \gamma(p_0 \sqcap^{\mathfrak{A}} p_1).$$

Because $\gamma p_0 \sqcap^{\mathfrak{Z}/\sim} \gamma p_1$ and $\gamma(p_0 \sqcap^{\mathfrak{A}} p_1)$ are equivalence classes, thus follows $\gamma p_0 \sqcap^{\mathfrak{Z}/\sim} \gamma p_1 = \gamma(p_0 \sqcap^{\mathfrak{A}} p_1)$.

To finish the proof it is enough to show that $\forall Q \in q : q = \gamma(Q \sqcap^{\mathfrak{A}} \mathcal{A})$ for every $q \in \mathfrak{Z}/\sim$. (From this it follows that γ is surjective because q is not empty and thus $\exists Q \in q : q = \gamma(Q \sqcap^{\mathfrak{A}} \mathcal{A})$.) Really,

$$\gamma(Q \sqcap^{\mathfrak{A}} \mathcal{A}) = \left\{ \frac{X \in \mathfrak{Z}}{X \sqcap^{\mathfrak{A}} \mathcal{A} = Q \sqcap^{\mathfrak{A}} \mathcal{A}} \right\} = [Q] = q.$$

□

This isomorphism is useful in both directions to reveal properties of both lattices $Z(D\mathcal{A})$ and $q \in \mathfrak{Z}/\sim$.

Corollary 630. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.

2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.

3°. $\mathfrak{Z}/\sim$ is a boolean lattice

Proof. Because $Z(D\mathcal{A})$ is a boolean lattice (theorem 98). □

5.32. Pseudodifference of filters

Proposition 631. The following is an implications tuple:

1°. $\mathfrak{A}$ is a lattice of filters on a set.

2°. $\mathfrak{A}$ is a lattice of filters over a boolean lattice.

3°. $\mathfrak{A}$ is an atomistic co-brouwerian lattice.

4°. For every $a, b \in \mathfrak{A}$ the following expressions are always equal:

(a) $a \setminus^* b = \bigsqcap \left\{ \frac{z \in \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$ (quasidifference of a and b);

(b) $a \# b = \bigsqcup\left\{ \frac{z \in \mathfrak{A}}{z \sqsubseteq a \wedge z \sqcap b = \bot} \right\}$ (second quasidifference of a and b);
(c) $\bigsqcup(\operatorname{atoms} a \setminus \operatorname{atoms} b)$.

Proof.
1°⇒2°. Obvious.
2°⇒3°. By corollary 531 and theorem 581.
3°⇒4°. Theorem 247.

Conjecture 632. $a \setminus^* b = a \# b$ for arbitrary filters a, b on powersets is not provable in ZF (without axiom of choice).

□

5.33. Function spaces of posets

Definition 633. Let $\mathfrak{A}_i$ be a family of posets indexed by some set $\operatorname{dom} \mathfrak{A}$. We will define order of indexed families of elements of posets by the formula

$$a \sqsubseteq b \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A} : a_i \sqsubseteq b_i.$$

I will call this new poset $\prod \mathfrak{A}$ *the function space* of posets and the above order *product order*.

Proposition 634. The function space for posets is also a poset.

Proof.
Reflexivity. Obvious.
Antisymmetry. Obvious.
Transitivity. Obvious.

□

Obvious 635. $\mathfrak{A}$ has least element iff each $\mathfrak{A}_i$ has a least element. In this case

$$\bot^{\prod \mathfrak{A}} = \prod_{i \in \operatorname{dom} \mathfrak{A}} \bot^{\mathfrak{A}_i}.$$

Proposition 636. $a \not\asymp b \Leftrightarrow \exists i \in \operatorname{dom} \mathfrak{A} : a_i \not\asymp b_i$ for every $a, b \in \prod \mathfrak{A}$ if every $\mathfrak{A}_i$ has least element.

Proof. If $\operatorname{dom} \mathfrak{A} = \emptyset$, then $a = b = \bot$, $a \asymp b$ and thus the theorem statement holds. Assume $\operatorname{dom} \mathfrak{A} \neq \emptyset$.

$$\begin{aligned}
& a \not\asymp b \Leftrightarrow \\
\exists c \in \prod \mathfrak{A} \setminus \{\bot^{\prod \mathfrak{A}}\} : (c \sqsubseteq a \wedge c \sqsubseteq b) & \Leftrightarrow \\
\exists c \in \prod \mathfrak{A} \setminus \{\bot^{\prod \mathfrak{A}}\} \forall i \in \operatorname{dom} \mathfrak{A} : (c_i \sqsubseteq a_i \wedge c_i \sqsubseteq b_i) & \Leftrightarrow \\
(\text{for the reverse implication take } c_j = \bot^{\mathfrak{A}_j} \text{ for } i \neq j) & \\
\exists i \in \operatorname{dom} \mathfrak{A}, c \in \mathfrak{A}_i \setminus \{\bot^{\mathfrak{A}_i}\} : (c \sqsubseteq a_i \wedge c \sqsubseteq b_i) & \Leftrightarrow \\
\exists i \in \operatorname{dom} \mathfrak{A} : a_i \not\asymp b_i. &
\end{aligned}$$

□

Proposition 637.

1°. If $\mathfrak{A}_i$ are join-semilattices then $\mathfrak{A}$ is a join-semilattice and

$$A \sqcup B = \lambda i \in \operatorname{dom} \mathfrak{A} : Ai \sqcup Bi. \tag{2}$$

2°. If $\mathfrak{A}_i$ are meet-semilattices then $\mathfrak{A}$ is a meet-semilattice and

$$A \sqcap B = \lambda i \in \operatorname{dom} \mathfrak{A} : Ai \sqcap Bi.$$

Proof. It is enough to prove the formula (2).

It's obvious that $\lambda i \in \operatorname{dom} \mathfrak{A} : Ai \sqcup Bi \sqsupseteq A, B$.

Let $C \sqsupseteq A, B$. Then (for every $i \in \operatorname{dom} \mathfrak{A}$) $Ci \sqsupseteq Ai$ and $Ci \sqsupseteq Bi$. Thus $Ci \sqsupseteq Ai \sqcup Bi$ that is $C \sqsupseteq \lambda i \in \operatorname{dom} \mathfrak{A} : Ai \sqcup Bi$. □

Corollary 638. If $\mathfrak{A}_i$ are lattices then $\prod \mathfrak{A}$ is a lattice.

Obvious 639. If $\mathfrak{A}_i$ are distributive lattices then $\prod \mathfrak{A}$ is a distributive lattice.

Proposition 640. If $\mathfrak{A}_i$ are boolean lattices then $\prod \mathfrak{A}$ is a boolean lattice.

Proof. We need to prove only that every element $a \in \prod \mathfrak{A}$ has a complement. But this complement is evidently $\lambda i \in \operatorname{dom} a : \overline{a_i}$. □

Proposition 641. If every $\mathfrak{A}_i$ is a poset then for every $S \in \mathscr{P} \prod \mathfrak{A}$

1°. $\bigsqcup S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i$ whenever every $\bigsqcup_{x \in S} x_i$ exists;
2°. $\bigsqcap S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcap_{x \in S} x_i$ whenever every $\bigsqcap_{x \in S} x_i$ exists.

Proof. It's enough to prove the first formula.

$\left(\lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i\right)_i = \bigsqcup_{x \in S} x_i \sqsupseteq x_i$ for every $x \in S$ and $i \in \operatorname{dom} \mathfrak{A}$.

Let $y \sqsupseteq x$ for every $x \in S$. Then $y_i \sqsupseteq x_i$ for every $i \in \operatorname{dom} \mathfrak{A}$ and thus $y_i \sqsupseteq \bigsqcup_{x \in S} x_i = \left(\lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i\right)_i$ that is $y \sqsupseteq \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i$.

Thus $\bigsqcup S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i$ by the definition of join. □

Corollary 642. If $\mathfrak{A}_i$ are posets then for every $S \in \mathscr{P} \prod \mathfrak{A}$

1°. $\bigsqcup S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcup_{x \in S} x_i$ whenever $\bigsqcup S$ exists;
2°. $\bigsqcap S = \lambda i \in \operatorname{dom} \mathfrak{A} : \bigsqcap_{x \in S} x_i$ whenever $\bigsqcap S$ exists.

Proof. It is enough to prove that (for every i) $\bigsqcup_{x \in S} x_i$ exists whenever $\bigsqcup S$ exists.

Fix $i \in \operatorname{dom} \mathfrak{A}$.

Take $y_i = \left(\bigsqcup S\right)_i$ and let prove that y_i is the least upper bound of $\left\{ \frac{x_i}{x \in S} \right\}$.

y_i is it's upper bound because $\bigsqcup S \sqsupseteq x$ and thus $\left(\bigsqcup S\right)_i \sqsupseteq x_i$ for every $x \in S$.

Let $x \in S$ and for some $t \in \mathfrak{A}_i$

$$T(t) = \lambda j \in \operatorname{dom} \mathfrak{A} : \begin{cases} t & \text{if } i = j \\ x_j & \text{if } i \neq j. \end{cases}$$

Let $t \sqsupseteq x_i$. Then $T(t) \sqsupseteq x$ for every $x \in S$. So $T(t) \sqsupseteq \bigsqcup S$ and consequently $t = T(t)_i \sqsupseteq y_i$.

So y_i is the least upper bound of $\left\{ \frac{x_i}{x \in S} \right\}$. □

Corollary 643. If $\mathfrak{A}_i$ are complete lattices then $\mathfrak{A}$ is a complete lattice.

Obvious 644. If $\mathfrak{A}_i$ are complete (co-)brouwerian lattices then $\mathfrak{A}$ is a (co-)brouwerian lattice.

Proposition 645. If each $\mathfrak{A}_i$ is a separable poset with least element (for some index set n) then $\prod \mathfrak{A}$ is a separable poset.

Proof. Let $a \neq b$. Then $\exists i \in \operatorname{dom} \mathfrak{A} : a_i \neq b_i$. So $\exists x \in \mathfrak{A}_i : (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = \lambda j \in \operatorname{dom} \mathfrak{A} : \begin{cases} x & \text{if } j = i; \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i. \end{cases}$ Then $y \not\asymp a$ and $y \asymp b$. □

OBVIOUS 646. If every $\mathfrak{A}_i$ is a poset with least element, then the set of atoms of $\prod\mathfrak{A}$ is

$$\left\{ \frac{\lambda i \in \operatorname{dom}\mathfrak{A} : \left(\begin{cases} a & \text{if } i = k; \\ \bot^{\mathfrak{A}_i} & \text{if } i \neq k \end{cases} \right)}{k \in \operatorname{dom}\mathfrak{A}, a \in \operatorname{atoms}^{\mathfrak{A}_k}} \right\}.$$

PROPOSITION 647. If every $\mathfrak{A}_i$ is an atomistic poset with least element, then $\prod\mathfrak{A}$ is an atomistic poset.

PROOF. $x_i = \bigsqcup \operatorname{atoms} x_i$ for every $x_i \in \mathfrak{A}_i$. Thus

$$x = \lambda i \in \operatorname{dom} x : x_i = \lambda i \in \operatorname{dom} x : \bigsqcup \operatorname{atoms} x_i =$$
$$\bigsqcup_{i \in \operatorname{dom} x} \lambda j \in \operatorname{dom} x : \begin{cases} x_i & \text{if } j = i \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases} =$$
$$\bigsqcup_{i \in \operatorname{dom} x} \lambda j \in \operatorname{dom} x : \begin{cases} \bigsqcup \operatorname{atoms} x_i & \text{if } j = i \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases} =$$
$$\bigsqcup_{i \in \operatorname{dom} x} \bigsqcup_{q \in \operatorname{atoms} x_i} \lambda j \in \operatorname{dom} x : \begin{cases} q & \text{if } j = i \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases}.$$

Thus x is a join of atoms of $\prod\mathfrak{A}$. □

COROLLARY 648. If $\mathfrak{A}_i$ are atomistic posets with least elements, then $\prod\mathfrak{A}$ is atomically separable.

PROOF. Proposition 230. □

PROPOSITION 649. Let $(\mathfrak{A}_{i\in n}, \mathfrak{Z}_{i\in n})$ be a family of filtrators. Then $(\prod\mathfrak{A}, \prod\mathfrak{Z})$ is a filtrator.

PROOF. We need to prove that $\prod\mathfrak{Z}$ is a sub-poset of $\prod\mathfrak{A}$. First $\prod\mathfrak{Z} \subseteq \prod\mathfrak{A}$ because $\mathfrak{Z}_i \subseteq \mathfrak{A}_i$ for each $i \in n$.

Let $A, B \in \prod\mathfrak{Z}$ and $A \sqsubseteq^{\prod\mathfrak{Z}} B$. Then $\forall i \in n : A_i \sqsubseteq^{\mathfrak{Z}_i} B_i$; consequently $\forall i \in n : A_i \sqsubseteq^{\mathfrak{A}_i} B_i$ that is $A \sqsubseteq^{\prod\mathfrak{A}} B$. □

PROPOSITION 650. Let $(\mathfrak{A}_{i\in n}, \mathfrak{Z}_{i\in n})$ be a family of filtrators.

1°. The filtrator $(\prod\mathfrak{A}, \prod\mathfrak{Z})$ is (binarily) join-closed if every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is (binarily) join-closed.
2°. The filtrator $(\prod\mathfrak{A}, \prod\mathfrak{Z})$ is (binarily) meet-closed if every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is (binarily) meet-closed.

PROOF. Let every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be binarily join-closed. Let $A, B \in \prod\mathfrak{Z}$ and $A \sqcup^{\prod\mathfrak{Z}} B$ exist. Then (by corollary 642)

$$A \sqcup^{\prod\mathfrak{Z}} B = \lambda i \in n : A_i \sqcup^{\mathfrak{Z}_i} B_i = \lambda i \in n : A_i \sqcup^{\mathfrak{A}_i} B_i = A \sqcup^{\prod\mathfrak{A}} B.$$

Let now every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be join-closed. Let $S \in \mathscr{P}\prod\mathfrak{Z}$ and $\bigsqcup^{\prod\mathfrak{Z}} S$ exist. Then (by corollary 642)

$$\bigsqcup^{\prod\mathfrak{Z}} S = \lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcup^{\mathfrak{Z}_i} \left\{ \frac{x_i}{x \in S} \right\} = \lambda i \in \operatorname{dom}\mathfrak{A} : \bigsqcup^{\mathfrak{A}_i} \left\{ \frac{x_i}{x \in S} \right\} = \bigsqcup^{\prod\mathfrak{A}} S.$$

The rest follows from symmetry. □

PROPOSITION 651. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ where $i \in n$ (for some index set n) is a down-aligned filtrator with separable core then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is with separable core.

PROOF. Let $a \neq b$. Then $\exists i \in n : a_i \neq b_i$. So $\exists x \in \mathfrak{Z}_i : (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = \lambda j \in n : \begin{cases} x & \text{if } j = i \\ \bot^{\mathfrak{A}_j} & \text{if } j \neq i \end{cases}$. Then we have $y \not\asymp a$ and $y \asymp b$ and $y \in \mathfrak{Z}$. □

PROPOSITION 652. Let every $\mathfrak{A}_i$ be a bounded lattice. Every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a central filtrator iff $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a central filtrator.

PROOF.

$$x \in Z\left(\prod \mathfrak{A}\right) \Leftrightarrow$$
$$\exists y \in \prod \mathfrak{A} : (x \sqcap y = \bot^{\prod \mathfrak{A}} \wedge x \sqcup y = \top^{\prod \mathfrak{A}}) \Leftrightarrow$$
$$\exists y \in \prod \mathfrak{A} \forall i \in \operatorname{dom} \mathfrak{A} : (x_i \sqcap y_i = \bot^{\mathfrak{A}_i} \wedge x_i \sqcup y_i = \top^{\mathfrak{A}_i}) \Leftrightarrow$$
$$\forall i \in \operatorname{dom} \mathfrak{A} \exists y \in \mathfrak{A}_i : (x_i \sqcap y = \bot^{\mathfrak{A}_i} \wedge x_i \sqcup y = \top^{\mathfrak{A}_i}) \Leftrightarrow$$
$$\forall i \in \operatorname{dom} \mathfrak{A} : x_i \in Z(\mathfrak{A}_i).$$

So

$$Z\left(\prod \mathfrak{A}\right) = \prod \mathfrak{Z} \Leftrightarrow \prod_{i \in \operatorname{dom} \mathfrak{A}} Z(\mathfrak{A}_i) = \prod \mathfrak{Z} \Leftrightarrow$$
$$(\text{because every } \mathfrak{Z}_i \text{ is nonempty}) \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A} : Z(\mathfrak{A}_i) = \mathfrak{Z}_i.$$

□

PROPOSITION 653. For every element a of a product filtrator $(\prod \mathfrak{A}, \prod \mathfrak{Z})$:

1°. $\operatorname{up} a = \prod_{i \in \operatorname{dom} a} \operatorname{up} a_i$;
2°. $\operatorname{down} a = \prod_{i \in \operatorname{dom} a} \operatorname{down} a_i$.

PROOF. We will prove only the first as the second is dual.

$$\operatorname{up} a = \left\{ \frac{c \in \prod \mathfrak{Z}}{c \sqsupseteq a} \right\} = \left\{ \frac{c \in \prod \mathfrak{Z}}{\forall i \in \operatorname{dom} a : c_i \sqsupseteq a_i} \right\} =$$
$$\left\{ \frac{c \in \prod \mathfrak{Z}}{\forall i \in \operatorname{dom} a : c_i \in \operatorname{up} a_i} \right\} = \prod_{i \in \operatorname{dom} a} \operatorname{up} a_i.$$

□

PROPOSITION 654. If every $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ is a prefiltered filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a prefiltered filtrator.

PROOF. Let $a, b \in \prod \mathfrak{A}$ and $a \neq b$. Then there exists $i \in n$ such that $a_i \neq b_i$ and so $\operatorname{up} a_i \neq \operatorname{up} b_i$. Consequently $\prod_{i \in \operatorname{dom} a} \operatorname{up} a_i \neq \prod_{i \in \operatorname{dom} a} \operatorname{up} b_i$ that is $\operatorname{up} a \neq \operatorname{up} b$. □

PROPOSITION 655. Let every $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ be a filtered filtrator with $\operatorname{up} x \neq \emptyset$ for every $x \in \mathfrak{A}_i$ (for every $i \in n$). Then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a filtered filtrator.

PROOF. Let every $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be a filtered filtrator. Let $\operatorname{up} a \supseteq \operatorname{up} b$ for some $a, b \in \prod \mathfrak{A}$. Then $\prod_{i \in \operatorname{dom} a} \operatorname{up} a_i \supseteq \prod_{i \in \operatorname{dom} a} \operatorname{up} b_i$ and consequently (taking into account that $\operatorname{up} x \neq \emptyset$ for every $x \in \mathfrak{A}_i$) $\operatorname{up} a_i \supseteq \operatorname{up} b_i$ for every $i \in n$. Then $\forall i \in n : a_i \sqsubseteq b_i$ that is $a \sqsubseteq b$. □

PROPOSITION 656. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be filtrators and each $\mathfrak{Z}_i$ be a complete lattice with $\operatorname{up} x \neq \emptyset$ for every $x \in \mathfrak{A}_i$ (for every $i \in n$). For $a \in \prod \mathfrak{A}$:

1°. $\operatorname{Cor} a = \lambda i \in \operatorname{dom} a : \operatorname{Cor} a_i$;
2°. $\operatorname{Cor}' a = \lambda i \in \operatorname{dom} a : \operatorname{Cor}' a_i$.

PROOF. We will prove only the first, because the second is dual.

$$\begin{aligned} &\operatorname{Cor} a = \\ &\bigsqcap^{\prod \mathfrak{Z}} \operatorname{up} a = \\ \lambda i \in \operatorname{dom} a : &\bigsqcap^{\mathfrak{Z}_i} \left\{ \frac{x_i}{x \in \operatorname{up} a} \right\} = (\operatorname{up} x \neq \emptyset \text{ taken into account}) \\ \lambda i \in \operatorname{dom} a : &\bigsqcap^{\mathfrak{Z}_i} \left\{ \frac{x}{x \in \operatorname{up} a_i} \right\} = \\ \lambda i \in \operatorname{dom} a : &\bigsqcap^{\mathfrak{Z}_i} \operatorname{up} a_i = \\ \lambda i \in \operatorname{dom} a : &\operatorname{Cor} a_i. \end{aligned}$$

□

PROPOSITION 657. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a filtrator with (co)separable core and each $\mathfrak{A}_i$ has a least (greatest) element, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a filtrator with (co)separable core.

PROOF. We will prove only for separable core, as co-separable core is dual.

$$\begin{gathered} x \asymp^{\prod \mathfrak{A}} y \Leftrightarrow \\ (\text{used the fact that } \mathfrak{A}_i \text{ has a least element}) \\ \forall i \in \operatorname{dom} \mathfrak{A} : x_i \asymp^{\mathfrak{A}_i} y_i \Rightarrow \\ \forall i \in \operatorname{dom} \mathfrak{A} \exists X \in \operatorname{up} x_i : X \asymp^{\mathfrak{A}_i} y_i \Leftrightarrow \\ \exists X \in \operatorname{up} x \forall i \in \operatorname{dom} \mathfrak{A} : X_i \asymp^{\mathfrak{A}_i} y_i \Leftrightarrow \\ \exists X \in \operatorname{up} x : X \asymp^{\prod \mathfrak{A}} y \end{gathered}$$

for every $x, y \in \prod \mathfrak{A}$. □

OBVIOUS 658.

1°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a down-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a down-aligned filtrator.
2°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is an up-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is an up-aligned filtrator.

OBVIOUS 659.

1°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a weakly down-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a weakly down-aligned filtrator.
2°. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a weakly up-aligned filtrator, then $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ is a weakly up-aligned filtrator.

PROPOSITION 660. If every b_i is substractive from a_i where a and b are n-indexed families of elements of distributive lattices with least elements (where n is an index set), then $a \setminus b = \lambda i \in n : a_i \setminus b_i$.

PROOF. We need to prove $(\lambda i \in n : a_i \setminus b_i) \sqcap b = \bot$ and $a \sqcup b = b \sqcup (\lambda i \in n : a_i \setminus b_i)$. Really

$$(\lambda i \in n : a_i \setminus b_i) \sqcap b = \lambda i \in n : (a_i \setminus b_i) \sqcap b_i = \bot;$$
$$b \sqcup (\lambda i \in n : a_i \setminus b_i) = \lambda i \in n : b_i \sqcup (a_i \setminus b_i) = \lambda i \in n : b_i \sqcup a_i = a \sqcup b.$$

□

PROPOSITION 661. If every $\mathfrak{A}_i$ is a distributive lattice, then $a \setminus^* b = \lambda i \in \operatorname{dom} \mathfrak{A} : a_i \setminus^* b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \setminus^* b_i$ is defined.

PROOF. We need to prove that $\lambda i \in \operatorname{dom} \mathfrak{A} : a_i \setminus^* b_i = \bigsqcap \left\{ \frac{z \in \prod \mathfrak{A}}{a \sqsubseteq b \sqcup z} \right\}$.

To prove it is enough to show $a_i \setminus^* b_i = \bigsqcap \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, a \sqsubseteq b \sqcup z} \right\}$ that is $a_i \setminus^* b_i = \bigsqcap \left\{ \frac{z \in \mathfrak{A}_i}{a_i \sqsubseteq b_i \sqcup z} \right\}$ because $z' \in \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, a \sqsubseteq b \sqcup z} \right\} \Leftrightarrow z' \in \left\{ \frac{z \in \mathfrak{A}_i}{a_i \sqsubseteq b_i \sqcup z} \right\}$ (for the reverse implication take $z_j = a_i$ for $j \neq i$), but $a_i \setminus^* b_i = \bigsqcap \left\{ \frac{z \in \mathfrak{A}_i}{a_i \sqsubseteq b_i \sqcup z} \right\}$ is true by definition. □

PROPOSITION 662. If every $\mathfrak{A}_i$ is a distributive lattice with least element, then $a \# b = \lambda i \in \operatorname{dom} \mathfrak{A} : a_i \# b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \# b_i$ is defined.

PROOF. We need to prove that $\lambda i \in \operatorname{dom} \mathfrak{A} : a_i \# b_i = \bigsqcup \left\{ \frac{z \in \prod \mathfrak{A}}{z \sqsubseteq a \wedge z \asymp b} \right\}$.

To prove it is enough to show $a_i \# b_i = \bigsqcup \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, z \sqsubseteq a \wedge z \asymp b} \right\}$ that is $a_i \# b_i = \bigsqcup \left\{ \frac{z_i}{z \in \prod \mathfrak{A}, z_i \sqsubseteq a_i \wedge \forall j \in \operatorname{dom} \mathfrak{A} : z_j \asymp b_j} \right\}$ that is $a_i \# b_i = \bigsqcup \left\{ \frac{z \in \mathfrak{A}_i}{z \sqsubseteq a_i \wedge z \asymp b_i} \right\}$ (take $z_j = \bot^{\mathfrak{A}_j}$ for $j \neq i$) what is true by definition. □

PROPOSITION 663. Let every $\mathfrak{A}_i$ be a poset with least element and a_i^* is defined. Then $a^* = \lambda i \in \operatorname{dom} \mathfrak{A} : a_i^*$.

PROOF. We need to prove that $\lambda i \in \operatorname{dom} \mathfrak{A} : a_i^* = \bigsqcup \left\{ \frac{c \in \prod \mathfrak{A}}{c \asymp a} \right\}$. To prove this it is enough to show that $a_i^* = \bigsqcup \left\{ \frac{c_i}{c \in \prod \mathfrak{A}, c \asymp a} \right\}$ that is $a_i^* = \bigsqcup \left\{ \frac{c_i}{c \in \prod \mathfrak{A}, \forall j \in \operatorname{dom} \mathfrak{A} : c_j \asymp a_j} \right\}$ that is $a_i^* = \bigsqcup \left\{ \frac{c_i}{c \in \prod \mathfrak{A}, c_i \asymp a_i} \right\}$ (take $c_j = \bot^{\mathfrak{A}_j}$ for $j \neq i$) that is $a_i^* = \bigsqcup \left\{ \frac{c \in \mathfrak{A}_i}{c \asymp a_i} \right\}$ what is true by definition. □

COROLLARY 664. Let every $\mathfrak{A}_i$ be a poset with greatest element and a_i^+ is defined. Then $a^+ = \lambda i \in \operatorname{dom} \mathfrak{A} : a_i^+$.

PROOF. By duality. □

5.34. Filters on a Set

In this section we will fix a powerset filtrator $(\mathfrak{A}, \mathfrak{Z}) = (\mathfrak{A}, \mathscr{P}\mathfrak{U})$ for some set $\mathfrak{U}$.

The consideration below is about filters on a set $\mathfrak{U}$, but this can be generalized for filters on complete atomic boolean algebras due complete atomic boolean algebras are isomorphic to algebras of sets on some set $\mathfrak{U}$.

5.34.1. Fréchet Filter.

DEFINITION 665. $\Omega = \left\{ \frac{\mho \setminus X}{X \text{ is a finite subset of } \mho} \right\}$ is called either *Fréchet filter* or *cofinite filter*.

It is trivial that Fréchet filter is a filter.

PROPOSITION 666. $\operatorname{Cor} \Omega = \bot^3$; $\bigcap \Omega = \emptyset$.

PROOF. This can be deduced from the formula $\forall \alpha \in \mho \exists X \in \Omega : \alpha \notin X$. □

THEOREM 667. $\max\left\{ \frac{\mathcal{X} \in \mathfrak{A}}{\operatorname{Cor} \mathcal{X} = \bot^3} \right\} = \max\left\{ \frac{\mathcal{X} \in \mathfrak{A}}{\bigcap \mathcal{X} = \emptyset} \right\} = \Omega$.

PROOF. Due the last proposition, it is enough to show that $\operatorname{Cor} \mathcal{X} = \bot^3 \Rightarrow \mathcal{X} \sqsubseteq \Omega$ for every filter $\mathcal{X}$.

Let $\operatorname{Cor} \mathcal{X} = \bot^3$ for some filter $\mathcal{X}$. Let $X \in \Omega$. We need to prove that $X \in \mathcal{X}$.

$X = \mho \setminus \{\alpha_0, \ldots, \alpha_n\}$. $\mho \setminus \{\alpha_i\} \in \mathcal{X}$ because otherwise $\alpha_i \in \uparrow^{-1} \operatorname{Cor} \mathcal{X}$. So $X \in \mathcal{X}$. □

THEOREM 668. $\Omega = \bigsqcup^{\mathfrak{A}} \left\{ \frac{x}{x \text{ is a non-trivial ultrafilter}} \right\}$.

PROOF. It follows from the facts that $\operatorname{Cor} x = \bot^3$ for every non-trivial ultrafilter x, that $\mathfrak{A}$ is an atomistic lattice, and the previous theorem. □

THEOREM 669. Cor is the lower adjoint of $\Omega \sqcup^{\mathfrak{A}} -$.

PROOF. Because both Cor and $\Omega \sqcup^{\mathfrak{A}} -$ are monotone, it is enough (theorem 126) to prove (for every filters $\mathcal{X}$ and $\mathcal{Y}$)

$$\mathcal{X} \sqsubseteq \Omega \sqcup^{\mathfrak{A}} \operatorname{Cor} \mathcal{X} \quad \text{and} \quad \operatorname{Cor}(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) \sqsubseteq \mathcal{Y}.$$

$\operatorname{Cor}(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) = \operatorname{Cor} \Omega \sqcup^3 \operatorname{Cor} \mathcal{Y} = \bot^3 \sqcup^3 \operatorname{Cor} \mathcal{Y} = \operatorname{Cor} \mathcal{Y} \sqsubseteq \mathcal{Y}$.

$\Omega \sqcup^{\mathfrak{A}} \operatorname{Cor} \mathcal{X} \sqsupseteq \operatorname{Edg} \mathcal{X} \sqcup^{\mathfrak{A}} \operatorname{Cor} \mathcal{X} = \mathcal{X}$. □

COROLLARY 670. $\operatorname{Cor} \mathcal{X} = \mathcal{X} \setminus^* \Omega$ for every filter on a set.

PROOF. By theorem 154. □

COROLLARY 671. $\operatorname{Cor} \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} \langle \operatorname{Cor} \rangle^* S$ for any set S of filters on a powerset.

This corollary can be rewritten in elementary terms and proved elementarily:

PROPOSITION 672. $\bigcap \bigcap S = \bigcup_{F \in S} \bigcap F$ for a set S of filters on some set.

PROOF. (by ANDREAS BLASS) The $\supseteq$ direction is rather formal. Consider any one of the sets being intersected on the left side, i.e., any set X that is in all the filters in S, and consider any of the sets being unioned (that's not a word, but you know what I mean) on the right, i.e., $\bigcap F$ for some $F \in S$. Then, since $X \in F$, we have $\bigcap F \subseteq X$. Taking the union over all $F \in S$ (while keeping X fixed), we get that the right side of your equation is $\subseteq X$. Since that's true for all $X \in \bigcap S$, we infer that the right side is a subset of the left side. (This argument seems to work in much greater generality; you just need that the relevant infima (in place of intersections) exist in your poset.)

For the $\subseteq$ direction, consider any element $x \in \bigcap \bigcap S$, and suppose, toward a contradiction, that it is not an element of the union on the right side of your equation. So, for each $F \in S$, we have $x \notin \bigcap F$, and therefore we can find a set $A_F \in F$ with $x \notin A_F$. Let $B = \bigcup_{F \in S} A_F$ and notice that $B \in F$ for every $F \in S$ (because $B \supseteq A_F$). So $B \in \bigcap S$. But, by choice of the A_F's, we have $x \notin B$, contrary to the assumption that $x \in \bigcap \bigcap S$. □

PROPOSITION 673. $\partial \Omega(U)$ is the set of infinite subsets of U.

Proof. $\partial\Omega(U) = \neg\langle\neg\rangle^*\Omega(U)$.

$\langle\neg\rangle^*\Omega$ is the set of finite subsets of U. Thus $\neg\langle\neg\rangle^*\Omega(U)$ is the set of infinite subsets of U. ☐

5.34.2. Number of Filters on a Set.

Definition 674. A collection Y of sets has finite intersection property iff intersection of any finite subcollection of Y is non-empty.

The following was borrowed from [7]. Thanks to Andreas Blass for email support about his proof.

Lemma 675. (by Hausdorff) For an infinite set X there is a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that given any disjoint finite subfamilies $\mathcal{A}$, $\mathcal{B}$, the intersection of sets in $\mathcal{A}$ and complements of sets in $\mathcal{B}$ is nonempty.

Proof. Let
$$X' = \left\{ \frac{(P,Q)}{P \in \mathscr{P}X \text{ is finite}, Q \in \mathscr{P}\mathscr{P}P} \right\}.$$

It's easy to show that $\operatorname{card} X' = \operatorname{card} X$. So it is enough to show this for X' instead of X. Let
$$\mathcal{F} = \left\{ \frac{\left\{ \frac{(P,Q)\in X'}{Y\cap P\in Q} \right\}}{Y \in \mathscr{P}X} \right\}.$$

To finish the proof we show that for every disjoint finite $Y_+ \in \mathscr{P}\mathscr{P}X$ and finite $Y_- \in \mathscr{P}\mathscr{P}X$ there exist $(P,Q) \in X'$ such that
$$\forall Y \in Y_+ : (P,Q) \in \left\{ \frac{(P,Q) \in X'}{Y \cap P \in Q} \right\} \quad \text{and} \quad \forall Y \in Y_- : (P,Q) \notin \left\{ \frac{(P,Q) \in X'}{Y \cap P \in Q} \right\}$$
what is equivalent to existence $(P,Q) \in X'$ such that
$$\forall Y \in Y_+ : Y \cap P \in Q \quad \text{and} \quad \forall Y \in Y_- : Y \cap P \notin Q.$$

For existence of this (P,Q), it is enough existence of P such that intersections $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$.

Really, for each pair of distinct $Y_0, Y_1 \in Y_+ \cup Y_-$ choose a point which lies in one of the sets Y_0, Y_1 and not in an other, and call the set of such points P. Then $Y \cap P$ are different for different $Y \in Y_+ \cup Y_-$. ☐

Corollary 676. For an infinite set X there is a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that for arbitrary disjoint subfamilies $\mathcal{A}$ and $\mathcal{B}$ the set $\mathcal{A} \cup \left\{ \frac{X\setminus A}{A\in\mathcal{B}} \right\}$ has finite intersection property.

Theorem 677. Let X be a set. The number of ultrafilters on X is $2^{2^{\operatorname{card} X}}$ if X is infinite and $\operatorname{card} X$ if X is finite.

Proof. The finite case follows from the fact that every ultrafilter on a finite set is trivial. Let X be infinite. From the lemma, there exists a family $\mathcal{F}$ of $2^{\operatorname{card} X}$ many subsets of X such that for every $\mathcal{G} \in \mathscr{P}\mathcal{F}$ we have $\Phi(\mathcal{F},\mathcal{G}) = \bigsqcap^{\mathfrak{A}} \mathcal{G} \sqcap \bigsqcap^{\mathfrak{A}} \left\{ \frac{X\setminus A}{A\in\mathcal{F}\setminus\mathcal{G}} \right\} \neq \bot^{\mathfrak{A}(X)}$.

This filter contains all sets from $\mathcal{G}$ and does not contain any sets from $\mathcal{F} \setminus \mathcal{G}$. So for every suitable pairs $(\mathcal{F}_0, \mathcal{G}_0)$ and $(\mathcal{F}_1, \mathcal{G}_1)$ there is $A \in \Phi(\mathcal{F}_0, \mathcal{G}_0)$ such that $\overline{A} \in \Phi(\mathcal{F}_1, \mathcal{G}_1)$. Consequently all filters $\Phi(\mathcal{F},\mathcal{G})$ are disjoint. So for every pair $(\mathcal{F},\mathcal{G})$ where $\mathcal{G} \in \mathscr{P}\mathcal{F}$ there exist a distinct ultrafilter under $\Phi(\mathcal{F},\mathcal{G})$, but the number of such pairs $(\mathcal{F},\mathcal{G})$ is $2^{2^{\operatorname{card} X}}$. Obviously the number of all filters is not above $2^{2^{\operatorname{card} X}}$. ☐

COROLLARY 678. The number of filters on $\mho$ is $2^{2^{\operatorname{card} X}}$ if $\mho$ us infinite and $2^{\operatorname{card}\mho}$ if $\mho$ is finite.

PROOF. The finite case is obvious. The infinite case follows from the theorem and the fact that filters are collections of sets and there cannot be more than $2^{2^{\operatorname{card}\mho}}$ collections of sets on $\mho$. □

5.35. Bases on filtrators

DEFINITION 679. A set S of binary relations is a *base* on a filtrator $(\mathfrak{A}, \mathfrak{Z})$ of $f \in \mathfrak{A}$ when all elements of S are above f and $\forall X \in \operatorname{up} f \exists T \in S : T \sqsubseteq X$.

OBVIOUS 680. Every base on an up-aligned filtrator is nonempty.

PROPOSITION 681. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. A set $S \in \mathscr{P}\mathfrak{Z}$ is a base of a filtrator element iff $\bigsqcap^{\mathfrak{A}} S$ exists and S is a base of $\bigsqcap^{\mathfrak{A}} S$.

PROOF.

1°⇒2°, 2°⇒3°. Obvious.

3°⇒4°.

⇐. Obvious.

⇒. Let S be a base of an $f \in \mathfrak{A}$. f is obviously a lower bound of S. Let g be a lower bound of S. Then for every $X \in \operatorname{up} f$ we have $g \sqsubseteq X$ that is $X \in \operatorname{up} g$. Thus $\operatorname{up} f \subseteq \operatorname{up} g$ and thus $f \sqsupseteq g$ that is f is the greatest upper bound of S.

□

PROPOSITION 682. There exists an $f \in \mathfrak{A}$ such that $\operatorname{up} f = S$ iff S is a base and is an upper set (for every set $S \in \mathscr{P}\mathfrak{Z}$).

PROOF.

⇒. If $\operatorname{up} f = S$ then S is an upper set and S is a base of f because $\forall X \in \operatorname{up} f \exists T \in S : T = X$.

⇐. Let S be a base of some filtrator element f and is an upper set. Then for every $X \in \operatorname{up} f$ there is $T \in S$ such that $T \sqsubseteq X$. Thus $X \in S$. We have $\operatorname{up} f \subseteq S$. But $S \subseteq \operatorname{up} f$ is obvious. We have $\operatorname{up} f = S$.

□

PROPOSITION 683. $\operatorname{up} f$ is a base of f for every $f \in \mathfrak{A}$.

PROOF. Denote $S = \operatorname{up} f$. That f is a lower bound of S is obvious.
If $X \in \operatorname{up} f$ then $\exists T \in S : T = X$. Thus S is a base of f. □

PROPOSITION 684. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. $f = \bigsqcap^{\mathfrak{A}} S$ for every base S of an $f \in \mathfrak{A}$.

PROOF.

1°⇒2°, 2°⇒3°. Obvious.

3°⇒4°. f is a lower bound of S by definition.

Let g be a lower bound of S. Then for every $X \in \operatorname{up} f$ there we have $g \sqsubseteq X$ that is $X \in \operatorname{up} g$. Thus $\operatorname{up} f \subseteq \operatorname{up} g$ and thus $f \sqsupseteq g$ that is f is the greatest lower bound of S.

□

PROPOSITION 685. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtered filtrator.
4°. If S is a base on a filtrator, then $\prod^{\mathfrak{A}} S$ exists and $\operatorname{up} \prod^{\mathfrak{A}} S = \bigcup_{K \in S} \operatorname{up} K$.

PROOF.

1°⇒2°, 2°⇒3°. Obvious.

3°⇒4°. $\prod^{\mathfrak{A}} S$ exists because our filtrator is filtered. Above we proved that S is a base of $\prod^{\mathfrak{A}} S$. That $\bigcup_{K \in S} \operatorname{up} K \subseteq \operatorname{up} \prod^{\mathfrak{A}} S$ is obvious. If $X \in \operatorname{up} \prod^{\mathfrak{A}} S$ then by properties of bases we have $K \in S$ such that $K \sqsubseteq X$. Thus $X \in \operatorname{up} K$ and so $X \in \bigcup_{K \in S} \operatorname{up} K$. So $\operatorname{up} \prod^{\mathfrak{A}} S \subseteq \bigcup_{K \in S} \operatorname{up} K$.

□

PROPOSITION 686. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with binarily meet-closed core such that $\forall a \in \mathfrak{A} : \operatorname{up} a \neq \emptyset$.
4°. A base on the filtrator $(\mathfrak{A}; \mathfrak{Z})$ is the same as base of a filter (on $\mathfrak{Z}$).

PROOF.

1°⇒2°. Obvious.

2°⇒3°. Corollary 536.

3°⇒4°.

⇒. Let S be a base of f on the filtrator $(\mathfrak{A}; \mathfrak{Z})$. Then for every $a, b \in S$ we have $a, b \in \operatorname{up} f$ and thus $a \sqcap^{\mathfrak{Z}} b = a \sqcap^{\mathfrak{A}} b \in \operatorname{up} f$. Thus $\exists x \in S : x \sqsubseteq a \sqcap^{\mathfrak{Z}} b$ that is $x \sqsubseteq a \wedge x \sqsubseteq b$. It remains to show that S is nonempty, but this follows from $\operatorname{up} a$ being nonempty.

⇐. Let S be a base of filter f (on $\mathfrak{Z}$). Let $X \in \operatorname{up} f$. Then there is $T \in S$ such that $T \sqsubseteq X$.

□

5.36. Some Counter-Examples

EXAMPLE 687. There exist a bounded distributive lattice which is not lattice with separable center.

PROOF. The lattice with the Hasse diagram[2] on figure 3 is bounded and distributive because it does not contain "diamond lattice" nor "pentagon lattice" as a sublattice [43].

It's center is $\{0, 1\}$. $x \sqcap y = 0$ despite $\operatorname{up} x = \{x, a, 1\}$ but $y \sqcap 1 \neq 0$ consequently the lattice is not with separable center. □

In this section $\mathfrak{A}$ denotes the set of filters on a set.

EXAMPLE 688. There is a separable poset (that is a set with $\star$ being an injection) which is not strongly separable (that is $\star$ isn't order reflective).

[2]See Wikipedia for a definition of Hasse diagrams.

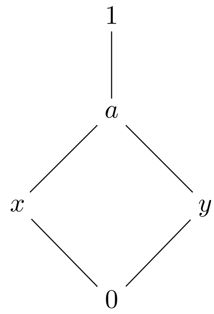

FIGURE 5

PROOF. (with help of sci.math partakers) Consider a poset with the Hasse diagram 4.

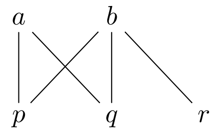

FIGURE 6

Then $\star p = \{p, a, b\}$, $\star q = \{q, a, b\}$, $\star r = \{r, b\}$, $\star a = \{p, q, a, b\}$, $\star b = \{p, q, a, b, r\}$.

Thus $\star x = \star y \Rightarrow x = y$ for any x, y in our poset.

$\star a \subseteq \star b$ but not $a \sqsubseteq b$. $\square$

EXAMPLE 689. There is a prefiltered filtrator which is not filtered.

PROOF. (MATTHIAS KLUPSCH) Take $\mathfrak{A} = \{a, b\}$ with the order being equality and $\mathfrak{Z} = \{b\}$. Then $\operatorname{up} a = \emptyset \sqsubseteq \{b\} = \operatorname{up} b$, so up is injective, hence the filtrator is prefiltered, but because of $a \not\sqsubseteq b$ the filtrator is not filtered. $\square$

For further examples we will use the filter Δ defined by the formula

$$\Delta = \prod^{\mathfrak{A}} \left\{ \frac{]-\epsilon; \epsilon[}{\epsilon \in \mathbb{R}, \epsilon > 0} \right\}$$

and more general

$$\Delta + a = \prod^{\mathfrak{A}} \left\{ \frac{]a-\epsilon; a+\epsilon[}{\epsilon \in \mathbb{R}, \epsilon > 0} \right\}.$$

EXAMPLE 690. There exists $A \in \mathscr{P}U$ such that $\prod^{\mathfrak{A}} A \neq \prod A$.

PROOF. $\prod^{\mathfrak{Z}} \left\{ \frac{]-\epsilon;\epsilon[}{\epsilon \in \mathbb{R}, \epsilon > 0} \right\} = \uparrow \{0\} \neq \Delta$. $\square$

EXAMPLE 691. There exists a set U and a filter a and a set S of filters on the set U such that $a \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S \neq \bigsqcup^{\mathfrak{A}} \langle a \sqcap^{\mathfrak{A}} \rangle^* S$.

PROOF. Let $a = \Delta$ and $S = \left\{ \frac{\uparrow^{\mathbb{R}}]\epsilon; +\infty[}{\epsilon > 0} \right\}$. Then $a \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} S = \Delta \sqcap^{\mathfrak{A}}]0; +\infty[) \neq \bot^{\mathfrak{A}}$ while $\bigsqcup^{\mathfrak{A}} \langle a \sqcap^{\mathfrak{A}} \rangle^* S = \bigsqcup^{\mathfrak{A}} \{\bot^{\mathfrak{A}}\} = \bot^{\mathfrak{A}}$. $\square$

EXAMPLE 692. There are tornings which are not weak partitions.

PROOF. $\left\{\frac{\Delta+a}{a\in\mathbb{R}}\right\}$ is a torning but not weak partition of the real line. $\square$

LEMMA 693. Let $\mathfrak{A}$ be the set of filters on a set U. Then $X\sqcap^{\mathfrak{A}}\Omega\sqsubseteq Y\sqcap^{\mathfrak{A}}\Omega$ iff $X\setminus Y$ is a finite set, having fixed sets $X,Y\in\mathscr{P}U$.

PROOF. Let M be the set of finite subsets of U.

$$\begin{aligned}X\sqcap^{\mathfrak{A}}\Omega\sqsubseteq Y\sqcap^{\mathfrak{A}}\Omega&\Leftrightarrow\\ \left\{\frac{X\cap K_X}{K_X\in\Omega}\right\}\supseteq\left\{\frac{Y\cap K_Y}{K_Y\in\Omega}\right\}&\Leftrightarrow\\ \forall K_Y\in\Omega\exists K_X\in\Omega:Y\cap K_Y=X\cap K_X&\Leftrightarrow\\ \forall L_Y\in M\exists L_X\in M:Y\setminus L_Y=X\setminus L_X&\Leftrightarrow\\ \forall L_Y\in M:X\setminus(Y\setminus L_Y)\in M&\Leftrightarrow\\ X\setminus Y\in M.&\end{aligned}$$

$\square$

EXAMPLE 694. There exists a filter $\mathcal{A}$ on a set U such that $(\mathscr{P}U)/\sim$ and $Z(D\mathcal{A})$ are not complete lattices.

PROOF. Due to the isomorphism it is enough to prove for $(\mathscr{P}U)/\sim$.

Let take $U=\mathbb{N}$ and $\mathcal{A}=\Omega$ be the Fréchet filter on $\mathbb{N}$.

Partition $\mathbb{N}$ into infinitely many infinite sets $A_0,A_1,\ldots$. To withhold our example we will prove that the set $\{[A_0],[A_1],\ldots\}$ has no supremum in $(\mathscr{P}U)/\sim$.

Let $[X]$ be an upper bound of $[A_0],[A_1],\ldots$ that is $\forall i\in\mathbb{N}:X\sqcap^{\mathfrak{A}}\Omega\sqsupseteq A_i\sqcap^{\mathfrak{A}}\Omega$ that is $A_i\setminus X$ is finite. Consequently X is infinite. So $X\cap A_i\neq\emptyset$.

Choose for every $i\in\mathbb{N}$ some $z_i\in X\cap A_i$. The $\{z_0,z_1,\ldots\}$ is an infinite subset of X (take into account that $z_i\neq z_j$ for $i\neq j$). Let $Y=X\setminus\{z_0,z_1,\ldots\}$. Then $Y\sqcap^{\mathfrak{A}}\Omega\sqsupseteq A_i\sqcap^{\mathfrak{A}}\Omega$ because $A_i\setminus Y=A_i\setminus(X\setminus\{z_i\})=(A_i\setminus X)\cup\{z_i\}$ which is finite because $A_i\setminus X$ is finite. Thus $[Y]$ is an upper bound for $\{[A_0],[A_1],\ldots\}$.

Suppose $Y\sqcap^{\mathfrak{A}}\Omega=X\sqcap^{\mathfrak{A}}\Omega$. Then $Y\setminus X$ is finite what is not true. So $Y\sqcap^{\mathfrak{A}}\Omega\sqsubset X\sqcap^{\mathfrak{A}}\Omega$ that is $[Y]$ is below $[X]$. $\square$

5.36.1. Weak and Strong Partition.

DEFINITION 695. A family S of subsets of a countable set is *independent* iff the intersection of any finitely many members of S and the complements of any other finitely many members of S is infinite.

LEMMA 696. The "infinite" at the end of the definition could be equivalently replaced with "nonempty" if we assume that S is infinite.

PROOF. Suppose that some sets from the above definition has a finite intersection J of cardinality n. Then (thanks S is infinite) get one more set $X\in S$ and we have $J\cap X\neq\emptyset$ and $J\cap(\mathbb{N}\setminus X)\neq\emptyset$. So $\operatorname{card}(J\cap X)<n$. Repeating this, we prove that for some finite family of sets we have empty intersection what is a contradiction. $\square$

LEMMA 697. There exists an independent family on $\mathbb{N}$ of cardinality $\mathfrak{c}$.

PROOF. Let C be the set of finite subsets of $\mathbb{Q}$. Since $\operatorname{card}C=\operatorname{card}\mathbb{N}$, it suffices to find $\mathfrak{c}$ independent subsets of C. For each $r\in\mathbb{R}$ let

$$E_r=\left\{\frac{F\in C}{\operatorname{card}(F\cap]-\infty;r[)\text{ is even}}\right\}.$$

All E_{r_1} and E_{r_2} are distinct for distinct $r_1,r_2\in\mathbb{R}$ since we may consider $F=\{r'\}\in C$ where a rational number r' is between r_1 and r_2 and thus F is a member of exactly one of the sets E_{r_1} and E_{r_2}. Thus $\operatorname{card}\left\{\frac{E_r}{r\in\mathbb{R}}\right\}=\mathfrak{c}$.

We will show that $\left\{ \frac{E_r}{r \in \mathbb{R}} \right\}$ is independent. Let $r_1, \ldots, r_k, s_1, \ldots, s_k$ be distinct reals. It is enough to show that these have a nonempty intersection, that is existence of some F such that F belongs to all the E_r and none of E_s.

But this can be easily accomplished taking F having zero or one element in each of intervals to which $r_1, \ldots, r_k, s_1, \ldots, s_k$ split the real line. □

EXAMPLE 698. There exists a weak partition of a filter on a set which is not a strong partition.

PROOF. (suggested by ANDREAS BLASS) Let $\left\{ \frac{X_r}{r \in \mathbb{R}} \right\}$ be an independent family of subsets of $\mathbb{N}$. We can assume $a \neq b \Rightarrow X_a \neq X_b$ due the above lemma.

Let $\mathcal{F}_a$ be a filter generated by X_a and the complements $\mathbb{N} \setminus X_b$ for all $b \in \mathbb{R}$, $b \neq a$. Independence implies that $\mathcal{F}_a \neq \bot^{\mathfrak{A}}$ (by properties of filter bases).

Let $S = \left\{ \frac{\mathcal{F}_r}{r \in \mathbb{R}} \right\}$. We will prove that S is a weak partition but not a strong partition.

Let $a \in \mathbb{R}$. Then $X_a \in \mathcal{F}_a$ while $\forall b \in \mathbb{R} \setminus \{a\} : \mathbb{N} \setminus X_a \in \mathcal{F}_b$ and therefore $\mathbb{N} \setminus X_a \in \bigsqcup^{\mathfrak{A}} \left\{ \frac{\mathcal{F}_b}{\mathbb{R} \ni b \neq a} \right\}$. Therefore $\mathcal{F}_a \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} \left\{ \frac{\mathcal{F}_b}{\mathbb{R} \ni b \neq a} \right\} = \bot^{\mathfrak{A}}$. Thus S is a weak partition.

Suppose S is a strong partition. Then for each set $Z \in \mathscr{P}\mathbb{R}$

$$\bigsqcup^{\mathfrak{A}} \left\{ \frac{\mathcal{F}_b}{b \in Z} \right\} \sqcap^{\mathfrak{A}} \bigsqcup^{\mathfrak{A}} \left\{ \frac{\mathcal{F}_b}{b \in \mathbb{R} \setminus Z} \right\} = \bot^{\mathfrak{A}}$$

what is equivalent to existence of $M(Z) \in \mathscr{P}\mathbb{N}$ such that

$$M(Z) \in \bigsqcup^{\mathfrak{A}} \left\{ \frac{\mathcal{F}_b}{b \in Z} \right\} \quad \text{and} \quad \mathbb{N} \setminus M(Z) \in \bigsqcup^{\mathfrak{A}} \left\{ \frac{\mathcal{F}_b}{b \in \mathbb{R} \setminus Z} \right\}$$

that is

$$\forall b \in Z : M(Z) \in \mathcal{F}_b \quad \text{and} \quad \forall b \in \mathbb{R} \setminus Z : \mathbb{N} \setminus M(Z) \in \mathcal{F}_b.$$

Suppose $Z \neq Z' \in \mathscr{P}\mathbb{N}$. Without loss of generality we may assume that some $b \in Z$ but $b \notin Z'$. Then $M(Z) \in \mathcal{F}_b$ and $\mathbb{N} \setminus M(Z') \in \mathcal{F}_b$. If $M(Z) = M(Z')$ then $\mathcal{F}_b = \bot^{\mathfrak{A}}$ what contradicts to the above.

So M is an injective function from $\mathscr{P}\mathbb{R}$ to $\mathscr{P}\mathbb{N}$ what is impossible due cardinality issues. □

LEMMA 699. (by NIELS DIEPEVEEN, with help of KARL KRONENFELD) Let K be a collection of nontrivial ultrafilters. We have $\bigsqcup K = \Omega$ iff $\exists \mathcal{G} \in K : A \in \operatorname{up} \mathcal{G}$ for every infinite set A.

PROOF.

$\Rightarrow$. Suppose $\bigsqcup K = \Omega$ and let A be a set such that $\nexists \mathcal{G} \in K : A \in \operatorname{up} \mathcal{G}$. Let's prove A is finite.

Really, $\forall \mathcal{G} \in K : \mathfrak{U} \setminus A \in \operatorname{up} \mathcal{G}$; $\mathfrak{U} \setminus A \in \operatorname{up} \Omega$; A is finite.

$\Leftarrow$. Let $\exists \mathcal{G} \in K : A \in \operatorname{up} \mathcal{G}$. Suppose A is a set in $\operatorname{up} \bigsqcup K$.

To finish the proof it's enough to show that $\mathfrak{U} \setminus A$ is finite.

Suppose $\mathfrak{U} \setminus A$ is infinite. Then $\exists \mathcal{G} \in K : \mathfrak{U} \setminus A \in \operatorname{up} \mathcal{G}$; $\exists \mathcal{G} \in K : A \notin \operatorname{up} \mathcal{G}$; $A \notin \operatorname{up} \bigsqcup K$, contradiction.

□

LEMMA 700. (by NIELS DIEPEVEEN) If K is a non-empty set of ultrafilters such that $\bigsqcup K = \Omega$, then for every $\mathcal{G} \in K$ we have $\bigsqcup (K \setminus \{\mathcal{G}\}) = \Omega$.

PROOF. $\exists \mathcal{F} \in K : A \in \operatorname{up} \mathcal{F}$ for every infinite set A.

The set A can be partitioned into two infinite sets A_1, A_2.

Take $\mathcal{F}_1, \mathcal{F}_2 \in K$ such that $A_1 \in \mathcal{F}_1$, $A_2 \in \mathcal{F}_2$.

$\mathcal{F}_1 \neq \mathcal{F}_2$ because otherwise A_1 and A_2 are not disjoint.

Obviously $A \in \mathcal{F}_1$ and $A \in \mathcal{F}_2$.

So there exist two different $\mathcal{F} \in K$ such that $A \in \operatorname{up} \mathcal{F}$. Consequently $\exists \mathcal{F} \in K \setminus \{\mathcal{G}\} : A \in \operatorname{up} \mathcal{F}$ that is $\bigsqcup (K \setminus \{\mathcal{G}\}) = \Omega$. $\square$

Example 701. There exists a filter on a set which cannot be weakly partitioned into ultrafilters.

Proof. Consider cofinite filter Ω on any infinite set.

Suppose K is its weak partition into ultrafilters. Then $x \asymp \bigsqcup (K \setminus \{x\})$ for some ultrafilter $x \in K$.

We have $\bigsqcup (K \setminus \{x\}) \sqsubset \bigsqcup K$ (otherwise $x \sqsubseteq \bigsqcup (K \setminus \{x\})$) what is impossible due the last lemma. $\square$

Corollary 702. There exists a filter on a set which cannot be strongly partitioned into ultrafilters.

5.37. Open problems about filters

Under which conditions $a \setminus^* b$ and $a \# b$ are complementive to a?

Generalize straight maps for arbitrary posets.

5.38. Further notation

Below to define funcoids and reloids we need a fixed powerset filtrator.

Let $(\mathscr{F} A, \mathscr{T} A)$ be an arbitrary but fixed powerset filtrator. This filtrator exists by the theorem 462.

I will call elements of $\mathscr{F}$ *filter objects*.

For brevity we will denote lattice operations on $\mathscr{F} A$ without indexes (for example, take $\bigsqcap S = \bigsqcap^{\mathscr{F} A} S$ for $S \in \mathscr{P} \mathscr{F} A$).

Note that above we also took operations on $\mathscr{T} A$ without indexes (for example, take $\bigsqcap S = \bigsqcap^{\mathscr{T} A} S$ for $S \in \mathscr{P} \mathscr{T} A$).

Because we identify $\mathscr{T} A$ with principal elements of $\mathscr{F} A$, the notation like $\bigsqcap S$ for $S \in \mathscr{P} \mathscr{T} A$ would be inconsistent (it can mean both $\bigsqcap^{\mathscr{T} A} S$ or $\bigsqcap^{\mathscr{F} A} S$). We explicitly state that $\bigsqcap S$ in this case does *not* mean $\bigsqcap^{\mathscr{F} A} S$.

For $\mathcal{X} \in \mathscr{F}$ we will denote $\operatorname{GR} \mathcal{X}$ the corresponding filter on $\mathscr{P} A$. It is a convenient notation to describe relations between filters and sets, consider for example the formula: $\{x\} \subseteq \bigcap \operatorname{GR} \mathcal{X}$.

We will denote lattice operations without pointing a specific set like $\bigsqcap^{\mathscr{F}} S = \bigsqcap^{\mathscr{F}(A)} S$ for a set $S \in \mathscr{P} \mathscr{F}(A)$.

5.39. Equivalent filters and rebase of filters

Throughout this section we will assume that $\mathfrak{Z}$ is a lattice.

An important example: $\mathfrak{Z}$ is the lattice of all small (regarding some Grothendieck universe) sets. (This $\mathfrak{Z}$ is not a powerset, and even not a complete lattice.)

Throughout this section I will use the word *filter* to denote a filter on a sublattice DA where $A \in \mathfrak{Z}$ (if not told explicitly to be a filter on some other set).

The following is an embedding from filters $\mathcal{A}$ on a lattice DA into the lattice of filters on $\mathfrak{Z}$: $\mathscr{S} \mathcal{A} = \left\{ \frac{K \in \mathfrak{Z}}{\exists X \in \mathcal{A} : X \sqsubseteq K} \right\}$.

Proposition 703. Values of this embedding are filters on the lattice $\mathfrak{Z}$.

Proof. That $\mathscr{S} \mathcal{A}$ is an upper set is obvious.

Let $P, Q \in \mathscr{S} \mathcal{A}$. Then $P, Q \in \mathfrak{Z}$ and there is an $X \in \mathcal{A}$ such that $X \sqsubseteq P$ and $Y \in \mathcal{A}$ such that $Y \sqsubseteq Q$. So $X \sqcap Y \in \mathcal{A}$ and $P \sqcap Q \sqsupseteq X \sqcap Y \in \mathcal{A}$, so $P \sqcap Q \in \mathscr{S} \mathcal{A}$. $\square$

5.39.1. Rebase of filters.

Definition 704. *Rebase* for every filter $\mathcal{A}$ and every $A \in \mathfrak{Z}$ is $\mathcal{A} \div A = \bigsqcap \left\{ \frac{\uparrow^A (X \sqcap A)}{X \in \mathcal{A}} \right\}$.

Obvious 705. $\langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}$ is a filter on A.

Proposition 706. The rebase conforms to the formula

$$\mathcal{A} \div A = \langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}.$$

Proof. We know that $\langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}$ is a filter.

If $P \in \langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}$ then $P \in \mathscr{P}A$ and $Y \sqcap A \sqsubseteq P$ for some $Y \in \mathcal{A}$. Thus $P \sqsupseteq Y \sqcap A \in \bigsqcap \left\{ \frac{\uparrow^A (Y \sqcap A)}{Y \in \mathcal{A}} \right\}$.

If $P \in \bigsqcap \left\{ \frac{\uparrow^A (X \sqcap A)}{X \in \mathcal{A}} \right\}$ then by properties of generalized filter bases, there exists $X \in \mathcal{A}$ such that $P \sqsupseteq X \sqcap A$. Also $P \in \mathscr{P}A$. Thus $P \in \langle A\sqcap\rangle^* \mathscr{S}\mathcal{A}$. □

Proposition 707. $\mathcal{X} \div \operatorname{Base}(\mathcal{X}) = \mathcal{X}$.

Proof. Because $X \sqcap \operatorname{Base}(\mathcal{X}) = X$ for $X \in \mathcal{X}$. □

Proposition 708. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \sqsubseteq A$.

Proof. $(\mathcal{X} \div A) \div B = \bigsqcap \left\{ \frac{\uparrow^B (Y \sqcap B)}{Y \in \bigsqcap \left\{ \frac{\uparrow^A (X \sqcap A)}{X \in \mathcal{X}} \right\}} \right\} = \bigsqcap \left\{ \frac{\uparrow^B (X \sqcap A)}{X \in \mathcal{X}} \right\} \sqcap \uparrow^B B = \bigsqcap \left\{ \frac{\uparrow^B (X \sqcap A \sqcap B)}{X \in \mathcal{X}} \right\} = \bigsqcap \left\{ \frac{\uparrow^B (X \sqcap B)}{X \in \mathcal{X}} \right\} = \mathcal{X} \div B$. □

Proposition 709. If $A \in \mathcal{A}$ then $\mathcal{A} \div A = \mathcal{A} \cap \mathscr{P}A$.

Proof. $\mathcal{A} \div A = \langle A\sqcap\rangle^* \mathscr{S}\mathcal{A} = \langle A\sqcap\rangle^* \left\{ \frac{K \in \mathfrak{Z}}{\exists X \in \mathcal{A} : X \sqsubseteq K} \right\} = \left\{ \frac{K \in \mathfrak{Z}}{K \in \mathcal{A} \wedge K \in \mathscr{P}A} \right\} = \mathcal{A} \cap \mathscr{P}A$. □

Proposition 710. Let filters $\mathcal{X}$ and $\mathcal{Y}$ be such that $\operatorname{Base}(\mathcal{X}) = \operatorname{Base}(\mathcal{Y}) = B$. Then $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} = \mathcal{Y}$ for every $\mathfrak{Z} \ni C \sqsupseteq B$.

Proof. $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} = \mathcal{Y} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} \Leftrightarrow \mathcal{X} = \mathcal{Y}$. □

5.39.2. Equivalence of filters.

Definition 711. Two filters $\mathcal{A}$ and $\mathcal{B}$ (with possibly different base sets) are equivalent ($\mathcal{A} \sim \mathcal{B}$) iff there exists an $X \in \mathfrak{Z}$ such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$.

Proposition 712. $\mathcal{X}$ and $\mathcal{Y}$ are equivalent iff ($\mathcal{X} \sim \mathcal{Y}$) iff $\mathcal{Y} = \mathcal{X} \div \operatorname{Base}(\mathcal{Y})$ and $\mathcal{X} = \mathcal{Y} \div \operatorname{Base}(\mathcal{X})$.

Proof.

$\Rightarrow$. Suppose $\mathcal{X} \sim \mathcal{Y}$ that is there exists a set P such that $\mathscr{P}P \cap \mathcal{X} = \mathscr{P}P \cap \mathcal{Y}$ and $P \in \mathcal{X}$, $P \in \mathcal{Y}$. Then $\mathcal{X} \div \operatorname{Base}(\mathcal{Y}) = (\mathscr{P}P \cap \mathcal{X}) \cup \left\{ \frac{K \in \mathscr{P} \operatorname{Base}(\mathcal{Y})}{K \sqsupseteq P} \right\} = (\mathscr{P}P \cap \mathcal{Y}) \cup \left\{ \frac{K \in \mathscr{P} \operatorname{Base}(\mathcal{Y})}{K \sqsupseteq P} \right\} = \mathcal{Y}$. So $\mathcal{X} \div \operatorname{Base}(\mathcal{Y}) = \mathcal{Y}$, $\mathcal{Y} \div \operatorname{Base}(\mathcal{X}) = \mathcal{X}$ is similar.

$\Leftarrow$. We have $\mathcal{Y} = (\mathcal{Y} \div \operatorname{Base}(\mathcal{X})) \div \operatorname{Base}(\mathcal{Y})$.

Thus as easy to show $\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}) \in \mathcal{Y}$ and similarly $\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}) \in \mathcal{X}$.

It's enough to show $\mathcal{X} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y})) = \mathcal{Y} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}))$ because for every $P \in \mathcal{X}, \mathcal{Y}$ we have $\mathcal{X} \cap \mathscr{P}P = \mathcal{X} \div P = (\mathcal{X} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}))) \div P$ and similarly $\mathcal{Y} \cap \mathscr{P}P = (\mathcal{Y} \div (\operatorname{Base}(\mathcal{X}) \sqcap \operatorname{Base}(\mathcal{Y}))) \div P$. But it follows from the conditions and proposition 708.

□

PROPOSITION 713. If two filters with the same base are equivalent they are equal.

PROOF. Let $\mathcal{A}$ and $\mathcal{B}$ be two filters and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$ for some set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$, and $\operatorname{Base}(\mathcal{A}) = \operatorname{Base}(\mathcal{B})$. Then

$$\mathcal{A} = (\mathscr{P}X \cap \mathcal{A}) \cup \left\{ \frac{Y \in D\operatorname{Base}(\mathcal{A})}{Y \sqsupseteq X} \right\} = (\mathscr{P}X \cap \mathcal{B}) \cup \left\{ \frac{Y \in D\operatorname{Base}(\mathcal{B})}{Y \sqsupseteq X} \right\} = \mathcal{B}.$$

□

PROPOSITION 714. If $A \in \mathscr{S}\mathcal{A}$ then $\mathcal{A} \div A \sim \mathcal{A}$.

PROOF.

$$(\mathcal{A} \div A) \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})) = \mathscr{S}\mathcal{A} \cap \mathscr{P}A \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})) = \mathscr{S}\mathcal{A} \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})) = \mathcal{A} \cap \mathscr{P}(A \sqcap \operatorname{Base}(\mathcal{A})).$$

Thus $\mathcal{A} \div A \sim \mathcal{A}$ because $A \sqcap \operatorname{Base}(\mathcal{A}) \sqsupseteq X \in \mathcal{A}$ for some $X \in \mathcal{A}$ and

$$A \sqcap \operatorname{Base}(\mathcal{A}) \sqsupseteq X \sqcap \operatorname{Base}(\mathcal{A}) \in \mathcal{A} \div A.$$

□

PROPOSITION 715. $\sim$ is an equivalence relation.

PROOF.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ for some filters $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Then there exist a set X such that $X \in \mathcal{A}$ and $X \in \mathcal{B}$ and $\mathscr{P}X \cap \mathcal{A} = \mathscr{P}X \cap \mathcal{B}$ and a set Y such that $Y \in \mathcal{B}$ and $Y \in \mathcal{C}$ and $\mathscr{P}Y \cap \mathcal{B} = \mathscr{P}Y \cap \mathcal{C}$. So $X \sqcap Y \in \mathcal{A}$ because

$$\mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{B} = \mathscr{P}(X \sqcap Y) \cap \mathcal{B} \supseteq \{X \sqcap Y\} \cap \mathcal{B} \ni X \sqcap Y.$$

Similarly we have $X \sqcap Y \in \mathcal{C}$. Finally

$$\mathscr{P}(X \sqcap Y) \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{A} = \mathscr{P}Y \cap \mathscr{P}X \cap \mathcal{B} = \mathscr{P}X \cap \mathscr{P}Y \cap \mathcal{B} = \mathscr{P}X \cap \mathscr{P}Y \cap \mathcal{C} = \mathscr{P}(X \sqcap Y) \cap \mathcal{C}.$$

□

DEFINITION 716. I will call equivalence classes as *unfixed filters*.

REMARK 717. The word “unfixed” is meant to negate “fixed” (having a particular base) filters.

PROPOSITION 718. $\mathcal{A} \sim \mathcal{B}$ iff $\mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$ for every filters $\mathcal{A}$, $\mathcal{B}$ on sets.[3]

[3]Use this proposition to shorten proofs of other theorem about equivalence of filters? (Our proof uses transitivity of equivalence of filters. So we can’t use it to prove that it is an equivalence relation, to avoid circular proof.)

PROOF. Let $\mathcal{A} \sim \mathcal{B}$. Then there is a set P such that $P \in \mathcal{A}$, $P \in \mathcal{B}$ and $\mathcal{A} \cap \mathscr{P}P = \mathcal{B} \cap \mathscr{P}P$. So $\mathscr{S}\mathcal{A} = (\mathcal{A} \cap \mathscr{P}P) \cup \left\{ \frac{K \in \mathfrak{Z}}{K \sqsupseteq P} \right\}$. Similarly $\mathscr{S}\mathcal{B} = (\mathcal{B} \cap \mathscr{P}P) \cup \left\{ \frac{K \in \mathfrak{Z}}{K \sqsupseteq P} \right\}$. Combining, we have $\mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$.

Let now $\mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$. Take $K \in \mathscr{S}\mathcal{A} = \mathscr{S}\mathcal{B}$. Then $\mathcal{A} \div K = \mathcal{B} \div K$ and thus (proposition 714) $\mathcal{A} \sim \mathcal{A} \div K = \mathcal{B} \div K \sim \mathcal{B}$, so having $\mathcal{A} \sim \mathcal{B}$. □

PROPOSITION 719. $\mathcal{A} \sim \mathcal{B} \Rightarrow \mathcal{A} \div B = \mathcal{B} \div B$ for every filters $\mathcal{A}$ and $\mathcal{B}$ and set B.

PROOF. $\mathcal{A} \div B = \langle B \sqcap \rangle^* \mathscr{S}\mathcal{A} = \langle B \sqcap \rangle^* \mathscr{S}\mathcal{B} = \mathcal{B} \div B$. □

5.39.3. Poset of unfixed filters.

LEMMA 720. Let filters $\mathcal{X}$ and $\mathcal{Y}$ be such that $\operatorname{Base}(\mathcal{X}) = \operatorname{Base}(\mathcal{Y}) = B$. Then $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$ for every set $C \supseteq B$.

PROOF. $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \div C \supseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} \supseteq \mathcal{Y} \cup \left\{ \frac{K \in \mathscr{P}C}{K \sqsupseteq B} \right\} \Leftrightarrow \mathcal{X} \supseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$. □

PROPOSITION 721. $\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow \mathcal{X} \div B \sqsubseteq \mathcal{Y} \div B$ for every filters $\mathcal{X}$, $\mathcal{Y}$ with the same base and set B.

PROOF. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \supseteq \mathcal{Y} \Rightarrow \mathcal{X} \div B \supseteq \mathcal{Y} \div B \Leftrightarrow \mathcal{X} \div B \sqsubseteq \mathcal{Y} \div B$. □

Define order of unfixed filters using already defined order of filters of a fixed base:

DEFINITION 722. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \exists x \in \mathcal{X}, y \in \mathcal{Y} : (\operatorname{Base}(x) = \operatorname{Base}(y) \wedge x \sqsubseteq y)$ for unfixed filters $\mathcal{X}$, $\mathcal{Y}$.

Proposition 718 allows to define:

DEFINITION 723. $\mathscr{S}\mathcal{A} = \mathscr{S}a$ for every $a \in \mathcal{A}$ for every unfixed filter $\mathcal{A}$.

THEOREM 724. $\mathscr{S}$ is an order-isomorphism from the poset of unfixed filters to the poset of filters on $\mathfrak{Z}$.

PROOF. We already know that $\mathscr{S}$ is an order embedding. It remains to prove that it is a surjection.

Let $\mathcal{Y}$ be a filter on $\mathfrak{Z}$. Take $\mathfrak{Z} \ni X \in \mathcal{Y}$. Then $\langle X \sqcap \rangle^* \mathcal{Y}$ is a filter on X and $\mathscr{S}[\langle X \sqcap \rangle^* \mathcal{Y}] = \mathscr{S}\langle X \sqcap \rangle^* \mathcal{Y} = \mathcal{Y}$. We have proved that it is a surjection. □

LEMMA 725. $\mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathscr{S}\mathcal{X} \sqsubseteq \mathscr{S}\mathcal{Y}$ for every unfixed filters $\mathcal{X}$, $\mathcal{Y}$.

PROOF.

$\Rightarrow$. Suppose $\mathcal{X} \sqsubseteq \mathcal{Y}$. Then there exist $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\operatorname{Base}(x) = \operatorname{Base}(y)$ and $x \sqsubseteq y$. Then $\mathscr{S}\mathcal{X} = \mathscr{S}x \sqsubseteq \mathscr{S}y = \mathscr{S}\mathcal{Y}$.

$\Leftarrow$. Suppose $\mathscr{S}\mathcal{X} \sqsubseteq \mathscr{S}\mathcal{Y}$. Then there are $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\mathscr{S}x \sqsubseteq \mathscr{S}y$. Consequently $\mathscr{S}x' \sqsubseteq \mathscr{S}y'$ for $x' = x \div (\operatorname{Base}(x) \sqcup \operatorname{Base}(y))$, $y' = y \div (\operatorname{Base}(x) \sqcup \operatorname{Base}(y))$. So we have $x' \in \mathcal{X}$, $y' \in \mathcal{Y}$, $\operatorname{Base}(x') = \operatorname{Base}(y')$ and $x' \sqsubseteq y'$, thus $\mathcal{X} \sqsubseteq \mathcal{Y}$.

□

THEOREM 726. $\sqsubseteq$ on the set of unfixed filters is a poset.

PROOF.

Reflexivity. From the previous theorem.
Transitivity. From the previous theorem.

Antisymmetry. Suppose $\mathcal{X} \sqsubseteq \mathcal{Y}$ and $\mathcal{Y} \sqsubseteq \mathcal{X}$. Then $\mathscr{S}\mathcal{X} \sqsubseteq \mathscr{S}\mathcal{Y}$ and $\mathscr{S}\mathcal{Y} \sqsubseteq \mathscr{S}\mathcal{X}$. Thus $\mathscr{S}\mathcal{X} = \mathscr{S}\mathcal{Y}$ and so $\mathscr{S}x = \mathscr{S}y$ for some $x \in \mathcal{X}$, $y \in \mathcal{Y}$. Consequently $\mathscr{S}(x \div B) = \mathscr{S}(y \div B)$ for $B = \operatorname{Base}(x) \sqcup \operatorname{Base}(y)$. Thus $x \div B = y \div B$ and so $x \sim y$, thus $\mathcal{X} = \mathcal{Y}$.

□

THEOREM 727. $[x] \sqsubseteq [y] \Leftrightarrow x \sqsubseteq y$ for filters x and y with the same base set.

PROOF.

$\Leftarrow$. Obvious.

$\Rightarrow$. Let $\operatorname{Base}(x) = \operatorname{Base}(y) = B$. Suppose $[x] \sqsubseteq [y]$. Then there exist $x' \sim x$ and $y' \sim y$ such that $C = \operatorname{Base}(x') = \operatorname{Base}(y')$ (for some set C) and $x' \sqsubseteq y'$.

We have by the lemma $x' \div (B \sqcup C) \sqsubseteq y' \div (B \sqcup C)$.

But $x' \div (B \sqcup C) = x \div (B \sqcup C)$ and $y' \div (B \sqcup C) = y \div (B \sqcup C)$. So $x \div (B \sqcup C) \sqsubseteq y \div (B \sqcup C)$ and thus again applying the lemma $x \sqsubseteq y$.

□

5.39.4. Rebase of unfixed filters.

Proposition 719 allows to define:

DEFINITION 728. $\mathcal{A} \div B = a \div B$ for an unfixed filter $\mathcal{A}$ and arbitrary $a \in \mathcal{A}$.

PROPOSITION 729. $\mathcal{X} \sqsubseteq \mathcal{Y} \Rightarrow \mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$ for every unfixed filters $\mathcal{X}$, $\mathcal{Y}$ and set C.

PROOF. Let $\mathcal{X} \sqsubseteq \mathcal{Y}$. Then there are $x \in \mathcal{X}$, $y \in \mathcal{Y}$ such that $\operatorname{Base}(x) = \operatorname{Base}(y)$ and $x \sqsubseteq y$. Then by proved above $x \div C \sqsubseteq y \div C$ what is equivalent to $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$. □

PROPOSITION 730. If $C \in \mathscr{S}\mathcal{X}$ and $C \in \mathscr{S}\mathcal{Y}$ for unfixed filters $\mathcal{X}$ and $\mathcal{Y}$ then $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$.

PROOF.

$\Leftarrow$. Previous proposition.

$\Rightarrow$. Let $\mathcal{X} \div C \sqsubseteq \mathcal{Y} \div C$. We have some $x \in \mathcal{X}$, $y \in \mathcal{Y}$, such that $\operatorname{Base}(x) = \operatorname{Base}(y)$ and $x \div C \sqsubseteq y \div C$. So $\mathscr{S}(x \div C) \sqsubseteq \mathscr{S}(y \div C)$. But $\mathscr{S}(x \div C) \sim x$ and $\mathscr{S}(y \div C) \sim y$. Thus $\mathscr{S}x \sqsubseteq \mathscr{S}y$ that is $x \sqsubseteq y$ and so $\mathcal{X} \sqsubseteq \mathcal{Y}$.

□

OBVIOUS 731. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \sqsubseteq A$ for every unfixed filter $\mathcal{X}$ and sets A, B.

OBVIOUS 732. $\mathcal{A} \div B = \langle B \sqcap \rangle^* \mathscr{S}\mathcal{A}$ for every unfixed filter $\mathcal{A}$.

OBVIOUS 733. If $A \in \mathscr{S}\mathcal{A}$ then $\mathcal{A} \div A \in \mathcal{A}$ for every unfixed filter $\mathcal{A}$.

PROPOSITION 734. If $C \in \mathscr{S}\mathcal{X}$ and $C \in \mathscr{S}\mathcal{Y}$ for unfixed filters $\mathcal{X}$ and $\mathcal{Y}$ then $\mathcal{X} \div C = \mathcal{Y} \div C \Leftrightarrow \mathcal{X} = \mathcal{Y}$.

PROOF. The backward implication is obvious. Let now $\mathcal{X} \div C = \mathcal{Y} \div C$. Take $x \in \mathcal{X}$, $y \in \mathcal{Y}$. We have $\mathcal{X} \div C = x \div C = (x \div B) \div C$ for $B = C \sqcup \operatorname{Base}(x) \sqcup \operatorname{Base}(y)$. Similary $\mathcal{Y} \div C = (y \div B) \div C$. Thus $(x \div B) \div C = (y \div B) \div C$ and thus $x \div B = y \div B$, so $x \sim y$ that is $\mathcal{X} = \mathcal{Y}$. □

PROPOSITION 735. $\mathcal{A} \div A = \bigsqcap \left\{ \frac{\uparrow^A (X \sqcap A)}{X \in \mathscr{S}\mathcal{A}} \right\}$ for every unfixed filter $\mathcal{A}$.

PROOF. Take $a \in \mathcal{A}$.

$$\bigsqcap\left\{\frac{\uparrow^A (X \sqcap A)}{X \in \mathscr{S}\mathcal{A}}\right\} = \bigsqcap\left\{\frac{\uparrow^A (X \sqcap A \sqcap \operatorname{Base}(a))}{X \in \mathscr{S}\mathcal{A}}\right\} = \bigsqcap\left\{\frac{\uparrow^A (X \sqcap A)}{X \in \mathscr{S}\mathcal{A} \cap \mathscr{P}\operatorname{Base}(a)}\right\} = \bigsqcap\left\{\frac{\uparrow^A (X \sqcap A)}{X \in \mathscr{S}a \cap \mathscr{P}\operatorname{Base}(a)}\right\} = \bigsqcap\left\{\frac{\uparrow^A (X \sqcap A)}{X \in a}\right\} = a \div A = \mathcal{A} \div A.$$

□

5.39.5. The diagram for unfixed filters.

Fix a set B.

LEMMA 736. $\mathcal{X} \mapsto \mathcal{X} \div B$ and $x \mapsto [x]$ are mutually inverse order isomorphisms between $\left\{\frac{\text{unfixed filter } \mathcal{X}}{B \in \mathscr{S}\mathcal{X}}\right\}$ and $\mathfrak{F}(DB)$.

PROOF. First, $\mathcal{X} \div B \in \mathfrak{F}(DB)$ for $\mathcal{X} \in \left\{\frac{\text{unfixed filter } \mathcal{X}}{B \in \mathscr{S}\mathcal{X}}\right\}$ and $[x] \in \left\{\frac{\text{unfixed filter } \mathcal{X}}{B \in \mathscr{S}\mathcal{X}}\right\}$ for $x \in \mathfrak{F}(DB)$.

Suppose $\mathcal{X}_0 \in \left\{\frac{\text{unfixed filter } \mathcal{X}}{B \in \mathscr{S}\mathcal{X}}\right\}$, $x = \mathcal{X}_0 \div B$, and $\mathcal{X}_1 = [x]$. We will prove $\mathcal{X}_0 = \mathcal{X}_1$. Really, $x \in \mathcal{X}_1$, $x = k \div B$ for $k \in \mathcal{X}_0$, $x \sim k$, thus $x \in \mathcal{X}_0$. So $\mathcal{X}_0 = \mathcal{X}_1$.

Suppose $x_0 \in \mathfrak{F}(DB)$, $\mathcal{X} = [x_0]$, $x_1 = \mathcal{X} \div B$. We will prove $x_0 = x_1$. Really, $x_1 = x_0 \div B$. So $x_1 = x_0$ because $\operatorname{Base}(x_0) = \operatorname{Base}(x_1) = B$.

So we proved that they are mutually inverse bijections. That they are order preserving is obvious. □

LEMMA 737. $\mathscr{S}$ and $\mathcal{X} \mapsto \langle B\sqcap\rangle^* \mathcal{X} = \mathcal{X} \cap \mathscr{P}B$ are mutually inverse order isomorphisms between $\mathfrak{F}(DB)$ and $\left\{\frac{\mathcal{X} \in \mathfrak{F}(\mathfrak{Z})}{B \in \mathcal{X}}\right\}$.

PROOF. First, $\mathscr{S}x \in \left\{\frac{\mathcal{X} \in \mathfrak{F}(\mathfrak{Z})}{B \in \mathcal{X}}\right\}$ for $x \in \mathfrak{F}(DB)$ because of theorem 724 and $\langle B\sqcap\rangle^* \mathcal{X} \in \mathfrak{F}(DB)$ obviously.

Let's prove $\langle B\sqcap\rangle^* \mathcal{X} = \mathcal{X} \cap \mathscr{P}B$. If $X \in \langle B\sqcap\rangle^* \mathcal{X}$ then $X \in \mathcal{X}$ (because $B \in \mathcal{X}$) and $X \in \mathscr{P}B$. So $X \in \mathcal{X} \cap \mathscr{P}B$. If $X \in \mathcal{X} \cap \mathscr{P}B$ then $X = B \cap X \in \langle B\sqcap\rangle^* \mathcal{X}$.

Let $x_0 \in \mathfrak{F}(DB)$, $\mathcal{X} = \mathscr{S}x$, and $x_1 = \langle B\sqcap\rangle^* \mathcal{X}$. Then obviously $x_0 = x_1$.

Let now $\mathcal{X}_0 \in \left\{\frac{\mathcal{X} \in \mathfrak{F}(\mathfrak{Z})}{B \in \mathcal{X}}\right\}$, $x = \langle B\sqcap\rangle^* \mathcal{X}_0$, and $\mathcal{X}_1 = \mathscr{S}x$. Then $\mathcal{X}_1 = \mathcal{X}_0 \cup \left\{\frac{K \in \mathfrak{Z}}{K \sqsupseteq B}\right\} = \mathcal{X}_0$.

So we proved that they are mutually inverse bijections. That they are order preserving is obvious. □

THEOREM 738. The diagram at the figure 5 (with the horizontal "unnamed" arrow *defined* as the inverse isomorphism of its opposite arrow) is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity (therefore "parallel" arrows are mutually inverse). The arrows preserve order.

PROOF. It's proved above, that all morphisms (except the "unnamed" arrow, which is the inverse morphism by definition) depicted on the diagram are bijections and the depicted "opposite" morphisms are mutually inverse.

That arrows preserve order is obvious.

It remains to apply lemma 196 (taking into account the proof of theorem 724). □

5.39.6. The lattice of unfixed filters.

THEOREM 739. Every nonempty set of unfixed filters has an infimum, provided that the lattice $\mathfrak{Z}$ is distributive.

$$\begin{tikzcd}
 & \mathfrak{F}(DB) \arrow[dl, shift left, "\mathscr{S}"] \arrow[dr, shift left, "x\mapsto[x]"] & \\
\left\{\frac{\mathcal{X}\in\mathfrak{F}(\mathfrak{Z})}{B\in\mathcal{X}}\right\} \arrow[ur, shift left, "\mathcal{X}\mapsto\langle B\sqcap\rangle^*\mathcal{X}=\mathcal{X}\sqcap\mathscr{P}B"] \arrow[rr, shift left] & & \left\{\frac{\text{unfixed filter }\mathcal{X}}{B\in\mathscr{S}\mathcal{X}}\right\} \arrow[ul, shift left, "\mathcal{X}\mapsto\mathcal{X}\div B"] \arrow[ll, shift left, "\mathscr{S}"]
\end{tikzcd}$$

FIGURE 7

PROOF. Theorem 520. □

THEOREM 740. Every bounded above set of unfixed filters has a supremum.

PROOF. Theorem 515 for nonempty sets of unfixed filters. The join $\bigsqcup\emptyset = [\bot]$ for the least filter $\bot \in \mathfrak{Z}(DA)$ for arbitrary $A \in \mathfrak{Z}$. □

COROLLARY 741. If $\mathfrak{Z}$ is the set of small sets, then every small set of unfixed filters has a supremum.

PROOF. Let S be a set of filters on $\mathfrak{Z}$. Then $T_{\mathcal{X}} \in \mathcal{X}$ is a small set for every $\mathcal{X} \in S$. Thus $\left\{\frac{T_{\mathcal{X}}}{\mathcal{X}\in S}\right\}$ is small set and thus $T = \bigcup\left\{\frac{T_{\mathcal{X}}}{\mathcal{X}\in S}\right\}$ is small set. Take the filter $\mathcal{T} = \uparrow T$. Then $\mathcal{T}$ is an upper bound of S and we can apply the theorem. □

OBVIOUS 742. The poset of unfixed filters for the lattice of small sets is bounded below (but not above).

PROPOSITION 743. The set of unfixed filters forms a co-brouwerian (and thus distributive) lattice, provided that $\mathfrak{Z}$ is distributive lattice which is an ideal base.

PROOF. Corollary 531. □

5.39.7. Principal unfixed filters and filtrator of unfixed filters.

DEFINITION 744. *Principal* unfixed filter is an unfixed filter corresponding to a principal filter on the poset $\mathfrak{Z}$.

DEFINITION 745. The *filtrator of unfixed filters* is the filtrator whose base are unfixed filters and whose core are principal unfixed filters.

We will equate principal unfixed filters with corresponding sets.

THEOREM 746. If we add principal filters on DB, principal filters on $\mathfrak{Z}$ containing B, and above defined principal unfixed filters corresponding to them to appropriate nodes of the diagram 5, then the diagram turns into a commutative diagram of isomorphisms between filtrators. (I will not draw the modified diagram for brevity.)

Every arrow of this diagram is an isomorphism between filtrators, every cycle in the diagram is identity.

PROOF. We need to prove only that principal filters on B and principal filters on $\mathfrak{Z}$ containing B correspond to each other by the isomorphisms of the diagram. But that's obvious. □

OBVIOUS 747. The filtrator of unfixed filters is a primary filtrator.

OBVIOUS 748. The filtrator of unfixed filters is down-aligned.

PROPOSITION 749. The filtrator of unfixed filters is

1°. filtered;
2°. with join-closed core.

PROOF. Theorem 534. □

PROPOSITION 750. The filtrator of unfixed filters is with binarily meet-closed core.

PROOF. Corollary 536. □

PROPOSITION 751. The filtrator of unfixed filters is with separable core.

PROOF. Theorem 537. □

PROPOSITION 752. $\operatorname{Cor}\mathcal{X}$ and $\operatorname{Cor}'\mathcal{X}$ are defined for every unfixed filter $\mathcal{X}$ and $\operatorname{Cor}\mathcal{X} = \operatorname{Cor}'\mathcal{X}$, provided that every DA is a complete lattice.

PROOF. $\operatorname{Cor}\mathcal{X}$ and $\operatorname{Cor}'\mathcal{X}$ exists because of the above isomorphism.
$\operatorname{Cor}'\mathcal{X} = \operatorname{Cor}\mathcal{X}$ by theorem 545. □

OBVIOUS 753. $\operatorname{Cor}\mathcal{X} = \operatorname{Cor}'\mathcal{X} = \bigcap\mathcal{X}$ for every filter $\mathcal{X} \in \mathfrak{F}(\text{small sets})$.

PROPOSITION 754. $\operatorname{atoms}\bigsqcap S = \bigcap\langle\operatorname{atoms}\rangle^* S$ whenever $\bigsqcap S$ is defined.

PROOF. Theorem 108. □

PROPOSITION 755. $\operatorname{atoms}(\mathcal{A}\sqcup\mathcal{B}) = \operatorname{atoms}\mathcal{A}\cup\operatorname{atoms}\mathcal{B}$ for unfixed filters $\mathcal{A}$, $\mathcal{B}$, whenever $\mathfrak{Z}$ is a distributive lattice which is an ideal base.

PROOF. Proposition 557. □

PROPOSITION 756. $\partial\mathcal{X}$ is a free star for every unfixed filter $\mathcal{X}$, whenever $\mathfrak{Z}$ is a distributive lattice which is an ideal base which has a least element.

PROOF. Theorem 566. □

PROPOSITION 757. The poset of unfixed filters is an atomistic lattice if every DA (for $A \in \mathfrak{A}$) is an atomistic lattice.

PROOF. Easily follows from 738 by isomorphism. □

PROPOSITION 758. The poset of unfixed filters is a strongly separable lattice if every DA (for $A \in \mathfrak{A}$) is an atomistic lattice.

PROOF. Theorem 234. □

PROPOSITION 759. $\operatorname{Cor}\mathcal{X} = \bigsqcup(\mathfrak{Z}\cap\operatorname{atoms}^{\text{unfixed filters}})$ for every unfixed filter $\mathcal{X}$ if every DA (for $A \in \mathfrak{A}$) is an atomistic lattice.

PROOF. Theorem 599. □

PROPOSITION 760. $\operatorname{Cor}(\mathcal{A}\sqcap\mathcal{B}) = \operatorname{Cor}\mathcal{A}\sqcap\operatorname{Cor}\mathcal{B}$ for every unfixed filters $\mathcal{A}$, $\mathcal{B}$, provided every DA (for $A \in \mathfrak{A}$) is a complete lattice.

PROOF. Theorem 601. □

PROPOSITION 761. $\operatorname{Cor}\bigsqcap^{\mathfrak{A}} S = \bigsqcap^{\mathfrak{Z}}\langle\operatorname{Cor}\rangle^* S$ for the filtrator of unfixed filters for every nonempty set S of unfixed filters, provided every DA (for $A \in \mathfrak{A}$) is a complete lattice.

PROOF. Theorem 602. □

PROPOSITION 762. $\operatorname{Cor}(\mathcal{A}\sqcup^{\mathfrak{A}}\mathcal{B}) = \operatorname{Cor}\mathcal{A}\sqcup^{\mathfrak{Z}}\operatorname{Cor}\mathcal{B}$ for the filtrator of unfixed filters for every unfixed filters $\mathcal{A}$, and $\mathcal{B}$, provided every DA (for $A \in \mathfrak{A}$) is a complete atomistic distributive lattice.

Proof. Can be easily deduced from theorem 603 and the triangular diagram (above) of isomorphic filtrators. □

Conjecture 763. The theorem 614 holds for unfixed filters, too.

It is expected to be easily provable using isomorphisms from the triangular diagram.

CHAPTER 6

Common knowledge, part 2 (topology)

In this chapter I describe basics of the theory known as *general topology*. Starting with the next chapter after this one I will describe generalizations of customary objects of general topology described in this chapter.

The reason why I've written this chapter is to show to the reader kinds of objects which I generalize below in this book. For example, funcoids and a generalization of proximity spaces, and funcoids are a generalization of pretopologies. To understand the intuitive meaning of funcoids one needs first know what are proximities and what are pretopologies.

Having said that, customary topology is *not* used in my definitions and proofs below. It is just to feed your intuition.

6.1. Metric spaces

The theory of topological spaces started immediately with the definition would be completely non-intuitive for the reader. It is the reason why I first describe metric spaces and show that metric spaces give rise for a topology (see below). Topological spaces are understandable as a generalization of topologies induced by metric spaces.

Metric spaces is a formal way to express the notion of *distance*. For example, there are distance $|x-y|$ between real numbers x and y, distance between points of a plane, etc.

DEFINITION 764. A *metric space* is a set U together with a function $d : U\times U \to \mathbb{R}$ (*distance* or *metric*) such that for every $x, y, z \in U$:

1°. $d(x, y) \geq 0$;
2°. $d(x, y) = 0 \Leftrightarrow x = y$;
3°. $d(x, y) = d(y, x)$ (*symmetry*);
4°. $d(x, z) \leq d(x, y) + d(y, z)$ (*triangle inequality*).

EXERCISE 765. Show that the Euclid space $\mathbb{R}^n$ (with the standard distance) is a metric space for every $n \in \mathbb{N}$.

DEFINITION 766. *Open ball* of *radius* $r > 0$ centered at point $a \in U$ is the set

$$B_r(a) = \left\{ \frac{x \in U}{d(a, x) < r} \right\}.$$

DEFINITION 767. *Closed ball* of *radius* $r > 0$ centered at point $a \in U$ is the set

$$B_r[a] = \left\{ \frac{x \in U}{d(a, x) \leq r} \right\}.$$

One example of use of metric spaces: *Limit* of a sequence x in a metric space can be defined as a point y in this space such that

$$\forall \epsilon > 0 \exists N \in \mathbb{N} \forall n > N : d(x_n, y) < \epsilon.$$

6.1.1. Open and closed sets.

Definition 768. A set A in a metric space is called *open* when $\forall a \in A \exists r > 0 : B_r(a) \subseteq A$.

Definition 769. A set A in a metric space is closed when its complement $U \setminus A$ is open.

Exercise 770. Show that: closed intervals on real line are closed sets, open intervals are open sets.

Exercise 771. Show that open balls are open and closed balls are closed.

Definition 772. Closure $\operatorname{cl}(A)$ of a set A in a metric space is the set of points y such that

$$\forall \epsilon > 0 \exists a \in A : d(y, a) < \epsilon.$$

Proposition 773. $\operatorname{cl}(A) \supseteq A$.

Proof. It follows from $d(a, a) = 0 < \epsilon$. □

Exercise 774. Prove $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$ for every subsets A and B of a metric space.

6.2. Pretopological spaces

Pretopological space can be defined in two equivalent ways: a *neighborhood system* or a *preclosure operator*. To be more clear I will call *pretopological space* only the first (neighborhood system) and the second call a *preclosure space*.

Definition 775. *Pretopological space* is a set U together with a filter $\Delta(x)$ on U for every $x \in U$, such that $\uparrow^U \{x\} \sqsubseteq \Delta(x)$. Δ is called a *pretopology* on U. Elements of $\operatorname{up} \Delta(x)$ are called *neighborhoods* of point x.

Definition 776. *Preclosure* on a set U is a unary operation cl on $\mathscr{P}U$ such that for every $A, B \in \mathscr{P}U$:

1°. $\operatorname{cl}(\emptyset) = \emptyset$;
2°. $\operatorname{cl}(A) \supseteq A$;
3°. $\operatorname{cl}(A \cup B) = \operatorname{cl}(A) \cup \operatorname{cl}(B)$.

I call a preclosure together with a set U as *preclosure space*.

Theorem 777. Small pretopological spaces and small preclosure spaces bijectively correspond to each other by the formulas:

$$\operatorname{cl}(A) = \left\{ \frac{x \in U}{A \in \partial\Delta(x)} \right\}; \tag{3}$$

$$\operatorname{up} \Delta(x) = \left\{ \frac{A \in \mathscr{P}U}{x \notin \operatorname{cl}(U \setminus A)} \right\}. \tag{4}$$

Proof. First let's prove that cl defined by formula (3) is really a preclosure.

$\operatorname{cl}(\emptyset) = \emptyset$ is obvious. If $x \in A$ then $A \in \partial\Delta(x)$ and so $\operatorname{cl}(A) \supseteq A$. $\operatorname{cl}(A \cup B) = \left\{ \frac{x \in U}{A \cup B \in \partial\Delta(x)} \right\} = \left\{ \frac{x \in U}{A \in \partial\Delta(x) \vee B \in \partial\Delta(x)} \right\} = \operatorname{cl}(A) \cup \operatorname{cl}(B)$. So, it is really a preclosure.

Next let's prove that Δ defined by formula (4) is a pretopology. That $\operatorname{up} \Delta(x)$ is an upper set is obvious. Let $A, B \in \operatorname{up} \Delta(x)$. Then $x \notin \operatorname{cl}(U \setminus A) \wedge x \notin \operatorname{cl}(U \setminus B)$; $x \notin \operatorname{cl}(U \setminus A) \cup \operatorname{cl}(U \setminus B) = \operatorname{cl}((U \setminus A) \cup (U \setminus B)) = \operatorname{cl}(U \setminus (A \cap B))$; $A \cap B \in \operatorname{up} \Delta(x)$. We have proved that $\Delta(x)$ is a filter object.

Let's prove $\uparrow^U \{x\} \sqsubseteq \Delta(x)$. If $A \in \operatorname{up} \Delta(x)$ then $x \notin \operatorname{cl}(U \setminus A)$ and consequently $x \notin U \setminus A$; $x \in A$; $A \in \operatorname{up} \uparrow^U \{x\}$. So $\uparrow^U \{x\} \sqsubseteq \Delta(x)$ and thus Δ is a pretopology.

It is left to prove that the functions defined by the above formulas are mutually inverse.

Let cl_0 be a preclosure, let Δ be the pretopology induced by cl_0 by the formula (4), let cl_1 be the preclosure induced by Δ by the formula (3). Let's prove $\mathrm{cl}_1 = \mathrm{cl}_0$. Really,

$$\begin{aligned} x \in \mathrm{cl}_1(A) \Leftrightarrow \\ \Delta(x) \not\asymp \uparrow^U A \Leftrightarrow \\ \forall X \in \operatorname{up} \Delta(x) : X \cap A \neq \emptyset \Leftrightarrow \\ \forall X \in \mathscr{P}U : (x \notin \mathrm{cl}_0(U \setminus X) \Rightarrow X \cap A \neq \emptyset) \Leftrightarrow \\ \forall X' \in \mathscr{P}U : (x \notin \mathrm{cl}_0(X') \Rightarrow A \setminus X' \neq \emptyset) \Leftrightarrow \\ \forall X' \in \mathscr{P}U : (A \setminus X' = \emptyset \Rightarrow x \in \mathrm{cl}_0(X')) \Leftrightarrow \\ \forall X' \in \mathscr{P}U : (A \subseteq X' \Rightarrow x \in \mathrm{cl}_0(X')) \Leftrightarrow \\ x \in \mathrm{cl}_0(A). \end{aligned}$$

So $\mathrm{cl}_1(A) = \mathrm{cl}_0(A)$.

Let now Δ_0 be a pretopology, let cl be the closure induced by Δ_0 by the formula (3), let Δ_1 be the pretopology induced by cl by the formula (4). Really

$$\begin{aligned} & A \in \operatorname{up} \Delta_1(x) \Leftrightarrow \\ & x \notin \mathrm{cl}(U \setminus A) \Leftrightarrow \\ \Delta_0(x) \asymp & \uparrow^U (U \setminus A) \Leftrightarrow \text{(proposition 551)} \\ & \uparrow^U A \sqsupseteq \Delta_0(x) \Leftrightarrow \\ & A \in \operatorname{up} \Delta_0(x). \end{aligned}$$

So $\Delta_1(x) = \Delta_0(x)$.

That these functions are mutually inverse, is now proved. □

6.2.1. Pretopology induced by a metric. Every metric space induces a pretopology by the formula:

$$\Delta(x) = \prod^{\mathscr{F}U} \left\{ \frac{B_r(x)}{r \in \mathbb{R}, r > 0} \right\}.$$

EXERCISE 778. Show that it is a pretopology.

PROPOSITION 779. The preclosure corresponding to this pretopology is the same as the preclosure of the metric space.

PROOF. I denote the preclosure of the metric space as cl_M and the preclosure corresponding to our pretopology as cl_P. We need to show $\mathrm{cl}_P = \mathrm{cl}_M$. Really:

$$\begin{aligned} \mathrm{cl}_P(A) = \\ \left\{ \frac{x \in U}{A \in \partial\Delta(x)} \right\} = \\ \left\{ \frac{x \in U}{\forall \epsilon > 0 : B_\epsilon(x) \not\asymp A} \right\} = \\ \left\{ \frac{y \in U}{\forall \epsilon > 0 \exists a \in A : d(y, a) < \epsilon} \right\} = \\ \mathrm{cl}_M(A) \end{aligned}$$

for every set $A \in \mathscr{P}U$. □

6.3. Topological spaces

PROPOSITION 780. For the set of open sets of a metric space (U, d) it holds:

1°. Union of any (possibly infinite) number of open sets is an open set.
2°. Intersection of a finite number of open sets is an open set.
3°. U is an open set.

PROOF. Let S be a set of open sets. Let $a \in \bigcup S$. Then there exists $A \in S$ such that $a \in A$. Because A is open we have $B_r(a) \subseteq A$ for some $r > 0$. Consequently $B_r(a) \subseteq \bigcup S$ that is $\bigcup S$ is open.

Let $A_0, \ldots, A_n$ be open sets. Let $a \in A_0 \cap \cdots \cap A_n$ for some $n \in \mathbb{N}$. Then there exist r_i such that $B_{r_i}(a) \subseteq A_i$. So $B_r(a) \subseteq A_0 \cap \cdots \cap A_n$ for $r = \min\{r_0, \ldots, r_n\}$ that is $A_0 \cap \cdots \cap A_n$ is open.

That U is an open set is obvious. □

The above proposition suggests the following definition:

DEFINITION 781. A *topology* on a set U is a collection $\mathcal{O}$ (called the set of *open sets*) of subsets of U such that:

1°. Union of any (possibly infinite) number of open sets is an open set.
2°. Intersection of a finite number of open sets is an open set.
3°. U is an open set.

The pair $(U, \mathcal{O})$ is called a *topological space*.

REMARK 782. From the above it is clear that every metric induces a topology.

PROPOSITION 783. Empty set is always open.

PROOF. Empty set is union of an empty set. □

DEFINITION 784. A *closed set* is a complement of an open set.

Topology can be equivalently expresses in terms of closed sets:

A *topology* on a set U is a collection (called the set of *closed sets*) of subsets of U such that:

1°. Intersection of any (possibly infinite) number of closed sets is a closed set.
2°. Union of a finite number of closed sets is a closed set.
3°. $\emptyset$ is a closed set.

EXERCISE 785. Show that the definitions using open and closed sets are equivalent.

6.3.1. Relationships between pretopologies and topologies.

6.3.1.1. *Topological space induced by preclosure space.* Having a preclosure space (U, cl) we define a topological space whose closed sets are such sets $A \in \mathscr{P}U$ that $\mathrm{cl}(A) = A$.

PROPOSITION 786. This really defines a topology.

PROOF. Let S be a set of closed sets. First, we need to prove that $\bigcap S$ is a closed set. We have $\mathrm{cl}(\bigcap S) \subseteq A$ for every $A \in S$. Thus $\mathrm{cl}(\bigcap S) \subseteq \bigcap S$ and consequently $\mathrm{cl}(\bigcap S) = \bigcap S$. So $\bigcap S$ is a closed set.

Let now $A_0, \ldots, A_n$ be closed sets, then

$$\mathrm{cl}(A_0 \cup \cdots \cup A_n) = \mathrm{cl}(A_0) \cup \cdots \cup \mathrm{cl}(A_n) = A_0 \cup \cdots \cup A_n$$

that is $A_0 \cup \cdots \cup A_n$ is a closed set.

That $\emptyset$ is a closed set is obvious. □

Having a pretopological space (U, Δ) we define a topological space whose open sets are
$$\left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \operatorname{up} \Delta(x)} \right\}.$$

PROPOSITION 787. This really defines a topology.

PROOF. Let set $S \subseteq \left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \operatorname{up} \Delta(x)} \right\}$. Then $\forall X \in S \forall x \in X : X \in \operatorname{up} \Delta(x)$. Thus
$$\forall x \in \bigcup S \exists X \in S : X \in \operatorname{up} \Delta(x)$$
and so $\forall x \in \bigcup S : \bigcup S \in \operatorname{up} \Delta(x)$. So $\bigcup S$ is an open set.

Let now $A_0, \dots, A_n \in \left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \operatorname{up} \Delta(x)} \right\}$ for $n \in \mathbb{N}$. Then $\forall x \in A_i : A_i \in \operatorname{up} \Delta(x)$ and so
$$\forall x \in A_0 \cap \dots \cap A_n : A_i \in \operatorname{up} \Delta(x);$$
thus $\forall x \in A_0 \cap \dots \cap A_n : A_0 \cap \dots \cap A_n \in \operatorname{up} \Delta(x)$. So $A_0 \cap \dots \cap A_n \in \left\{ \frac{X \in \mathscr{P}U}{\forall x \in X : X \in \operatorname{up} \Delta(x)} \right\}$.

That U is an open set is obvious. □

PROPOSITION 788. Topology τ defined by a pretopology and topology ρ defined by the corresponding preclosure, are the same.

PROOF. Let $A \in \mathscr{P}U$.

A is ρ-closed $\Leftrightarrow \operatorname{cl}(A) = A \Leftrightarrow \operatorname{cl}(A) \subseteq A \Leftrightarrow \forall x \in U : (A \in \partial \Delta(x) \Rightarrow x \in A)$;
$$\begin{aligned} A \text{ is } \tau\text{-open} \Leftrightarrow \\ \forall x \in A : A \in \operatorname{up} \Delta(x) \Leftrightarrow \\ \forall x \in U : (x \in A \Rightarrow A \in \operatorname{up} \Delta(x)) \Leftrightarrow \\ \forall x \in U : (x \notin U \setminus A \Rightarrow U \setminus A \notin \partial \Delta(x)). \end{aligned}$$

So ρ-closed and τ-open sets are complements of each other. It follows $\rho = \tau$. □

6.3.1.2. *Preclosure space induced by topological space.* We define a preclosure and a pretopology induced by a topology and then show these two are equivalent.

Having a topological space we define a preclosure space by the formula
$$\operatorname{cl}(A) = \bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\}.$$

PROPOSITION 789. It is really a preclosure.

PROOF. $\operatorname{cl}(\emptyset) = \emptyset$ because $\emptyset$ is a closed set. $\operatorname{cl}(A) \supseteq A$ is obvious.
$$\begin{aligned} \operatorname{cl}(A \cup B) = \\ \bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A \cup B} \right\} = \\ \bigcap \left\{ \frac{X_1 \cup X_2}{X_1, X_2 \in \mathscr{P}U \text{ are closed sets}, X_1 \supseteq A, X_2 \supseteq B} \right\} = \\ \bigcap \left\{ \frac{X_1 \in \mathscr{P}U}{X_1 \text{ is a closed set}, X_1 \supseteq A} \right\} \cup \bigcap \left\{ \frac{X_2 \in \mathscr{P}U}{X_2 \text{ is a closed set}, X_2 \supseteq B} \right\} = \\ \operatorname{cl}(A) \cup \operatorname{cl}(B). \end{aligned}$$

Thus cl is a preclosure. □

Or: $\Delta(x) = \prod^{\mathscr{F}} \left\{ \frac{X \in \mathcal{O}}{x \in X} \right\}$.

It is trivially a pretopology (used the fact that $U \in \mathcal{O}$).

Proposition 790. The preclosure and the pretopology defined in this section above correspond to each other (by the formulas from theorem 777).

Proof. We need to prove $\operatorname{cl}(A) = \left\{ \frac{x\in U}{\Delta(x) \not\asymp \uparrow^U A} \right\}$, that is

$$\bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\} = \left\{ \frac{x \in U}{\bigsqcap^{\mathscr{F}U} \left\{ \frac{X\in\mathcal{O}}{x\in X} \right\} \not\asymp \uparrow^U A} \right\}.$$

Equivalently transforming it, we get:

$$\bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\} = \left\{ \frac{x \in U}{\forall X \in \mathcal{O} : (x \in X \Rightarrow \uparrow^U X \not\asymp \uparrow^U A)} \right\};$$
$$\bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\} = \left\{ \frac{x \in U}{\forall X \in \mathcal{O} : (x \in X \Rightarrow X \not\asymp A)} \right\}.$$

We have

$$\begin{aligned} x \in \bigcap \left\{ \frac{X \in \mathscr{P}U}{X \text{ is a closed set}, X \supseteq A} \right\} &\Leftrightarrow \\ \forall X \in \mathscr{P}U : (X \text{ is a closed set} \wedge X \supseteq A \Rightarrow x \in X) &\Leftrightarrow \\ \forall X' \in \mathcal{O} : (U \setminus X' \supseteq A \Rightarrow x \in U \setminus X') &\Leftrightarrow \\ \forall X' \in \mathcal{O} : (X' \asymp A \Rightarrow x \notin X') &\Leftrightarrow \\ \forall X \in \mathcal{O} : (x \in X \Rightarrow X \not\asymp A). & \end{aligned}$$

So our equivalence holds. □

Proposition 791. If τ is the topology induced by pretopology π, in turn induced by topology ρ, then $\tau = \rho$.

Proof. The set of closed sets of τ is

$$\begin{aligned} \left\{ \frac{A \in \mathscr{P}U}{\operatorname{cl}_\pi(A) = A} \right\} &= \\ \left\{ \frac{A \in \mathscr{P}U}{\bigcap \left\{ \frac{X\in\mathscr{P}U}{X \text{ is a closed set in } \rho, X\supseteq A} \right\} = A} \right\} &= \\ \left\{ \frac{A \in \mathscr{P}U}{A \text{ is a closed set in } \rho} \right\} & \end{aligned}$$

(taken into account that intersecting closed sets is a closed set). □

Definition 792. Idempotent closures are called *Kuratowski closures*.

Theorem 793. The above defined correspondences between topologies and pretopologies, restricted to Kuratowski closures, is a bijection.

Proof. Taking into account the above proposition, it's enough to prove that:

If τ is the pretopology induced by topology π, in turn induced by a Kuratowski closure ρ, then $\tau = \rho$.

$$\begin{aligned}&\operatorname{cl}_\tau(A) = \\ \bigcap\left\{\frac{X \in \mathscr{P}U}{X \text{ is a closed set in } \pi, X \supseteq A}\right\} &= \\ \bigcap\left\{\frac{X \in \mathscr{P}U}{\operatorname{cl}_\rho(X) = X, X \supseteq A}\right\} &= \\ \bigcap\left\{\frac{\operatorname{cl}_\rho(X)}{X \in \mathscr{P}U, \operatorname{cl}_\rho(X) = X, X \supseteq \operatorname{cl}_\rho(A)}\right\} &= \\ \bigcap\left\{\frac{\operatorname{cl}_\rho(\operatorname{cl}_\rho(X))}{X = A}\right\} &= \\ \operatorname{cl}_\rho(\operatorname{cl}_\rho(A)) &= \\ \operatorname{cl}_\rho(A)&.\end{aligned}$$

□

6.3.1.3. *Topology induced by a metric.*

DEFINITION 794. Every metric space induces a topology in this way: A set X is open iff

$$\forall x \in X \exists \epsilon > 0 : B_r(x) \subseteq X.$$

EXERCISE 795. Prove it is really a topology and this topology is the same as the topology, induced by the pretopology, in turn induced by our metric space.

6.4. Proximity spaces

Let (U, d) be metric space. We will define *distance* between sets $A, B \in \mathscr{P}U$ by the formula

$$d(A, B) = \inf\left\{\frac{d(a, b)}{a \in A, b \in B}\right\}.$$

(Here "inf" denotes infimum on the real line.)

DEFINITION 796. Sets $A, B \in \mathscr{P}U$ are *near* (denoted $A\ \delta\ B$) iff $d(A, B) = 0$.

δ defined in this way (for a metric space) is an example of proximity as defined below.

DEFINITION 797. A *proximity space* is a set (U, δ) conforming to the following axioms (for every $A, B, C \in \mathscr{P}U$):

1°. $A \cap B \neq \emptyset \Rightarrow A\ \delta\ B$;
2°. if $A\ \delta\ B$ then $A \neq \emptyset$ and $B \neq \emptyset$;
3°. $A\ \delta\ B \Rightarrow B\ \delta\ A$ (*symmetry*);
4°. $(A \cup B)\ \delta\ C \Leftrightarrow A\ \delta\ C \vee B\ \delta\ C$;
5°. $C\ \delta\ (A \cup B) \Leftrightarrow C\ \delta\ A \vee C\ \delta\ B$;
6°. $A\ \bar{\delta}\ B$ implies existence of $P, Q \in \mathscr{P}U$ with $A\ \bar{\delta}\ P$, $B\ \bar{\delta}\ Q$ and $P \cup Q = U$.

EXERCISE 798. Show that proximity generated by a metric space is really a proximity (conforms to the above axioms).

DEFINITION 799. *Quasi-proximity* is defined as the above but without the symmetry axiom.

DEFINITION 800. Closure is generated by a proximity by the following formula:

$$\operatorname{cl}(A) = \left\{\frac{a \in U}{\{a\}\ \delta\ A}\right\}.$$

PROPOSITION 801. Every closure generated by a proximity is a Kuratowski closure.

PROOF. First prove it is a preclosure. $\operatorname{cl}(\emptyset) = \emptyset$ is obvious. $\operatorname{cl}(A) \supseteq A$ is obvious.

$$\operatorname{cl}(A \cup B) = \\ \left\{ \frac{a \in U}{\{a\} \ \delta \ A \cup B} \right\} = \\ \left\{ \frac{a \in U}{\{a\} \ \delta \ A \vee \{a\} \ \delta \ B} \right\} = \\ \left\{ \frac{a \in U}{\{a\} \ \delta \ A} \right\} \cup \left\{ \frac{a \in U}{\{a\} \ \delta \ B} \right\} = \\ \operatorname{cl}(A) \cup \operatorname{cl}(B).$$

It is remained to prove that cl is idempotent, that is $\operatorname{cl}(\operatorname{cl}(A)) = \operatorname{cl}(A)$. It is enough to show $\operatorname{cl}(\operatorname{cl}(A)) \subseteq \operatorname{cl}(A)$ that is if $x \notin \operatorname{cl}(A)$ then $x \notin \operatorname{cl}(\operatorname{cl}(A))$.

If $x \notin \operatorname{cl}(A)$ then $\{x\} \ \bar{\delta} \ A$. So there are $P, Q \in \mathscr{P}U$ such that $\{x\} \ \bar{\delta} \ P$, $A \ \bar{\delta} \ Q$, $P \cup Q = U$. Then $U \setminus Q \subseteq P$, so $\{x\} \ \bar{\delta} \ U \setminus Q$ and hence $x \in Q$. Hence $U \setminus \operatorname{cl}(A) \subseteq Q$, and so $\operatorname{cl}(A) \subseteq U \setminus Q \subseteq P$. Consequently $\{x\} \ \bar{\delta} \ \operatorname{cl}(A)$ and hence $x \notin \operatorname{cl}(\operatorname{cl}(A))$. □

6.5. Definition of uniform spaces

Here I will present the traditional definition of uniform spaces. Below in the chapter about reloids I will present a shortened and more algebraic (however a little less elementary) definition of uniform spaces.

DEFINITION 802. *Uniform space* is a pair (U, D) of a set U and filter $D \in \mathfrak{F}(U \times U)$ (called *uniformity* or the set of *entourages*) such that:

1°. If $F \in D$ then $\mathrm{id}_U \subseteq F$.
2°. If $F \in D$ then there exists $G \in D$ such that $G \circ G \subseteq F$.
3°. If $F \in D$ then $F^{-1} \in D$.

Part 2

Funcoids and reloids

CHAPTER 7

Funcoids

In this chapter (and several following chapters) the word *filter* will refer to a filter (or equivalently any filter object) on a set (rather than a filter on an arbitrary poset).

7.1. Informal introduction into funcoids

Funcoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also funcoids are a generalization of binary relations.

That funcoids are a common generalization of "spaces" (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement "f is a continuous function from a space μ to a space ν" can be described in terms of funcoids as the formula $f \circ \mu \sqsubseteq \nu \circ f$ (see below for details).

Most naturally funcoids appear as a generalization of proximity spaces.[1]

Let δ be a proximity. We will extend the relation δ from sets to filters by the formula:

$$\mathcal{A}\ \delta'\ \mathcal{B} \Leftrightarrow \forall A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B} : A\ \delta\ B.$$

Then (as it will be proved below) there exist two functions $\alpha, \beta \in \mathscr{F}^{\mathscr{F}}$ such that

$$\mathcal{A}\ \delta'\ \mathcal{B} \Leftrightarrow \mathcal{B} \sqcap \alpha\mathcal{A} \neq \bot^{\mathscr{F}} \Leftrightarrow \mathcal{A} \sqcap \beta\mathcal{B} \neq \bot^{\mathscr{F}}.$$

The pair (α, β) is called *funcoid* when $\mathcal{B} \sqcap \alpha\mathcal{A} \neq \bot^{\mathscr{F}} \Leftrightarrow \mathcal{A} \sqcap \beta\mathcal{B} \neq \bot^{\mathscr{F}}$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first α and the second β. The first component of a funcoid f is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of principal funcoids (see below) these coincide.)

One of the most important properties of a funcoid is that it is uniquely determined by just one of its components. That is a funcoid f is uniquely determined by the function $\langle f \rangle$. Moreover a funcoid f is uniquely determined by values of $\langle f \rangle$ on principal filters.

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha \in \mathscr{F}^{\mho}$ for some set $\mho$. Then we define $\alpha' X = \bigsqcup_{x \in X} \alpha x$ for every set $X \in \mathscr{P}\mho$. We will prove that there exists a unique funcoid f such that $\alpha' = \langle f \rangle|_{\mathfrak{P}} \circ \uparrow$ where $\mathfrak{P}$ is the set of principal filters on $\mho$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator p on a set $\mho$ it exists a unique funcoid f such that $\langle f \rangle|_{\mathfrak{P}} \circ \uparrow = \uparrow \circ p$.

[1] In fact I discovered funcoids pondering on topological spaces, not on proximity spaces, but this is only of a historic interest.

For every binary relation p on a set $\mho$ there exists unique funcoid f such that

$$\forall X \in \mathscr{P}\mho : \langle f \rangle \uparrow X = \uparrow \langle p \rangle^* X$$

(where $\langle p \rangle^*$ is defined in the introduction), recall that a funcoid is uniquely determined by the values of its first component on sets. I will call such funcoids *principal*. So funcoids are a generalization of binary relations.

Composition of binary relations (i.e. of principal funcoids) complies with the formulas:

$$\langle g \circ f \rangle^* = \langle g \rangle^* \circ \langle f \rangle^* \quad \text{and} \quad \left\langle (g \circ f)^{-1} \right\rangle^* = \left\langle f^{-1} \right\rangle^* \circ \left\langle g^{-1} \right\rangle^*.$$

By similar formulas we can define composition of every two funcoids. Funcoids with this composition form a category (*the category of funcoids*).

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha, \beta)^{-1} = (\beta, \alpha)$. In the particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filters instead of acting on sets. Below there will be defined domain and image of a funcoid (the domain and the image of a funcoid are filters).

7.2. Basic definitions

Definition 803. Let us call a *funcoid* from a set A to a set B a quadruple (A, B, α, β) where $\alpha \in \mathscr{F}(B)^{\mathscr{F}(A)}$, $\alpha \in \mathscr{F}(A)^{\mathscr{F}(B)}$ such that

$$\forall \mathcal{X} \in \mathscr{F}(A), \mathcal{Y} \in \mathscr{F}(B) : (\mathcal{Y} \not\asymp \alpha\mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \beta\mathcal{Y}).$$

Definition 804. *Source* and *destination* of every funcoid (A, B, α, β) are defined as:

$$\operatorname{Src}(A, B, \alpha, \beta) = A \quad \text{and} \quad \operatorname{Dst}(A, B, \alpha, \beta) = B.$$

I will denote $\mathsf{FCD}(A, B)$ the set of funcoids from A to B.

I will denote FCD the set of all funcoids (for small sets).

Definition 805. I will call an *endofuncoid* a funcoid whose source is the same as it's destination.

Definition 806. $\langle (A, B, \alpha, \beta) \rangle \stackrel{\text{def}}{=} \alpha$ for a funcoid (A, B, α, β).

Definition 807. The *reverse* funcoid $(A, B, \alpha, \beta)^{-1} = (B, A, \beta, \alpha)$ for a funcoid (A, B, α, β).

Note 808. The reverse funcoid is *not* an inverse in the sense of group theory or category theory.

Proposition 809. If f is a funcoid then f^{-1} is also a funcoid.

Proof. It follows from symmetry in the definition of funcoid. □

Obvious 810. $(f^{-1})^{-1} = f$ for a funcoid f.

Definition 811. The relation $[f] \in \mathscr{P}(\mathscr{F}(\operatorname{Src} f) \times \mathscr{F}(\operatorname{Dst} f))$ is defined (for every funcoid f and $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$ by the formula $\mathcal{X} \mathrel{[f]} \mathcal{Y} \Leftrightarrow \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$.

Obvious 812. $\mathcal{X} \mathrel{[f]} \mathcal{Y} \Leftrightarrow \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \left\langle f^{-1} \right\rangle \mathcal{Y}$ for every funcoid f and $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$.

Obvious 813. $\left[f^{-1}\right] = [f]^{-1}$ for a funcoid f.

Theorem 814. Let A, B be sets.

1°. For given value of $\langle f\rangle \in \mathscr{F}(B)^{\mathscr{F}(A)}$ there exists no more than one funcoid $f \in \mathsf{FCD}(A, B)$.
2°. For given value of $[f] \in \mathscr{P}(\mathscr{F}(A) \times \mathscr{F}(B))$ there exists no more than one funcoid $f \in \mathsf{FCD}(A, B)$.

Proof. Let $f, g \in \mathsf{FCD}(A, B)$.

Obviously, $\langle f\rangle = \langle g\rangle \Rightarrow [f] = [g]$ and $\langle f^{-1}\rangle = \langle g^{-1}\rangle \Rightarrow [f] = [g]$. So it's enough to prove that $[f] = [g] \Rightarrow \langle f\rangle = \langle g\rangle$.

Provided that $[f] = [g]$ we have $\mathcal{Y} \not\asymp \langle f\rangle\mathcal{X} \Leftrightarrow \mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \mathcal{X}\ [g]\ \mathcal{Y} \Leftrightarrow \mathcal{Y} \not\asymp \langle g\rangle\mathcal{X}$ and consequently $\langle f\rangle\mathcal{X} = \langle g\rangle\mathcal{X}$ for every $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Y} \in \mathscr{F}(B)$ because a set of filters is separable, thus $\langle f\rangle = \langle g\rangle$. □

Proposition 815. $\langle f\rangle\bot = \bot$ for every funcoid f.

Proof. $\mathcal{Y} \not\asymp \langle f\rangle\bot \Leftrightarrow \bot \not\asymp \langle f^{-1}\rangle\mathcal{Y} \Leftrightarrow 0 \Leftrightarrow \mathcal{Y} \not\asymp \bot$. Thus $\langle f\rangle\bot = \bot$ by separability of filters. □

Proposition 816. $\langle f\rangle(\mathcal{I} \sqcup \mathcal{J}) = \langle f\rangle\mathcal{I} \sqcup \langle f\rangle\mathcal{J}$ for every funcoid f and $\mathcal{I}, \mathcal{J} \in \mathscr{F}(\operatorname{Src} f)$.

Proof.

$$\begin{aligned}
&\star\langle f\rangle(\mathcal{I} \sqcup \mathcal{J}) = \\
&\left\{\frac{\mathcal{Y} \in \mathscr{F}}{\mathcal{Y} \not\asymp \langle f\rangle(\mathcal{I} \sqcup \mathcal{J})}\right\} = \\
&\left\{\frac{\mathcal{Y} \in \mathscr{F}}{\mathcal{I} \sqcup \mathcal{J} \not\asymp \langle f^{-1}\rangle\mathcal{Y}}\right\} = \\
&\left\{\frac{\mathcal{Y} \in \mathscr{F}}{\mathcal{I} \not\asymp \langle f^{-1}\rangle\mathcal{Y} \vee \mathcal{J} \not\asymp \langle f^{-1}\rangle\mathcal{Y}}\right\} = \\
&\left\{\frac{\mathcal{Y} \in \mathscr{F}}{\mathcal{Y} \not\asymp \langle f\rangle\mathcal{I} \vee \mathcal{Y} \not\asymp \langle f\rangle\mathcal{J}}\right\} = \\
&\left\{\frac{\mathcal{Y} \in \mathscr{F}}{\mathcal{Y} \not\asymp \langle f\rangle\mathcal{I} \sqcup \langle f\rangle\mathcal{J}}\right\} = \\
&\star(\langle f\rangle\mathcal{I} \sqcup \langle f\rangle\mathcal{J}).
\end{aligned}$$

Thus $\langle f\rangle(\mathcal{I} \sqcup \mathcal{J}) = \langle f\rangle\mathcal{I} \sqcup \langle f\rangle\mathcal{J}$ because $\mathscr{F}(\operatorname{Dst} f)$ is separable. □

Proposition 817. For every $f \in \mathsf{FCD}(A, B)$ for every sets A and B we have:

1°. $\mathcal{K}\ [f]\ \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K}\ [f]\ \mathcal{I} \vee \mathcal{K}\ [f]\ \mathcal{J}$ for every $\mathcal{I}, \mathcal{J} \in \mathscr{F}(B)$, $\mathcal{K} \in \mathscr{F}(A)$.
2°. $\mathcal{I} \sqcup \mathcal{J}\ [f]\ \mathcal{K} \Leftrightarrow \mathcal{I}\ [f]\ \mathcal{K} \vee \mathcal{J}\ [f]\ \mathcal{K}$ for every $\mathcal{I}, \mathcal{J} \in \mathscr{F}(A)$, $\mathcal{K} \in \mathscr{F}(B)$.

Proof.

1°.

$$\begin{aligned}
&\mathcal{K}\ [f]\ \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \\
&(\mathcal{I} \sqcup \mathcal{J}) \sqcap \langle f\rangle\mathcal{K} \neq \bot^{\mathscr{F}(B)} \Leftrightarrow \\
&\mathcal{I} \sqcap \langle f\rangle\mathcal{K} \neq \bot^{\mathscr{F}(B)} \vee \mathcal{J} \sqcap \langle f\rangle\mathcal{K} \neq \bot^{\mathscr{F}(B)} \Leftrightarrow \\
&\mathcal{K}\ [f]\ \mathcal{I} \vee \mathcal{K}\ [f]\ \mathcal{J}.
\end{aligned}$$

2°. Similar.

□

7.2.1. Composition of funcoids.

DEFINITION 818. Funcoids f and g are *composable* when $\operatorname{Dst} f = \operatorname{Src} g$.

DEFINITION 819. *Composition* of composable funcoids is defined by the formula

$$(B, C, \alpha_2, \beta_2) \circ (A, B, \alpha_1, \beta_1) = (A, C, \alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2).$$

PROPOSITION 820. If f, g are composable funcoids then $g \circ f$ is a funcoid.

PROOF. Let $f = (A, B, \alpha_1, \beta_1)$, $g = (B, C, \alpha_2, \beta_2)$. For every $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Y} \in \mathscr{F}(C)$ we have

$$\mathcal{Y} \not\asymp (\alpha_2 \circ \alpha_1)\mathcal{X} \Leftrightarrow \mathcal{Y} \not\asymp \alpha_2\alpha_1\mathcal{X} \Leftrightarrow \alpha_1\mathcal{X} \not\asymp \beta_2\mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp \beta_1\beta_2\mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp (\beta_1 \circ \beta_2)\mathcal{Y}.$$

So $(A, C, \alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2)$ is a funcoid. □

OBVIOUS 821. $\langle g \circ f\rangle = \langle g\rangle \circ \langle f\rangle$ for every composable funcoids f and g.

PROPOSITION 822. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable funcoids f, g, h.

PROOF.

$$\begin{aligned}
\langle (h \circ g) \circ f\rangle &= \\
\langle h \circ g\rangle \circ \langle f\rangle &= \\
(\langle h\rangle \circ \langle g\rangle) \circ \langle f\rangle &= \\
\langle h\rangle \circ (\langle g\rangle \circ \langle f\rangle) &= \\
\langle h\rangle \circ \langle g \circ f\rangle &= \\
\langle h \circ (g \circ f)\rangle&.
\end{aligned}$$

□

THEOREM 823. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every composable funcoids f and g.

PROOF. $\left\langle (g \circ f)^{-1}\right\rangle = \left\langle f^{-1}\right\rangle \circ \left\langle g^{-1}\right\rangle = \left\langle f^{-1} \circ g^{-1}\right\rangle$. □

7.3. Funcoid as continuation

Let f be a funcoid.

DEFINITION 824. $\langle f\rangle^*$ is the function $\mathscr{T}(\operatorname{Src} f) \to \mathscr{F}(\operatorname{Dst} f)$ defined by the formula

$$\langle f\rangle^* X = \langle f\rangle \uparrow X.$$

DEFINITION 825. $[f]^*$ is the relation between $\mathscr{T}(\operatorname{Src} f)$ and $\mathscr{T}(\operatorname{Dst} f)$ defined by the formula

$$X\ [f]^*\ Y \Leftrightarrow \uparrow X\ [f] \uparrow Y.$$

OBVIOUS 826.

1°. $\langle f\rangle^* = \langle f\rangle \circ \uparrow$;
2°. $[f]^* = \uparrow^{-1} \circ [f] \circ \uparrow$.

OBVIOUS 827. $\langle g\rangle\langle f\rangle^* X = \langle g \circ f\rangle^* X$ for every $X \in \mathscr{T}(\operatorname{Src} f)$.

THEOREM 828. For every funcoid f and $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$

1°. $\langle f\rangle\mathcal{X} = \prod\left\langle \langle f\rangle^*\right\rangle^* \operatorname{up} \mathcal{X}$;
2°. $\mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X\ [f]^*\ Y$.

PROOF.

2°.

$$\begin{aligned}\mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \\ \mathcal{Y} \sqcap \langle f\rangle\mathcal{X} \neq \bot \Leftrightarrow \\ \forall Y \in \operatorname{up}\mathcal{Y} : \uparrow Y \sqcap \langle f\rangle\mathcal{X} \neq \bot \Leftrightarrow \\ \forall Y \in \operatorname{up}\mathcal{Y} : \mathcal{X}\ [f] \uparrow Y.\end{aligned}$$

Analogously $\mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X} : \uparrow X\ [f]\ \mathcal{Y}$. Combining these two equivalences we get

$$\mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : \uparrow X\ [f] \uparrow Y \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : X\ [f]^*\ Y.$$

1°.

$$\begin{aligned}\mathcal{Y} \sqcap \langle f\rangle\mathcal{X} \neq \bot \Leftrightarrow \\ \mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \\ \forall X \in \operatorname{up}\mathcal{X} : \uparrow X\ [f]\ \mathcal{Y} \Leftrightarrow \\ \forall X \in \operatorname{up}\mathcal{X} : \mathcal{Y} \sqcap \langle f\rangle^* X \neq \bot.\end{aligned}$$

Let's denote $W = \left\{ \frac{\mathcal{Y} \sqcap \langle f\rangle^* X}{X \in \operatorname{up}\mathcal{X}} \right\}$. We will prove that W is a generalized filter base. To prove this it is enough to show that $V = \left\{ \frac{\langle f\rangle^* X}{X \in \operatorname{up}\mathcal{X}} \right\}$ is a generalized filter base.

Let $\mathcal{P}, \mathcal{Q} \in V$. Then $\mathcal{P} = \langle f\rangle^* A$, $\mathcal{Q} = \langle f\rangle^* B$ where $A, B \in \operatorname{up}\mathcal{X}$; $A \sqcap B \in \operatorname{up}\mathcal{X}$ and $\mathcal{R} \sqsubseteq \mathcal{P} \sqcap \mathcal{Q}$ for $\mathcal{R} = \langle f\rangle^*(A \sqcap B) \in V$. So V is a generalized filter base and thus W is a generalized filter base.

$\bot \notin W \Leftrightarrow \bigsqcap W \neq \bot$ by properties of generalized filter bases. That is

$$\forall X \in \operatorname{up}\mathcal{X} : \mathcal{Y} \sqcap \langle f\rangle^* X \neq \bot^{\mathscr{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{Y} \sqcap \bigsqcap \langle \langle f\rangle^* \rangle^* \operatorname{up}\mathcal{X} \neq \bot.$$

Comparing with the above, $\mathcal{Y} \sqcap \langle f\rangle\mathcal{X} \neq \bot^{\mathscr{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{Y} \sqcap \bigsqcap \langle \langle f\rangle^* \rangle^* \operatorname{up}\mathcal{X} \neq \bot$. So $\langle f\rangle\mathcal{X} = \bigsqcap \langle \langle f\rangle^* \rangle^* \operatorname{up}\mathcal{X}$ because the lattice of filters is separable.

□

Corollary 829. Let f be a funcoid.

1°. The value of f can be restored from the value of $\langle f\rangle^*$.
2°. The value of f can be restored from the value of $[f]^*$.

Proposition 830. For every $f \in \mathsf{FCD}(A, B)$ we have (for every $I, J \in \mathscr{T} A$)

$$\langle f\rangle^* \bot = \bot, \quad \langle f\rangle^*(I \sqcup J) = \langle f\rangle^* I \sqcup \langle f\rangle^* J$$

and

$$\neg(I\ [f]^*\ \bot), I \sqcup J\ [f]^*\ K \Leftrightarrow I\ [f]^*\ K \vee J\ [f]^*\ K \qquad \text{(for every } I, J \in \mathscr{T} A,\ K \in \mathscr{T} B\text{)},$$

$$\neg(\bot\ [f]^*\ I), K\ [f]^*\ I \sqcup J \Leftrightarrow K\ [f]^*\ I \vee K\ [f]^*\ J \qquad \text{(for every } I, J \in \mathscr{T} B,\ K \in \mathscr{T} A\text{)}.$$

Proof. $\langle f\rangle^* \bot = \langle f\rangle \bot = \langle f\rangle \bot = \bot$;

$$\langle f\rangle^*(I \sqcup J) = \langle f\rangle \uparrow (I \sqcup J) = \langle f\rangle \uparrow I \sqcup \langle f\rangle \uparrow J = \langle f\rangle^* I \sqcup \langle f\rangle^* J.$$

$I\,[f]^*\,\bot \Leftrightarrow \bot \not\asymp \langle f\rangle \uparrow I \Leftrightarrow 0;$

$$\begin{aligned} I \sqcup J\,[f]^*\,K &\Leftrightarrow \\ \uparrow(I \sqcup J)\,[f]{\uparrow}\,K &\Leftrightarrow \\ \uparrow K \not\asymp \langle f\rangle \uparrow (I \sqcup J) &\Leftrightarrow \\ \uparrow K \not\asymp \langle f\rangle^*(I \sqcup J) &\Leftrightarrow \\ \uparrow K \not\asymp \langle f\rangle^* I \sqcup \langle f\rangle^* J &\Leftrightarrow \\ \uparrow K \not\asymp \langle f\rangle^* I \vee \uparrow K \not\asymp \langle f\rangle^* J &\Leftrightarrow \\ I\,[f]^*\,K \vee J\,[f]^*\,K. \end{aligned}$$

The rest follows from symmetry. □

THEOREM 831. (fundamental theorem of theory of funcoids) Fix sets A and B. Let $L_F = \lambda f \in \mathsf{FCD}(A,B) : \langle f\rangle^*$ and $L_R = \lambda f \in \mathsf{FCD}(A,B) : [f]^*$.

1°. L_F is a bijection from the set $\mathsf{FCD}(A,B)$ to the set of functions $\alpha \in \mathscr{F}(B)^{\mathscr{T}A}$ that obey the conditions (for every $I,J \in \mathscr{T}A$)

$$\alpha\bot = \bot, \quad \alpha(I \sqcup J) = \alpha I \sqcup \alpha J. \tag{5}$$

For such α it holds (for every $\mathcal{X} \in \mathscr{F}(A)$)

$$\langle L_F^{-1}\alpha\rangle \mathcal{X} = \bigsqcap \langle\alpha\rangle^* \operatorname{up} \mathcal{X}. \tag{6}$$

2°. L_R is a bijection from the set $\mathsf{FCD}(A,B)$ to the set of binary relations $\delta \in \mathscr{P}(\mathscr{T}A \times \mathscr{T}B)$ that obey the conditions

$$\begin{aligned} &\neg(I\ \delta\ \bot), \quad I \sqcup J\ \delta\ K \Leftrightarrow I\ \delta\ K \vee J\ \delta\ K \quad (\text{for every } I,J \in \mathscr{T}A,\ K \in \mathscr{T}B), \\ &\neg(\bot\ \delta\ I), \quad K\ \delta\ I \sqcup J \Leftrightarrow K\ \delta\ I \vee K\ \delta\ J \quad (\text{for every } I,J \in \mathscr{T}B,\ K \in \mathscr{T}A). \end{aligned} \tag{7}$$

For such δ it holds (for every $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Y} \in \mathscr{F}(B)$)

$$\mathcal{X}\ \left[L_R^{-1}\delta\right]\ \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : X\ \delta\ Y. \tag{8}$$

PROOF. Injectivity of L_F and L_R, formulas (6) (for $\alpha \in \operatorname{im} L_F$) and (8) (for $\delta \in \operatorname{im} L_R$), formulas (5) and (7) follow from two previous theorems. The only thing remaining to prove is that for every α and δ that obey the above conditions a corresponding funcoid f exists.

2°. Let define $\alpha \in \mathscr{F}(B)^{\mathscr{T}A}$ by the formula $\partial(\alpha X) = \left\{\frac{Y \in \mathscr{T}B}{X \delta Y}\right\}$ for every $X \in \mathscr{T}A$. (It is obvious that $\left\{\frac{Y \in \mathscr{T}B}{X \delta Y}\right\}$ is a free star.) Analogously it can be defined $\beta \in \mathscr{F}(A)^{\mathscr{T}B}$ by the formula $\partial(\beta Y) = \left\{\frac{X \in \mathscr{T}A}{X \delta Y}\right\}$. Let's continue α and β to $\alpha' \in \mathscr{F}(B)^{\mathscr{F}(A)}$ and $\beta' \in \mathscr{F}(A)^{\mathscr{F}(B)}$ by the formulas

$$\alpha'\mathcal{X} = \bigsqcap \langle\alpha\rangle^* \operatorname{up}\mathcal{X} \quad \text{and} \quad \beta'\mathcal{Y} = \bigsqcap \langle\beta\rangle^* \operatorname{up}\mathcal{Y}$$

and δ to δ' by the formula

$$\mathcal{X}\ \delta'\ \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : X\ \delta\ Y.$$

$\mathcal{Y} \sqcap \alpha'\mathcal{X} \neq \bot \Leftrightarrow \mathcal{Y} \sqcap \bigsqcap \langle\alpha\rangle^* \operatorname{up}\mathcal{X} \neq \bot \Leftrightarrow \bigsqcap \langle \mathcal{Y}\sqcap\rangle^* \langle\alpha\rangle^* \operatorname{up}\mathcal{X} \neq \bot$. Let's prove that

$$W = \langle \mathcal{Y}\sqcap\rangle^* \langle\alpha\rangle^* \operatorname{up}\mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle\alpha\rangle^* \operatorname{up}\mathcal{X}$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \langle\alpha\rangle^* \operatorname{up}\mathcal{X}$ then exist $X_1, X_2 \in \operatorname{up}\mathcal{X}$ such that $\mathcal{A} = \alpha X_1$, $\mathcal{B} = \alpha X_2$.

Then $\alpha(X_1 \sqcap X_2) \in \langle\alpha\rangle^* \operatorname{up}\mathcal{X}$. So $\langle\alpha\rangle^* \operatorname{up}\mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

By properties of generalized filter bases, $\bigsqcap \langle\mathcal{Y}\sqcap\rangle^* \langle\alpha\rangle^* \mathcal{X} \neq \bot$ is equivalent to

$$\forall X \in \operatorname{up}\mathcal{X} : \mathcal{Y} \sqcap \alpha X \neq \bot,$$

what is equivalent to

$$\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : \uparrow Y \sqcap \alpha X \neq \bot \Leftrightarrow$$
$$\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : Y \in \partial(\alpha X) \Leftrightarrow$$
$$\forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \ \delta \ Y.$$

Combining the equivalencies we get $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \bot \Leftrightarrow \mathcal{X} \ \delta' \ \mathcal{Y}$. Analogously $\mathcal{X} \sqcap \beta' \mathcal{Y} \neq \bot \Leftrightarrow \mathcal{X} \ \delta' \ \mathcal{Y}$. So $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \bot \Leftrightarrow \mathcal{X} \sqcap \beta' \mathcal{Y} \neq \bot$, that is (A, B, α', β') is a funcoid. From the formula $\mathcal{Y} \sqcap \alpha' \mathcal{X} \neq \bot^{\mathscr{F}(B)} \Leftrightarrow \mathcal{X} \ \delta' \ \mathcal{Y}$ it follows that

$$X \ [(A, B, \alpha', \beta')]^* \ Y \Leftrightarrow \uparrow Y \sqcap \alpha' \uparrow X \neq \bot \Leftrightarrow \uparrow X \ \delta' \uparrow Y \Leftrightarrow X \ \delta \ Y.$$

1°. Let define the relation $\delta \in \mathscr{P}(\mathscr{T} A \times \mathscr{T} B)$ by the formula $X \ \delta \ Y \Leftrightarrow \uparrow Y \sqcap \alpha X \neq \bot$.

That $\neg(I \ \delta \ \bot)$ and $\neg(\bot \ \delta \ I)$ is obvious. We have

$$I \sqcup J \ \delta \ K \Leftrightarrow$$
$$\uparrow K \sqcap \alpha(I \sqcup J) \neq \bot \Leftrightarrow$$
$$\uparrow K \sqcap (\alpha I \sqcup \alpha J) \neq \bot \Leftrightarrow$$
$$\uparrow K \sqcap \alpha I \neq \bot \vee \uparrow K \sqcap \alpha J \neq \bot \Leftrightarrow$$
$$I \ \delta \ K \vee J \ \delta \ K$$

and

$$K \ \delta \ I \sqcup J \Leftrightarrow$$
$$\uparrow (I \sqcup J) \sqcap \alpha K \neq \bot \Leftrightarrow$$
$$(\uparrow I \sqcup \uparrow J) \sqcap \alpha K \neq \bot \Leftrightarrow$$
$$\uparrow I \sqcap \alpha K \neq \bot \vee \uparrow J \sqcap \alpha K \neq \bot \Leftrightarrow$$
$$K \ \delta \ I \vee K \ \delta \ J.$$

That is the formulas (7) are true.

Accordingly to the above there exists a funcoid f such that

$$\mathcal{X} \ [f] \ \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \ \delta \ Y.$$

For every $X \in \mathscr{T} A$, $Y \in \mathscr{T} B$ we have:

$$\uparrow Y \sqcap \langle f \rangle \uparrow X \neq \bot \Leftrightarrow \uparrow X \ [f] \uparrow Y \Leftrightarrow X \ \delta \ Y \Leftrightarrow \uparrow Y \sqcap \alpha X \neq \bot,$$

consequently $\forall X \in \mathscr{T} A : \alpha X = \langle f \rangle \uparrow X = \langle f \rangle^* X$.

□

Note that by the last theorem to every (quasi-)proximity δ corresponds a unique funcoid. So funcoids are a generalization of (quasi-)proximity structures. Reverse funcoids can be considered as a generalization of conjugate quasi-proximity.

Corollary 832. If $\alpha \in \mathscr{F}(B)^{\mathscr{T} A}$, $\beta \in \mathscr{F}(A)^{\mathscr{T} B}$ are functions such that $Y \not\asymp \alpha X \Leftrightarrow X \not\asymp \beta Y$ for every $X \in \mathscr{T} A$, $Y \in \mathscr{T} B$, then there exists exactly one funcoid f such that $\langle f \rangle^* = \alpha$, $\langle f^{-1} \rangle^* = \beta$.

Proof. Prove $\alpha(I \sqcup J) = \alpha I \sqcup \alpha J$. Really, $Y \not\asymp \alpha(I \sqcup J) \Leftrightarrow I \sqcup J \not\asymp \beta Y \Leftrightarrow I \not\asymp \beta Y \vee J \not\asymp \beta Y \Leftrightarrow Y \not\asymp \alpha I \vee Y \not\asymp \alpha J \Leftrightarrow Y \not\asymp \alpha I \sqcup \alpha J$. So $\alpha(I \sqcup J) = \alpha I \sqcup \alpha J$ by star-separability. Similarly $\beta(I \sqcup J) = \beta I \sqcup \beta J$.

Thus by the theorem there exists a funcoid f such that $\langle f \rangle^* = \alpha$, $\langle f^{-1} \rangle^* = \beta$.

That this funcoid is unique, follows from the above. □

Definition 833. Any **Rel**-morphism $F : A \to B$ corresponds to a funcoid $\uparrow^{\mathsf{FCD}} F \in \mathsf{FCD}(A, B)$, where by definition $\langle \uparrow^{\mathsf{FCD}} F \rangle \mathcal{X} = \bigsqcap^{\mathscr{F}} \langle \langle F \rangle^* \rangle^* \operatorname{up} \mathcal{X}$ for every $\mathcal{X} \in \mathscr{F}(A)$.

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff. (Take $\alpha = \uparrow \circ \langle F\rangle^*$.)

PROPOSITION 834. $\left\langle \uparrow^{\mathsf{FCD}} f\right\rangle^* X = \langle f\rangle^* X$ for a **Rel**-morphism f and $X \in \mathscr{T} \operatorname{Src} f$.

PROOF. $\left\langle \uparrow^{\mathsf{FCD}} f\right\rangle^* X = \min \langle\uparrow\rangle^* \left\langle \langle f\rangle^*\right\rangle^* \operatorname{up} X = \uparrow \langle f\rangle^* X = \langle f\rangle^* X$. ∎

COROLLARY 835. $\left[\uparrow^{\mathsf{FCD}} f\right]^* = [f]^*$ for every **Rel**-morphism f.

PROOF. $X \left[\uparrow^{\mathsf{FCD}} f\right]^* Y \Leftrightarrow Y \not\asymp \left\langle \uparrow^{\mathsf{FCD}} f\right\rangle^* X \Leftrightarrow Y \not\asymp \langle f\rangle^* X \Leftrightarrow X\ [f]^*\ Y$ for $X \in \mathscr{T} \operatorname{Src} f$, $Y \in \mathscr{T} \operatorname{Dst} f$. ∎

DEFINITION 836. $\uparrow^{\mathsf{FCD}(A,B)} f = \uparrow^{\mathsf{FCD}} (A, B, f)$ for every binary relation f between sets A and B.

DEFINITION 837. Funcoids corresponding to a binary relation (= multivalued function) are called *principal funcoids*.

PROPOSITION 838. $\uparrow^{\mathsf{FCD}} g \circ \uparrow^{\mathsf{FCD}} f = \uparrow^{\mathsf{FCD}} (g \circ f)$ for composable morphisms f, g of category **Rel**.

PROOF. For every $X \in \mathscr{T} \operatorname{Src} f$

$$\left\langle \uparrow^{\mathsf{FCD}} g \circ \uparrow^{\mathsf{FCD}} f\right\rangle^* X = \left\langle \uparrow^{\mathsf{FCD}} g\right\rangle^* \left\langle \uparrow^{\mathsf{FCD}} f\right\rangle^* X = \langle g\rangle^* \langle f\rangle^* X = \langle g \circ f\rangle^* X = \left\langle \uparrow^{\mathsf{FCD}} (g \circ f)\right\rangle^* X.$$

∎

We may equate principal funcoids with corresponding binary relations by the method of appendix A. This is useful for describing relationships of funcoids and binary relations, such as for the formulas of continuous functions and continuous funcoids (see below).

Thus $(\mathsf{FCD}(A, B), \mathbf{Rel}(A, B))$ is a filtrator. I call it *filtrator of funcoids*.

THEOREM 839. If S is a generalized filter base on $\operatorname{Src} f$ then $\langle f\rangle \bigsqcap S = \bigsqcap \langle \langle f\rangle\rangle^* S$ for every funcoid f.

PROOF. $\langle f\rangle \bigsqcap S \sqsubseteq \langle f\rangle X$ for every $X \in S$ and thus $\langle f\rangle \bigsqcap S \sqsubseteq \bigsqcap \langle \langle f\rangle\rangle^* S$.

By properties of generalized filter bases:

$$\begin{aligned} \langle f\rangle \bigsqcap S &= \\ \bigsqcap \left\langle \langle f\rangle^*\right\rangle^* \operatorname{up} \bigsqcap S &= \\ \bigsqcap \left\langle \langle f\rangle^*\right\rangle^* \left\{ \frac{X}{\exists \mathcal{P} \in S : X \in \operatorname{up} \mathcal{P}} \right\} &= \\ \bigsqcap \left\{ \frac{\langle f\rangle^* X}{\exists \mathcal{P} \in S : X \in \operatorname{up} \mathcal{P}} \right\} &\sqsupseteq \\ \bigsqcap_{\mathcal{P} \in S} \langle f\rangle \mathcal{P} &= \\ \bigsqcap \langle \langle f\rangle\rangle^* S. \end{aligned}$$

∎

PROPOSITION 840. $\mathcal{X}\ [f] \bigsqcap S \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X}\ [f]\ \mathcal{Y}$ if f is a funcoid and S is a generalized filter base on $\operatorname{Dst} f$.

PROOF.

$$\mathcal{X}\ [f] \bigsqcap S \Leftrightarrow \bigsqcap S \sqcap \langle f\rangle\mathcal{X} \neq \bot \Leftrightarrow \bigsqcap \langle \langle f\rangle\mathcal{X}\sqcap\rangle^* S \neq \bot \Leftrightarrow$$
$$\text{(by properties of generalized filter bases)} \Leftrightarrow$$
$$\exists \mathcal{Y} \in \langle \langle f\rangle\mathcal{X}\sqcap\rangle^* S : \mathcal{Y} \neq \bot \Leftrightarrow \exists \mathcal{Y} \in S : \langle f\rangle\mathcal{X} \sqcap \mathcal{Y} \neq \bot \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X}\ [f]\ \mathcal{Y}.$$

□

DEFINITION 841. A function f between two posets is said to *preserve filtered meets*, when $f \bigsqcap S = \bigsqcap \langle f\rangle^* S$ whenever $\bigsqcap S$ is defined for a filter base S on the first of the two posets.

THEOREM 842. (discovered by TODD TRIMBLE) A function $\varphi : \mathscr{F}(A) \to \mathscr{F}(B)$ preserves finite joins (including nullary joins) and filtered meets iff there exists a funcoid f such that $\langle f\rangle = \varphi$.

PROOF. Backward implication follows from above.

Let $\psi = \varphi|_{\mathscr{T} A}$. Then ψ preserves bottom element and binary joins. Thus there exists a funcoid f such that $\langle f\rangle^* = \psi$.

It remains to prove that $\langle f\rangle = \varphi$.

Really, $\langle f\rangle\mathcal{X} = \bigsqcap \langle \langle f\rangle^*\rangle^* \operatorname{up}\mathcal{X} = \bigsqcap \langle\psi\rangle^* \operatorname{up}\mathcal{X} = \bigsqcap \langle\varphi\rangle^* \operatorname{up}\mathcal{X} = \varphi \bigsqcap \operatorname{up}\mathcal{X} = \varphi\mathcal{X}$ for every $\mathcal{X} \in \mathscr{F}(A)$. □

COROLLARY 843. Funcoids f from A to B bijectively correspond by the formula $\langle f\rangle = \varphi$ to functions $\varphi : \mathscr{F}(A) \to \mathscr{F}(B)$ preserving finite joins and filtered meets.

7.4. Another way to represent funcoids as binary relations

This is based on a TODD TRIMBLE's idea.

DEFINITION 844. The binary relation $\xi^\circledast \in \mathscr{P}(\mathscr{F}(\operatorname{Src}\xi) \times \mathscr{F}(\operatorname{Dst}\xi))$ for a funcoid ξ is defined by the formula $\mathcal{A}\ \xi^\circledast\ \mathcal{B} \Leftrightarrow \mathcal{B} \sqsupseteq \langle\xi\rangle\mathcal{A}$.

DEFINITION 845. The binary relation $\xi^* \in \mathscr{P}(\mathscr{T}\operatorname{Src}\xi \times \mathscr{T}\operatorname{Dst}\xi)$ for a funcoid ξ is defined by the formula

$$A\ \xi^*\ B \Leftrightarrow B \sqsupseteq \langle\xi\rangle A \Leftrightarrow B \in \operatorname{up}\langle\xi\rangle A.$$

PROPOSITION 846. Funcoid ξ can be restored from

1°. the value of $\xi^\circledast$;
2°. the value of ξ^*.

PROOF.

1°. The value of $\langle\xi\rangle$ can be restored from $\xi^\circledast$.
2°. The value of $\langle\xi\rangle^*$ can be restored from ξ^*.

□

THEOREM 847. Let ν and ξ be composable funcoids. Then:

1°. $\xi^\circledast \circ \nu^\circledast = (\xi\circ\nu)^\circledast$;
2°. $\xi^* \circ \nu^* = (\xi\circ\nu)^*$.

PROOF.

1°.

$$\mathcal{A}\ (\xi^\circledast \circ \nu^\circledast)\ \mathcal{C} \Leftrightarrow \exists \mathcal{B} : \left(\mathcal{A}\ \nu^\circledast\ \mathcal{B} \wedge \mathcal{B}\ \xi^\circledast\ \mathcal{C}\right) \Leftrightarrow$$
$$\exists \mathcal{B} \in \mathscr{F}(\operatorname{Dst}\nu) : (\mathcal{B} \sqsupseteq \langle\nu\rangle\mathcal{A} \wedge \mathcal{C} \sqsupseteq \langle\xi\rangle\mathcal{B}) \Leftrightarrow$$
$$\mathcal{C} \sqsupseteq \langle\xi\rangle\langle\nu\rangle\mathcal{A} \Leftrightarrow \mathcal{C} \sqsupseteq \langle\xi\circ\nu\rangle\mathcal{A} \Leftrightarrow \mathcal{A}\ (\xi\circ\nu)^\circledast\ \mathcal{C}.$$

2°.

$$A\ (\xi^* \circ \nu^*)\ C \Leftrightarrow \exists B : (A\ \nu^*\ B \wedge B\ \xi^*\ C) \Leftrightarrow$$
$$\exists B : (B \in \operatorname{up}\langle \nu \rangle A \wedge C \in \operatorname{up}\langle \xi \rangle B) \Leftrightarrow \exists B \in \operatorname{up}\langle \nu \rangle A : C \in \operatorname{up}\langle \xi \rangle B.$$

$A\ (\xi \circ \nu)^*\ C \Leftrightarrow C \in \operatorname{up}\langle \xi \circ \nu \rangle B \Leftrightarrow C \in \operatorname{up}\langle \xi \rangle \langle \nu \rangle B$.

It remains to prove

$$\exists B \in \operatorname{up}\langle \nu \rangle A : C \in \operatorname{up}\langle \xi \rangle B \Leftrightarrow C \in \operatorname{up}\langle \xi \rangle \langle \nu \rangle A.$$

$\exists B \in \operatorname{up}\langle \nu \rangle A : C \in \operatorname{up}\langle \xi \rangle B \Rightarrow C \in \operatorname{up}\langle \xi \rangle \langle \nu \rangle A$ is obvious.

Let $C \in \operatorname{up}\langle \xi \rangle \langle \nu \rangle A$. Then $C \in \operatorname{up} \bigsqcap \langle \langle \xi \rangle \rangle^* \operatorname{up}\langle \nu \rangle A$; so by properties of generalized filter bases, $\exists P \in \langle \langle \xi \rangle \rangle^* \operatorname{up}\langle \nu \rangle A : C \in \operatorname{up} P$; $\exists B \in \operatorname{up}\langle \nu \rangle A : C \in \operatorname{up}\langle \xi \rangle B$. □

REMARK 848. The above theorem is interesting by the fact that composition of funcoids is represented as relational composition of binary relations.

7.5. Lattices of funcoids

DEFINITION 849. $f \sqsubseteq g \overset{\text{def}}{=} [f] \subseteq [g]$ for $f, g \in \mathsf{FCD}(A, B)$ for every sets A, B.

Thus every $\mathsf{FCD}(A, B)$ is a poset. (It's taken into account that $[f] \neq [g]$ when $f \neq g$.)

We will consider filtrators (*filtrators of funcoids*) whose base is $\mathsf{FCD}(A, B)$ and whose core are principal funcoids from A to B.

LEMMA 850. $\langle f \rangle^* X = \bigsqcap^{\mathscr{F}}_{F \in \operatorname{up} f} \langle F \rangle^* X$ for every funcoid f and typed set $X \in \mathscr{T}(\operatorname{Src} f)$.

PROOF. Obviously $\langle f \rangle^* X \sqsubseteq \bigsqcap^{\mathscr{F}}_{F \in \operatorname{up} f} \langle F \rangle^* X$.

Let $B \in \operatorname{up}\langle f \rangle^* X$. Let $F_B = X \times B \sqcup \overline{X} \times \top$.

$\langle F_B \rangle^* X = B$.

Let $P \in \mathscr{T}(\operatorname{Src} f)$. We have

$$\bot \neq P \sqsubseteq X \Rightarrow \langle F_B \rangle^* P = B \sqsupseteq \langle f \rangle^* P$$

and

$$P \not\sqsubseteq X \Rightarrow \langle F_B \rangle^* P = \top \sqsupseteq \langle f \rangle^* P.$$

Thus $\langle F_B \rangle^* P \sqsupseteq \langle f \rangle^* P$ for every P and so $F_B \sqsupseteq f$ that is $F_B \in \operatorname{up} f$.

Thus $\forall B \in \operatorname{up}\langle f \rangle^* X : B \in \operatorname{up} \bigsqcap^{\mathscr{F}}_{F \in \operatorname{up} f} \langle F \rangle^* X$ because $B \in \operatorname{up}\langle F_B \rangle^* X$.

So $\bigsqcap_{F \in \operatorname{up} f} \langle F \rangle^* X \sqsubseteq \langle f \rangle^* X$. □

THEOREM 851. $\langle f \rangle \mathcal{X} = \bigsqcap^{\mathscr{F}}_{F \in \operatorname{up} f} \langle F \rangle \mathcal{X}$ for every funcoid f and $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$.

PROOF.

$$\begin{aligned}
&\bigsqcap_{F\in\operatorname{up} f}^{\mathscr{F}}\langle F\rangle\mathcal{X}=\\
&\bigsqcap_{F\in\operatorname{up} f}^{\mathscr{F}}\bigsqcap^{\mathscr{F}}\langle\langle F\rangle^*\rangle^*\operatorname{up}\mathcal{X}=\\
&\bigsqcap_{F\in\operatorname{up} f}^{\mathscr{F}}\bigsqcap_{X\in\operatorname{up}\mathcal{X}}^{\mathscr{F}}\langle F\rangle^*X=\\
&\bigsqcap_{X\in\operatorname{up}\mathcal{X}}^{\mathscr{F}}\bigsqcap_{F\in\operatorname{up} f}^{\mathscr{F}}\langle F\rangle^*X=\\
&\bigsqcap_{X\in\operatorname{up}\mathcal{X}}^{\mathscr{F}}\langle f\rangle^*X=\\
&\langle f\rangle\mathcal{X}
\end{aligned}$$

(the lemma used). □

Below it is shown that $\mathsf{FCD}(A,B)$ are complete lattices for every sets A and B. We will apply lattice operations to subsets of such sets without explicitly mentioning $\mathsf{FCD}(A,B)$.

THEOREM 852. $\mathsf{FCD}(A,B)$ is a complete lattice (for every sets A and B). For every $R\in\mathscr{P}\mathsf{FCD}(A,B)$ and $X\in\mathscr{T}A$, $Y\in\mathscr{T}B$

1°. $X\ [\bigsqcup R]^*\ Y\Leftrightarrow\exists f\in R:X\ [f]^*\ Y$;
2°. $\langle\bigsqcup R\rangle^*X=\bigsqcup_{f\in R}\langle f\rangle^*X$.

PROOF. Accordingly [27] to prove that it is a complete lattice it's enough to prove existence of all joins.

2°. $\alpha X\overset{\text{def}}{=}\bigsqcup_{f\in R}\langle f\rangle^*X$. We have $\alpha\bot=\bot$;

$$\begin{aligned}
&\alpha(I\sqcup J)=\\
&\bigsqcup_{f\in R}\langle f\rangle^*(I\sqcup J)=\\
&\bigsqcup_{f\in R}(\langle f\rangle^*I\sqcup\langle f\rangle^*J)=\\
&\bigsqcup_{f\in R}\langle f\rangle^*I\sqcup\bigsqcup\langle f\rangle^*J=\\
&\alpha I\sqcup\alpha J.
\end{aligned}$$

So $\langle h\rangle^*=\alpha$ for some funcoid h. Obviously

$$\forall f\in R:h\sqsupseteq f. \tag{9}$$

And h is the least funcoid for which holds the condition (9). So $h=\bigsqcup R$.

1°.

$$\begin{aligned} X \left[\bigsqcup R\right]^* Y &\Leftrightarrow \\ \uparrow Y \sqcap \left\langle\bigsqcup R\right\rangle^* X \neq \bot &\Leftrightarrow \\ \uparrow Y \sqcap \bigsqcup_{f\in R} \langle f\rangle^* X \neq \bot &\Leftrightarrow \\ \exists f \in R : \uparrow Y \sqcap \langle f\rangle^* X \neq \bot &\Leftrightarrow \\ \exists f \in R : X \, [f]^* \, Y & \end{aligned}$$

(used proposition 610).

□

In the next theorem, compared to the previous one, the class of infinite joins is replaced with lesser class of binary joins and simultaneously class of sets is changed to more wide class of filters.

THEOREM 853. For every $f, g \in \mathsf{FCD}(A, B)$ and $\mathcal{X} \in \mathscr{F}(A)$ (for every sets A, B)

1°. $\langle f \sqcup g\rangle\mathcal{X} = \langle f\rangle\mathcal{X} \sqcup \langle g\rangle\mathcal{X}$;
2°. $[f \sqcup g] = [f] \cup [g]$.

PROOF.

1°. Let $\alpha\mathcal{X} \overset{\text{def}}{=} \langle f\rangle\mathcal{X} \sqcup \langle g\rangle\mathcal{X}$; $\beta\mathcal{Y} \overset{\text{def}}{=} \langle f^{-1}\rangle\mathcal{Y} \sqcup \langle g^{-1}\rangle\mathcal{Y}$ for every $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Y} \in \mathscr{F}(B)$. Then

$$\begin{aligned} \mathcal{Y} \sqcap \alpha\mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle f\rangle\mathcal{X} \neq \bot \vee \mathcal{Y} \sqcap \langle g\rangle\mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{X} \sqcap \langle f^{-1}\rangle\mathcal{Y} \neq \bot \vee \mathcal{X} \sqcap \langle g^{-1}\rangle\mathcal{Y} \neq \bot &\Leftrightarrow \\ \mathcal{X} \sqcap \beta\mathcal{Y} \neq \bot. & \end{aligned}$$

So $h = (A, B, \alpha, \beta)$ is a funcoid. Obviously $h \sqsupseteq f$ and $h \sqsupseteq g$. If $p \sqsupseteq f$ and $p \sqsupseteq g$ for some funcoid p then $\langle p\rangle\mathcal{X} \sqsupseteq \langle f\rangle\mathcal{X} \sqcup \langle g\rangle\mathcal{X} = \langle h\rangle\mathcal{X}$ that is $p \sqsupseteq h$. So $f \sqcup g = h$.

2°. For every $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Y} \in \mathscr{F}(B)$ we have

$$\begin{aligned} \mathcal{X} \, [f \sqcup g] \, \mathcal{Y} &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle f \sqcup g\rangle\mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{Y} \sqcap (\langle f\rangle\mathcal{X} \sqcup \langle g\rangle\mathcal{X}) \neq \bot &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle f\rangle\mathcal{X} \neq \bot \vee \mathcal{Y} \sqcap \langle g\rangle\mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{X} \, [f] \, \mathcal{Y} \vee \mathcal{X} \, [g] \, \mathcal{Y}. & \end{aligned}$$

□

7.6. More on composition of funcoids

PROPOSITION 854. $[g \circ f] = [g] \circ \langle f\rangle = \left\langle g^{-1}\right\rangle^{-1} \circ [f]$ for every composable funcoids f and g.

PROOF. For every $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} g)$ we have

$$\begin{aligned} \mathcal{X} [g \circ f] \mathcal{Y} &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle g \circ f\rangle \mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{Y} \sqcap \langle g\rangle\langle f\rangle \mathcal{X} \neq \bot &\Leftrightarrow \\ \langle f\rangle \mathcal{X} [g] \mathcal{Y} &\Leftrightarrow \\ \mathcal{X} ([g] \circ \langle f\rangle) \mathcal{Y} & \end{aligned}$$

and

$$\begin{aligned} [g \circ f] &= \\ \left[(f^{-1} \circ g^{-1})^{-1}\right] &= \\ \left[f^{-1} \circ g^{-1}\right]^{-1} &= \\ \left(\left[f^{-1}\right] \circ \left\langle g^{-1}\right\rangle\right)^{-1} &= \\ \left\langle g^{-1}\right\rangle^{-1} \circ [f]. & \end{aligned}$$

□

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x \ (g \circ f) \ z \Leftrightarrow \exists y : (x \ f \ y \wedge y \ g \ z)$ for every x and z and every binary relations f and g.

THEOREM 855. For every sets A, B, C and $f \in \mathsf{FCD}(A, B)$, $g \in \mathsf{FCD}(B, C)$ and $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Z} \in \mathscr{F}(C)$

$$\mathcal{X} [g \circ f] \mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}).$$

PROOF.

$$\begin{aligned} \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X} [f] y \wedge y [g] \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{Z} \sqcap \langle g\rangle y \neq \bot \wedge y \sqcap \langle f\rangle \mathcal{X} \neq \bot) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{Z} \sqcap \langle g\rangle y \neq \bot \wedge y \sqsubseteq \langle f\rangle \mathcal{X}) &\Rightarrow \\ \mathcal{Z} \sqcap \langle g\rangle\langle f\rangle \mathcal{X} \neq \bot &\Leftrightarrow \\ \mathcal{X} [g \circ f] \mathcal{Z}. & \end{aligned}$$

Reversely, if $\mathcal{X} [g \circ f] \mathcal{Z}$ then $\langle f\rangle \mathcal{X} [g] \mathcal{Z}$, consequently there exists $y \in \operatorname{atoms}\langle f\rangle \mathcal{X}$ such that $y [g] \mathcal{Z}$; we have $\mathcal{X} [f] y$. □

THEOREM 856. For every sets A, B, C

1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for $g, h \in \mathsf{FCD}(A, B)$, $f \in \mathsf{FCD}(B, C)$;
2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$ for $g, h \in \mathsf{FCD}(B, C)$, $f \in \mathsf{FCD}(A, B)$.

PROOF. I will prove only the first equality because the other is analogous.
For every $\mathcal{X} \in \mathscr{F}(A)$, $\mathcal{Z} \in \mathscr{F}(C)$

$$\begin{aligned} \mathcal{X} [f \circ (g \sqcup h)] \mathcal{Z} &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X} [g \sqcup h] y \wedge y [f] \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : ((\mathcal{X} [g] y \vee \mathcal{X} [h] y) \wedge y [f] \mathcal{Z}) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : ((\mathcal{X} [g] y \wedge y [f] \mathcal{Z}) \vee (\mathcal{X} [h] y \wedge y [f] \mathcal{Z})) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X} [g] y \wedge y [f] \mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathscr{F}(B)} : (\mathcal{X} [h] y \wedge y [f] \mathcal{Z}) &\Leftrightarrow \\ \mathcal{X} [f \circ g] \mathcal{Z} \vee \mathcal{X} [f \circ h] \mathcal{Z} &\Leftrightarrow \\ \mathcal{X} [f \circ g \sqcup f \circ h] \mathcal{Z}. & \end{aligned}$$

□

Another proof of the above theorem (without atomic filters):

PROOF.

$$\begin{aligned} \langle f \circ (g \sqcup h)\rangle \mathcal{X} &= \\ \langle f\rangle \langle g \sqcup h\rangle \mathcal{X} &= \\ \langle f\rangle (\langle g\rangle \mathcal{X} \sqcup \langle h\rangle \mathcal{X}) &= \\ \langle f\rangle \langle g\rangle \mathcal{X} \sqcup \langle f\rangle \langle h\rangle \mathcal{X} &= \\ \langle f \circ g\rangle \mathcal{X} \sqcup \langle f \circ h\rangle \mathcal{X} &= \\ \langle f \circ g \sqcup f \circ h\rangle \mathcal{X}. \end{aligned}$$

□

7.7. Domain and range of a funcoid

DEFINITION 857. Let A be a set. The *identity funcoid* $1^{\mathsf{FCD}}_A = (A, A, \mathrm{id}_{\mathscr{F}(A)}, \mathrm{id}_{\mathscr{F}(A)})$.

OBVIOUS 858. The identity funcoid is a funcoid.

PROPOSITION 859. $[f] = [1_{\mathrm{Dst}\, f}] \circ \langle f\rangle$ for every funcoid f.

PROOF. From proposition 854. □

DEFINITION 860. Let A be a set, $\mathcal{A} \in \mathscr{F}(A)$. The *restricted identity funcoid*

$$\mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}} = (A, A, \mathcal{A}\sqcap, \mathcal{A}\sqcap).$$

PROPOSITION 861. The restricted identity funcoid is a funcoid.

PROOF. We need to prove that $(\mathcal{A} \sqcap \mathcal{X}) \sqcap \mathcal{Y} \neq \bot \Leftrightarrow (\mathcal{A} \sqcap \mathcal{Y}) \sqcap \mathcal{X} \neq \bot$ what is obvious. □

OBVIOUS 862.

1°. $(1^{\mathsf{FCD}}_A)^{-1} = 1^{\mathsf{FCD}}_A$;
2°. $(\mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}})^{-1} = \mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}}$.

OBVIOUS 863. For every $\mathcal{X}, \mathcal{Y} \in \mathscr{F}(A)$

1°. $\mathcal{X} \left[1^{\mathsf{FCD}}_A\right] \mathcal{Y} \Leftrightarrow \mathcal{X} \sqcap \mathcal{Y} \neq \bot$;
2°. $\mathcal{X} \left[\mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}}\right] \mathcal{Y} \Leftrightarrow \mathcal{A} \sqcap \mathcal{X} \sqcap \mathcal{Y} \neq \bot$.

DEFINITION 864. I will define *restricting* of a funcoid f to a filter $\mathcal{A} \in \mathscr{F}(\mathrm{Src}\, f)$ by the formula

$$f|_{\mathcal{A}} = f \circ \mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}}.$$

DEFINITION 865. *Image* of a funcoid f will be defined by the formula $\mathrm{im}\, f = \langle f\rangle \top^{\mathscr{F}(\mathrm{Src}\, f)}$.

Domain of a funcoid f is defined by the formula $\mathrm{dom}\, f = \mathrm{im}\, f^{-1}$.

OBVIOUS 866. For every morphism $f \in \mathbf{Rel}(A, B)$ for sets A and B

1°. $\mathrm{im} \uparrow^{\mathsf{FCD}} f = \uparrow \mathrm{im}\, f$;
2°. $\mathrm{dom} \uparrow^{\mathsf{FCD}} f = \uparrow \mathrm{dom}\, f$.

PROPOSITION 867. $\langle f\rangle \mathcal{X} = \langle f\rangle (\mathcal{X} \sqcap \mathrm{dom}\, f)$ for every funcoid f, $\mathcal{X} \in \mathscr{F}(\mathrm{Src}\, f)$.

Proof. For every $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$ we have

$$\begin{aligned}\mathcal{Y} \sqcap \langle f \rangle (\mathcal{X} \sqcap \operatorname{dom} f) \neq \bot \Leftrightarrow \\ \mathcal{X} \sqcap \operatorname{dom} f \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq \bot \Leftrightarrow \\ \mathcal{X} \sqcap \operatorname{im} f^{-1} \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq \bot \Leftrightarrow \\ \mathcal{X} \sqcap \langle f^{-1} \rangle \mathcal{Y} \neq \bot \Leftrightarrow \\ \mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq \bot.\end{aligned}$$

Thus $\langle f \rangle (\mathcal{X} \sqcap \operatorname{dom} f) = \langle f \rangle \mathcal{X}$ because the lattice of filters is separable. □

Proposition 868. $\langle f \rangle \mathcal{X} = \operatorname{im}(f|_{\mathcal{X}})$ for every funcoid f, $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$.

Proof.

$$\begin{aligned}\operatorname{im}(f|_{\mathcal{X}}) = \\ \left\langle f \circ \operatorname{id}^{\mathsf{FCD}}_{\mathcal{X}} \right\rangle \top = \\ \langle f \rangle \left\langle \operatorname{id}^{\mathsf{FCD}}_{\mathcal{X}} \right\rangle \top = \\ \langle f \rangle (\mathcal{X} \sqcap \top) = \\ \langle f \rangle \mathcal{X}.\end{aligned}$$

□

Proposition 869. $\mathcal{X} \sqcap \operatorname{dom} f \neq \bot \Leftrightarrow \langle f \rangle \mathcal{X} \neq \bot$ for every funcoid f and $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$.

Proof.

$$\begin{aligned}\mathcal{X} \sqcap \operatorname{dom} f \neq \bot \Leftrightarrow \\ \mathcal{X} \sqcap \langle f^{-1} \rangle \top^{\mathscr{F}(\operatorname{Dst} f)} \neq \bot \Leftrightarrow \\ \top \sqcap \langle f \rangle \mathcal{X} \neq \bot \Leftrightarrow \\ \langle f \rangle \mathcal{X} \neq \bot.\end{aligned}$$

□

Corollary 870. $\operatorname{dom} f = \bigsqcup \left\{ \frac{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}}{\langle f \rangle a \neq \bot} \right\}$.

Proof. This follows from the fact that $\mathscr{F}(\operatorname{Src} f)$ is an atomistic lattice. □

Proposition 871. $\operatorname{dom}(f|_{\mathcal{A}}) = \mathcal{A} \sqcap \operatorname{dom} f$ for every funcoid f and $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$.

Proof.

$$\begin{aligned}\operatorname{dom}(f|_{\mathcal{A}}) = \\ \operatorname{im}(\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}} \circ f^{-1}) = \\ \left\langle \operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}} \right\rangle \langle f^{-1} \rangle \top = \\ \mathcal{A} \sqcap \langle f^{-1} \rangle \top = \\ \mathcal{A} \sqcap \operatorname{dom} f.\end{aligned}$$

□

Theorem 872. $\operatorname{im} f = \bigsqcap^{\mathscr{F}} \langle \operatorname{im} \rangle^* \operatorname{up} f$ and $\operatorname{dom} f = \bigsqcap^{\mathscr{F}} \langle \operatorname{dom} \rangle^* \operatorname{up} f$ for every funcoid f.

PROOF.

$$\begin{aligned} &\operatorname{im} f = \\ &\langle f\rangle\top = \\ &\prod_{F\in\operatorname{up} f}^{\mathscr{F}} \langle F\rangle\top = \\ &\prod_{F\in\operatorname{up} f}^{\mathscr{F}} \operatorname{im} F = \\ &\prod^{\mathscr{F}} \langle\operatorname{im}\rangle^* \operatorname{up} f. \end{aligned}$$

The second formula follows from symmetry. □

PROPOSITION 873. For every composable funcoids f, g:

1°. If $\operatorname{im} f \sqsupseteq \operatorname{dom} g$ then $\operatorname{im}(g\circ f) = \operatorname{im} g$.
2°. If $\operatorname{im} f \sqsubseteq \operatorname{dom} g$ then $\operatorname{dom}(g\circ f) = \operatorname{dom} f$.

PROOF.

1°.

$$\begin{aligned} &\operatorname{im}(g\circ f) = \\ &\langle g\circ f\rangle\top = \\ &\langle g\rangle\langle f\rangle\top = \\ &\langle g\rangle \operatorname{im} f = \\ &\langle g\rangle(\operatorname{im} f \sqcap \operatorname{dom} g) = \\ &\langle g\rangle \operatorname{dom} g = \\ &\langle g\rangle\top = \\ &\operatorname{im} g. \end{aligned}$$

2°. $\operatorname{dom}(g\circ f) = \operatorname{im}(f^{-1}\circ g^{-1})$ what by proved above is equal to $\operatorname{im} f^{-1}$ that is $\operatorname{dom} f$.

□

7.8. Categories of funcoids

I will define two categories, the *category of funcoids* and the *category of funcoid triples*.

The *category of funcoids* is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\mathsf{FCD}(A,B)$.
- The composition is the composition of funcoids.
- Identity morphism for a set is the identity funcoid for that set.

To show it is really a category is trivial.

The *category of funcoid triples* is defined as follows:

- Objects are filters on small sets.
- The morphisms from a filter $\mathcal{A}$ to a filter $\mathcal{B}$ are triples $(\mathcal{A},\mathcal{B},f)$ where $f\in\mathsf{FCD}(\operatorname{Base}(\mathcal{A}),\operatorname{Base}(\mathcal{B}))$ and $\operatorname{dom} f\sqsubseteq\mathcal{A}\wedge\operatorname{im} f\sqsubseteq\mathcal{B}$.
- The composition is defined by the formula $(\mathcal{B},\mathcal{C},g)\circ(\mathcal{A},\mathcal{B},f) = (\mathcal{A},\mathcal{C},g\circ f)$.
- Identity morphism for a filter $\mathcal{A}$ is $\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$.

To prove that it is really a category is trivial.

PROPOSITION 874. $\uparrow^{\mathsf{FCD}}$ is a functor from **Rel** to FCD.

PROOF. $\uparrow^{\mathsf{FCD}}(g\circ f)=\uparrow^{\mathsf{FCD}} g\circ\uparrow^{\mathsf{FCD}} f$ was proved above. $\uparrow^{\mathsf{FCD}} 1^{\mathbf{Rel}}_A=1^{\mathsf{FCD}}_A$ is obvious. □

7.9. Specifying funcoids by functions or relations on atomic filters

THEOREM 875. For every funcoid f and $\mathcal{X}\in\mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y}\in\mathscr{F}(\operatorname{Dst} f)$

1°. $\langle f\rangle\mathcal{X}=\bigsqcup\langle\langle f\rangle\rangle^*\operatorname{atoms}\mathcal{X}$;
2°. $\mathcal{X}\ [f]\ \mathcal{Y}\Leftrightarrow\exists x\in\operatorname{atoms}\mathcal{X},y\in\operatorname{atoms}\mathcal{Y}:x\ [f]\ y$.

PROOF.

1°.

$$\begin{aligned}&\mathcal{Y}\sqcap\langle f\rangle\mathcal{X}\neq\bot\Leftrightarrow\\&\mathcal{X}\sqcap\langle f^{-1}\rangle\mathcal{Y}\neq\bot\Leftrightarrow\\&\exists x\in\operatorname{atoms}\mathcal{X}:x\sqcap\langle f^{-1}\rangle\mathcal{Y}\neq\bot\Leftrightarrow\\&\exists x\in\operatorname{atoms}\mathcal{X}:\mathcal{Y}\sqcap\langle f\rangle x\neq\bot.\end{aligned}$$

$\partial\langle f\rangle\mathcal{X}=\bigsqcup\langle\partial\rangle^*\langle\langle f\rangle\rangle^*\operatorname{atoms}\mathcal{X}=\partial\bigsqcup\langle\langle f\rangle\rangle^*\operatorname{atoms}\mathcal{X}$. So $\langle f\rangle\mathcal{X}=\bigsqcup\langle\langle f\rangle\rangle^*\operatorname{atoms}\mathcal{X}$ by corollary 568.

2°. If $\mathcal{X}\ [f]\ \mathcal{Y}$, then $\mathcal{Y}\sqcap\langle f\rangle\mathcal{X}\neq\bot$, consequently there exists $y\in\operatorname{atoms}\mathcal{Y}$ such that $y\sqcap\langle f\rangle\mathcal{X}\neq\bot$, $\mathcal{X}\ [f]\ y$. Repeating this second time we get that there exists $x\in\operatorname{atoms}\mathcal{X}$ such that $x\ [f]\ y$. From this it follows

$$\exists x\in\operatorname{atoms}\mathcal{X},y\in\operatorname{atoms}\mathcal{Y}:x\ [f]\ y.$$

The reverse is obvious.

□

COROLLARY 876. Let f be a funcoid.

- The value of f can be restored from the value of $\langle f\rangle|_{\operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}}$.
- The value of f can be restored from the value of $[f]|_{\operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}\times\operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}}$.

THEOREM 877. Let A and B be sets.

1°. A function $\alpha\in\mathscr{F}(B)^{\operatorname{atoms}^{\mathscr{F}(A)}}$ such that (for every $a\in\operatorname{atoms}^{\mathscr{F}(A)}$)

$$\alpha a\sqsubseteq\bigsqcap\left\langle\bigsqcup\circ\langle\alpha\rangle^*\circ\operatorname{atoms}\circ\uparrow\right\rangle^*\operatorname{up}a\tag{10}$$

can be continued to the function $\langle f\rangle$ for a unique $f\in\mathsf{FCD}(A,B)$;

$$\langle f\rangle\mathcal{X}=\bigsqcup\langle\alpha\rangle^*\operatorname{atoms}\mathcal{X}\tag{11}$$

for every $\mathcal{X}\in\mathscr{F}(A)$.

2°. A relation $\delta\in\mathscr{P}(\operatorname{atoms}^{\mathscr{F}(A)}\times\operatorname{atoms}^{\mathscr{F}(B)})$ such that (for every $a\in\operatorname{atoms}^{\mathscr{F}(A)}$, $b\in\operatorname{atoms}^{\mathscr{F}(B)}$)

$$\forall X\in\operatorname{up}a,Y\in\operatorname{up}b\exists x\in\operatorname{atoms}\uparrow X,y\in\operatorname{atoms}\uparrow Y:x\ \delta\ y\Rightarrow a\ \delta\ b\tag{12}$$

can be continued to the relation $[f]$ for a unique $f\in\mathsf{FCD}(A,B)$;

$$\mathcal{X}\ [f]\ \mathcal{Y}\Leftrightarrow\exists x\in\operatorname{atoms}\mathcal{X},y\in\operatorname{atoms}\mathcal{Y}:x\ \delta\ y\tag{13}$$

for every $\mathcal{X}\in\mathscr{F}(A)$, $\mathcal{Y}\in\mathscr{F}(B)$.

PROOF. Existence of no more than one such funcoids and formulas (11) and (13) follow from the previous theorem.

1°. Consider the function $\alpha' \in \mathscr{F}(B)^{\mathscr{T} A}$ defined by the formula (for every $X \in \mathscr{T} A$)

$$\alpha' X = \bigsqcup \langle \alpha \rangle^* \operatorname{atoms} \uparrow X.$$

Obviously $\alpha' \bot^{\mathscr{T} A} = \bot^{\mathscr{F}(B)}$. For every $I, J \in \mathscr{T} A$

$$\begin{aligned} &\alpha'(I \sqcup J) = \\ &\bigsqcup \langle \alpha \rangle^* \operatorname{atoms} \uparrow (I \sqcup J) = \\ &\bigsqcup \langle \alpha \rangle^* (\operatorname{atoms} \uparrow \cup \operatorname{atoms} \uparrow J) = \\ &\bigsqcup (\langle \alpha \rangle^* \operatorname{atoms} \uparrow I \cup \langle \alpha \rangle^* \operatorname{atoms} \uparrow J) = \\ &\bigsqcup \langle \alpha \rangle^* \operatorname{atoms} \uparrow I \sqcup \bigsqcup \langle \alpha \rangle^* \operatorname{atoms} \uparrow J = \\ &\alpha' I \sqcup \alpha' J. \end{aligned}$$

Let continue α' till a funcoid f (by the theorem 831): $\langle f \rangle \mathcal{X} = \bigsqcap \langle \alpha' \rangle^* \operatorname{up} \mathcal{X}$. Let's prove the reverse of (10):

$$\begin{aligned} &\bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \circ \operatorname{atoms} \circ \uparrow \right\rangle^* \operatorname{up} a = \\ &\bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \right\rangle^* \langle \operatorname{atoms} \rangle^* \langle \uparrow \rangle^* \operatorname{up} a \sqsubseteq \\ &\bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \right\rangle^* \{\{a\}\} = \\ &\bigsqcap \left\{ \left(\bigsqcup \circ \langle \alpha \rangle^* \right) \{a\} \right\} = \\ &\bigsqcap \left\{ \bigsqcup \langle \alpha \rangle^* \{a\} \right\} = \\ &\bigsqcap \left\{ \bigsqcup \{\alpha a\} \right\} = \\ &\bigsqcap \{\alpha a\} = \\ &\alpha a. \end{aligned}$$

Finally,

$$\alpha a = \bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \circ \operatorname{atoms} \circ \uparrow \right\rangle^* \operatorname{up} a = \bigsqcap \langle \alpha' \rangle^* \operatorname{up} a = \langle f \rangle a,$$

so $\langle f \rangle$ is a continuation of α.

2°. Consider the relation $\delta' \in \mathscr{P}(\mathscr{T} A \times \mathscr{T} B)$ defined by the formula (for every $X \in \mathscr{T} A$, $Y \in \mathscr{T} B$)

$$X \; \delta' \; Y \Leftrightarrow \exists x \in \operatorname{atoms} \uparrow X, y \in \operatorname{atoms} \uparrow Y : x \; \delta \; y.$$

Obviously $\neg (X \; \delta' \; \bot^{\mathscr{F}(B)})$ and $\neg (\bot^{\mathscr{F}(A)} \; \delta' \; Y)$.

For suitable I and J we have:

$$\begin{aligned} &I \sqcup J \; \delta' \; Y \Leftrightarrow \\ &\exists x \in \operatorname{atoms} \uparrow (I \sqcup J), y \in \operatorname{atoms} \uparrow Y : x \; \delta \; y \Leftrightarrow \\ &\exists x \in \operatorname{atoms} \uparrow I \cup \operatorname{atoms} \uparrow J, y \in \operatorname{atoms} \uparrow Y : x \; \delta \; y \Leftrightarrow \\ &\exists x \in \operatorname{atoms} \uparrow I, y \in \operatorname{atoms} \uparrow Y : x \; \delta \; y \vee \exists x \in \operatorname{atoms} \uparrow J, y \in \operatorname{atoms} \uparrow Y : x \; \delta \; y \Leftrightarrow \\ &I \; \delta' \; Y \vee J \; \delta' \; Y; \end{aligned}$$

similarly $X \; \delta' \; I \sqcup J \Leftrightarrow X \; \delta' \; I \vee X \; \delta' \; J$ for suitable I and J. Let's continue δ' till a funcoid f (by the theorem 831):

$$\mathcal{X} \; [f] \; \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \; \delta' \; Y.$$

The reverse of (12) implication is trivial, so

$$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow X, y \in \operatorname{atoms} \uparrow Y : x \ \delta \ y \Leftrightarrow a \ \delta \ b.$$

Also

$$\begin{aligned}\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow X, y \in \operatorname{atoms} \uparrow Y : x \ \delta \ y \Leftrightarrow \\ \forall X \in \operatorname{up} a, Y \in \operatorname{up} b : X \ \delta' \ Y \Leftrightarrow \\ a \ [f] \ b.\end{aligned}$$

So $a \ \delta \ b \Leftrightarrow a \ [f] \ b$, that is $[f]$ is a continuation of δ.

□

One of uses of the previous theorem is the proof of the following theorem:

THEOREM 878. If A and B are sets, $R \in \mathscr{P}\mathsf{FCD}(A, B)$, $x \in \operatorname{atoms}^{\mathscr{F}(A)}$, $y \in \operatorname{atoms}^{\mathscr{F}(B)}$, then

1°. $\langle \bigsqcap R \rangle x = \bigsqcap_{f \in R} \langle f \rangle x$;
2°. $x \ [\bigsqcap R] \ y \Leftrightarrow \forall f \in R : x \ [f] \ y$.

PROOF.
2°. Let denote $x \ \delta \ y \Leftrightarrow \forall f \in R : x \ [f] \ y$. For every $a \in \operatorname{atoms}^{\mathscr{F}(A)}$, $b \in \operatorname{atoms}^{\mathscr{F}(B)}$

$$\begin{aligned}\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow X, y \in \operatorname{atoms} \uparrow Y : x \ \delta \ y \Rightarrow \\ \forall f \in R, X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} \uparrow X, y \in \operatorname{atoms} \uparrow Y : x \ [f] \ y \Rightarrow \\ \forall f \in R, X \in \operatorname{up} a, Y \in \operatorname{up} b : X \ [f]^* \ Y \Rightarrow \\ \forall f \in R : a \ [f] \ b \Leftrightarrow \\ a \ \delta \ b.\end{aligned}$$

So by theorem 877, δ can be continued till $[p]$ for some funcoid $p \in \mathsf{FCD}(A, B)$. For every funcoid $q \in \mathsf{FCD}(A, B)$ such that $\forall f \in R : q \sqsubseteq f$ we have

$$x \ [q] \ y \Rightarrow \forall f \in R : x \ [f] \ y \Leftrightarrow x \ \delta \ y \Leftrightarrow x \ [p] \ y,$$

so $q \sqsubseteq p$. Consequently $p = \bigsqcap R$.
From this $x \ [\bigsqcap R] \ y \Leftrightarrow \forall f \in R : x \ [f] \ y$.
1°. From the former

$$\begin{aligned}y \in \operatorname{atoms}\left\langle \bigsqcap R \right\rangle x \Leftrightarrow \\ y \sqcap \left\langle \bigsqcap R \right\rangle x \neq \bot \Leftrightarrow \\ \forall f \in R : y \sqcap \langle f \rangle x \neq \bot \Leftrightarrow \\ y \in \bigsqcap \langle \operatorname{atoms} \rangle^* \left\{ \frac{\langle f \rangle x}{f \in R} \right\} \Leftrightarrow \\ y \in \operatorname{atoms} \bigsqcap_{f \in R} \langle f \rangle x\end{aligned}$$

for every $y \in \operatorname{atoms}^{\mathscr{F}(A)}$. From this it follows $\langle \bigsqcap R \rangle x = \bigsqcap_{f \in R} \langle f \rangle x$.

□

THEOREM 879. $g \circ f = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\}$ for every composable funcoids f and g.

PROOF. Let $x \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}$. Then

$$\begin{aligned}
\langle g \circ f \rangle x &= \\
\langle g \rangle \langle f \rangle x &= \quad (\text{theorem } 851) \\
\prod^{\mathscr{F}}_{G \in \operatorname{up} g} \langle G \rangle \langle f \rangle x &= \quad (\text{theorem } 851) \\
\prod^{\mathscr{F}}_{G \in \operatorname{up} g} \langle G \rangle \prod^{\mathscr{F}}_{F \in \operatorname{up} f} \langle F \rangle x &= \quad (\text{theorem } 839) \\
\prod^{\mathscr{F}}_{G \in \operatorname{up} g} \prod^{\mathscr{F}}_{F \in \operatorname{up} f} \langle G \rangle \langle F \rangle x &= \\
\prod^{\mathscr{F}} \left\{ \frac{\langle G \rangle \langle F \rangle x}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\} &= \\
\prod^{\mathscr{F}} \left\{ \frac{\langle G \circ F \rangle x}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\} &= \quad (\text{theorem } 878) \\
\left\langle \prod^{\mathsf{FCD}} \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\} \right\rangle x. &
\end{aligned}$$

Thus $g \circ f = \prod^{\mathsf{FCD}} \left\{ \frac{G \circ F}{F \in \operatorname{up} f . G \in \operatorname{up} g} \right\}$. □

PROPOSITION 880. For $f \in \mathsf{FCD}(A, B)$, a finite set $X \in \mathscr{P} A$ and a function $t \in \mathscr{F}(B)^X$ there exists (obviously unique) $g \in \mathsf{FCD}(A, B)$ such that $\langle g \rangle p = \langle f \rangle p$ for $p \in \operatorname{atoms}^{\mathscr{F}(A)} \setminus \operatorname{atoms} X$ and $\langle g \rangle @\{x\} = t(x)$ for $x \in X$.

This funcoid g is determined by the formula

$$g = (f \setminus (@X \times^{\mathsf{FCD}} \top)) \sqcup \bigsqcup_{x \in X} (@\{x\} \times^{\mathsf{FCD}} t(x)).$$

PROOF. Take $g = (f \setminus (@X \times^{\mathsf{FCD}} \top)) \sqcup \bigsqcup_{q \in X} (@\{q\} \times^{\mathsf{FCD}} t(x))$ that is $g = \left(f \sqcap \overline{X \times \top}\right) \sqcup \bigsqcup_{q \in X} (@\{q\} \times^{\mathsf{FCD}} t(x)) = \left(f \sqcap \left(\overline{X} \times \top\right)\right) \sqcup \bigsqcup_{q \in X} (@\{q\} \times^{\mathsf{FCD}} t(x))$.

$\langle g \rangle p =$ (theorem 853) $= \left\langle f \sqcap \left(\overline{X} \times \top\right) \right\rangle p \sqcup \bigsqcup_{q \in X} \langle @\{q\} \times^{\mathsf{FCD}} t(x) \rangle p =$ (theorem 878) $= \left(\langle f \rangle p \sqcap \left\langle \overline{X} \times \top \right\rangle p\right) \sqcup \bigsqcup_{q \in X} \langle @\{q\} \times^{\mathsf{FCD}} t(x) \rangle p$.

So $\langle g \rangle @\{x\} = (\langle f \rangle^* @\{x\} \sqcap \bot) \sqcup t(x) = t(x)$ for $x \in X$.

If $p \in \operatorname{atoms}^{\mathscr{F}(A)} \setminus \operatorname{atoms} X$ then we have $\langle g \rangle p = (\langle f \rangle p \sqcap \top) \sqcup \bot = \langle f \rangle p$. □

COROLLARY 881. If $f \in \mathsf{FCD}(A, B)$, $x \in A$, and $\mathcal{Y} \in \mathscr{F}(B)$, then there exists an (obviously unique) $g \in \mathsf{FCD}(A, B)$ such that $\langle g \rangle p = \langle f \rangle p$ for all ultrafilters p except of $p = @\{x\}$ and $\langle g \rangle @\{x\} = \mathcal{Y}$.

This funcoid g is determined by the formula

$$g = (f \setminus (@\{x\} \times^{\mathsf{FCD}} \top)) \sqcup (\{x\} \times^{\mathsf{FCD}} \mathcal{Y}).$$

THEOREM 882. Let A, B, C be sets, $f \in \mathsf{FCD}(A, B)$, $g \in \mathsf{FCD}(B, C)$, $h \in \mathsf{FCD}(A, C)$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

PROOF.

$$\begin{aligned}
g\circ f\not\asymp h\Leftrightarrow\\
\exists a\in\operatorname{atoms}^{\mathscr{F}(A)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:a\left[(g\circ f)\sqcap h\right]c\Leftrightarrow\\
\exists a\in\operatorname{atoms}^{\mathscr{F}(A)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:(a\left[g\circ f\right]c\wedge a\left[h\right]c)\Leftrightarrow\\
\exists a\in\operatorname{atoms}^{\mathscr{F}(A)},b\in\operatorname{atoms}^{\mathscr{F}(B)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:(a\left[f\right]b\wedge b\left[g\right]c\wedge a\left[h\right]c)\Leftrightarrow\\
\exists b\in\operatorname{atoms}^{\mathscr{F}(B)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:(b\left[g\right]c\wedge b\left[h\circ f^{-1}\right]c)\Leftrightarrow\\
\exists b\in\operatorname{atoms}^{\mathscr{F}(B)},c\in\operatorname{atoms}^{\mathscr{F}(C)}:b\left[g\sqcap(h\circ f^{-1})\right]c\Leftrightarrow\\
g\not\asymp h\circ f^{-1}.
\end{aligned}$$

□

7.10. Funcoidal product of filters

A generalization of Cartesian product of two sets is funcoidal product of two filters:

DEFINITION 883. *Funcoidal product* of filters $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\in\mathsf{FCD}(\operatorname{Base}(\mathcal{A}),\operatorname{Base}(\mathcal{B}))$ that for every $\mathcal{X}\in\operatorname{Base}(\mathcal{A})$, $\mathcal{Y}\in\operatorname{Base}(\mathcal{B})$

$$\mathcal{X}\left[\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\right]\mathcal{Y}\Leftrightarrow\mathcal{X}\not\asymp\mathcal{A}\wedge\mathcal{Y}\not\asymp\mathcal{B}.$$

PROPOSITION 884. $\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ is really a funcoid and

$$\left\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\right\rangle\mathcal{X}=\begin{cases}\mathcal{B} & \text{if } \mathcal{X}\not\asymp\mathcal{A}\\ \bot^{\mathscr{F}(\operatorname{Base}(\mathcal{B}))} & \text{if } \mathcal{X}\asymp\mathcal{A}.\end{cases}$$

PROOF. Obvious. □

OBVIOUS 885.

- $\uparrow^{\mathsf{FCD}(U,V)}(A\times B)=\uparrow^U A\times\uparrow^V B$ for sets $A\subseteq U$ and $B\subseteq V$.
- $\uparrow^{\mathsf{FCD}}(A\times B)=\uparrow A\times\uparrow B$ for typed sets A and B.

PROPOSITION 886. $f\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\Leftrightarrow\operatorname{dom}f\sqsubseteq\mathcal{A}\wedge\operatorname{im}f\sqsubseteq\mathcal{B}$ for every $f\in\mathsf{FCD}(A,B)$ and $\mathcal{A}\in\mathscr{F}(A)$, $\mathcal{B}\in\mathscr{F}(B)$.

PROOF. If $f\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ then $\operatorname{dom}f\sqsubseteq\operatorname{dom}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\sqsubseteq\mathcal{A}$, $\operatorname{im}f\sqsubseteq\operatorname{im}(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})\sqsubseteq\mathcal{B}$. If $\operatorname{dom}f\sqsubseteq\mathcal{A}\wedge\operatorname{im}f\sqsubseteq\mathcal{B}$ then

$$\forall\mathcal{X}\in\mathscr{F}(A),\mathcal{Y}\in\mathscr{F}(B):(\mathcal{X}\left[f\right]\mathcal{Y}\Rightarrow\mathcal{X}\sqcap\mathcal{A}\neq\bot\wedge\mathcal{Y}\sqcap\mathcal{B}\neq\bot);$$

consequently $f\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of a meet on the lattice of funcoids:

THEOREM 887. $f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})=\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\circ f\circ\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$ for every funcoid f and $\mathcal{A}\in\mathscr{F}(\operatorname{Src}f)$, $\mathcal{B}\in\mathscr{F}(\operatorname{Dst}f)$.

PROOF. $h\stackrel{\text{def}}{=}\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\circ f\circ\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$. For every $\mathcal{X}\in\mathscr{F}(\operatorname{Src}f)$

$$\langle h\rangle\mathcal{X}=\left\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\right\rangle\langle f\rangle\left\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}\right\rangle\mathcal{X}=\mathcal{B}\sqcap\langle f\rangle(\mathcal{A}\sqcap\mathcal{X}).$$

From this, as easy to show, $h\sqsubseteq f$ and $h\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$. If $g\sqsubseteq f\wedge g\sqsubseteq\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ for a $g\in\mathsf{FCD}(\operatorname{Src}f,\operatorname{Dst}f)$ then $\operatorname{dom}g\sqsubseteq\mathcal{A}$, $\operatorname{im}g\sqsubseteq\mathcal{B}$,

$$\langle g\rangle\mathcal{X}=\mathcal{B}\sqcap\langle g\rangle(\mathcal{A}\sqcap\mathcal{X})\sqsubseteq\mathcal{B}\sqcap\langle f\rangle(\mathcal{A}\sqcap\mathcal{X})=\left\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\right\rangle\langle f\rangle\left\langle\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}\right\rangle\mathcal{X}=\langle h\rangle\mathcal{X},$$

$g\sqsubseteq h$. So $h=f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})$. □

COROLLARY 888. $f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \top^{\mathscr{F}(\operatorname{Dst} f)})$ for every funcoid f and $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$.

PROOF. $f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \top^{\mathscr{F}(\operatorname{Dst} f)}) = \operatorname{id}^{\mathsf{FCD}}_{\top^{\mathscr{F}(\operatorname{Dst} f)}} \circ f \circ \operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}} = f \circ \operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}} = f|_{\mathcal{A}}$. □

COROLLARY 889. $f \not\asymp \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \mathcal{A}\ [f]\ \mathcal{B}$ for every funcoid f and $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathscr{F}(\operatorname{Dst} f)$.

PROOF.

$$\begin{aligned}
f \not\asymp \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} &\Leftrightarrow \\
\left\langle f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \right\rangle^{*} \top \neq \bot &\Leftrightarrow \\
\left\langle \operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}} \circ f \circ \operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}} \right\rangle^{*} \top \neq \bot &\Leftrightarrow \\
\left\langle \operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}} \right\rangle \langle f \rangle \left\langle \operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}} \right\rangle^{*} \top \neq \bot &\Leftrightarrow \\
\mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \top) \neq \bot &\Leftrightarrow \\
\mathcal{B} \sqcap \langle f \rangle \mathcal{A} \neq \bot &\Leftrightarrow \\
\mathcal{A}\ [f]\ \mathcal{B}.&
\end{aligned}$$

□

COROLLARY 890. Every filtrator of funcoids is star-separable.

PROOF. The set of funcoidal products of principal filters is a separation subset of the lattice of funcoids. □

THEOREM 891. Let A, B be sets. If $S \in \mathscr{P}(\mathscr{F}(A) \times \mathscr{F}(B))$ then

$$\bigsqcap_{(\mathcal{A},\mathcal{B}) \in S} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \bigsqcap \operatorname{dom} S \times^{\mathsf{FCD}} \bigsqcap \operatorname{im} S.$$

PROOF. If $x \in \operatorname{atoms}^{\mathscr{F}(A)}$ then by theorem 878

$$\left\langle \bigsqcap_{(\mathcal{A},\mathcal{B}) \in S} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \right\rangle x = \bigsqcap_{(\mathcal{A},\mathcal{B}) \in S} \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x.$$

If $x \not\asymp \bigsqcap \operatorname{dom} S$ then

$$\forall (\mathcal{A},\mathcal{B}) \in S : (x \sqcap \mathcal{A} \neq \bot \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \mathcal{B});$$
$$\left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x}{(\mathcal{A},\mathcal{B}) \in S} \right\} = \operatorname{im} S;$$

if $x \asymp \bigsqcap \operatorname{dom} S$ then

$$\exists (\mathcal{A},\mathcal{B}) \in S : (x \sqcap \mathcal{A} = \bot \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \bot);$$
$$\left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x}{(\mathcal{A},\mathcal{B}) \in S} \right\} \ni \bot.$$

So

$$\left\langle \bigsqcap_{(\mathcal{A},\mathcal{B}) \in S} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \right\rangle x = \begin{cases} \bigsqcap \operatorname{im} S & \text{if } x \not\asymp \bigsqcap \operatorname{dom} S \\ \bot^{\mathscr{F}(B)} & \text{if } x \asymp \bigsqcap \operatorname{dom} S. \end{cases}$$

From this the statement of the theorem follows. □

COROLLARY 892. For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathscr{F}(A)$, $\mathcal{B}_0, \mathcal{B}_1 \in \mathscr{F}(B)$ (for every sets A, B)

$$(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1).$$

PROOF. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = \bigsqcap\{\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1\}$ what is by the last theorem equal to $(\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1)$. □

THEOREM 893. If A, B are sets and $\mathcal{A} \in \mathscr{F}(A)$ then $\mathcal{A}\times^{\mathsf{FCD}}$ is a complete homomorphism from the lattice $\mathscr{F}(B)$ to the lattice $\mathsf{FCD}(A, B)$, if also $\mathcal{A} \neq \bot^{\mathscr{F}(A)}$ then it is an order embedding.

PROOF. Let $S \in \mathscr{P}\mathscr{F}(B)$, $X \in \mathscr{T}A$, $x \in \operatorname{atoms}^{\mathscr{F}(A)}$.

$$\left\langle \bigsqcup \langle \mathcal{A}\times^{\mathsf{FCD}}\rangle^* S \right\rangle^* X =$$
$$\bigsqcup_{\mathcal{B}\in S} \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\rangle^* X =$$
$$\begin{cases} \bigsqcup S & \text{if } X \in \partial\mathcal{A} \\ \bot^{\mathscr{F}(B)} & \text{if } X \notin \partial\mathcal{A} \end{cases} =$$
$$\left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup S \right\rangle^* X;$$
$$\left\langle \bigsqcap \langle \mathcal{A}\times^{\mathsf{FCD}}\rangle^* S \right\rangle x =$$
$$\bigsqcap_{\mathcal{B}\in S} \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\rangle x =$$
$$\begin{cases} \bigsqcap S & \text{if } x \not\asymp \mathcal{A} \\ \bot^{\mathscr{F}(B)} & \text{if } x \asymp \mathcal{A}. \end{cases}$$

Thus $\bigsqcup \langle \mathcal{A}\times^{\mathsf{FCD}}\rangle^* S = \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup S$ and $\bigsqcap \langle \mathcal{A}\times^{\mathsf{FCD}}\rangle^* S = \mathcal{A} \times^{\mathsf{FCD}} \bigsqcap S$.
If $\mathcal{A} \neq \bot$ then obviously $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{X} \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$. □

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a funcoidal product of filters) funcoid (of atomic width).

PROPOSITION 894. If f is a funcoid and a is an atomic filter on $\operatorname{Src} f$ then

$$f|_a = a \times^{\mathsf{FCD}} \langle f\rangle a.$$

PROOF. Let $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$.

$$\mathcal{X} \not\asymp a \Rightarrow \langle f|_a\rangle \mathcal{X} = \langle f\rangle a, \quad \mathcal{X} \asymp a \Rightarrow \langle f|_a\rangle \mathcal{X} = \bot^{\mathscr{F}(\operatorname{Dst} f)}.$$

□

LEMMA 895. $\lambda\mathcal{B} \in \mathscr{F}(B) : \top^{\mathscr{F}} \times^{\mathsf{FCD}} \mathcal{B}$ is an upper adjoint of $\lambda f \in \mathsf{FCD}(A, B) : \operatorname{im} f$ (for every sets A, B).

PROOF. We need to prove $\operatorname{im} f \sqsubseteq \mathcal{B} \Leftrightarrow f \sqsubseteq \top \times^{\mathsf{FCD}} \mathcal{B}$ what is obvious. □

COROLLARY 896. Image and domain of funcoids preserve joins.

PROOF. By properties of Galois connections and duality. □

PROPOSITION 897. $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$ for every funcoid f and filters $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.

PROOF. $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Rightarrow \operatorname{dom} f \sqsubseteq \mathcal{A}$ because $\operatorname{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \sqsubseteq \mathcal{A}$.
Let now $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$. Then $\langle f\rangle \mathcal{X} \neq \bot \Rightarrow \mathcal{X} \not\asymp \mathcal{A}$ that is $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \top$. Similarly $f \sqsubseteq \top \times^{\mathsf{FCD}} \mathcal{B}$. Thus $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

7.11. Atomic funcoids

THEOREM 898. An $f \in \mathsf{FCD}(A, B)$ is an atom of the lattice $\mathsf{FCD}(A, B)$ (for some sets A, B) iff it is a funcoidal product of two atomic filter objects.

PROOF.

$\Rightarrow$. Let $f \in \mathsf{FCD}(A, B)$ be an atom of the lattice $\mathsf{FCD}(A, B)$. Let's get elements $a \in \operatorname{atoms} \operatorname{dom} f$ and $b \in \operatorname{atoms}\langle f\rangle a$. Then for every $\mathcal{X} \in \mathscr{F}(A)$

$$\mathcal{X} \asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b\right\rangle \mathcal{X} = \bot \sqsubseteq \langle f\rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b\right\rangle \mathcal{X} = b \sqsubseteq \langle f\rangle \mathcal{X}.$$

So $a \times^{\mathsf{FCD}} b \sqsubseteq f$; because f is atomic we have $f = a \times^{\mathsf{FCD}} b$.

$\Leftarrow$. Let $a \in \operatorname{atoms}^{\mathscr{F}(A)}$, $b \in \operatorname{atoms}^{\mathscr{F}(B)}$, $f \in \mathsf{FCD}(A, B)$. If $b \asymp \langle f\rangle a$ then $\neg(a\ [f]\ b)$, $f \asymp a \times^{\mathsf{FCD}} b$; if $b \sqsubseteq \langle f\rangle a$ then $\forall \mathcal{X} \in \mathscr{F}(A) : (\mathcal{X} \not\asymp a \Rightarrow \langle f\rangle \mathcal{X} \sqsupseteq b)$, $f \sqsupseteq a \times^{\mathsf{FCD}} b$. Consequently $f \asymp a \times^{\mathsf{FCD}} b \vee f \sqsupseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atom.

□

THEOREM 899. The lattice $\mathsf{FCD}(A, B)$ is atomic (for every fixed sets A, B).

PROOF. Let f be a non-empty funcoid from A to B. Then $\operatorname{dom} f \neq \bot$, thus by theorem 576 there exists $a \in \operatorname{atoms} \operatorname{dom} f$. So $\langle f\rangle a \neq \bot$ thus it exists $b \in \operatorname{atoms}\langle f\rangle a$. Finally the atomic funcoid $a \times^{\mathsf{FCD}} b \sqsubseteq f$. □

THEOREM 900. The lattice $\mathsf{FCD}(A, B)$ is separable (for every fixed sets A, B).

PROOF. Let $f, g \in \mathsf{FCD}(A, B)$, $f \sqsubset g$. Then there exists $a \in \operatorname{atoms}^{\mathscr{F}(A)}$ such that $\langle f\rangle a \sqsubset \langle g\rangle a$. So because the lattice $\mathscr{F}(B)$ is atomically separable, there exists $b \in \operatorname{atoms}$ such that $\langle f\rangle a \sqcap b = \bot$ and $b \sqsubseteq \langle g\rangle a$. For every $x \in \operatorname{atoms}^{\mathscr{F}(A)}$

$$\langle f\rangle a \sqcap \left\langle a \times^{\mathsf{FCD}} b\right\rangle a = \langle f\rangle a \sqcap b = \bot,$$
$$x \neq a \Rightarrow \langle f\rangle x \sqcap \left\langle a \times^{\mathsf{FCD}} b\right\rangle x = \langle f\rangle x \sqcap \bot = \bot.$$

Thus $\langle f\rangle x \sqcap \left\langle a \times^{\mathsf{FCD}} b\right\rangle x = \bot$ and consequently $f \asymp a \times^{\mathsf{FCD}} b$.

$$\left\langle a \times^{\mathsf{FCD}} b\right\rangle a = b \sqsubseteq \langle g\rangle a,$$
$$x \neq a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b\right\rangle x = \bot \sqsubseteq \langle g\rangle x.$$

Thus $\left\langle a \times^{\mathsf{FCD}} b\right\rangle x \sqsubseteq \langle g\rangle x$ and consequently $a \times^{\mathsf{FCD}} b \sqsubseteq g$.
So the lattice $\mathsf{FCD}(A, B)$ is separable by theorem 225. □

COROLLARY 901. The lattice $\mathsf{FCD}(A, B)$ is:

1°. separable;
2°. strongly separable;
3°. atomically separable;
4°. conforming to Wallman's disjunction property.

PROOF. By theorem 233. □

REMARK 902. For more ways to characterize (atomic) separability of the lattice of funcoids see subsections "Separation subsets and full stars" and "Atomically separable lattices".

COROLLARY 903. The lattice $\mathsf{FCD}(A, B)$ is an atomistic lattice.

PROOF. By theorem 231. □

PROPOSITION 904. $\operatorname{atoms}(f \sqcup g) = \operatorname{atoms} f \cup \operatorname{atoms} g$ for every funcoids $f, g \in \mathsf{FCD}(A, B)$ (for every sets A, B).

Proof. $a \times^{\mathsf{FCD}} b \not\asymp f \sqcup g \Leftrightarrow a \; [f \sqcup g] \; b \Leftrightarrow a \; [f] \; b \vee a \; [g] \; b \Leftrightarrow a \times^{\mathsf{FCD}} b \not\asymp f \vee a \times^{\mathsf{FCD}} b \not\asymp g$ for every atomic filters a and b. □

Theorem 905. The set of funcoids between sets A and B is a co-frame.

Proof. Theorems 831 and 533. □

Remark 906. The above proof does not use axiom of choice (unlike the below proof).

See also an older proof of the set of funcoids being co-brouwerian:

Theorem 907. For every $f, g, h \in \mathsf{FCD}(A, B)$, $R \in \mathscr{P}\mathsf{FCD}(A, B)$ (for every sets A and B)

1°. $f \sqcap (g \sqcup h) = (f \sqcap g) \sqcup (f \sqcap h)$;
2°. $f \sqcup \bigsqcap R = \bigsqcap \langle f \sqcup \rangle^* R$.

Proof. We will take into account that the lattice of funcoids is an atomistic lattice.

1°.

$$\begin{aligned}
&\operatorname{atoms}(f \sqcap (g \sqcup h)) = \\
&\operatorname{atoms} f \cap \operatorname{atoms}(g \sqcup h) = \\
&\operatorname{atoms} f \cap (\operatorname{atoms} g \cup \operatorname{atoms} h) = \\
&(\operatorname{atoms} f \cap \operatorname{atoms} g) \cup (\operatorname{atoms} f \cap \operatorname{atoms} h) = \\
&\operatorname{atoms}(f \sqcap g) \cup \operatorname{atoms}(f \sqcap h) = \\
&\operatorname{atoms}((f \sqcap g) \sqcup (f \sqcap h)).
\end{aligned}$$

2°.

$$\begin{aligned}
\operatorname{atoms}\left(f \sqcup \bigsqcap R\right) &= \\
\operatorname{atoms} f \cup \operatorname{atoms} \bigsqcap R &= \\
\operatorname{atoms} f \cup \bigcap \langle \operatorname{atoms} \rangle^* R &= \\
\bigcap \langle (\operatorname{atoms} f) \cup \rangle^* \langle \operatorname{atoms} \rangle^* R &= \quad \text{(use the following equality)} \\
\bigcap \langle \operatorname{atoms} \rangle^* \langle f \sqcup \rangle^* R &= \\
\operatorname{atoms} \bigsqcap \langle f \sqcup \rangle^* R.
\end{aligned}$$

$$\begin{aligned}
\langle (\operatorname{atoms} f) \cup \rangle^* \langle \operatorname{atoms} \rangle^* R &= \\
\left\{ \frac{(\operatorname{atoms} f) \cup A}{A \in \langle \operatorname{atoms} \rangle^* R} \right\} &= \\
\left\{ \frac{(\operatorname{atoms} f) \cup A}{\exists C \in R : A = \operatorname{atoms} C} \right\} &= \\
\left\{ \frac{(\operatorname{atoms} f) \cup (\operatorname{atoms} C)}{C \in R} \right\} &= \\
\left\{ \frac{\operatorname{atoms}(f \sqcup C)}{C \in R} \right\} &= \\
\left\{ \frac{\operatorname{atoms} B}{\exists C \in R : B = f \sqcup C} \right\} &= \\
\left\{ \frac{\operatorname{atoms} B}{B \in \langle f \sqcup \rangle^* C} \right\} &= \\
\langle \operatorname{atoms} \rangle^* \langle f \sqcup \rangle^* R.
\end{aligned}$$

□

CONJECTURE 908. $f \sqcap \bigsqcup S = \bigsqcup \langle f \sqcap \rangle^* S$ for principal funcoid f and a set S of funcoids of appropriate sources and destinations.

REMARK 909. See also example 1336 below.

The next proposition is one more (among the theorem 855) generalization for funcoids of composition of relations.

PROPOSITION 910. For every composable funcoids f, g

$$\operatorname{atoms}(g \circ f) = \left\{ \frac{x \times^{\mathsf{FCD}} z}{\begin{array}{l} x \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, z \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} g)}, \\ \exists y \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)} : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms} g) \end{array}} \right\}.$$

PROOF. Using the theorem 855,

$$x \times^{\mathsf{FCD}} z \not\asymp g \circ f \Leftrightarrow x \, [g \circ f] \, z \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)} : (x \times^{\mathsf{FCD}} y \not\asymp f \wedge y \times^{\mathsf{FCD}} z \not\asymp g).$$

□

COROLLARY 911. $g \circ f = \bigsqcup \left\{ \frac{G \circ F}{F \in \operatorname{atoms} f, G \in \operatorname{atoms} g} \right\}$ for every composable funcoids f, g.

THEOREM 912. Let f be a funcoid.

1°. $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y}$ for every $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$;

2°. $\langle f \rangle \mathcal{X} = \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$.

PROOF.

1°.

$$\begin{aligned} \exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y} \Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge \mathcal{X} \left[a \times^{\mathsf{FCD}} b\right] \mathcal{Y}) \Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge a \times^{\mathsf{FCD}} b \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \Leftrightarrow \\ \exists F \in \operatorname{atoms} f : (F \not\asymp f \wedge F \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \Leftrightarrow \\ f \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} \Leftrightarrow \\ \mathcal{X} \, [f] \, \mathcal{Y}. \end{aligned}$$

2°. Let $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$. Suppose $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$. Then $\mathcal{X} \, [f] \, \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{Y} \not\asymp \langle F \rangle \mathcal{X}$; $\mathcal{Y} \not\asymp \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. So $\langle f \rangle \mathcal{X} \sqsubseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. The contrary $\langle f \rangle \mathcal{X} \sqsupseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ is obvious.

□

7.12. Complete funcoids

DEFINITION 913. I will call *co-complete* such a funcoid f that $\langle f \rangle^* X$ is a principal filter for every $X \in \mathscr{T}(\operatorname{Src} f)$.

OBVIOUS 914. Funcoid f is co-complete iff $\langle f \rangle \mathcal{X} \in \mathfrak{P}(\operatorname{Dst} f)$ for every $\mathcal{X} \in \mathfrak{P}(\operatorname{Src} f)$.

DEFINITION 915. I will call *generalized closure* such a function $\alpha \in (\mathscr{T} B)^{\mathscr{T} A}$ (for some sets A, B) that

1°. $\alpha\bot = \bot$;
2°. $\forall I, J \in \mathscr{T} A : \alpha(I \sqcup J) = \alpha I \sqcup \alpha J$.

Obvious 916. A funcoid f is co-complete iff $\langle f\rangle^* = \uparrow \circ \alpha$ for a generalized closure α.

Remark 917. Thus funcoids can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of funcoids.

Definition 918. I will call a *complete funcoid* a funcoid whose reverse is co-complete.

Theorem 919. The following conditions are equivalent for every funcoid f:
1°. funcoid f is complete;
2°. $\forall S \in \mathscr{P}\mathscr{F}(\operatorname{Src} f), J \in \mathscr{T}(\operatorname{Dst} f) : (\bigsqcup S\ [f]\ J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I}\ [f]\ J)$;
3°. $\forall S \in \mathscr{P}\mathscr{T}(\operatorname{Src} f), J \in \mathscr{T}(\operatorname{Dst} f) : \left(\bigsqcup S\ [f]^*\ J \Leftrightarrow \exists I \in S : I\ [f]^*\ J\right)$;
4°. $\forall S \in \mathscr{P}\mathscr{F}(\operatorname{Src} f) : \langle f\rangle \bigsqcup S = \bigsqcup \langle\langle f\rangle\rangle^* S$;
5°. $\forall S \in \mathscr{P}\mathscr{T}(\operatorname{Src} f) : \langle f\rangle^* \bigsqcup S = \bigsqcup \langle\langle f\rangle^*\rangle^* S$;
6°. $\forall A \in \mathscr{T}(\operatorname{Src} f) : \langle f\rangle^* A = \bigsqcup_{a \in \operatorname{atoms} A} \langle f\rangle^* a$.

Proof.

3°⇒1°. For every $S \in \mathscr{P}\mathscr{T}(\operatorname{Src} f)$, $J \in \mathscr{T}(\operatorname{Dst} f)$
$$\bigsqcup S \sqcap \left\langle f^{-1}\right\rangle^* J \neq \bot \Leftrightarrow \exists I \in S : I \sqcap \left\langle f^{-1}\right\rangle^* J \neq \bot,$$
consequently by theorem 583 we have that $\left\langle f^{-1}\right\rangle^* J$ is a principal filter.

1°⇒2°. For every $S \in \mathscr{P}\mathscr{F}(\operatorname{Src} f)$, $J \in \mathscr{T}(\operatorname{Dst} f)$ we have that $\left\langle f^{-1}\right\rangle^* J$ is a principal filter, consequently
$$\bigsqcup S \sqcap \left\langle f^{-1}\right\rangle^* J \neq \bot \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I} \sqcap \left\langle f^{-1}\right\rangle^* J \neq \bot.$$
From this follows 2°.

6°⇒5°.
$$\langle f\rangle^* \bigsqcup S = \bigsqcup_{a \in \operatorname{atoms} \bigsqcup S} \langle f\rangle^* a = \bigsqcup \bigcup_{A \in S} \left\{ \frac{\langle f\rangle^* a}{a \in \operatorname{atoms} A} \right\} = \bigsqcup_{A \in S} \bigsqcup_{a \in \operatorname{atoms} A} \langle f\rangle^* a = \bigsqcup_{A \in S} \langle f\rangle^* A = \bigsqcup \langle\langle f\rangle^*\rangle^* S.$$

2°⇒4°. Using theorem 583,
$$J \not\asymp \langle f\rangle \bigsqcup S \Leftrightarrow \bigsqcup S\ [f]\ J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I}\ [f]\ J \Leftrightarrow \exists \mathcal{I} \in S : J \not\asymp \langle f\rangle \mathcal{I} \Leftrightarrow J \not\asymp \bigsqcup \langle\langle f\rangle\rangle^* S.$$

2°⇒3°, 4° ⇒5°, 5°⇒3°, 5°⇒6°. Obvious.

□

The following proposition shows that complete funcoids are a direct generalization of pretopological spaces.

PROPOSITION 920. To specify a complete funcoid f it is enough to specify $\langle f\rangle^*$ on one-element sets, values of $\langle f\rangle^*$ on one element sets can be specified arbitrarily.

PROOF. From the above theorem is clear that knowing $\langle f\rangle^*$ on one-element sets $\langle f\rangle^*$ can be found on every set and then the value of $\langle f\rangle$ can be inferred for every filter.

Choosing arbitrarily the values of $\langle f\rangle^*$ on one-element sets we can define a complete funcoid the following way: $\langle f\rangle^* X = \bigsqcup_{\alpha\in\mathrm{atoms}\, X}\langle f\rangle^*\alpha$ for every $X\in\mathscr{T}(\operatorname{Src} f)$. Obviously it is really a complete funcoid. □

THEOREM 921. A funcoid is principal iff it is both complete and co-complete.

PROOF.

⇒. Obvious.

⇐. Let f be both a complete and co-complete funcoid. Consider the relation g defined by that $\uparrow\langle g\rangle^*\alpha = \langle f\rangle^*\alpha$ for one-element sets α (g is correctly defined because f corresponds to a generalized closure). Because f is a complete funcoid f is the funcoid corresponding to g.

□

THEOREM 922. If $R\in\mathscr{P}\mathsf{FCD}(A,B)$ is a set of (co-)complete funcoids then $\bigsqcup R$ is a (co-)complete funcoid (for every sets A and B).

PROOF. It is enough to prove for co-complete funcoids. Let $R\in\mathscr{P}\mathsf{FCD}(A,B)$ be a set of co-complete funcoids. Then for every $X\in\mathscr{T}(\operatorname{Src} f)$

$$\left\langle\bigsqcup R\right\rangle^* X = \bigsqcup_{f\in R}\langle f\rangle^* X$$

is a principal filter (used theorem 852). □

COROLLARY 923. If R is a set of binary relations between sets A and B then $\bigsqcup\langle\uparrow^{\mathsf{FCD}(A,B)}\rangle^* R = \uparrow^{\mathsf{FCD}(A,B)}\bigcup R$.

PROOF. From two last theorems. □

LEMMA 924. Every funcoid is representable as meet (on the lattice of funcoids) of binary relations of the form $X\times Y\sqcup\overline{X}\times\top^{\mathscr{T}(B)}$ (where X, Y are typed sets).

PROOF. Let $f\in\mathsf{FCD}(A,B)$, $X\in\mathscr{T}A$, $Y\in\operatorname{up}\langle f\rangle X$, $g(X,Y)\overset{\text{def}}{=} X\times Y\sqcup\overline{X}\times\top^{\mathscr{T}(B)}$. Then $g(X,Y) = X\times^{\mathsf{FCD}} Y\sqcup\overline{X}\times^{\mathsf{FCD}}\top^{\mathscr{F}(B)}$. For every $K\in\mathscr{T}A$

$$\langle g(X,Y)\rangle^* K = \left\langle X\times^{\mathsf{FCD}} Y\right\rangle^* K\sqcup\left\langle\overline{X}\times^{\mathsf{FCD}}\top^{\mathscr{F}(B)}\right\rangle^* K = \left(\begin{cases}\bot^{\mathscr{F}(B)} & \text{if } K=\bot^{\mathscr{T}A}\\ Y & \text{if } \bot^{\mathscr{T}A}\neq K\sqsubseteq X\\ \top^{\mathscr{F}(B)} & \text{if } K\not\sqsubseteq X\end{cases}\right)\sqsupseteq\langle f\rangle^* K;$$

so $g(X,Y)\sqsupseteq f$. For every $X\in\mathscr{T}A$

$$\bigsqcap_{Y\in\operatorname{up}\langle f\rangle^* X}\langle g(X,Y)\rangle^* X = \bigsqcap^{\mathscr{F}}_{Y\in\operatorname{up}\langle f\rangle^* X} Y = \langle f\rangle^* X;$$

consequently

$$\left\langle \bigsqcap \left\{ \frac{g(X,Y)}{X \in \mathscr{T}A, Y \in \operatorname{up}\langle f\rangle^* X} \right\} \right\rangle^* X \sqsubseteq \langle f\rangle^* X$$

that is

$$\bigsqcap \left\{ \frac{g(X,Y)}{X \in \mathscr{T}A, Y \in \operatorname{up}\langle f\rangle^* X} \right\} \sqsubseteq f$$

and finally

$$f = \bigsqcap \left\{ \frac{g(X,Y)}{X \in \mathscr{T}A, Y \in \operatorname{up}\langle f\rangle^* X} \right\}.$$

□

COROLLARY 925. Filtrators of funcoids are filtered.

THEOREM 926.

1°. g is metacomplete if g is a complete funcoid.
2°. g is co-metacomplete if g is a co-complete funcoid.

PROOF.

1°. Let R be a set of funcoids from a set A to a set B and g be a funcoid from B to some C. Then

$$\begin{aligned}
\left\langle g \circ \bigsqcup R \right\rangle^* X &= \\
\langle g\rangle \left\langle \bigsqcup R \right\rangle^* X &= \\
\langle g\rangle \bigsqcup_{f \in R} \langle f\rangle^* X &= \\
\bigsqcup_{f \in R} \langle g\rangle \langle f\rangle^* X &= \\
\bigsqcup_{f \in R} \langle g \circ f\rangle^* X &= \\
\left\langle \bigsqcup_{f \in R} (g \circ f) \right\rangle^* X &= \\
\left\langle \bigsqcup \langle g\circ\rangle^* R \right\rangle^* X &
\end{aligned}$$

for every typed set $X \in \mathscr{T}A$. So $g \circ \bigsqcup R = \bigsqcup \langle g\circ\rangle^* R$.

2°. By duality.

□

CONJECTURE 927. g is complete if g is a metacomplete funcoid.

I will denote $\mathsf{ComplFCD}$ and $\mathsf{CoComplFCD}$ the sets of small complete and co-complete funcoids correspondingly. $\mathsf{ComplFCD}(A, B)$ are complete funcoids from A to B and likewise with $\mathsf{CoComplFCD}(A, B)$.

OBVIOUS 928. $\mathsf{ComplFCD}$ and $\mathsf{CoComplFCD}$ are closed regarding composition of funcoids.

PROPOSITION 929. $\mathsf{ComplFCD}$ and $\mathsf{CoComplFCD}$ (with induced order) are complete lattices.

PROOF. It follows from theorem 922. □

THEOREM 930. Atoms of the lattice $\mathsf{ComplFCD}(A, B)$ are exactly funcoidal products of the form $\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ where $\alpha \in A$ and b is an ultrafilter on B.

PROOF. First, it's easy to see that $\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ are elements of $\mathsf{ComplFCD}(A,B)$. Also $\bot^{\mathsf{FCD}(A,B)}$ is an element of $\mathsf{ComplFCD}(A,B)$.

$\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ are atoms of $\mathsf{ComplFCD}(A,B)$ because they are atoms of $\mathsf{FCD}(A,B)$.

It remains to prove that if f is an atom of $\mathsf{ComplFCD}(A,B)$ then $f =\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ for some $\alpha \in A$ and an ultrafilter b on B.

Suppose $f \in \mathsf{FCD}(A,B)$ is a non-empty complete funcoid. Then there exists $\alpha \in A$ such that $\langle f\rangle^* @\{\alpha\} \neq \bot^{\mathscr{F}(B)}$. Thus $\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b \sqsubseteq f$ for some ultrafilter b on B. If f is an atom then $f =\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$. □

THEOREM 931. $G \mapsto \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ is an order isomorphism from the set of functions $G \in \mathscr{F}(B)^A$ to the set $\mathsf{ComplFCD}(A,B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \langle f\rangle^* @\{\alpha\}$ where f is a complete funcoid.

PROOF. $\bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ is complete because $G(\alpha) = \bigsqcup \operatorname{atoms} G(\alpha)$ and thus

$$\bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha)) = \bigsqcup\left\{ \frac{\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b}{\alpha \in A, b \in \operatorname{atoms} G(\alpha)} \right\}$$

is complete. So $G \mapsto \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ is a function from $G \in \mathscr{F}(B)^A$ to $\mathsf{ComplFCD}(A,B)$.

Let f be complete. Then take

$$G(\alpha) = \bigsqcup\left\{ \frac{b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}}{\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b \sqsubseteq f} \right\}$$

and we have $f = \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ obviously. So $G \mapsto \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$ is surjection onto $\mathsf{ComplFCD}(A,B)$.

Let now prove that it is an injection:

Let

$$f = \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} F(\alpha)) = \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$$

for some $F, G \in \mathscr{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$. We need to prove $F = G$. Let $\beta \in \operatorname{Src} f$.

$$\langle f\rangle^* @\{\beta\} = \bigsqcup_{\alpha\in A} \left\langle \uparrow^A \{\alpha\} \times^{\mathsf{FCD}} F(\alpha)\right\rangle^* @\{\beta\} = F(\beta).$$

Similarly $\langle f\rangle^* @\{\beta\} = G(\beta)$. So $F(\beta) = G(\beta)$.

We have proved that it is a bijection. To show that it is monotone is trivial.

Denote $f = \bigsqcup_{\alpha\in A}(\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha))$. Then

$$\langle f\rangle^* @\{\alpha'\} = (\text{because } \uparrow^A \{\alpha'\} \text{ is principal}) = \\ \bigsqcup_{\alpha\in A} \left\langle \uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha)\right\rangle @\{\alpha'\} = \left\langle \uparrow^A \{\alpha'\} \times^{\mathsf{FCD}} G(\alpha')\right\rangle @\{\alpha'\} = G(\alpha').$$

□

COROLLARY 932. $G \mapsto \bigsqcup_{\alpha\in A}(G(\alpha) \times^{\mathsf{FCD}} \uparrow^A \{\alpha\})$ is an order isomorphism from the set of functions $G \in \mathscr{F}(B)^A$ to the set $\mathsf{CoComplFCD}(A,B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \left\langle f^{-1}\right\rangle^* @\{\alpha\}$ where f is a co-complete funcoid.

COROLLARY 933. $\mathsf{ComplFCD}(A,B)$ and $\mathsf{CoComplFCD}(A,B)$ are co-frames.

7.13. Funcoids corresponding to pretopologies

Let Δ be a pretopology on a set U and cl the preclosure corresponding to it (see theorem 777).

Both induce a funcoid, I will show that these two funcoids are reverse of each other:

THEOREM 934. Let f be a complete funcoid defined by the formula $\langle f\rangle^* @\{x\} = \Delta(x)$ for every $x \in U$, let g be a co-complete funcoid defined by the formula $\langle g\rangle^* X = \uparrow^U \operatorname{cl}(\operatorname{GR} X)$ for every $X \in \mathscr{T} U$. Then $g = f^{-1}$.

REMARK 935. It is obvious that funcoids f and g exist.

PROOF. For $X, Y \in \mathscr{T} U$ we have

$$X\ [g]^*\ Y \Leftrightarrow$$
$$\uparrow Y \not\asymp \langle g\rangle \uparrow X \Leftrightarrow$$
$$Y \not\asymp \operatorname{cl}(\operatorname{GR} X) \Leftrightarrow$$
$$\exists y \in Y : \Delta(y) \not\asymp \uparrow X \Leftrightarrow$$
$$\exists y \in Y : \langle f\rangle^* \uparrow^U \{y\} \not\asymp \uparrow X \Leftrightarrow$$

(proposition 610 and properties of complete funcoids)

$$\langle f\rangle^* Y \not\asymp \uparrow X \Leftrightarrow$$
$$Y\ [f]^*\ X.$$

So $g = f^{-1}$. □

7.14. Completion of funcoids

THEOREM 936. $\operatorname{Cor} f = \operatorname{Cor}' f$ for an element f of a filtrator of funcoids.

PROOF. By theorem 545 and corollary 925. □

DEFINITION 937. *Completion* of a funcoid $f \in \mathsf{FCD}(A, B)$ is the complete funcoid $\operatorname{Compl} f \in \mathsf{FCD}(A, B)$ defined by the formula $\langle \operatorname{Compl} f\rangle^* @\{\alpha\} = \langle f\rangle^* @\{\alpha\}$ for $\alpha \in \operatorname{Src} f$.

DEFINITION 938. *Co-completion* of a funcoid f is defined by the formula

$$\operatorname{CoCompl} f = (\operatorname{Compl} f^{-1})^{-1}.$$

OBVIOUS 939. $\operatorname{Compl} f \sqsubseteq f$ and $\operatorname{CoCompl} f \sqsubseteq f$.

PROPOSITION 940. The filtrator $(\mathsf{FCD}(A, B), \mathsf{ComplFCD}(A, B))$ is filtered.

PROOF. Because the filtrator of funcoids is filtered. □

THEOREM 941. $\operatorname{Compl} f = \operatorname{Cor}^{\mathsf{ComplFCD}(A,B)} f = \operatorname{Cor}'^{\mathsf{ComplFCD}(A,B)} f$ for every funcoid $f \in \mathsf{FCD}(A, B)$.

PROOF. $\operatorname{Cor}^{\mathsf{ComplFCD}(A,B)} f = \operatorname{Cor}'^{\mathsf{ComplFCD}(A,B)} f$ using theorem 545 since the filtrator $(\mathsf{FCD}(A, B), \mathsf{ComplFCD}(A, B))$ is filtered.

Let $g \in \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f$. Then $g \in \mathsf{ComplFCD}(A, B)$ and $g \sqsupseteq f$. Thus $g = \operatorname{Compl} g \sqsupseteq \operatorname{Compl} f$.

Thus $\forall g \in \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f : g \sqsupseteq \operatorname{Compl} f$.

Let $\forall g \in \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f : h \sqsubseteq g$ for some $h \in \mathsf{ComplFCD}(A, B)$.

Then $h \sqsubseteq \bigsqcap \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f = f$ and consequently $h = \operatorname{Compl} h \sqsubseteq \operatorname{Compl} f$.

Thus

$$\operatorname{Compl} f = \bigsqcap^{\mathsf{ComplFCD}(A,B)} \operatorname{up}^{\mathsf{ComplFCD}(A,B)} f = \operatorname{Cor}^{\mathsf{ComplFCD}(A,B)} f.$$

$\square$

THEOREM 942. $\langle \operatorname{CoCompl} f\rangle^* X = \operatorname{Cor}\langle f\rangle^* X$ for every funcoid f and typed set $X \in \mathscr{T}(\operatorname{Src} f)$.

PROOF. $\operatorname{CoCompl} f \sqsubseteq f$ thus $\langle \operatorname{CoCompl} f\rangle^* X \sqsubseteq \langle f\rangle^* X$ but $\langle \operatorname{CoCompl} f\rangle^* X$ is a principal filter thus $\langle \operatorname{CoCompl} f\rangle^* X \sqsubseteq \operatorname{Cor}\langle f\rangle^* X$.

Let $\alpha X = \operatorname{Cor}\langle f\rangle^* X$. Then $\alpha \bot^{\mathscr{T}(\operatorname{Src} f)} = \bot^{\mathscr{F}(\operatorname{Dst} f)}$ and

$$\alpha(X \sqcup Y) = \operatorname{Cor}\langle f\rangle^*(X \sqcup Y) = \operatorname{Cor}(\langle f\rangle^* X \sqcup \langle f\rangle^* Y) = \operatorname{Cor}\langle f\rangle^* X \sqcup \operatorname{Cor}\langle f\rangle^* Y = \alpha X \sqcup \alpha Y$$

(used theorem 603). Thus α can be continued till $\langle g\rangle$ for some funcoid g. This funcoid is co-complete.

Evidently g is the greatest co-complete element of $\mathsf{FCD}(\operatorname{Src} f, \operatorname{Dst} f)$ which is lower than f.

Thus $g = \operatorname{CoCompl} f$ and $\operatorname{Cor}\langle f\rangle^* X = \alpha X = \langle g\rangle^* X = \langle \operatorname{CoCompl} f\rangle^* X$. $\square$

THEOREM 943. $\mathsf{ComplFCD}(A, B)$ is an atomistic lattice.

PROOF. Let $f \in \mathsf{ComplFCD}(A, B)$, $X \in \mathscr{T}(\operatorname{Src} f)$.

$$\langle f\rangle^* X = \bigsqcup_{x \in \operatorname{atoms} X} \langle f\rangle^* x = \bigsqcup_{x \in \operatorname{atoms} X} \langle f|_x\rangle^* x = \bigsqcup_{x \in \operatorname{atoms} X} \langle f|_x\rangle^* X,$$

thus $f = \bigsqcup_{x \in \operatorname{atoms} X}(f|_x)$. It is trivial that every $f|_x$ is a join of atoms of $\mathsf{ComplFCD}(A, B)$. $\square$

THEOREM 944. A funcoid is complete iff it is a join (on the lattice $\mathsf{FCD}(A, B)$) of atomic complete funcoids.

PROOF. It follows from the theorem 922 and the previous theorem. $\square$

COROLLARY 945. $\mathsf{ComplFCD}(A, B)$ is join-closed.

THEOREM 946. $\operatorname{Compl} \bigsqcup R = \bigsqcup \langle \operatorname{Compl}\rangle^* R$ for every $R \in \mathscr{P}\mathsf{FCD}(A, B)$ (for every sets A, B).

PROOF. For every typed set X

$$\begin{aligned}&\left\langle \operatorname{Compl} \bigsqcup R\right\rangle^* X = \\ &\bigsqcup_{x \in \operatorname{atoms} X} \left\langle \bigsqcup R\right\rangle^* x = \\ &\bigsqcup_{x \in \operatorname{atoms} X} \bigsqcup_{f \in R} \langle f\rangle^* x = \\ &\bigsqcup_{f \in R} \bigsqcup_{x \in \operatorname{atoms} X} \langle f\rangle^* x = \\ &\bigsqcup_{f \in R} \langle \operatorname{Compl} f\rangle^* X = \\ &\left\langle \bigsqcup \langle \operatorname{Compl}\rangle^* R\right\rangle^* X.\end{aligned}$$

$\square$

COROLLARY 947. Compl is a lower adjoint.

CONJECTURE 948. Compl is not an upper adjoint (in general).

PROPOSITION 949. $\operatorname{Compl} f = \bigsqcup_{\alpha \in \operatorname{Src} f}(f|_{\uparrow\{\alpha\}})$ for every funcoid f.

PROOF. Let denote R the right part of the equality to prove.

$\langle R\rangle^* @\{\beta\} = \bigsqcup_{\alpha\in\operatorname{Src} f} \langle f|_{\uparrow\{\alpha\}}\rangle^* @\{\beta\} = \langle f\rangle^* @\{\beta\}$ for every $\beta \in \operatorname{Src} f$ and R is complete as a join of complete funcoids.

Thus R is the completion of f. $\square$

CONJECTURE 950. $\operatorname{Compl} f = f \setminus^* (\Omega \times^{\mathsf{FCD}} \mho)$.

This conjecture may be proved by considerations similar to these in the section "Fréchet filter".

LEMMA 951. Co-completion of a complete funcoid is complete.

PROOF. Let f be a complete funcoid.

$$\langle \operatorname{CoCompl} f\rangle^* X = \operatorname{Cor}\langle f\rangle^* X = \operatorname{Cor} \bigsqcup_{x\in\operatorname{atoms} X} \langle f\rangle^* x = \bigsqcup_{x\in\operatorname{atoms} X} \operatorname{Cor}\langle f\rangle^* x = \bigsqcup_{x\in\operatorname{atoms} X} \langle \operatorname{CoCompl} f\rangle^* x$$

for every set typed $X \in \mathscr{T}(\operatorname{Src} f)$. Thus $\operatorname{CoCompl} f$ is complete. $\square$

THEOREM 952. $\operatorname{Compl}\operatorname{CoCompl} f = \operatorname{CoCompl}\operatorname{Compl} f = \operatorname{Cor} f$ for every funcoid f.

PROOF. $\operatorname{Compl}\operatorname{CoCompl} f$ is co-complete since (used the lemma) $\operatorname{CoCompl} f$ is co-complete. Thus $\operatorname{Compl}\operatorname{CoCompl} f$ is a principal funcoid. $\operatorname{CoCompl} f$ is the greatest co-complete funcoid under f and $\operatorname{Compl}\operatorname{CoCompl} f$ is the greatest complete funcoid under $\operatorname{CoCompl} f$. So $\operatorname{Compl}\operatorname{CoCompl} f$ is greater than any principal funcoid under $\operatorname{CoCompl} f$ which is greater than any principal funcoid under f. Thus $\operatorname{Compl}\operatorname{CoCompl} f$ is the greatest principal funcoid under f. Thus $\operatorname{Compl}\operatorname{CoCompl} f = \operatorname{Cor} f$. Similarly $\operatorname{CoCompl}\operatorname{Compl} f = \operatorname{Cor} f$. $\square$

7.14.1. More on completion of funcoids.

PROPOSITION 953. For every composable funcoids f and g

1°. $\operatorname{Compl}(g\circ f) \sqsupseteq \operatorname{Compl} g \circ \operatorname{Compl} f$;
2°. $\operatorname{CoCompl}(g\circ f) \sqsupseteq \operatorname{CoCompl} g \circ \operatorname{CoCompl} f$.

PROOF.

1°. $\operatorname{Compl} g \circ \operatorname{Compl} f = \operatorname{Compl}(\operatorname{Compl} g \circ \operatorname{Compl} f) \sqsubseteq \operatorname{Compl}(g\circ f)$.
2°. $\operatorname{CoCompl} g \circ \operatorname{CoCompl} f = \operatorname{CoCompl}(\operatorname{CoCompl} g \circ \operatorname{CoCompl} f) \sqsubseteq \operatorname{CoCompl}(g\circ f)$.

$\square$

PROPOSITION 954. For every composable funcoids f and g

1°. $\operatorname{CoCompl}(g\circ f) = (\operatorname{CoCompl} g)\circ f$ if f is a co-complete funcoid.
2°. $\operatorname{Compl}(f\circ g) = f\circ\operatorname{Compl} g$ if f is a complete funcoid.

PROOF.

1°. For every $X \in \mathscr{T}(\operatorname{Src} f)$

$$\begin{aligned}\langle \operatorname{CoCompl}(g\circ f)\rangle^* X &= \\ \operatorname{Cor}\langle g\circ f\rangle^* X &= \\ \operatorname{Cor}\langle g\rangle\langle f\rangle^* X &= \\ \langle \operatorname{CoCompl} g\rangle\langle f\rangle^* X &= \\ \langle(\operatorname{CoCompl} g)\circ f\rangle^* X.&\end{aligned}$$

2°. $(\operatorname{CoCompl}(g\circ f))^{-1} = f^{-1}\circ(\operatorname{CoCompl} g)^{-1}$; $\operatorname{Compl}(g\circ f)^{-1} = f^{-1}\circ\operatorname{Compl} g^{-1}$; $\operatorname{Compl}(f^{-1}\circ g^{-1}) = f^{-1}\circ\operatorname{Compl} g^{-1}$. After variable replacement we get $\operatorname{Compl}(f\circ g) = f\circ\operatorname{Compl} g$ (after the replacement f is a complete funcoid).

□

COROLLARY 955. For every composable funcoids f and g

1°. $\operatorname{Compl} f\circ\operatorname{Compl} g = \operatorname{Compl}(\operatorname{Compl} f\circ g)$.
2°. $\operatorname{CoCompl} g\circ\operatorname{CoCompl} f = \operatorname{CoCompl}(g\circ\operatorname{CoCompl} f)$.

PROPOSITION 956. For every composable funcoids f and g

1°. $\operatorname{Compl}(g\circ f) = \operatorname{Compl}(g\circ(\operatorname{Compl} f))$;
2°. $\operatorname{CoCompl}(g\circ f) = \operatorname{CoCompl}((\operatorname{CoCompl} g)\circ f)$.

PROOF.

1°.

$$\langle g\circ(\operatorname{Compl} f)\rangle^*@\{x\} = \langle g\rangle\langle\operatorname{Compl} f\rangle^*@\{x\} = \langle g\rangle\langle f\rangle^*@\{x\} = \langle g\circ f\rangle^*@\{x\}.$$

Thus $\operatorname{Compl}(g\circ(\operatorname{Compl} f)) = \operatorname{Compl}(g\circ f)$.

2°. $(\operatorname{Compl}(g\circ(\operatorname{Compl} f))^{-1} = (\operatorname{Compl}(g\circ f))^{-1}$; $\operatorname{CoCompl}(g\circ(\operatorname{Compl} f))^{-1} = \operatorname{CoCompl}(g\circ f)^{-1}$; $\operatorname{CoCompl}((\operatorname{Compl} f)^{-1}\circ g^{-1}) = \operatorname{CoCompl}(f^{-1}\circ g^{-1})$; $\operatorname{CoCompl}((\operatorname{CoCompl} f^{-1})\circ g^{-1}) = \operatorname{CoCompl}(f^{-1}\circ g^{-1})$. After variable replacement $\operatorname{CoCompl}((\operatorname{CoCompl} g)\circ f) = \operatorname{CoCompl}(g\circ f)$.

□

THEOREM 957. The filtrator of funcoids (from a given set A to a given set B) is with co-separable core.

PROOF. Let $f,g\in\mathsf{FCD}(A,B)$ and $f\sqcup g=\top$. Then for every $X\in\mathscr{T}A$ we have

$$\langle f\rangle^* X\sqcup\langle g\rangle^* X=\top\Leftrightarrow\operatorname{Cor}\langle f\rangle^* X\sqcup\operatorname{Cor}\langle g\rangle^* X=\top\Leftrightarrow \langle\operatorname{CoCompl} f\rangle^* X\sqcup\langle\operatorname{CoCompl} g\rangle^* X=\top.$$

Thus $\langle\operatorname{CoCompl} f\sqcup\operatorname{CoCompl} g\rangle^* X=\top$;

$$f\sqcup g=\top\Rightarrow\operatorname{CoCompl} f\sqcup\operatorname{CoCompl} g=\top. \tag{14}$$

Applying the dual of the formulas (14) to the formula (14) we get:

$$f\sqcup g=\top\Rightarrow\operatorname{Compl}\operatorname{CoCompl} f\sqcup\operatorname{Compl}\operatorname{CoCompl} g=\top$$

that is $f\sqcup g=\top\Rightarrow\operatorname{Cor} f\sqcup\operatorname{Cor} g=\top$. So $\mathsf{FCD}(A,B)$ is with co-separable core. □

COROLLARY 958. The filtrator of complete funcoids is also with co-separable core.

7.15. Monovalued and injective funcoids

Following the idea of definition of monovalued morphism let's call *monovalued* such a funcoid f that $f\circ f^{-1}\sqsubseteq\mathrm{id}^{\mathsf{FCD}}_{\operatorname{im} f}$.

Similarly, I will call a funcoid injective when $f^{-1}\circ f\sqsubseteq\mathrm{id}^{\mathsf{FCD}}_{\operatorname{dom} f}$.

OBVIOUS 959. A funcoid f is:

1°. monovalued iff $f\circ f^{-1}\sqsubseteq 1^{\mathsf{FCD}}_{\operatorname{Dst} f}$;
2°. injective iff $f^{-1}\circ f\sqsubseteq 1^{\mathsf{FCD}}_{\operatorname{Src} f}$.

In other words, a funcoid is monovalued (injective) when it is a monovalued (injective) morphism of the category of funcoids. Monovaluedness is dual of injectivity.

OBVIOUS 960.

1°. A morphism $(\mathcal{A}, \mathcal{B}, f)$ of the category of funcoid triples is monovalued iff the funcoid f is monovalued.
2°. A morphism $(\mathcal{A}, \mathcal{B}, f)$ of the category of funcoid triples is injective iff the funcoid f is injective.

THEOREM 961. The following statements are equivalent for a funcoid f:

1°. f is monovalued.
2°. It is metamonovalued.
3°. It is weakly metamonovalued.
4°. $\forall a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)} : \langle f\rangle a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)} \cup \{\bot^{\mathscr{F}(\operatorname{Dst} f)}\}$.
5°. $\forall \mathcal{I}, \mathcal{J} \in \mathscr{F}(\operatorname{Dst} f) : \langle f^{-1}\rangle(\mathcal{I} \sqcap \mathcal{J}) = \langle f^{-1}\rangle\mathcal{I} \sqcap \langle f^{-1}\rangle\mathcal{J}$.
6°. $\forall I, J \in \mathscr{T}(\operatorname{Dst} f) : \langle f^{-1}\rangle^{*}(I \sqcap J) = \langle f^{-1}\rangle^{*} I \sqcap \langle f^{-1}\rangle^{*} J$.

PROOF.

4°⇒5°. Let $a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}$, $\langle f\rangle a = b$. Then because $b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)} \cup \{\bot^{\mathscr{F}(\operatorname{Dst} f)}\}$

$$(\mathcal{I} \sqcap \mathcal{J}) \sqcap b \neq \bot \Leftrightarrow \mathcal{I} \sqcap b \neq \bot \wedge \mathcal{J} \sqcap b \neq \bot;$$
$$a\, [f]\, \mathcal{I} \sqcap \mathcal{J} \Leftrightarrow a\, [f]\, \mathcal{I} \wedge a\, [f]\, \mathcal{J};$$
$$\mathcal{I} \sqcap \mathcal{J}\, \left[f^{-1}\right]\, a \Leftrightarrow \mathcal{I}\, \left[f^{-1}\right]\, a \wedge \mathcal{J}\, \left[f^{-1}\right]\, a;$$
$$a \sqcap \langle f^{-1}\rangle(\mathcal{I} \sqcap \mathcal{J}) \neq \bot \Leftrightarrow a \sqcap \langle f^{-1}\rangle\mathcal{I} \neq \bot \wedge a \sqcap \langle f^{-1}\rangle\mathcal{J} \neq \bot;$$
$$\langle f^{-1}\rangle(\mathcal{I} \sqcap \mathcal{J}) = \langle f^{-1}\rangle\mathcal{I} \sqcap \langle f^{-1}\rangle\mathcal{J}.$$

5°⇒1°. $\langle f^{-1}\rangle a \sqcap \langle f^{-1}\rangle b = \langle f^{-1}\rangle(a \sqcap b) = \langle f^{-1}\rangle\bot = \bot$ for every two distinct atomic filter objects a and b on $\operatorname{Dst} f$. This is equivalent to $\neg(\langle f^{-1}\rangle a\, [f]\, b)$; $b \asymp \langle f\rangle\langle f^{-1}\rangle a$; $b \asymp \langle f \circ f^{-1}\rangle a$; $\neg(a\, \left[f \circ f^{-1}\right]\, b)$. So $a\, \left[f \circ f^{-1}\right]\, b \Rightarrow a = b$ for every ultrafilters a and b. This is possible only when $f \circ f^{-1} \sqsubseteq 1^{\mathsf{FCD}}_{\operatorname{Dst} f}$.

6°⇒5°.

$$\begin{aligned}
&\langle f^{-1}\rangle(\mathcal{I} \sqcap \mathcal{J}) = \\
&\bigsqcap \langle\langle f\rangle^{*}\rangle^{*} \operatorname{up}(\mathcal{I} \sqcap \mathcal{J}) = \\
&\bigsqcap \langle\langle f\rangle^{*}\rangle^{*} \left\{ \frac{I \sqcap J}{I \in \operatorname{up}\mathcal{I}, J \in \operatorname{up}\mathcal{J}} \right\} = \\
&\bigsqcap \left\{ \frac{\langle f\rangle^{*}(I \sqcap J)}{I \in \operatorname{up}\mathcal{I}, J \in \operatorname{up}\mathcal{J}} \right\} = \\
&\bigsqcap \left\{ \frac{\langle f\rangle^{*} I \sqcap \langle f\rangle^{*} J}{I \in \operatorname{up}\mathcal{I}, J \in \operatorname{up}\mathcal{J}} \right\} = \\
&\bigsqcap \left\{ \frac{\langle f\rangle^{*} I}{I \in \operatorname{up}\mathcal{I}} \right\} \sqcap \bigsqcap \left\{ \frac{\langle f\rangle^{*} J}{J \in \operatorname{up}\mathcal{J}} \right\} = \\
&\langle f^{-1}\rangle\mathcal{I} \sqcap \langle f^{-1}\rangle\mathcal{J}.
\end{aligned}$$

5°⇒6°. Obvious.

¬4°⇒¬1°. Suppose $\langle f\rangle a \notin \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)} \cup \{\bot^{\mathscr{F}(\operatorname{Dst} f)}\}$ for some $a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}$. Then there exist two atomic filters p and q on $\operatorname{Dst} f$ such that $p \neq q$ and $\langle f\rangle a \sqsupseteq p \wedge \langle f\rangle a \sqsupseteq q$. Consequently $p \not\asymp \langle f\rangle a$; $a \not\asymp \langle f^{-1}\rangle p$; $a \sqsubseteq \langle f^{-1}\rangle p$;

$\langle f \circ f^{-1} \rangle p = \langle f \rangle \langle f^{-1} \rangle p \sqsupseteq \langle f \rangle a \sqsupseteq q$; $\langle f \circ f^{-1} \rangle p \not\sqsubseteq p$ and $\langle f \circ f^{-1} \rangle p \neq \bot^{\mathscr{F}(\mathrm{Dst}\, f)}$. So it cannot be $f \circ f^{-1} \sqsubseteq 1^{\mathsf{FCD}}_{\mathrm{Dst}\, f}$.

2°⇒3°. Obvious.

1°⇒2°.

$$\left\langle \left(\bigsqcap G \right) \circ f \right\rangle x = \left\langle \bigsqcap G \right\rangle \langle f \rangle x = \bigsqcap_{g \in G} \langle g \rangle \langle f \rangle x = \bigsqcap_{g \in G} \langle g \circ f \rangle x = \left\langle \bigsqcap_{g \in G} (g \circ f) \right\rangle x$$

for every atomic filter object $x \in \operatorname{atoms}^{\mathscr{F}(\mathrm{Src}\, f)}$. Thus $(\bigsqcap G) \circ f = \bigsqcap_{g \in G} (g \circ f)$.

3°⇒1°. Take $g = a \times^{\mathsf{FCD}} y$ and $h = b \times^{\mathsf{FCD}} y$ for arbitrary atomic filter objects $a \neq b$ and y. We have $g \sqcap h = \bot$; thus $(g \circ f) \sqcap (h \circ f) = (g \sqcap h) \circ f = \bot$ and thus impossible $x \, [f] \, a \wedge x \, [f] \, b$ as otherwise $x \, [g \circ f] \, y$ and $x \, [h \circ f] \, y$ so $x \, [(g \circ f) \sqcap (h \circ f)] \, y$. Thus f is monovalued.

□

Corollary 962. A binary relation corresponds to a monovalued funcoid iff it is a function.

Proof. Because $\forall I, J \in \mathscr{P}(\operatorname{im} f) : \langle f^{-1} \rangle^* (I \sqcap J) = \langle f^{-1} \rangle^* I \sqcap \langle f^{-1} \rangle^* J$ is true for a funcoid f corresponding to a binary relation if and only if it is a function (see proposition 388). □

Remark 963. This corollary can be reformulated as follows: For binary relations (principal funcoids) the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a funcoid are the same.

Theorem 964. If f, g are funcoids, $f \sqsubseteq g$ and g is monovalued then $g|_{\mathrm{dom}\, f} = f$.

Proof. Obviously $g|_{\mathrm{dom}\, f} \sqsupseteq f$. Suppose for contrary that $g|_{\mathrm{dom}\, f} \sqsubset f$. Then there exists an atom $a \in \operatorname{atoms} \operatorname{dom} f$ such that $\langle g|_{\mathrm{dom}\, f} \rangle a \neq \langle f \rangle a$ that is $\langle g \rangle a \sqsubset \langle f \rangle a$ what is impossible. □

7.16. Open maps

Definition 965. An *open map* from a topological space to a topological space is a function which maps open sets into open sets.

An obvious generalization of this is *open map* f from an endofuncoid μ to an endofuncoid ν, which is by definition a function (or rather a principal, entirely defined, monovalued funcoid) from $\operatorname{Ob} \mu$ to $\operatorname{Ob} \nu$ such that

$$\forall x \in \operatorname{Ob} \mu, V \in \langle \mu \rangle^* \{x\} : \langle f \rangle^* V \sqsupseteq \langle \nu \rangle \langle f \rangle^* @\{x\}.$$

This formula is equivalent (exercise!) to

$$\forall x \in \operatorname{Ob} \mu : \langle f \rangle \langle \mu \rangle^* @\{x\} \sqsupseteq \langle \nu \rangle \langle f \rangle^* @\{x\}.$$

It can be abstracted/simplified further (now for an *arbitrary* funcoid f from $\operatorname{Ob} \mu$ to $\operatorname{Ob} \nu$):

$$\operatorname{Compl}(f \circ \mu) \sqsupseteq \operatorname{Compl}(\nu \circ f).$$

Definition 966. An *open funcoid* from an endofuncoid μ to an endofuncoid ν is a funcoid f from $\operatorname{Ob} \mu$ to $\operatorname{Ob} \nu$ such that $\operatorname{Compl}(f \circ \mu) \sqsupseteq \operatorname{Compl}(\nu \circ f)$.

Obvious 967. A funcoid f is open iff $f \circ \mu \sqsupseteq \operatorname{Compl}(\nu \circ f)$.

Theorem 968. Let μ, ν, π be endofuncoids. Let f be an principal monovalued open funcoid from $\operatorname{Ob} \mu$ to $\operatorname{Ob} \nu$ and g is a open funcoid from $\operatorname{Ob} \nu$ to $\operatorname{Ob} \pi$. Then $g \circ f$ is an open funcoid from $\operatorname{Ob} \mu$ to $\operatorname{Ob} \pi$.

PROOF.

$$\begin{aligned}
&\langle g \circ f \rangle \langle \mu \rangle^* @\{x\} = \\
&\langle g \rangle \langle f \rangle \langle \mu \rangle^* @\{x\} \sqsupseteq \\
&\langle g \rangle \langle \nu \rangle \langle f \rangle^* @\{x\} \sqsupseteq \quad \text{(using that } f \text{ is monovalued and principal)} \\
&\langle \pi \rangle \langle g \rangle \langle f \rangle^* @\{x\} = \\
&\langle \pi \rangle \langle g \circ f \rangle @\{x\}.
\end{aligned}$$

□

PROBLEM 969. Devise a pointfree (not using a particular point x) proof of the above theorem. It should refer to a lemma which may use a particular point, but the proof of the theorem itself should be without a particular point.

7.17. T_0-, T_1-, T_2-, T_3-, and T_4-separable funcoids

For funcoids it can be generalized T_0-, T_1-, T_2-, and T_3- separability. Worthwhile note that T_0 and T_2 separability is defined through T_1 separability.

DEFINITION 970. Let call T_1*-separable* such endofuncoid f that for every $\alpha, \beta \in \operatorname{Ob} f$ is true

$$\alpha \neq \beta \Rightarrow \neg(@\{\alpha\} \ [f]^* \ @\{\beta\}).$$

PROPOSITION 971. An endofuncoid f is T_1-separable iff $\operatorname{Cor} f \sqsubseteq 1^{\mathsf{FCD}}_{\operatorname{Ob} f}$.

PROOF.

$$\forall x, y \in \operatorname{Ob} f : (@\{x\} \ [f]^* \ @\{y\} \Rightarrow x = y) \Leftrightarrow$$
$$\forall x, y \in \operatorname{Ob} f : (@\{x\} \ [\operatorname{Cor} f]^* \ @\{y\} \Rightarrow x = y) \Leftrightarrow \operatorname{Cor} f \sqsubseteq 1^{\mathsf{FCD}}_{\operatorname{Ob} f}.$$

□

PROPOSITION 972. An endofuncoid f is T_1-separable iff $\operatorname{Cor}\langle f \rangle^* \{x\} \sqsubseteq \{x\}$ for every $x \in \operatorname{Ob} f$.

PROOF. $\operatorname{Cor}\langle f \rangle^* \{x\} \sqsubseteq \{x\} \Leftrightarrow \langle \operatorname{CoCompl} f \rangle^* \{x\} \sqsubseteq \{x\} \Leftrightarrow \operatorname{Compl} \operatorname{CoCompl} f \sqsubseteq 1^{\mathsf{FCD}}_{\operatorname{Ob} f} \Leftrightarrow \operatorname{Cor} f \sqsubseteq 1^{\mathsf{FCD}}_{\operatorname{Ob} f}$. □

DEFINITION 973. Let call T_0*-separable* such funcoid $f \in \mathsf{FCD}(A, A)$ that $f \sqcap f^{-1}$ is T_1-separable.

DEFINITION 974. Let call T_2*-separable* such funcoid f that $f^{-1} \circ f$ is T_1-separable.

For symmetric transitive funcoids T_0-, T_1- and T_2-separability are the same (see theorem 255).

OBVIOUS 975. A funcoid f is T_2-separable iff $\alpha \neq \beta \Rightarrow \langle f \rangle^* @\{\alpha\} \not\asymp \langle f \rangle^* @\{\beta\}$ for every $\alpha, \beta \in \operatorname{Src} f$.

DEFINITION 976. Funcoid f is *regular* iff for every $C \in \mathscr{T} \operatorname{Dst} f$ and $p \in \operatorname{Src} f$

$$\langle f \rangle \langle f^{-1} \rangle C \asymp \langle f \rangle @\{p\} \Leftarrow \uparrow^{\operatorname{Src} f} \{p\} \asymp \langle f^{-1} \rangle C.$$

PROPOSITION 977. The following are pairwise equivalent:

1°. A funcoid f is regular.
2°. $\operatorname{Compl}(f \circ f^{-1} \circ f) \sqsubseteq \operatorname{Compl} f$.
3°. $\operatorname{Compl}(f \circ f^{-1} \circ f) \sqsubseteq f$.

Proof. Equivalently transform the defining formula for regular funcoids:
$\langle f\rangle\langle f^{-1}\rangle C \asymp \langle f\rangle @\{p\} \Leftarrow \uparrow^{\operatorname{Src} f} \{p\} \asymp \langle f^{-1}\rangle C$;
$\langle f\rangle\langle f^{-1}\rangle C \not\asymp \langle f\rangle @\{p\} \Rightarrow \uparrow^{\operatorname{Src} f} \{p\} \not\asymp \langle f^{-1}\rangle C$;
(by definition of funcoids)
$C \not\asymp \langle f\rangle\langle f^{-1}\rangle\langle f\rangle @\{p\} \Rightarrow C \not\asymp \langle f\rangle @\{p\}$;
$\langle f\rangle\langle f^{-1}\rangle\langle f\rangle @\{p\} \sqsubseteq \langle f\rangle @\{p\}$;
$\left\langle f \circ f^{-1} \circ f\right\rangle @\{p\} \sqsubseteq \langle f\rangle @\{p\}$;
$\operatorname{Compl}(f \circ f^{-1} \circ f) \sqsubseteq \operatorname{Compl} f$;
$\operatorname{Compl}(f \circ f^{-1} \circ f) \sqsubseteq f$. □

Proposition 978. If f is complete, regularity of funcoid f is equivalent to $f \circ \operatorname{Compl}(f^{-1} \circ f) \sqsubseteq f$.

Proof. By proposition 954. □

Remark 979. After seeing how it collapses into algebraic formulas about funcoids, the definition for a funcoid being regular seems quite arbitrary and sucked out of the finger (not an example of algebraic elegance). So I present these formulas only because they coincide with the traditional definition of regular topological spaces. However this is only my personal opinion and it may be wrong.

Definition 980. An endofuncoid is T_3- iff it is both T_2- and regular.

A topological space S is called T_4-separable when for any two disjoint closed sets $A, B \subseteq S$ there exist disjoint open sets U, V containing A and B respectively.

Let f be the complete funcoid corresponding to the topological space.

Since the closed sets are exactly sets of the form $\left\langle f^{-1}\right\rangle^* X$ and sets X and Y having non-intersecting open neighborhood is equivalent to $\langle f\rangle^* X \asymp \langle f\rangle^* Y$, the above is equivalent to:
$\left\langle f^{-1}\right\rangle^* A \asymp \left\langle f^{-1}\right\rangle^* B \Rightarrow \langle f\rangle^* \left\langle f^{-1}\right\rangle^* A \asymp \langle f\rangle^* \left\langle f^{-1}\right\rangle^* B$;
$\langle f\rangle^* \left\langle f^{-1}\right\rangle^* A \not\asymp \langle f\rangle^* \left\langle f^{-1}\right\rangle^* B \Rightarrow \langle f^{-1}\rangle^* A \not\asymp \left\langle f^{-1}\right\rangle^* B$;
$\langle f\rangle^* \left\langle f^{-1}\right\rangle^* \langle f\rangle^* \left\langle f^{-1}\right\rangle^* A \not\asymp B \Rightarrow \langle f\rangle^* \left\langle f^{-1}\right\rangle^* A \not\asymp B$;
$\langle f\rangle^* \left\langle f^{-1}\right\rangle^* \langle f\rangle^* \left\langle f^{-1}\right\rangle^* A \sqsubseteq \langle f\rangle^* \left\langle f^{-1}\right\rangle^* A$;
$f \circ f^{-1} \circ f \circ f^{-1} \sqsubseteq f \circ f^{-1}$.
Take the last formula as the definition of T_4-funcoid f.

7.18. Filters closed regarding a funcoid

Definition 981. Let's call *closed* regarding a funcoid $f \in \mathsf{FCD}(A, A)$ such filter $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$ that $\langle f\rangle\mathcal{A} \sqsubseteq \mathcal{A}$.

This is a generalization of closedness of a set regarding an unary operation.

Proposition 982. If I and J are closed (regarding some funcoid f), S is a set of closed filters on $\operatorname{Src} f$, then

1°. $\mathcal{I} \sqcup \mathcal{J}$ is a closed filter;
2°. $\bigsqcap S$ is a closed filter.

Proof. Let denote the given funcoid as f. $\langle f\rangle(\mathcal{I} \sqcup \mathcal{J}) = \langle f\rangle\mathcal{I} \sqcup \langle f\rangle\mathcal{J} \sqsubseteq \mathcal{I} \sqcup \mathcal{J}$, $\langle f\rangle \bigsqcap S \sqsubseteq \bigsqcap \langle\langle f\rangle\rangle^* S \sqsubseteq \bigsqcap S$. Consequently the filters $\mathcal{I} \sqcup \mathcal{J}$ and $\bigsqcap S$ are closed. □

Proposition 983. If S is a set of filters closed regarding a complete funcoid, then the filter $\bigsqcup S$ is also closed regarding our funcoid.

Proof. $\langle f\rangle \bigsqcup S = \bigsqcup \langle\langle f\rangle\rangle^* S \sqsubseteq \bigsqcup S$ where f is the given funcoid. □

7.19. Proximity spaces

Fix a set U. Let equate typed subsets of U with subsets of U.

We will prove that proximity spaces are essentially the same as reflexive, symmetric, transitive funcoids.

Our primary interest here is the last axiom (6°) in the definition 797 of proximity spaces.

PROPOSITION 984. If f is a transitive, symmetric funcoid, then the last axiom of proximity holds.

PROOF.

$$\neg\left(A\ [f]^*\ B\right) \Leftrightarrow \neg\left(A\ \left[f^{-1}\circ f\right]^*\ B\right) \Leftrightarrow \langle f\rangle^* B \asymp \langle f\rangle^* A \Leftrightarrow \\ \exists M \in U : M \asymp \langle f\rangle^* A \wedge \overline{M} \asymp \langle f\rangle^* B.$$

□

PROPOSITION 985. For a reflexive funcoid, the last axiom of proximity implies that it is transitive and symmetric.

PROOF. Let $\neg\left(A\ [f]^*\ B\right)$ implies $\exists M : M \asymp \langle f\rangle^* A \wedge \overline{M} \asymp \langle f\rangle^* B$. Then $\neg\left(A\ [f]^*\ B\right)$ implies $M \asymp \langle f\rangle^* A \wedge \langle f\rangle^* B \sqsubseteq M$, thus $\langle f\rangle^* A \asymp \langle f\rangle^* B$; $\neg\left(A\ \left[f^{-1}\circ f\right]^*\ B\right)$ that is $f \sqsupseteq f^{-1}\circ f$ and thus $f = f^{-1}\circ f$. By theorem 255 f is transitive and symmetric. □

THEOREM 986. Reflexive, symmetric, transitive funcoids endofuncoids on a set U are essentially the same as proximity spaces on U.

PROOF. Above and theorem 831. □

CHAPTER 8

Reloids

8.1. Basic definitions

DEFINITION 987. Let A, B be sets. $\mathsf{RLD}\sharp(A, B)$ is the base of an arbitrary but fixed primary filtrator over $\mathbf{Rel}(A, B)$.

OBVIOUS 988. $(\mathsf{RLD}\sharp(A, B), \mathbf{Rel}(A, B))$ is a powerset filtrator.

DEFINITION 989. I call a reloid from a set A to a set B a triple (A, B, F) where $F \in \mathsf{RLD}\sharp(A, B)$.

DEFINITION 990. *Source* and *destination* of every reloid (A, B, F) are defined as

$$\operatorname{Src}(A, B, F) = A \quad \text{and} \quad \operatorname{Dst}(A, B, F) = B.$$

I will denote $\mathsf{RLD}(A, B)$ the set of reloids from A to B.
I will denote RLD the set of all reloids (for small sets).

DEFINITION 991. I will call *endoreloids* reloids with the same source and destination.

DEFINITION 992.

- $\uparrow^{\mathsf{RLD}\sharp} f$ is the principal filter object corresponding to a $\mathbf{Rel}$-morphism f.
- $\uparrow^{\mathsf{RLD}\sharp(A,B)} f =\uparrow^{\mathsf{RLD}\sharp} (A, B, f)$ for every binary relation $f \in \mathscr{P}(A \times B)$.
- $\uparrow^{\mathsf{RLD}} f = (\operatorname{Src} f, \operatorname{Dst} f, \uparrow^{\mathsf{RLD}\sharp} f)$ for every $\mathbf{Rel}$-morphism f.
- $\uparrow^{\mathsf{RLD}(A,B)} f =\uparrow^{\mathsf{RLD}} (A, B, f)$ for every binary relation $f \in \mathscr{P}(A \times B)$.

DEFINITION 993. I call members of a set $\left\langle\uparrow^{\mathsf{RLD}}\right\rangle^* \mathbf{Rel}(A, B)$ as *principal* reloids.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations.

DEFINITION 994. $\operatorname{up} f^{-1} = \left\{ \frac{F^{-1}}{F \in \operatorname{up} f} \right\}$ for every $f \in \mathsf{RLD}\sharp(A, B)$.

PROPOSITION 995. f^{-1} exists and $f^{-1} \in \mathsf{RLD}\sharp(B, A)$.

PROOF. We need to prove that $\left\{ \frac{F^{-1}}{F \in \operatorname{up} f} \right\}$ is a filter, but that's obvious. □

DEFINITION 996. The *reverse* reloid of a reloid is defined by the formula

$$(A, B, F)^{-1} = (B, A, F^{-1}).$$

NOTE 997. The reverse reloid is *not* an inverse in the sense of group theory or category theory.

Reverse reloid is a generalization of conjugate quasi-uniformity.

DEFINITION 998. Every set $\mathsf{RLD}(A, B)$ is a poset by the formula $f \sqsubseteq g \Leftrightarrow \operatorname{GR} f \sqsubseteq \operatorname{GR} g$. We will apply lattice operations to subsets of $\mathsf{RLD}(A, B)$ without explicitly mentioning $\mathsf{RLD}(A, B)$.

Filtrators of reloids are $(\mathsf{RLD}(A, B), \mathbf{Rel}(A, B))$ (for all sets A, B). Here I equate principal reloids with corresponding $\mathbf{Rel}$-morphisms.

OBVIOUS 999. $(\mathsf{RLD}(A,B),\mathbf{Rel}(A,B))$ is a powerset filtrator isomorphic to the filtrator $(\mathsf{RLD}\sharp(A,B),\mathbf{Rel}(A,B))$. Thus $\mathsf{RLD}(A,B)$ is a special case of $\mathsf{RLD}\sharp(A,B)$.

8.2. Composition of reloids

DEFINITION 1000. Reloids f and g are *composable* when $\operatorname{Dst} f = \operatorname{Src} g$.

DEFINITION 1001. *Composition* of (composable) reloids is defined by the formula

$$g \circ f = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\}.$$

OBVIOUS 1002. Composition of reloids is a reloid.

OBVIOUS 1003. $\uparrow^{\mathsf{RLD}} g \circ \uparrow^{\mathsf{RLD}} f = \uparrow^{\mathsf{RLD}} (g \circ f)$ for composable morphisms f, g of category **Rel**.

THEOREM 1004. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable reloids f, g, h.

PROOF. For two nonempty collections A and B of sets I will denote

$$A \sim B \Leftrightarrow \forall K \in A \exists L \in B : L \subseteq K \wedge \forall K \in B \exists L \in A : L \subseteq K.$$

It is easy to see that $\sim$ is a transitive relation.

I will denote $B \circ A = \left\{ \frac{L \circ K}{K \in A, L \in B} \right\}$.

Let first prove that for every nonempty collections of relations A, B, C

$$A \sim B \Rightarrow A \circ C \sim B \circ C.$$

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B : K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for every $P \in A \circ C$ there exists $P' \in B \circ C$ such that $P' \subseteq P$; the vice versa is analogous. So $A \circ C \sim B \circ C$.

$\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ g) \circ \operatorname{up} f$, $\operatorname{up}(h \circ g) \sim (\operatorname{up} h) \circ (\operatorname{up} g)$. By proven above $\operatorname{up}((h \circ g) \circ f) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

Analogously $\operatorname{up}(h \circ (g \circ f)) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

So $\operatorname{up}(h \circ (g \circ f)) \sim \operatorname{up}((h \circ g) \circ f)$ what is possible only if $\operatorname{up}(h \circ (g \circ f)) = \operatorname{up}((h \circ g) \circ f)$. Thus $(h \circ g) \circ f = h \circ (g \circ f)$. □

EXERCISE 1005. Prove $f_n \circ \dots \circ f_0 = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F_n \circ \dots \circ F_0}{F_i \in \operatorname{up} f_i} \right\}$ for every composable reloids $f_0, \dots, f_n$ where n is an integer, independently of the inserted parentheses. (Hint: Use generalized filter bases.)

THEOREM 1006. For every reloid f:

1°. $f \circ f = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F \circ F}{F \in \operatorname{up} f} \right\}$ if $\operatorname{Src} f = \operatorname{Dst} f$;

2°. $f^{-1} \circ f = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F^{-1} \circ F}{F \in \operatorname{up} f} \right\}$;

3°. $f \circ f^{-1} = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F \circ F^{-1}}{F \in \operatorname{up} f} \right\}$.

PROOF. I will prove only 1° and 2° because 3° is analogous to 2°.

1°. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f : H \circ H \sqsubseteq G \circ F$. To prove it take $H = F \sqcap G$.

2°. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f : H^{-1} \circ H \sqsubseteq G^{-1} \circ F$. To prove it take $H = F \sqcap G$. Then $H^{-1} \circ H = (F \sqcap G)^{-1} \circ (F \sqcap G) \sqsubseteq G^{-1} \circ F$.

□

EXERCISE 1007. Prove $f^n = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F^n}{F \in \operatorname{up} f} \right\}$ for every endofuncoid f and positive integer n.

THEOREM 1008. For every sets A, B, C if $g, h \in \mathsf{RLD}(A, B)$ then

1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for every $f \in \mathsf{RLD}(B, C)$;
2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$ for every $f \in \mathsf{RLD}(C, A)$.

PROOF. We'll prove only the first as the second is dual.
By the infinite distributivity law for filters we have

$$\begin{aligned} &f \circ g \sqcup f \circ h = \\ \bigsqcap^{\mathsf{RLD}}\left\{\frac{F \circ G}{F \in \operatorname{up} f, G \in \operatorname{up} g}\right\} \sqcup \bigsqcap^{\mathsf{RLD}}\left\{\frac{F \circ H}{F \in \operatorname{up} f, H \in \operatorname{up} h}\right\} &= \\ \bigsqcap^{\mathsf{RLD}}\left\{\frac{(F_1 \circ G) \sqcup^{\mathsf{RLD}} (F_2 \circ H)}{F_1, F_2 \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h}\right\} &= \\ \bigsqcap^{\mathsf{RLD}}\left\{\frac{(F_1 \circ G) \sqcup (F_2 \circ H)}{F_1, F_2 \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h}\right\}&. \end{aligned}$$

Obviously

$$\begin{aligned} \bigsqcap^{\mathsf{RLD}}\left\{\frac{(F_1 \circ G) \sqcup (F_2 \circ H)}{F_1, F_2 \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h}\right\} &\sqsupseteq \\ \bigsqcap^{\mathsf{RLD}}\left\{\frac{(((F_1 \sqcap F_2) \circ G) \sqcup ((F_1 \sqcap F_2) \circ H))}{F_1, F_2 \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h}\right\} &= \\ \bigsqcap^{\mathsf{RLD}}\left\{\frac{(F \circ G) \sqcup (F \circ H)}{F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h}\right\} &= \\ \bigsqcap^{\mathsf{RLD}}\left\{\frac{F \circ (G \sqcup H)}{F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h}\right\}&. \end{aligned}$$

Because $G \in \operatorname{up} g \wedge H \in \operatorname{up} h \Rightarrow G \sqcup H \in \operatorname{up}(g \sqcup h)$ we have

$$\begin{aligned} \bigsqcap^{\mathsf{RLD}}\left\{\frac{F \circ (G \sqcup H)}{F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h}\right\} &\sqsupseteq \\ \bigsqcap^{\mathsf{RLD}}\left\{\frac{F \circ K}{F \in \operatorname{up} f, K \in \operatorname{up}(g \sqcup h)}\right\} &= \\ f \circ (g \sqcup h)&. \end{aligned}$$

Thus we have proved $f \circ g \sqcup f \circ h \sqsupseteq f \circ (g \sqcup h)$. But obviously $f \circ (g \sqcup h) \sqsupseteq f \circ g$ and $f \circ (g \sqcup h) \sqsupseteq f \circ h$ and so $f \circ (g \sqcup h) \sqsupseteq f \circ g \sqcup f \circ h$. Thus $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$. □

THEOREM 1009. Let A, B, C be sets, $f \in \mathsf{RLD}(A, B)$, $g \in \mathsf{RLD}(B, C)$, $h \in \mathsf{RLD}(A, C)$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

PROOF.

$$\begin{aligned} g \circ f \not\asymp h \Leftrightarrow \\ \prod^{\mathsf{RLD}} \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\} \sqcap \prod^{\mathsf{RLD}} \operatorname{up} h \neq \bot \Leftrightarrow \\ \prod^{\mathsf{RLD}} \left\{ \frac{(G \circ F) \sqcap^{\mathsf{RLD}} H}{F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h} \right\} \neq \bot \Leftrightarrow \\ \prod^{\mathsf{RLD}} \left\{ \frac{(G \circ F) \sqcap H}{F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h} \right\} \neq \bot \Leftrightarrow \\ \forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h :\uparrow^{\mathsf{RLD}} ((G \circ F) \sqcap H) \neq \bot \Leftrightarrow \\ \forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : G \circ F \not\asymp H \end{aligned}$$

(used properties of generalized filter bases).

Similarly $g \not\asymp h \circ f^{-1} \Leftrightarrow \forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : G \not\asymp H \circ F^{-1}$.

Thus $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}$ because $G \circ F \not\asymp H \Leftrightarrow G \not\asymp H \circ F^{-1}$ by proposition 283. □

THEOREM 1010. For every composable reloids f and g

1°. $g \circ f = \bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\}$.

2°. $g \circ f = \bigsqcup \left\{ \frac{G \circ f}{G \in \operatorname{atoms} g} \right\}$.

PROOF. We will prove only the first as the second is dual. □

Obviously $\bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\} \sqsubseteq g \circ f$. We need to prove $\bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\} \sqsupseteq g \circ f$. Really,

$$\begin{aligned} \bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\} \sqsupseteq g \circ f \Leftrightarrow \\ \forall x \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} g) : \left(x \not\asymp g \circ f \Rightarrow x \not\asymp \bigsqcup \left\{ \frac{g \circ F}{F \in \operatorname{atoms} f} \right\} \right) \Leftarrow \\ \forall x \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} g) : (x \not\asymp g \circ f \Rightarrow \exists F \in \operatorname{atoms} f : x \not\asymp g \circ F) \Leftrightarrow \\ \forall x \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} g) : (g^{-1} \circ x \not\asymp f \Rightarrow \exists F \in \operatorname{atoms} f : g^{-1} \circ x \not\asymp F) \end{aligned}$$

what is obviously true.

COROLLARY 1011. If f and g are composable reloids, then

$$g \circ f = \bigsqcup \left\{ \frac{G \circ F}{F \in \operatorname{atoms} f, G \in \operatorname{atoms} g} \right\}.$$

PROOF. $g \circ f = \bigsqcup_{F \in \operatorname{atoms} f} (g \circ F) = \bigsqcup_{F \in \operatorname{atoms} f} \bigsqcup_{G \in \operatorname{atoms} g} (G \circ F) = \bigsqcup \left\{ \frac{G \circ F}{F \in \operatorname{atoms} f, G \in \operatorname{atoms} g} \right\}$. □

8.3. Reloidal product of filters

DEFINITION 1012. Reloidal product of filters $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \stackrel{\text{def}}{=} \prod^{\mathsf{RLD}} \left\{ \frac{A \times B}{A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}} \right\}.$$

OBVIOUS 1013.

- $\uparrow^U A \times^{\mathsf{RLD}} \uparrow^V B = \uparrow^{\mathsf{RLD}(U,V)} (A \times B)$ for every sets $A \subseteq U$, $B \subseteq V$.
- $\uparrow A \times^{\mathsf{RLD}} \uparrow B = \uparrow^{\mathsf{RLD}} (A \times B)$ for every typed sets A, B.

THEOREM 1014. $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}=\bigsqcup\left\{\frac{a\times^{\mathsf{RLD}}b}{a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}}\right\}$ for every filters $\mathcal{A}$ and $\mathcal{B}$.

PROOF. Obviously $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\sqsupseteq\bigsqcup\left\{\frac{a\times^{\mathsf{RLD}}b}{a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}}\right\}$.

Reversely, let $K\in\operatorname{up}\bigsqcup\left\{\frac{a\times^{\mathsf{RLD}}b}{a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}}\right\}$. Then $K\in\operatorname{up}(a\times^{\mathsf{RLD}}b)$ for every $a\in\operatorname{atoms}\mathcal{A}$, $b\in\operatorname{atoms}\mathcal{B}$. $K\sqsupseteq X_a\times Y_b$ for some $X_a\in\operatorname{up}a$, $Y_b\in\operatorname{up}b$;

$$K\sqsupseteq\bigsqcup\left\{\frac{X_a\times Y_b}{a\in\operatorname{atoms}\mathcal{A},b\in\operatorname{atoms}\mathcal{B}}\right\}=\bigsqcup\left\{\frac{X_a}{a\in\operatorname{atoms}\mathcal{A}}\right\}\times\bigsqcup\left\{\frac{Y_b}{b\in\operatorname{atoms}\mathcal{B}}\right\}\sqsupseteq A\times B$$

where $A\in\operatorname{up}\mathcal{A}$, $B\in\operatorname{up}\mathcal{B}$; $K\in\operatorname{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$. □

THEOREM 1015. If $\mathcal{A}_0,\mathcal{A}_1\in\mathscr{F}(A)$, $\mathcal{B}_0,\mathcal{B}_1\in\mathscr{F}(B)$ for some sets A, B then

$$(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0)\sqcap(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1)=(\mathcal{A}_0\sqcap\mathcal{A}_1)\times^{\mathsf{RLD}}(\mathcal{B}_0\sqcap\mathcal{B}_1).$$

PROOF.

$$\begin{aligned}&(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0)\sqcap(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1)=\\ &\prod^{\mathsf{RLD}}\left\{\frac{P\sqcap Q}{P\in\operatorname{up}(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0),Q\in\operatorname{up}(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1)}\right\}=\\ &\prod^{\mathsf{RLD}}\left\{\frac{(A_0\times B_0)\sqcap(A_1\times B_1)}{A_0\in\operatorname{up}\mathcal{A}_0,B_0\in\operatorname{up}\mathcal{B}_0,A_1\in\operatorname{up}\mathcal{A}_1,B_1\in\operatorname{up}\mathcal{B}_1}\right\}=\\ &\prod^{\mathsf{RLD}}\left\{\frac{(A_0\sqcap A_1)\times(B_0\sqcap B_1)}{A_0\in\operatorname{up}\mathcal{A}_0,B_0\in\operatorname{up}\mathcal{B}_0,A_1\in\operatorname{up}\mathcal{A}_1,B_1\in\operatorname{up}\mathcal{B}_1}\right\}=\\ &\prod^{\mathsf{RLD}}\left\{\frac{K\times L}{K\in\operatorname{up}(\mathcal{A}_0\sqcap\mathcal{A}_1),L\in\operatorname{up}(\mathcal{B}_0\sqcap\mathcal{B}_1)}\right\}=\\ &(\mathcal{A}_0\sqcap\mathcal{A}_1)\times^{\mathsf{RLD}}(\mathcal{B}_0\sqcap\mathcal{B}_1).\end{aligned}$$

□

THEOREM 1016. If $S\in\mathscr{P}(\mathscr{F}(A)\times\mathscr{F}(B))$ for some sets A, B then

$$\prod\left\{\frac{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}}{(\mathcal{A},\mathcal{B})\in S}\right\}=\prod\operatorname{dom}S\times^{\mathsf{RLD}}\prod\operatorname{im}S.$$

PROOF. Let $\mathcal{P}=\prod\operatorname{dom}S$, $\mathcal{Q}=\prod\operatorname{im}S$; $l=\prod\left\{\frac{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}}{(\mathcal{A},\mathcal{B})\in S}\right\}$.

$\mathcal{P}\times^{\mathsf{RLD}}\mathcal{Q}\sqsubseteq l$ is obvious.

Let $F\in\operatorname{up}(\mathcal{P}\times^{\mathsf{RLD}}\mathcal{Q})$. Then there exist $P\in\operatorname{up}\mathcal{P}$ and $Q\in\operatorname{up}\mathcal{Q}$ such that $F\sqsupseteq P\times Q$.

$P=P_1\sqcap\dots\sqcap P_n$ where $P_i\in\operatorname{dom}S$ and $Q=Q_1\sqcap\dots\sqcap Q_m$ where $Q_j\in\operatorname{im}S$.

$P\times Q=\prod_{i,j}(P_i\times Q_j)$.

$P_i\times Q_j\in\operatorname{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$ for some $(\mathcal{A},\mathcal{B})\in S$. $P\times Q=\prod_{i,j}(P_i\times Q_j)\in\operatorname{up}l$. So $F\in\operatorname{up}l$. □

COROLLARY 1017. $\prod\langle\mathcal{A}\times^{\mathsf{RLD}}\rangle^*T=\mathcal{A}\times^{\mathsf{RLD}}\prod T$ if $\mathcal{A}$ is a filter and T is a set of filters with common base.

PROOF. Take $S=\{\mathcal{A}\}\times T$ where T is a set of filters.

Then $\prod\left\{\frac{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}}{\mathcal{B}\in T}\right\}=\mathcal{A}\times^{\mathsf{RLD}}\prod T$ that is $\prod\langle\mathcal{A}\times^{\mathsf{RLD}}\rangle^*T=\mathcal{A}\times^{\mathsf{RLD}}\prod T$. □

DEFINITION 1018. I will call a reloid *convex* iff it is a join of direct products.

8.4. Restricting reloid to a filter. Domain and image

DEFINITION 1019. *Identity reloid* for a set A is defined by the formula $1_A^{\mathsf{RLD}} = \uparrow^{\mathsf{RLD}(A,A)} \mathrm{id}_A$.

OBVIOUS 1020. $(1_A^{\mathsf{RLD}})^{-1} = 1_A^{\mathsf{RLD}}$.

DEFINITION 1021. I define *restricting* a reloid f to a filter $\mathcal{A}$ as $f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \top^{\mathscr{F}(\operatorname{Dst} f)})$.

DEFINITION 1022. *Domain* and *image* of a reloid f are defined as follows:

$$\operatorname{dom} f = \prod^{\mathscr{F}} \langle \operatorname{dom} \rangle^* \operatorname{up} f; \quad \operatorname{im} f = \prod^{\mathscr{F}} \langle \operatorname{im} \rangle^* \operatorname{up} f.$$

PROPOSITION 1023. $f \sqsubseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$ for every reloid f and filters $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathscr{F}(\operatorname{Dst} f)$.

PROOF.

$\Rightarrow$. It follows from $\operatorname{dom}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \sqsubseteq \mathcal{A} \wedge \operatorname{im}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \sqsubseteq \mathcal{B}$.

$\Leftarrow$. $\operatorname{dom} f \sqsubseteq \mathcal{A} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A} \exists F \in \operatorname{up} f : \operatorname{dom} F \sqsubseteq A$. Analogously $\operatorname{im} f \sqsubseteq \mathcal{B} \Leftrightarrow \forall B \in \operatorname{up} \mathcal{B} \exists G \in \operatorname{up} f : \operatorname{im} G \sqsubseteq B$.

Let $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$, $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$. Then there exist $F, G \in \operatorname{up} f$ such that $\operatorname{dom} F \sqsubseteq A \wedge \operatorname{im} G \sqsubseteq B$. Consequently $F \sqcap G \in \operatorname{up} f$, $\operatorname{dom}(F \sqcap G) \sqsubseteq A$, $\operatorname{im}(F \sqcap G) \sqsubseteq B$ that is $F \sqcap G \sqsubseteq A \times B$. So there exists $H \in \operatorname{up} f$ such that $H \sqsubseteq A \times B$ for every $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$. So $f \sqsubseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.

□

DEFINITION 1024. I call *restricted identity reloid* for a filter $\mathcal{A}$ the reloid

$$\mathrm{id}_{\mathcal{A}}^{\mathsf{RLD}} = (1_{\operatorname{Base}(\mathcal{A})}^{\mathsf{RLD}})|_{\mathcal{A}}.$$

THEOREM 1025. $\mathrm{id}_{\mathcal{A}}^{\mathsf{RLD}} = \prod_{A \in \operatorname{up} \mathcal{A}}^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} \mathrm{id}_A$ for every filter $\mathcal{A}$.

PROOF. Let $K \in \operatorname{up} \prod_{A \in \operatorname{up} \mathcal{A}}^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} \mathrm{id}_A$, then there exists $A \in \operatorname{up} \mathcal{A}$ such that $\operatorname{GR} K \supseteq \mathrm{id}_A$. Then

$$\begin{aligned}
&\mathrm{id}_{\mathcal{A}}^{\mathsf{RLD}} \sqsubseteq \\
&\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} \mathrm{id}_{\operatorname{Base}(\mathcal{A})} \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \top) \sqsubseteq \\
&\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} \mathrm{id}_{\operatorname{Base}(\mathcal{A})} \sqcap (A \times^{\mathsf{RLD}} \top) = \\
&\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} \mathrm{id}_{\operatorname{Base}(\mathcal{A})} \sqcap \uparrow^{\mathsf{RLD}} (A \times \top) = \\
&\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} (\mathrm{id}_{\operatorname{Base}(\mathcal{A})} \cap \operatorname{GR}(A \times \top)) = \\
&\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} \mathrm{id}_A \sqsubseteq K.
\end{aligned}$$

Thus $K \in \operatorname{up} \mathrm{id}_{\mathcal{A}}^{\mathsf{RLD}}$.

Reversely let $K \in \operatorname{up} \mathrm{id}_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{up}(1_{\operatorname{Base}(\mathcal{A})}^{\mathsf{RLD}} \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \top))$, then there exists $A \in \operatorname{up} \mathcal{A}$ such that

$$\begin{aligned}
K \in \operatorname{up} \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} (\mathrm{id}_{\operatorname{Base}(\mathcal{A})} \cap \operatorname{GR}(A \times \top)) = \\
\operatorname{up} \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} \mathrm{id}_A \supseteq \\
\operatorname{up} \prod_{A \in \operatorname{up} \mathcal{A}}^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{A}))} \mathrm{id}_A.
\end{aligned}$$

□

COROLLARY 1026. $(\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}})^{-1} = \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$.

THEOREM 1027. $f|_{\mathcal{A}} = f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$ for every reloid f and $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$.

PROOF. We need to prove that

$$f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \top) = f \circ \bigsqcap^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Src} f)} \left\{ \frac{\mathrm{id}_A}{A \in \operatorname{up} \mathcal{A}} \right\}.$$

We have

$$\begin{aligned} f \circ \bigsqcap^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Src} f)} \left\{ \frac{\mathrm{id}_A}{A \in \operatorname{up} \mathcal{A}} \right\} &= \\ \bigsqcap^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Src} f)} \left\{ \frac{\mathrm{GR}(F) \circ \mathrm{id}_A}{F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A}} \right\} &= \\ \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F|_A}{F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A}} \right\} &= \\ \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F \sqcap (A \times \top^{\mathscr{F}(\operatorname{Dst} f)})}{F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A}} \right\} &= \\ \bigsqcap^{\mathsf{RLD}} \left\{ \frac{F}{F \in \operatorname{up} f} \right\} \sqcap \bigsqcap^{\mathsf{RLD}} \left\{ \frac{A \times \top^{\mathscr{F}(\operatorname{Dst} f)}}{A \in \operatorname{up} \mathcal{A}} \right\} &= \\ f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \top). \end{aligned}$$

□

THEOREM 1028. $(g \circ f)|_{\mathcal{A}} = g \circ (f|_{\mathcal{A}})$ for every composable reloids f and g and $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$.

PROOF. $(g \circ f)|_{\mathcal{A}} = (g \circ f) \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = g \circ (f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}) = g \circ (f|_{\mathcal{A}})$. □

THEOREM 1029. $f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathrm{id}^{\mathsf{RLD}}_{\mathcal{B}} \circ f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$ for every reloid f and $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathscr{F}(\operatorname{Dst} f)$.

PROOF.

$$\begin{aligned} f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) &= \\ f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \top^{\mathscr{F}(\operatorname{Dst} f)}) \sqcap (\top^{\mathscr{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B}) &= \\ f|_{\mathcal{A}} \sqcap (\top^{\mathscr{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B}) &= \\ (f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}) \sqcap (\top^{\mathscr{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B}) &= \\ ((f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}})^{-1} \sqcap (\top^{\mathscr{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B})^{-1})^{-1} &= \\ ((\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} \circ f^{-1}) \sqcap (\mathcal{B} \times^{\mathsf{RLD}} \top^{\mathscr{F}(\operatorname{Src} f)}))^{-1} &= \\ (\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} \circ f^{-1} \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{B}})^{-1} &= \\ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{B}} \circ f \circ \mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}&. \end{aligned}$$

□

PROPOSITION 1030. $\mathrm{id}_{\mathcal{B}} \circ \mathrm{id}_{\mathcal{A}} = \mathrm{id}_{\mathcal{A} \sqcap \mathcal{B}}$ for all filters $\mathcal{A}$, $\mathcal{B}$ (on some set U).

PROOF. $\mathrm{id}_{\mathcal{B}} \circ \mathrm{id}_{\mathcal{A}} = (\mathrm{id}_{\mathcal{B}})|_{\mathcal{A}} = (1^{\mathsf{RLD}}_U|_{\mathcal{B}})|_{\mathcal{A}} = 1^{\mathsf{RLD}}_U|_{\mathcal{A} \sqcap \mathcal{B}} = \mathrm{id}_{\mathcal{A} \sqcap \mathcal{B}}$. □

THEOREM 1031. $f|_{\uparrow\{\alpha\}} = \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} \operatorname{im}(f|_{\uparrow\{\alpha\}})$ for every reloid f and $\alpha \in \operatorname{Src} f$.

Proof. First,

$$\begin{aligned}
&\operatorname{im}(f|_{\uparrow\{\alpha\}}) = \\
&\bigsqcap^{\mathsf{RLD}} \langle \operatorname{im}\rangle^* \operatorname{up}(f|_{\uparrow\{\alpha\}}) = \\
&\bigsqcap^{\mathsf{RLD}} \langle \operatorname{im}\rangle^* \operatorname{up}(f \sqcap (\uparrow^{\operatorname{Src} f} \{\alpha\} \times \top^{\mathscr{F}(\operatorname{Dst} f)})) = \\
&\bigsqcap^{\mathsf{RLD}} \left\{ \frac{\operatorname{im}(F \cap (\{\alpha\} \times \top^{\mathscr{T}(\operatorname{Dst} f)}))}{F \in \operatorname{up} f} \right\} = \\
&\bigsqcap^{\mathsf{RLD}} \left\{ \frac{\operatorname{im}(F|_{\uparrow\{\alpha\}})}{F \in \operatorname{up} f} \right\}.
\end{aligned}$$

Taking this into account we have:

$$\begin{aligned}
&\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} \operatorname{im}(f|_{\uparrow\{\alpha\}}) = \\
&\bigsqcap^{\mathsf{RLD}} \left\{ \frac{\uparrow^{\operatorname{Src} f} \{\alpha\} \times K}{K \in \operatorname{im}(f|_{\uparrow\{\alpha\}})} \right\} = \\
&\bigsqcap^{\mathsf{RLD}} \left\{ \frac{\uparrow^{\operatorname{Src} f} \{\alpha\} \times \operatorname{im}(F|_{\uparrow\{\alpha\}})}{F \in \operatorname{up} f} \right\} = \\
&\bigsqcap^{\mathsf{RLD}} \left\{ \frac{F|_{\uparrow\{\alpha\}}}{F \in \operatorname{up} f} \right\} = \\
&\bigsqcap^{\mathsf{RLD}} \left\{ \frac{F \sqcap (\uparrow^{\operatorname{Src} f} \{\alpha\} \times \top^{\mathscr{T}(\operatorname{Dst} f)})}{F \in \operatorname{up} f} \right\} = \\
&\bigsqcap^{\mathsf{RLD}} \left\{ \frac{F}{F \in \operatorname{up} f} \right\} \sqcap \uparrow^{\mathsf{RLD}} (\uparrow^{\operatorname{Src} f} \{\alpha\} \times \top^{\mathscr{T}(\operatorname{Dst} f)}) = \\
&f \sqcap \uparrow^{\mathsf{RLD}} (\uparrow^{\operatorname{Src} f} \{\alpha\} \times \top^{\mathscr{T}(\operatorname{Dst} f)}) = \\
&f|_{\uparrow\{\alpha\}}.
\end{aligned}$$

□

Lemma 1032. $\lambda \mathcal{B} \in \mathscr{F}(B) : \top^{\mathscr{F}} \times^{\mathsf{RLD}} \mathcal{B}$ is an upper adjoint of $\lambda f \in \mathsf{RLD}(A, B) : \operatorname{im} f$ (for every sets A, B).

Proof. We need to prove $\operatorname{im} f \sqsubseteq \mathcal{B} \Leftrightarrow f \sqsubseteq \top^{\mathscr{F}} \times^{\mathsf{RLD}} \mathcal{B}$ what is obvious. □

Corollary 1033. Image and domain of reloids preserve joins.

Proof. By properties of Galois connections and duality. □

8.5. Categories of reloids

I will define two categories, the *category of reloids* and the *category of reloid triples*.

The *category of reloids* is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\mathsf{RLD}(A, B)$.
- The composition is the composition of reloids.
- Identity morphism for a set is the identity reloid for that set.

To show it is really a category is trivial.

The *category of reloid triples* is defined as follows:

- Objects are small sets.

- The morphisms from a filter $\mathcal{A}$ to a filter $\mathcal{B}$ are triples $(\mathcal{A},\mathcal{B},f)$ where $f\in\mathsf{RLD}(\operatorname{Base}(\mathcal{A}),\operatorname{Base}(\mathcal{B}))$ and $\operatorname{dom} f\sqsubseteq\mathcal{A}$, $\operatorname{im} f\sqsubseteq\mathcal{B}$.
- The composition is defined by the formula $(\mathcal{B},\mathcal{C},g)\circ(\mathcal{A},\mathcal{B},f)=(\mathcal{A},\mathcal{C},g\circ f)$.
- Identity morphism for a filter $\mathcal{A}$ is $\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}$.

To prove that it is really a category is trivial.

Proposition 1034. $\uparrow^{\mathsf{RLD}}$ is a functor from **Rel** to RLD.

Proof. $\uparrow^{\mathsf{RLD}}(g\circ f)=\uparrow^{\mathsf{RLD}}g\circ\uparrow^{\mathsf{RLD}}f$ was proved above. $\uparrow^{\mathsf{RLD}}1^{\mathbf{Rel}}_A=1^{\mathsf{RLD}}_A$ is by definition. ∎

8.6. Monovalued and injective reloids

Following the idea of definition of monovalued morphism let's call *monovalued* such a reloid f that $f\circ f^{-1}\sqsubseteq\mathrm{id}^{\mathsf{RLD}}_{\operatorname{im} f}$.

Similarly, I will call a reloid *injective* when $f^{-1}\circ f\sqsubseteq\mathrm{id}^{\mathsf{RLD}}_{\operatorname{dom} f}$.

Obvious 1035. A reloid f is

- monovalued iff $f\circ f^{-1}\sqsubseteq 1^{\mathsf{RLD}}_{\operatorname{Dst} f}$;
- injective iff $f^{-1}\circ f\sqsubseteq 1^{\mathsf{RLD}}_{\operatorname{Src} f}$.

In other words, a reloid is monovalued (injective) when it is a monovalued (injective) morphism of the category of reloids.

Monovaluedness is dual of injectivity.

Obvious 1036.

1°. A morphism $(\mathcal{A},\mathcal{B},f)$ of the category of reloid triples is monovalued iff the reloid f is monovalued.

2°. A morphism $(\mathcal{A},\mathcal{B},f)$ of the category of reloid triples is injective iff the reloid f is injective.

Theorem 1037.

1°. A reloid f is a monovalued iff there exists a **Set**-morphism (monovalued **Rel**-morphism) $F\in\operatorname{up} f$.

2°. A reloid f is a injective iff there exists an injective **Rel**-morphism $F\in\operatorname{up} f$.

3°. A reloid f is a both monovalued and injective iff there exists an injection (a monovalued and injective **Rel**-morphism = injective **Set**-morphism) $F\in\operatorname{up} f$.

Proof. The reverse implications are obvious. Let's prove the direct implications:

1°. Let f be a monovalued reloid. Then $f\circ f^{-1}\sqsubseteq 1^{\mathsf{RLD}}_{\operatorname{Dst} f}$, that is

$$\bigsqcap^{\mathsf{RLD}}\left\{\frac{F\circ F^{-1}}{F\in\operatorname{up} f}\right\}\sqsubseteq 1^{\mathsf{RLD}}_{\operatorname{Dst} f}.$$

It's simple to show that $\left\{\frac{F\circ F^{-1}}{F\in\operatorname{up} f}\right\}$ is a filter base. Consequently there exists $F\in\operatorname{up} f$ such that $F\circ F^{-1}\sqsubseteq 1^{\mathsf{RLD}}_{\operatorname{Dst} f}$ that is F is monovalued.

2°. Similar.

3°. Let f be a both monovalued and injective reloid. Then by proved above there exist $F,G\in\operatorname{up} f$ such that F is monovalued and G is injective. Thus $F\sqcap G\in\operatorname{up} f$ is both monovalued and injective.

∎

CONJECTURE 1038. A reloid f is monovalued iff

$$\forall g \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f) : (g \sqsubseteq f \Rightarrow \exists \mathcal{A} \in \mathscr{F}(\operatorname{Src} f) : g = f|_{\mathcal{A}}).$$

8.7. Complete reloids and completion of reloids

DEFINITION 1039. A *complete* reloid is a reloid representable as a join of reloidal products $\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in A$ and b is an ultrafilter on B for some sets A and B.

DEFINITION 1040. A *co-complete* reloid is a reloid representable as a join of reloidal products $a \times^{\mathsf{RLD}} \uparrow^A \{\beta\}$ where $\beta \in B$ and a is an ultrafilter on A for some sets A and B.

I will denote the sets of complete and co-complete reloids from a set A to a set B as $\mathsf{ComplRLD}(A, B)$ and $\mathsf{CoComplRLD}(A, B)$ correspondingly and set of all (co-)complete reloids (for small sets) as $\mathsf{ComplRLD}$ and $\mathsf{CoComplRLD}$.

OBVIOUS 1041. Complete and co-complete are dual.

THEOREM 1042. $G \mapsto \bigsqcup\left\{\frac{\uparrow^A\{\alpha\}\times^{\mathsf{RLD}} G(\alpha)}{\alpha\in A}\right\}$ is an order isomorphism from the set of functions $G \in \mathscr{F}(B)^A$ to the set $\mathsf{ComplRLD}(A, B)$.

The inverse isomorphism is described by the formula $G(\alpha) = \operatorname{im}(f|_{\uparrow\{\alpha\}})$ where f is a complete reloid.

PROOF. $\bigsqcup\left\{\frac{\uparrow^A\{\alpha\}\times^{\mathsf{RLD}} G(\alpha)}{\alpha\in A}\right\}$ is complete because $G(\alpha) = \bigsqcup \operatorname{atoms} G(\alpha)$ and thus

$$\bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in A}\right\} = \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b}{\alpha \in A, b \in \operatorname{atoms} G(\alpha)}\right\}$$

is complete. So $G \mapsto \bigsqcup\left\{\frac{\uparrow^A\{\alpha\}\times^{\mathsf{RLD}} G(\alpha)}{\alpha\in A}\right\}$ is a function from $G \in \mathscr{F}(B)^A$ to $\mathsf{ComplRLD}(A, B)$.

Let f be complete. Then take

$$G(\alpha) = \bigsqcup\left\{\frac{b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}}{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b \sqsubseteq f}\right\}$$

and we have $f = \bigsqcup\left\{\frac{\uparrow^A\{\alpha\}\times^{\mathsf{RLD}} G(\alpha)}{\alpha\in A}\right\}$ obviously. So $G \mapsto \bigsqcup\left\{\frac{\uparrow^A\{\alpha\}\times^{\mathsf{RLD}} G(\alpha)}{\alpha\in A}\right\}$ is surjection onto $\mathsf{ComplRLD}(A, B)$.

Let now prove that it is an injection:

Let

$$f = \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} F(\alpha)}{\alpha \in A}\right\} = \bigsqcup\left\{\frac{\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in A}\right\}$$

for some $F, G \in \mathscr{F}(B)^A$. We need to prove $F = G$. Let $\beta \in \operatorname{Src} f$.

$$f \sqcap (\uparrow^A \{\beta\} \times^{\mathsf{RLD}} \top^{\mathscr{F}(B)}) = \text{(theorem 610)}$$
$$\bigsqcup\left\{\frac{(\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} F(\alpha)) \sqcap (\uparrow^A \{\beta\} \times^{\mathsf{RLD}} \top^{\mathscr{F}(B)})}{\alpha \in A}\right\} =$$
$$\uparrow^A \{\beta\} \times^{\mathsf{RLD}} F(\beta).$$

Similarly $f \sqcap (\uparrow^A \{\beta\} \times^{\mathsf{RLD}} \top^{\mathscr{F}(B)}) = \uparrow^A \{\beta\} \times^{\mathsf{RLD}} G(\beta)$. Thus $\uparrow^A \{\beta\} \times^{\mathsf{RLD}} F(\beta) = \uparrow^A \{\beta\} \times^{\mathsf{RLD}} G(\beta)$ and so $F(\beta) = G(\beta)$.

We have proved that it is a bijection. To show that it is monotone is trivial.

Denote $f = \bigsqcup\left\{\frac{\uparrow^A\{\alpha\}\times^{\mathsf{RLD}}G(\alpha)}{\alpha\in A}\right\}$. Then

$$\operatorname{im}(f|_{\uparrow\{\alpha'\}}) = \operatorname{im}(f\sqcap(\uparrow^A\{\alpha'\}\times\top^{\mathscr{F}(B)})) =$$
$$\text{(because } \uparrow^A\{\alpha'\}\times\top^{\mathscr{F}(B)} \text{ is principal)} =$$
$$\operatorname{im}\bigsqcup\left\{\frac{(\uparrow^A\{\alpha\}\times^{\mathsf{RLD}}G(\alpha))\sqcap(\uparrow^A\{\alpha'\}\times\top^{\mathscr{F}(B)})}{\alpha\in\operatorname{Src}f}\right\} =$$
$$\operatorname{im}(\uparrow^A\{\alpha'\}\times^{\mathsf{RLD}}G(\alpha')) = G(\alpha').$$

□

Corollary 1043. $G\mapsto\bigsqcup\left\{\frac{G(\alpha)\times^{\mathsf{RLD}}\uparrow^A\{\alpha\}}{\alpha\in A}\right\}$ is an order isomorphism from the set of functions $G\in\mathscr{F}(B)^A$ to the set $\mathsf{CoComplRLD}(A,B)$.

The inverse isomorphism is described by the formula $G(\alpha)=\operatorname{im}(f^{-1}|_{\uparrow\{\alpha\}})$ where f is a co-complete reloid.

Corollary 1044. $\mathsf{ComplRLD}(A,B)$ and $\mathsf{ComplFCD}(A,B)$ are a co-frames.

Obvious 1045. Complete and co-complete reloids are convex.

Obvious 1046. Principal reloids are complete and co-complete.

Obvious 1047. Join (on the lattice of reloids) of complete reloids is complete.

Theorem 1048. A reloid which is both complete and co-complete is principal.

Proof. Let f be a complete and co-complete reloid. We have

$$f=\bigsqcup\left\{\frac{\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}G(\alpha)}{\alpha\in\operatorname{Src}f}\right\}\quad\text{and}\quad f=\bigsqcup\left\{\frac{H(\beta)\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\}}{\beta\in\operatorname{Dst}f}\right\}$$

for some functions $G:\operatorname{Src}f\to\mathscr{F}(\operatorname{Dst}f)$ and $H:\operatorname{Dst}f\to\mathscr{F}(\operatorname{Src}f)$. For every $\alpha\in\operatorname{Src}f$ we have

$$G(\alpha)=$$
$$\operatorname{im}f|_{\uparrow\{\alpha\}}=$$
$$\operatorname{im}(f\sqcap(\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}\top^{\mathscr{F}(\operatorname{Dst}f)}))= \quad (*)$$
$$\operatorname{im}\bigsqcup\left\{\frac{(H(\beta)\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\})\sqcap(\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}\top^{\mathscr{F}(\operatorname{Dst}f)})}{\beta\in\operatorname{Dst}f}\right\}=$$
$$\operatorname{im}\bigsqcup\left\{\frac{(H(\beta)\sqcap\uparrow^{\operatorname{Src}f}\{\alpha\})\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\}}{\beta\in\operatorname{Dst}f}\right\}=$$
$$\operatorname{im}\bigsqcup\left\{\frac{\left(\begin{cases}\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\} & \text{if } H(\beta)\not\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}\\ \bot^{\mathsf{RLD}(\operatorname{Src}f,\operatorname{Dst}f)} & \text{if } H(\beta)\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}\end{cases}\right)}{\beta\in\operatorname{Dst}f}\right\}=$$
$$\operatorname{im}\bigsqcup\left\{\frac{\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\}}{\beta\in\operatorname{Dst}f,H(\beta)\not\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}}\right\}=$$
$$\operatorname{im}\bigsqcup\left\{\frac{\uparrow^{\mathsf{RLD}(\operatorname{Src}f,\operatorname{Dst}f)}\{(\alpha,\beta)\}}{\beta\in\operatorname{Dst}f,H(\beta)\not\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}}\right\}=$$
$$\bigsqcup\left\{\frac{\uparrow^{\operatorname{Dst}f}\{\beta\}}{\beta\in\operatorname{Dst}f,H(\beta)\not\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}}\right\}$$

* theorem 610 was used.

Thus $G(\alpha)$ is a principal filter that is $G(\alpha) =\uparrow^{\operatorname{Dst} f} g(\alpha)$ for some $g : \operatorname{Src} f \to \operatorname{Dst} f$; $\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} G(\alpha) =\uparrow^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)} (\{\alpha\} \times g(\alpha))$; f is principal as a join of principal reloids. □

DEFINITION 1049. *Completion* and *co-completion* of a reloid $f \in \mathsf{RLD}(A, B)$ are defined by the formulas:

$$\operatorname{Compl} f = \operatorname{Cor}^{\mathsf{ComplRLD}(A,B)} f; \quad \operatorname{CoCompl} f = \operatorname{Cor}^{\mathsf{CoComplRLD}(A,B)} f.$$

THEOREM 1050. Atoms of the lattice $\mathsf{ComplRLD}(A, B)$ are exactly reloidal products of the form $\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in A$ and b is an ultrafilter on B.

PROOF. First, it's easy to see that $\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ are elements of $\mathsf{ComplRLD}(A, B)$. Also $\perp^{\mathsf{RLD}(A,B)}$ is an element of $\mathsf{ComplRLD}(A, B)$.

$\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ are atoms of $\mathsf{ComplRLD}(A, B)$ because they are atoms of $\mathsf{RLD}(A, B)$.

It remains to prove that if f is an atom of $\mathsf{ComplRLD}(A, B)$ then $f =\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ for some $\alpha \in A$ and an ultrafilter b on B.

Suppose f is a non-empty complete reloid. Then $\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b \sqsubseteq f$ for some $\alpha \in A$ and an ultrafilter b on B. If f is an atom then $f =\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$. □

OBVIOUS 1051. $\mathsf{ComplRLD}(A, B)$ is an atomistic lattice.

PROPOSITION 1052. $\operatorname{Compl} f = \bigsqcup \left\{ \frac{f|_{\uparrow\{\alpha\}}}{\alpha \in \operatorname{Src} f} \right\}$ for every reloid f.

PROOF. Let's denote R the right part of the equality to be proven.

That R is a complete reloid follows from the equality

$$f|_{\uparrow\{\alpha\}} =\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} \operatorname{im}(f|_{\uparrow\{\alpha\}}).$$

Obviously, $R \sqsubseteq f$.

The only thing left to prove is that $g \sqsubseteq R$ for every complete reloid g such that $g \sqsubseteq f$.

Really let g be a complete reloid such that $g \sqsubseteq f$. Then

$$g = \bigsqcup \left\{ \frac{\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} G(\alpha)}{\alpha \in \operatorname{Src} f} \right\}$$

for some function $G : \operatorname{Src} f \to \mathscr{F}(\operatorname{Dst} f)$.

We have $\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} G(\alpha) = g|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} \sqsubseteq f|_{\uparrow\{\alpha\}}$. Thus $g \sqsubseteq R$. □

CONJECTURE 1053. $\operatorname{Compl} f \sqcap \operatorname{Compl} g = \operatorname{Compl}(f \sqcap g)$ for every $f, g \in \mathsf{RLD}(A, B)$.

PROPOSITION 1054. Conjecture 1053 is equivalent to the statement that meet of every two complete reloids is a complete reloid.

PROOF. Let conjecture 1053 holds. Then for complete funcoids f and g we have $f \sqcap g = \operatorname{Compl}(f \sqcap g)$ and thus $f \sqcap g$ is complete.

Let meet of every two complete reloid is complete. Then $\operatorname{Compl} f \sqcap \operatorname{Compl} g$ is complete and thus it is greatest complete reloid which is less $\operatorname{Compl} f$ and less $\operatorname{Compl} g$ what is the same as greatest complete reloid which is less than f and g that is $\operatorname{Compl}(f \sqcap g)$. □

THEOREM 1055. $\operatorname{Compl} \bigsqcup R = \bigsqcup \langle \operatorname{Compl} \rangle^* R$ for every set $R \in \mathscr{P}\mathsf{RLD}(A, B)$ for every sets A, B.

PROOF.

$$
\begin{aligned}
&\operatorname{Compl} \bigsqcup R = \\
&\bigsqcup \left\{ \frac{(\bigsqcup R)|_{\uparrow^A \{\alpha\}}}{\alpha \in A} \right\} = \quad \text{(theorem 610)} \\
&\bigsqcup \left\{ \frac{\bigsqcup \left\{ \frac{f|_{\uparrow\{\alpha\}}}{\alpha \in A} \right\}}{f \in R} \right\} = \\
&\bigsqcup \langle \operatorname{Compl} \rangle^* R.
\end{aligned}
$$

□

LEMMA 1056. Completion of a co-complete reloid is principal.

PROOF. Let f be a co-complete reloid. Then there is a function $F : \operatorname{Dst} f \to \mathscr{F}(\operatorname{Src} f)$ such that

$$f = \bigsqcup \left\{ \frac{F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f} \{\alpha\}}{\alpha \in \operatorname{Dst} f} \right\}.$$

So

$$
\begin{aligned}
&\operatorname{Compl} f = \\
&\bigsqcup \left\{ \frac{\left(\bigsqcup \left\{ \frac{F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f} \{\alpha\}}{\alpha \in \operatorname{Dst} f} \right\} \right) |_{\uparrow\{\beta\}}}{\beta \in \operatorname{Src} f} \right\} = \\
&\bigsqcup \left\{ \frac{\left(\bigsqcup \left\{ \frac{F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f} \{\alpha\}}{\alpha \in \operatorname{Dst} f} \right\} \right) \sqcap (\uparrow^{\operatorname{Src} f} \{\beta\} \times^{\mathsf{RLD}} \top^{\mathscr{F}(\operatorname{Dst} f)})}{\beta \in \operatorname{Src} f} \right\} = (*) \\
&\bigsqcup \left\{ \frac{\bigsqcup \left\{ \frac{(F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f} \{\alpha\}) \sqcap (\uparrow^{\operatorname{Src} f} \{\beta\} \times^{\mathsf{RLD}} \top^{\mathscr{F}(\operatorname{Dst} f)})}{\alpha \in \operatorname{Dst} f} \right\}}{\beta \in \operatorname{Src} f} \right\} = \\
&\bigsqcup \left\{ \frac{\bigsqcup \left\{ \frac{\uparrow^{\operatorname{Src} f} \{\beta\} \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f} \{\alpha\}}{\alpha \in \operatorname{Dst} f} \right\}}{\beta \in \operatorname{Src} f, \uparrow^{\operatorname{Src} f} \{\beta\} \sqsubseteq F(\alpha)} \right\}
\end{aligned}
$$

* theorem 610.

Thus $\operatorname{Compl} f$ is principal. □

THEOREM 1057. $\operatorname{Compl} \operatorname{CoCompl} f = \operatorname{CoCompl} \operatorname{Compl} f = \operatorname{Cor} f$ for every reloid f.

PROOF. We will prove only $\operatorname{Compl} \operatorname{CoCompl} f = \operatorname{Cor} f$. The rest follows from symmetry.

From the lemma $\operatorname{Compl} \operatorname{CoCompl} f$ is principal. It is obvious $\operatorname{Compl} \operatorname{CoCompl} f \sqsubseteq f$. So to finish the proof we need to show only that for every principal reloid $F \sqsubseteq f$ we have $F \sqsubseteq \operatorname{Compl} \operatorname{CoCompl} f$.

Really, obviously $F \sqsubseteq \operatorname{CoCompl} f$ and thus $F = \operatorname{Compl} F \sqsubseteq \operatorname{Compl} \operatorname{CoCompl} f$. □

CONJECTURE 1058. If f is a complete reloid, then it is metacomplete.

CONJECTURE 1059. If f is a metacomplete reloid, then it is complete.

CONJECTURE 1060. $\operatorname{Compl} f = f \setminus^* (\Omega^{\operatorname{Src} f} \times^{\mathsf{RLD}} \top^{\mathscr{F}(\operatorname{Dst} f)})$ for every reloid f.

By analogy with similar properties of funcoids described above:

PROPOSITION 1061. For composable reloids f and g it holds

1°. $\operatorname{Compl}(g \circ f) \sqsupseteq (\operatorname{Compl} g) \circ (\operatorname{Compl} f)$

2°. $\operatorname{CoCompl}(g \circ f) \sqsupseteq (\operatorname{CoCompl} g) \circ (\operatorname{CoCompl} f)$.

PROOF.

1°. $(\operatorname{Compl} g) \circ (\operatorname{Compl} f) \sqsubseteq \operatorname{Compl}((\operatorname{Compl} g) \circ (\operatorname{Compl} f)) \sqsubseteq \operatorname{Compl}(g \circ f)$.

2°. By duality.

□

CONJECTURE 1062. For composable reloids f and g it holds

1°. $\operatorname{Compl}(g \circ f) = (\operatorname{Compl} g) \circ f$ if f is a co-complete reloid;

2°. $\operatorname{CoCompl}(f \circ g) = f \circ \operatorname{CoCompl} g$ if f is a complete reloid;

3°. $\operatorname{CoCompl}((\operatorname{Compl} g) \circ f) = \operatorname{Compl}(g \circ (\operatorname{CoCompl} f)) = (\operatorname{Compl} g) \circ (\operatorname{CoCompl} f)$;

4°. $\operatorname{Compl}(g \circ (\operatorname{Compl} f)) = \operatorname{Compl}(g \circ f)$;

5°. $\operatorname{CoCompl}((\operatorname{CoCompl} g) \circ f) = \operatorname{CoCompl}(g \circ f)$.

8.8. What uniform spaces are

PROPOSITION 1063. Uniform spaces are exactly reflexive, symmetric, transitive endoreloids.

PROOF. Easy to prove using theorem 1006. □

CHAPTER 9

Relationships between funcoids and reloids

9.1. Funcoid induced by a reloid

Every reloid f induces a funcoid $(\mathsf{FCD})f \in \mathsf{FCD}(\operatorname{Src} f, \operatorname{Dst} f)$ by the following formulas (for every $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$):

$$\mathcal{X} \left[(\mathsf{FCD})f\right] \mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}} F\right] \mathcal{Y};$$

$$\langle(\mathsf{FCD})f\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{F \in \operatorname{up} f} \left\langle\uparrow^{\mathsf{FCD}} F\right\rangle\mathcal{X}.$$

We should prove that $(\mathsf{FCD})f$ is really a funcoid.

Proof. We need to prove that

$$\mathcal{X} \left[(\mathsf{FCD})f\right] \mathcal{Y} \Leftrightarrow \mathcal{Y} \sqcap \langle(\mathsf{FCD})f\rangle\mathcal{X} \neq \bot \Leftrightarrow \mathcal{X} \sqcap \left\langle(\mathsf{FCD})f^{-1}\right\rangle\mathcal{Y} \neq \bot.$$

The above formula is equivalent to:

$$\begin{aligned} &\forall F \in \operatorname{up} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}} F\right] \mathcal{Y} \Leftrightarrow \\ &\mathcal{Y} \sqcap \bigsqcap_{F \in \operatorname{up} f} \left\langle\uparrow^{\mathsf{FCD}} F\right\rangle\mathcal{X} \neq \bot \Leftrightarrow \\ &\mathcal{X} \sqcap \bigsqcap_{F \in \operatorname{up} f} \left\langle\uparrow^{\mathsf{FCD}} F^{-1}\right\rangle\mathcal{Y} \neq \bot. \end{aligned}$$

We have $\mathcal{Y} \sqcap \bigsqcap_{F \in \operatorname{up} f}\left\langle\uparrow^{\mathsf{FCD}} F\right\rangle\mathcal{X} = \bigsqcap_{F \in \operatorname{up} f}\left(\mathcal{Y} \sqcap \left\langle\uparrow^{\mathsf{FCD}} F\right\rangle\mathcal{X}\right)$.
Let's denote $W = \left\{ \frac{\mathcal{Y} \sqcap \left\langle\uparrow^{\mathsf{FCD}} F\right\rangle\mathcal{X}}{F \in \operatorname{up} f} \right\}$.

$$\forall F \in \operatorname{up} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}} F\right] \mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up} f : \mathcal{Y} \sqcap \left\langle\uparrow^{\mathsf{FCD}} F\right\rangle\mathcal{X} \neq \bot \Leftrightarrow \bot \notin W.$$

We need to prove only that $\bot \notin W \Leftrightarrow \bigsqcap W \neq \bot$. (The rest follows from symmetry.) To prove it is enough to show that W is a generalized filter base.

Let's prove that W is a generalized filter base. For this it's enough to prove that $V = \left\{ \frac{\left\langle\uparrow^{\mathsf{FCD}} F\right\rangle\mathcal{X}}{F \in \operatorname{up} f} \right\}$ is a generalized filter base. Let $\mathcal{A}, \mathcal{B} \in V$ that is $\mathcal{A} = \left\langle\uparrow^{\mathsf{FCD}} P\right\rangle\mathcal{X}$, $\mathcal{B} = \left\langle\uparrow^{\mathsf{FCD}} Q\right\rangle\mathcal{X}$ where $P, Q \in \operatorname{up} f$. Then for $\mathcal{C} = \left\langle\uparrow^{\mathsf{FCD}} (P \sqcap Q)\right\rangle\mathcal{X}$ is true both $\mathcal{C} \in V$ and $\mathcal{C} \sqsubseteq \mathcal{A}, \mathcal{B}$. So V is a generalized filter base and thus W is a generalized filter base. □

Proposition 1064. $(\mathsf{FCD}) \uparrow^{\mathsf{RLD}} f = \uparrow^{\mathsf{FCD}} f$ for every **Rel**-morphism f.

Proof. $\mathcal{X} \left[(\mathsf{FCD}) \uparrow^{\mathsf{RLD}} f\right] \mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up} \uparrow^{\mathsf{RLD}} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}} F\right] \mathcal{Y} \Leftrightarrow \mathcal{X} \left[\uparrow^{\mathsf{FCD}} f\right] \mathcal{Y}$ (for every $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$). □

Theorem 1065. $\mathcal{X} \left[(\mathsf{FCD})f\right] \mathcal{Y} \Leftrightarrow \mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \not\asymp f$ for every reloid f and $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$.

PROOF.

$$\begin{aligned}
\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \not\asymp f \Leftrightarrow \\
\forall F \in \operatorname{up} f, P \in \operatorname{up}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}) : P \not\asymp F \Leftrightarrow \\
\forall F \in \operatorname{up} f, X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \times Y \not\asymp F \Leftrightarrow \\
\forall F \in \operatorname{up} f, X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \left[\uparrow^{\mathsf{FCD}} F\right] Y \Leftrightarrow \\
\forall F \in \operatorname{up} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}} F\right] \mathcal{Y} \Leftrightarrow \\
\mathcal{X} \left[(\mathsf{FCD})f\right] \mathcal{Y}.
\end{aligned}$$

□

THEOREM 1066. $(\mathsf{FCD})f = \bigsqcap^{\mathsf{FCD}} \operatorname{up} f$ for every reloid f.

PROOF. Let a be an ultrafilter on $\operatorname{Src} f$.
$\langle (\mathsf{FCD})f \rangle a = \bigsqcap \left\{ \frac{\langle \uparrow^{\mathsf{FCD}} F \rangle a}{F \in \operatorname{up} f} \right\}$ by the definition of (FCD).
$\left\langle \bigsqcap^{\mathsf{FCD}} \operatorname{up} f \right\rangle a = \bigsqcap \left\{ \frac{\langle \uparrow^{\mathsf{FCD}} F \rangle a}{F \in \operatorname{up} f} \right\}$ by theorem 878.
So $\langle (\mathsf{FCD})f \rangle a = \left\langle \bigsqcap^{\mathsf{FCD}} \operatorname{up} f \right\rangle a$ for every ultrafilter a. □

LEMMA 1067. For every two filter bases S and T of morphisms $\mathbf{Rel}(U, V)$ and every typed set $A \in \mathscr{T} U$

$$\bigsqcap^{\mathsf{RLD}} S = \bigsqcap^{\mathsf{RLD}} T \Rightarrow \bigsqcap^{\mathscr{F}}_{F \in S} \langle F \rangle^* A = \bigsqcap^{\mathscr{F}}_{G \in T} \langle G \rangle^* A.$$

PROOF. Let $\bigsqcap^{\mathsf{RLD}} S = \bigsqcap^{\mathsf{RLD}} T$.

First let prove that $\left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$ is a filter base. Let $X, Y \in \left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$. Then $X = \langle F_X \rangle^* A$ and $Y = \langle F_Y \rangle^* A$ for some $F_X, F_Y \in S$. Because S is a filter base, we have $S \ni F_Z \sqsubseteq F_X \sqcap F_Y$. So $\langle F_Z \rangle^* A \sqsubseteq X \sqcap Y$ and $\langle F_Z \rangle^* A \in \left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$. So $\left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$ is a filter base.

Suppose $X \in \operatorname{up} \bigsqcap^{\mathscr{F}}_{F \in S} \langle F \rangle^* A$. Then there exists $X' \in \left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$ where $X \sqsupseteq X'$ because $\left\{ \frac{\langle F \rangle^* A}{F \in S} \right\}$ is a filter base. That is $X' = \langle F \rangle^* A$ for some $F \in S$. There exists $G \in T$ such that $G \sqsubseteq F$ because T is a filter base. Let $Y' = \langle G \rangle^* A$. We have $Y' \sqsubseteq X' \sqsubseteq X$; $Y' \in \left\{ \frac{\langle G \rangle^* A}{G \in T} \right\}$; $Y' \in \operatorname{up} \bigsqcap^{\mathscr{F}}_{G \in T} \langle G \rangle^* A$; $X \in \operatorname{up} \bigsqcap^{\mathscr{F}}_{G \in T} \langle G \rangle^* A$. The reverse is symmetric. □

LEMMA 1068. $\left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\}$ is a filter base for every reloids f and g.

PROOF. Let denote $D = \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\}$. Let $A \in D \wedge B \in D$. Then $A = G_A \circ F_A \wedge B = G_B \circ F_B$ for some $F_A, F_B \in \operatorname{up} f$, $G_A, G_B \in \operatorname{up} g$. So $A \sqcap B \sqsupseteq (G_A \sqcap G_B) \circ (F_A \sqcap F_B) \in D$ because $F_A \sqcap F_B \in \operatorname{up} f$ and $G_A \sqcap G_B \in \operatorname{up} g$. □

THEOREM 1069. $(\mathsf{FCD})(g \circ f) = ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)$ for every composable reloids f and g.

PROOF.

$$\langle (\mathsf{FCD})(g \circ f) \rangle^* X = \bigsqcap^{\mathscr{F}}_{H \in \operatorname{up}(g \circ f)} \langle H \rangle^* X = \bigsqcap^{\mathscr{F}}_{H \in \operatorname{up} \bigsqcap^{\mathsf{RLD}} \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\}} \langle H \rangle^* X.$$

Obviously

$$\bigsqcap^{\mathsf{RLD}}\left\{\frac{G\circ F}{F\in\operatorname{up}f,G\in\operatorname{up}g}\right\}=\bigsqcap^{\mathsf{RLD}}\operatorname{up}\bigsqcap^{\mathsf{RLD}}\left\{\frac{G\circ F}{F\in\operatorname{up}f,G\in\operatorname{up}g}\right\};$$

from this by lemma 1067 (taking into account that

$$\left\{\frac{G\circ F}{F\in\operatorname{up}f,G\in\operatorname{up}g}\right\}$$

and

$$\operatorname{up}\bigsqcap^{\mathsf{RLD}}\left\{\frac{G\circ F}{F\in\operatorname{up}f,G\in\operatorname{up}g}\right\}$$

are filter bases)

$$\bigsqcap^{\mathsf{RLD}}_{H\in\operatorname{up}\bigsqcap^{\mathsf{RLD}}\left\{\frac{G\circ F}{F\in\operatorname{up}f,G\in\operatorname{up}g}\right\}}\langle H\rangle^*X=\bigsqcap^{\mathscr{F}}\left\{\frac{\langle G\circ F\rangle^*X}{F\in\operatorname{up}f,G\in\operatorname{up}g}\right\}.$$

On the other side

$$\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle^*X=\langle(\mathsf{FCD})g\rangle\langle(\mathsf{FCD})f\rangle^*X=$$
$$\langle(\mathsf{FCD})g\rangle\bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}f}\langle F\rangle^*X=\bigsqcap_{G\in\operatorname{up}g}\langle\uparrow^{\mathsf{FCD}}G\rangle\bigsqcap^{\mathsf{RLD}}_{F\in\operatorname{up}f}\langle F\rangle^*X.$$

Let's prove that $\left\{\frac{\langle F\rangle^*X}{F\in\operatorname{up}f}\right\}$ is a filter base. If $A,B\in\left\{\frac{\langle F\rangle^*X}{F\in\operatorname{up}f}\right\}$ then $A=\langle F_1\rangle^*X$, $B=\langle F_2\rangle^*X$ where $F_1,F_2\in\operatorname{up}f$. $A\sqcap B\sqsupseteq\langle F_1\sqcap F_2\rangle^*X\in\left\{\frac{\langle F\rangle^*X}{F\in\operatorname{up}f}\right\}$. So $\left\{\frac{\langle F\rangle^*X}{F\in\operatorname{up}f}\right\}$ is really a filter base.

By theorem 839 we have

$$\langle\uparrow^{\mathsf{FCD}}G\rangle\bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}f}\langle F\rangle^*X=\bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}f}\langle G\rangle^*\langle F\rangle^*X.$$

So continuing the above equalities,

$$\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle^*X=$$
$$\bigsqcap^{\mathscr{F}}_{G\in\operatorname{up}g}\bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}f}\langle G\rangle^*\langle F\rangle^*X=$$
$$\bigsqcap^{\mathscr{F}}\left\{\frac{\langle G\rangle^*\langle F\rangle^*X}{F\in\operatorname{up}f,G\in\operatorname{up}g}\right\}=$$
$$\bigsqcap^{\mathscr{F}}\left\{\frac{\langle G\circ F\rangle^*X}{F\in\operatorname{up}f,G\in\operatorname{up}g}\right\}.$$

Combining these equalities we get $\langle(\mathsf{FCD})(g\circ f)\rangle^*X=\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle^*X$ for every typed set $X\in\mathscr{T}(\operatorname{Src}f)$. □

Proposition 1070. $(\mathsf{FCD})\operatorname{id}^{\mathsf{RLD}}_{\mathcal{A}}=\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$ for every filter $\mathcal{A}$.

Proof. Recall that $\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = \bigsqcap\left\{\frac{\uparrow^{\mathrm{Base}(\mathcal{A})}\mathrm{id}_A}{A\in\mathrm{up}\,\mathcal{A}}\right\}$. For every $\mathcal{X},\mathcal{Y}\in\mathscr{F}(\mathrm{Base}(\mathcal{A}))$ we have

$$\begin{aligned}\mathcal{X}\left[(\mathsf{FCD})\,\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}\right]\mathcal{Y}&\Leftrightarrow\\ \mathcal{X}\times^{\mathsf{RLD}}\mathcal{Y}\not\asymp\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}}&\Leftrightarrow\\ \forall A\in\mathrm{up}\,\mathcal{A}:\mathcal{X}\times^{\mathsf{RLD}}\mathcal{Y}\not\asymp\uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))}\mathrm{id}_A&\Leftrightarrow\\ \forall A\in\mathrm{up}\,\mathcal{A}:\mathcal{X}\left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A}),\mathrm{Base}(\mathcal{A}))}\mathrm{id}_A\right]\mathcal{Y}&\Leftrightarrow\\ \forall A\in\mathrm{up}\,\mathcal{A}:\mathcal{X}\sqcap\mathcal{Y}\not\asymp A&\Leftrightarrow\\ \mathcal{X}\sqcap\mathcal{Y}\not\asymp\mathcal{A}&\Leftrightarrow\\ \mathcal{X}\left[\mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}}\right]\mathcal{Y}&\end{aligned}$$

(used properties of generalized filter bases). □

Corollary 1071. $(\mathsf{FCD})1^{\mathsf{RLD}}_A = 1^{\mathsf{FCD}}_A$ for every set A.

Proposition 1072. (FCD) is a functor from RLD to FCD.

Proof. Preservation of composition and of identity is proved above. □

Proposition 1073.

1°. $(\mathsf{FCD})f$ is a monovalued funcoid if f is a monovalued reloid.
2°. $(\mathsf{FCD})f$ is an injective funcoid if f is an injective reloid.

Proof. We will prove only the first as the second is dual. Let f be a monovalued reloid. Then $f\circ f^{-1}\sqsubseteq 1^{\mathsf{RLD}}_{\mathrm{Dst}\,f}$; $(\mathsf{FCD})(f\circ f^{-1})\sqsubseteq 1^{\mathsf{FCD}}_{\mathrm{Dst}\,f}$; $(\mathsf{FCD})f\circ((\mathsf{FCD})f)^{-1}\sqsubseteq 1^{\mathsf{FCD}}_{\mathrm{Dst}\,f}$ that is $(\mathsf{FCD})f$ is a monovalued funcoid. □

Proposition 1074. $(\mathsf{FCD})(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}) = \mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}$ for every filters $\mathcal{A}$, $\mathcal{B}$.

Proof. $\mathcal{X}\left[(\mathsf{FCD})(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})\right]\mathcal{Y}\Leftrightarrow\forall F\in\mathrm{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}):\mathcal{X}\left[\uparrow^{\mathsf{FCD}}F\right]\mathcal{Y}$ (for every $\mathcal{X}\in\mathscr{F}(\mathrm{Base}(\mathcal{A}))$, $\mathcal{Y}\in\mathscr{F}(\mathrm{Base}(\mathcal{B}))$.

Evidently

$$\forall F\in\mathrm{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}):\mathcal{X}\left[\uparrow^{\mathsf{FCD}}F\right]\mathcal{Y}\Rightarrow\forall A\in\mathrm{up}\,\mathcal{A},B\in\mathrm{up}\,\mathcal{B}:\mathcal{X}\left[A\times B\right]\mathcal{Y}.$$

Let $\forall A\in\mathrm{up}\,\mathcal{A},B\in\mathrm{up}\,\mathcal{B}:\mathcal{X}\left[A\times B\right]\mathcal{Y}$. Then if $F\in\mathrm{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$, there are $A\in\mathrm{up}\,\mathcal{A}$, $B\in\mathrm{up}\,\mathcal{B}$ such that $F\sqsupseteq A\times B$. So $\mathcal{X}\left[\uparrow^{\mathsf{FCD}}F\right]\mathcal{Y}$. We have proved

$$\forall F\in\mathrm{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}):\mathcal{X}\left[\uparrow^{\mathsf{FCD}}F\right]\mathcal{Y}\Leftrightarrow\forall A\in\mathrm{up}\,\mathcal{A},B\in\mathrm{up}\,\mathcal{B}:\mathcal{X}\left[A\times B\right]\mathcal{Y}.$$

Further

$$\begin{aligned}&\forall A\in\mathrm{up}\,\mathcal{A},B\in\mathrm{up}\,\mathcal{B}:\mathcal{X}\left[A\times B\right]\mathcal{Y}\Leftrightarrow\\ &\qquad\forall A\in\mathrm{up}\,\mathcal{A},B\in\mathrm{up}\,\mathcal{B}:(\mathcal{X}\not\asymp A\wedge\mathcal{Y}\not\asymp B)\Leftrightarrow\\ &\qquad\qquad\mathcal{X}\not\asymp\mathcal{A}\wedge\mathcal{Y}\not\asymp\mathcal{B}\Leftrightarrow\mathcal{X}\left[\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\right]\mathcal{Y}.\end{aligned}$$

Thus $\mathcal{X}\left[(\mathsf{FCD})(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})\right]\mathcal{Y}\Leftrightarrow\mathcal{X}\left[\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\right]\mathcal{Y}$. □

Proposition 1075. $\mathrm{dom}(\mathsf{FCD})f = \mathrm{dom}\,f$ and $\mathrm{im}(\mathsf{FCD})f = \mathrm{im}\,f$ for every reloid f.

PROOF.

$$\operatorname{im}(\mathsf{FCD})f = \langle(\mathsf{FCD})f\rangle\top = \prod^{\mathscr{F}}_{F\in\operatorname{up}f}\langle F\rangle^*\top = \prod^{\mathscr{F}}_{F\in\operatorname{up}f}\operatorname{im}F = \prod^{\mathscr{F}}\langle\operatorname{im}\rangle^*\operatorname{up}f = \operatorname{im}f.$$

$\operatorname{dom}(\mathsf{FCD})f = \operatorname{dom}f$ is similar. □

PROPOSITION 1076. $(\mathsf{FCD})(f\sqcap(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})) = (\mathsf{FCD})f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B})$ for every reloid f and $\mathcal{A}\in\mathscr{F}(\operatorname{Src}f)$ and $\mathcal{B}\in\mathscr{F}(\operatorname{Dst}f)$.

PROOF.

$$\begin{aligned}(\mathsf{FCD})(f\sqcap(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})) &=\\ (\mathsf{FCD})(\operatorname{id}^{\mathsf{RLD}}_{\mathcal{B}}\circ f\circ\operatorname{id}^{\mathsf{RLD}}_{\mathcal{A}}) &=\\ (\mathsf{FCD})\operatorname{id}^{\mathsf{RLD}}_{\mathcal{B}}\circ(\mathsf{FCD})f\circ(\mathsf{FCD})\operatorname{id}^{\mathsf{RLD}}_{\mathcal{A}} &=\\ \operatorname{id}^{\mathsf{FCD}}_{\mathcal{B}}\circ(\mathsf{FCD})f\circ\operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}} &=\\ (\mathsf{FCD})f\sqcap(\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}).\end{aligned}$$

□

COROLLARY 1077. $(\mathsf{FCD})(f|_{\mathcal{A}}) = ((\mathsf{FCD})f)|_{\mathcal{A}}$ for every reloid f and a filter $\mathcal{A}\in\mathscr{F}(\operatorname{Src}f)$.

PROPOSITION 1078. $\langle(\mathsf{FCD})f\rangle\mathcal{X} = \operatorname{im}(f|_{\mathcal{X}})$ for every reloid f and a filter $\mathcal{X}\in\mathscr{F}(\operatorname{Src}f)$.

PROOF. $\operatorname{im}(f|_{\mathcal{X}}) = \operatorname{im}(\mathsf{FCD})(f|_{\mathcal{X}}) = \operatorname{im}(((\mathsf{FCD})f)|_{\mathcal{X}}) = \langle(\mathsf{FCD})f\rangle\mathcal{X}$. □

PROPOSITION 1079. $(\mathsf{FCD})f = \bigsqcup\left\{\frac{x\times^{\mathsf{FCD}}y}{x\in\operatorname{atoms}^{\mathscr{F}(\operatorname{Src}f)},y\in\operatorname{atoms}^{\mathscr{F}(\operatorname{Dst}f)},x\times^{\mathsf{RLD}}y\not\asymp f}\right\}$ for every reloid f.

PROOF. $(\mathsf{FCD})f = \bigsqcup\left\{\frac{x\times^{\mathsf{FCD}}y}{x\in\operatorname{atoms}^{\mathscr{F}(\operatorname{Src}f)},y\in\operatorname{atoms}^{\mathscr{F}(\operatorname{Dst}f)},x\times^{\mathsf{FCD}}y\not\asymp(\mathsf{FCD})f}\right\}$, but $x\times^{\mathsf{FCD}}y\not\asymp(\mathsf{FCD})f\Leftrightarrow x\mathrel{[(\mathsf{FCD})f]}y\Leftrightarrow x\times^{\mathsf{RLD}}y\not\asymp f$, thus follows the theorem. □

9.2. Reloids induced by a funcoid

Every funcoid $f\in\mathsf{FCD}(A,B)$ induces a reloid from A to B in two ways, intersection of *outward* relations and union of *inward* reloidal products of filters:

$$(\mathsf{RLD})_{\mathrm{out}}f = \prod^{\mathsf{RLD}}\operatorname{up}f;$$

$$(\mathsf{RLD})_{\mathrm{in}}f = \bigsqcup\left\{\frac{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}}{\mathcal{A}\in\mathscr{F}(A),\mathcal{B}\in\mathscr{F}(B),\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\sqsubseteq f}\right\}.$$

THEOREM 1080. $(\mathsf{RLD})_{\mathrm{in}}f = \bigsqcup\left\{\frac{a\times^{\mathsf{RLD}}b}{a\in\operatorname{atoms}^{\mathscr{F}(A)},b\in\operatorname{atoms}^{\mathscr{F}(B)},a\times^{\mathsf{FCD}}b\sqsubseteq f}\right\}$.

PROOF. It follows from theorem 1014. □

PROPOSITION 1081. $\operatorname{up}\uparrow^{\mathsf{RLD}}f = \operatorname{up}\uparrow^{\mathsf{FCD}}f$ for every **Rel**-morphism f.

PROOF. $X\in\operatorname{up}\uparrow^{\mathsf{RLD}}f\Leftrightarrow X\sqsupseteq f\Leftrightarrow X\in\operatorname{up}\uparrow^{\mathsf{FCD}}f$. □

PROPOSITION 1082. $(\mathsf{RLD})_{\mathrm{out}}\uparrow^{\mathsf{FCD}}f = \uparrow^{\mathsf{RLD}}f$ for every **Rel**-morphism f.

PROOF. $(\mathsf{RLD})_{\mathrm{out}}\uparrow^{\mathsf{FCD}}f = \prod^{\mathsf{RLD}}\operatorname{up}f = \uparrow^{\mathsf{RLD}}\min\operatorname{up}f = \uparrow^{\mathsf{RLD}}f$ taking into account the previous proposition. □

Surprisingly, a funcoid is greater inward than outward:

THEOREM 1083. $(\mathsf{RLD})_{\mathrm{out}} f \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f$ for every funcoid f.

PROOF. We need to prove

$$(\mathsf{RLD})_{\mathrm{out}} f \sqsubseteq \bigsqcup \left\{ \frac{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}}{\mathcal{A} \in \mathscr{F}(A), \mathcal{B} \in \mathscr{F}(B), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f} \right\}.$$

Let

$$K \in \operatorname{up} \bigsqcup \left\{ \frac{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}}{\mathcal{A} \in \mathscr{F}(A), \mathcal{B} \in \mathscr{F}(B), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f} \right\}.$$

Then

$$\begin{aligned} K &\in \operatorname{up} \uparrow^{\mathsf{RLD}} \bigsqcup \left\{ \frac{X_{\mathcal{A}} \times Y_{\mathcal{B}}}{\mathcal{A} \in \mathscr{F}(A), \mathcal{B} \in \mathscr{F}(B), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f} \right\} \\ &= (\mathsf{RLD})_{\mathrm{out}} \uparrow^{\mathsf{FCD}} \bigsqcup \left\{ \frac{X_{\mathcal{A}} \times Y_{\mathcal{B}}}{\mathcal{A} \in \mathscr{F}(A), \mathcal{B} \in \mathscr{F}(B), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f} \right\} \\ &= (\mathsf{RLD})_{\mathrm{out}} \bigsqcup^{\mathsf{FCD}} \left\{ \frac{\uparrow^{\mathsf{FCD}} (X_{\mathcal{A}} \times Y_{\mathcal{B}})}{\mathcal{A} \in \mathscr{F}(A), \mathcal{B} \in \mathscr{F}(B), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f} \right\} \\ &\sqsupseteq (\mathsf{RLD})_{\mathrm{out}} \bigsqcup \operatorname{atoms} f \\ &= (\mathsf{RLD})_{\mathrm{out}} f \end{aligned}$$

where $X_{\mathcal{A}} \in \operatorname{up} \mathcal{A}$, $X_{\mathcal{B}} \in \operatorname{up} \mathcal{B}$. $K \in \operatorname{up} (\mathsf{RLD})_{\mathrm{out}} f$. □

PROPOSITION 1084. $(\mathsf{RLD})_{\mathrm{out}} f \sqcup (\mathsf{RLD})_{\mathrm{out}} g = (\mathsf{RLD})_{\mathrm{out}} (f \sqcup g)$ for funcoids f, g.

PROOF. $(\mathsf{RLD})_{\mathrm{out}} f \sqcup (\mathsf{RLD})_{\mathrm{out}} g = \bigsqcap^{\mathsf{RLD}}_{F \in \operatorname{up} f} F \sqcup \bigsqcap^{\mathsf{RLD}}_{G \in \operatorname{up} g} G = \bigsqcap^{\mathsf{RLD}}_{F \in \operatorname{up} f, G \in \operatorname{up} g} (F \sqcup G) = \bigsqcap^{\mathsf{RLD}}_{H \in \operatorname{up}(f \sqcup g)} H = (\mathsf{RLD})_{\mathrm{out}} (f \sqcup g)$. □

THEOREM 1085. $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$ for every funcoid f.

PROOF. For every typed sets $X \in \mathscr{T}(\operatorname{Src} f)$, $Y \in \mathscr{T}(\operatorname{Dst} f)$

$$\begin{gathered} X \; [(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f]^* \; Y \Leftrightarrow \\ X \times^{\mathsf{RLD}} Y \not\asymp (\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow \\ \uparrow^{\mathsf{RLD}} (X \times Y) \not\asymp \bigsqcup \left\{ \frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq f} \right\} \Leftrightarrow (*) \\ \exists a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)} : (a \times^{\mathsf{FCD}} b \sqsubseteq f \wedge a \sqsubseteq X \wedge b \sqsubseteq Y) \Leftrightarrow \\ X \; [f]^* \; Y. \end{gathered}$$

* theorem 583.

Thus $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$. □

REMARK 1086. The above theorem allows to represent funcoids as reloids ($(\mathsf{RLD})_{\mathrm{in}} f$ is the reloid representing funcoid f). Refer to the section "Funcoidal reloids" below for more details.

OBVIOUS 1087. $(\mathsf{RLD})_{\mathrm{in}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for every filters $\mathcal{A}$, $\mathcal{B}$.

CONJECTURE 1088. $(\mathsf{RLD})_{\mathrm{out}} \operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}} = \operatorname{id}^{\mathsf{RLD}}_{\mathcal{A}}$ for every filter $\mathcal{A}$.

EXERCISE 1089. Prove that generally $(\mathsf{RLD})_{\mathrm{in}} \operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}} \neq \operatorname{id}^{\mathsf{RLD}}_{\mathcal{A}}$. I call $(\mathsf{RLD})_{\mathrm{in}} \operatorname{id}^{\mathsf{FCD}}_{\mathcal{A}}$ *thick identity* or *thick diagonal*, because it is greater ("thicker") than identity $\operatorname{id}^{\mathsf{RLD}}_{\mathcal{A}}$.

PROPOSITION 1090. $\operatorname{dom} (\mathsf{RLD})_{\mathrm{in}} f = \operatorname{dom} f$ and $\operatorname{im} (\mathsf{RLD})_{\mathrm{in}} f = \operatorname{im} f$ for every funcoid f.

PROOF. We will prove only $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}} f = \operatorname{dom} f$ as the other formula follows from symmetry. Really:
$\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}} f = \operatorname{dom} \bigsqcup \left\{ \frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}, a \times^{\mathsf{FCD}} b \sqsubseteq f} \right\}$.
By corollary 1033 we have

$$\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}} f = \bigsqcup \left\{ \frac{\operatorname{dom}(a \times^{\mathsf{RLD}} b)}{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}, a \times^{\mathsf{FCD}} b \sqsubseteq f} \right\} = \bigsqcup \left\{ \frac{\operatorname{dom}(a \times^{\mathsf{FCD}} b)}{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}, a \times^{\mathsf{FCD}} b \sqsubseteq f} \right\}.$$

By corollary 896 we have

$$\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}} f = \operatorname{dom} \bigsqcup \left\{ \frac{a \times^{\mathsf{FCD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}, a \times^{\mathsf{FCD}} b \sqsubseteq f} \right\} = \operatorname{dom} f.$$

□

PROPOSITION 1091. $\operatorname{dom}(f|_{\mathcal{A}}) = \mathcal{A} \sqcap \operatorname{dom} f$ for every reloid f and filter $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$.

PROOF. $\operatorname{dom}(f|_{\mathcal{A}}) = \operatorname{dom}(\mathsf{FCD})(f|_{\mathcal{A}}) = \operatorname{dom}((\mathsf{FCD})f)|_{\mathcal{A}} = \mathcal{A} \sqcap \operatorname{dom}(\mathsf{FCD})f = \mathcal{A} \sqcap \operatorname{dom} f$. □

THEOREM 1092. For every composable reloids f, g:

1°. If $\operatorname{im} f \sqsupseteq \operatorname{dom} g$ then $\operatorname{im}(g \circ f) = \operatorname{im} g$;
2°. If $\operatorname{im} f \sqsubseteq \operatorname{dom} g$ then $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

PROOF.

1°. $\operatorname{im}(g \circ f) = \operatorname{im}(\mathsf{FCD})(g \circ f) = \operatorname{im}((\mathsf{FCD})g \circ (\mathsf{FCD})f) = \operatorname{im}(\mathsf{FCD})g = \operatorname{im} g$.
2°. Similar.

□

LEMMA 1093. If a, b, c are filters on powersets and $b \neq \bot$, then

$$\bigsqcup^{\mathsf{RLD}} \left\{ \frac{G \circ F}{F \in \operatorname{atoms}(a \times^{\mathsf{RLD}} b), G \in \operatorname{atoms}(b \times^{\mathsf{RLD}} c)} \right\} = a \times^{\mathsf{RLD}} c.$$

PROOF.

$$a \times^{\mathsf{RLD}} c = (b \times^{\mathsf{RLD}} c) \circ (a \times^{\mathsf{RLD}} b) = (\text{corollary } 1011) = \bigsqcup^{\mathsf{RLD}} \left\{ \frac{G \circ F}{F \in \operatorname{atoms}(a \times^{\mathsf{RLD}} b), G \in \operatorname{atoms}(b \times^{\mathsf{RLD}} c)} \right\}.$$

□

THEOREM 1094. $a \times^{\mathsf{RLD}} b \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow a \times^{\mathsf{FCD}} b \sqsubseteq f$ for every funcoid f and $a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}$, $b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}$.

PROOF. $a \times^{\mathsf{FCD}} b \sqsubseteq f \Rightarrow a \times^{\mathsf{RLD}} b \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f$ is obvious.

$$a \times^{\mathsf{RLD}} b \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f \Rightarrow a \times^{\mathsf{RLD}} b \not\asymp (\mathsf{RLD})_{\mathrm{in}} f \Rightarrow$$
$$a \left[(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f\right] b \Rightarrow a [f] b \Rightarrow a \times^{\mathsf{FCD}} b \sqsubseteq f.$$

□

CONJECTURE 1095. If $\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f$ then $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f$ for every funcoid f and $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathscr{F}(\operatorname{Dst} f)$.

THEOREM 1096. $\operatorname{up}(\mathsf{FCD})g \supseteq \operatorname{up} g$ for every reloid g.

PROOF. Let $K \in \operatorname{up} g$. Then for every typed sets $X \in \mathscr{T} \operatorname{Src} g$, $Y \in \mathscr{T} \operatorname{Dst} g$

$$X [K]^* Y \Leftrightarrow X \left[\uparrow^{\mathsf{FCD}} K\right]^* Y \Leftrightarrow X \left[(\mathsf{FCD}) \uparrow^{\mathsf{RLD}} K\right]^* Y \Leftarrow X [(\mathsf{FCD})g]^* Y.$$

Thus $\uparrow^{\mathsf{FCD}} K \sqsupseteq (\mathsf{FCD})g$ that is $K \in \operatorname{up}(\mathsf{FCD})g$. □

THEOREM 1097. $g \circ (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \circ f = \left\langle (\mathsf{FCD})f^{-1} \right\rangle \mathcal{A} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})g \rangle \mathcal{B}$ for every reloids f, g and filters $\mathcal{A} \in \mathscr{F}(\operatorname{Dst} f)$, $\mathcal{B} \in \mathscr{F}(\operatorname{Src} g)$.

PROOF.

$$g \circ (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \circ f =$$
$$\bigsqcap^{\mathsf{RLD}} \left\{ \frac{G \circ (A \times B) \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g, A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}} \right\} =$$
$$\bigsqcap^{\mathsf{RLD}} \left\{ \frac{\left\langle F^{-1} \right\rangle^* A \times \langle G \rangle^* B}{F \in \operatorname{up} f, G \in \operatorname{up} g, A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}} \right\} =$$
$$\bigsqcap^{\mathsf{RLD}} \left\{ \frac{\left\langle F^{-1} \right\rangle^* A \times^{\mathsf{RLD}} \langle G \rangle^* B}{F \in \operatorname{up} f, G \in \operatorname{up} g, A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}} \right\} =$$
(theorem 1016)
$$\bigsqcap^{\mathscr{F}} \left\{ \frac{\left\langle F^{-1} \right\rangle^* A}{F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A}} \right\} \times^{\mathsf{RLD}} \bigsqcap^{\mathscr{F}} \left\{ \frac{\langle G \rangle^* B}{G \in \operatorname{up} g, B \in \operatorname{up} \mathcal{B}} \right\} =$$
$$\bigsqcap^{\mathscr{F}} \left\{ \frac{\left\langle \uparrow^{\mathsf{FCD}} F^{-1} \right\rangle^* A}{F \in \operatorname{up} f, A \in \operatorname{up} \mathcal{A}} \right\} \times^{\mathsf{RLD}} \bigsqcap^{\mathscr{F}} \left\{ \frac{\left\langle \uparrow^{\mathsf{FCD}} G \right\rangle^* B}{G \in \operatorname{up} g, B \in \operatorname{up} \mathcal{B}} \right\} =$$
$$\bigsqcap^{\mathscr{F}} \left\{ \frac{\left\langle \uparrow^{\mathsf{FCD}} F^{-1} \right\rangle \mathcal{A}}{F \in \operatorname{up} f} \right\} \times^{\mathsf{RLD}} \bigsqcap^{\mathscr{F}} \left\{ \frac{\left\langle \uparrow^{\mathsf{FCD}} G \right\rangle \mathcal{B}}{G \in \operatorname{up} g} \right\} =$$
(by definition of (FCD))
$$\left\langle (\mathsf{FCD})f^{-1} \right\rangle \mathcal{A} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})g \rangle \mathcal{B}.$$

□

COROLLARY 1098.
1°. $(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \circ f = \left\langle (\mathsf{FCD})f^{-1} \right\rangle \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$;
2°. $g \circ (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})g \rangle \mathcal{B}$.

9.3. Galois connections between funcoids and reloids

THEOREM 1099. $(\mathsf{FCD}) : \mathsf{RLD}(A, B) \to \mathsf{FCD}(A, B)$ is the lower adjoint of $(\mathsf{RLD})_{\mathrm{in}} : \mathsf{FCD}(A, B) \to \mathsf{RLD}(A, B)$ for every sets A, B.

PROOF. Because (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ are trivially monotone, it's enough to prove (for every $f \in \mathsf{RLD}(A, B)$, $g \in \mathsf{FCD}(A, B)$)

$$f \sqsubseteq (\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})f \quad \text{and} \quad (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}g \sqsubseteq g.$$

The second formula follows from the fact that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}g = g$.

$$(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})f =$$

$$\bigsqcup\left\{\frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq (\mathsf{FCD})f}\right\} =$$

$$\bigsqcup\left\{\frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)}, a \, [(\mathsf{FCD})f] \, b}\right\} =$$

$$\bigsqcup\left\{\frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)}, a \times^{\mathsf{RLD}} b \not\asymp f}\right\} \sqsupseteq$$

$$\bigsqcup\left\{\frac{p \in \operatorname{atoms}(a \times^{\mathsf{RLD}} b)}{a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)}, p \not\asymp f}\right\} =$$

$$\bigsqcup\left\{\frac{p \in \operatorname{atoms}^{\mathsf{RLD}(A,B)}}{p \not\asymp f}\right\} =$$

$$\bigsqcup\left\{\frac{p}{p \in \operatorname{atoms} f}\right\} = f.$$

□

COROLLARY 1100.

1°. $(\mathsf{FCD}) \bigsqcup S = \bigsqcup \langle (\mathsf{FCD}) \rangle^* S$ if $S \in \mathscr{P}\mathsf{RLD}(A, B)$.
2°. $(\mathsf{RLD})_{\mathrm{in}} \bigsqcap S = \bigsqcap \langle (\mathsf{RLD})_{\mathrm{in}} \rangle^* S$ if $S \in \mathscr{P}\mathsf{FCD}(A, B)$.

THEOREM 1101. $(\mathsf{RLD})_{\mathrm{in}}(f \sqcup g) = (\mathsf{RLD})_{\mathrm{in}} f \sqcup (\mathsf{RLD})_{\mathrm{in}} g$ for every funcoids $f, g \in \mathsf{FCD}(A, B)$.

PROOF.

$$(\mathsf{RLD})_{\mathrm{in}}(f \sqcup g) = \bigsqcup\left\{\frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq f \sqcup g}\right\} =$$

$$\bigsqcup\left\{\frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq f \vee a \times^{\mathsf{FCD}} b \sqsubseteq g}\right\} =$$

$$\bigsqcup\left\{\frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq f}\right\} \sqcup$$

$$\bigsqcup\left\{\frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(A)}, b \in \operatorname{atoms}^{\mathscr{F}(B)}, a \times^{\mathsf{FCD}} b \sqsubseteq g}\right\} =$$

$$(\mathsf{RLD})_{\mathrm{in}} f \sqcup (\mathsf{RLD})_{\mathrm{in}} g.$$

□

PROPOSITION 1102. $(\mathsf{RLD})_{\mathrm{in}}(f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})) = ((\mathsf{RLD})_{\mathrm{in}} f) \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$ for every funcoid f and $\mathcal{A} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathscr{F}(\operatorname{Dst} f)$.

PROOF.

$$(\mathsf{RLD})_{\mathrm{in}}(f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})) = ((\mathsf{RLD})_{\mathrm{in}} f) \sqcap (\mathsf{RLD})_{\mathrm{in}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = ((\mathsf{RLD})_{\mathrm{in}} f) \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}).$$

□

COROLLARY 1103. $(\mathsf{RLD})_{\mathrm{in}}(f|_{\mathcal{A}}) = ((\mathsf{RLD})_{\mathrm{in}} f)|_{\mathcal{A}}$.

CONJECTURE 1104. $(\mathsf{RLD})_{\mathrm{in}}$ is not a lower adjoint (in general).

CONJECTURE 1105. $(\mathsf{RLD})_{\mathrm{out}}$ is neither a lower adjoint nor an upper adjoint (in general).

EXERCISE 1106. Prove that $\operatorname{card}\mathsf{FCD}(A,B) = 2^{2^{\max\{A,B\}}}$ if A or B is an infinite set (provided that A and B are nonempty).

LEMMA 1107. $\uparrow^{\mathsf{FCD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\} \sqsubseteq (\mathsf{FCD})g \Leftrightarrow \uparrow^{\mathsf{RLD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\} \sqsubseteq g$ for every reloid g.

PROOF.

$$\uparrow^{\mathsf{FCD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\} \sqsubseteq (\mathsf{FCD})g \Leftrightarrow \uparrow^{\mathsf{FCD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\} \not\asymp (\mathsf{FCD})g \Leftrightarrow @\{x\}\ [(\mathsf{FCD})g]^*\ @\{y\} \Leftrightarrow \uparrow^{\mathsf{RLD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\} \not\asymp g \Leftrightarrow \uparrow^{\mathsf{RLD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\} \sqsubseteq g.$$

□

THEOREM 1108. $\operatorname{Cor}(\mathsf{FCD})g = (\mathsf{FCD})\operatorname{Cor} g$ for every reloid g.

PROOF.

$$\operatorname{Cor}(\mathsf{FCD})g = \bigsqcup\left\{\frac{\uparrow^{\mathsf{FCD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\}}{\uparrow^{\mathsf{FCD}}\{(x,y)\} \sqsubseteq (\mathsf{FCD})g}\right\} = \bigsqcup\left\{\frac{\uparrow^{\mathsf{FCD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\}}{\uparrow^{\mathsf{RLD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\} \sqsubseteq g}\right\} = \bigsqcup\left\{\frac{(\mathsf{FCD})\uparrow^{\mathsf{RLD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\}}{\uparrow^{\mathsf{RLD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\} \sqsubseteq g}\right\} = (\mathsf{FCD})\bigsqcup\left\{\frac{\uparrow^{\mathsf{RLD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\}}{\uparrow^{\mathsf{RLD}(\operatorname{Src} g,\operatorname{Dst} g)}\{(x,y)\} \sqsubseteq g}\right\} = (\mathsf{FCD})\operatorname{Cor} g.$$

□

CONJECTURE 1109.

1°. $\operatorname{Cor}(\mathsf{RLD})_{\mathrm{in}}g = (\mathsf{RLD})_{\mathrm{in}}\operatorname{Cor} g$;
2°. $\operatorname{Cor}(\mathsf{RLD})_{\mathrm{out}}g = (\mathsf{RLD})_{\mathrm{out}}\operatorname{Cor} g$.

THEOREM 1110. For every reloid f:

1°. $\operatorname{Compl}(\mathsf{FCD})f = (\mathsf{FCD})\operatorname{Compl} f$;
2°. $\operatorname{CoCompl}(\mathsf{FCD})f = (\mathsf{FCD})\operatorname{CoCompl} f$.

PROOF. We will prove only the first, because the second is dual.

$$\operatorname{Compl}(\mathsf{FCD})f = \bigsqcup_{\alpha\in\operatorname{Src} f}((\mathsf{FCD})f)|_{\uparrow\{\alpha\}} = (\text{proposition } 1076) = \bigsqcup_{\alpha\in\operatorname{Src} f}(\mathsf{FCD})(f|_{\uparrow\{\alpha\}}) = (\mathsf{FCD})\bigsqcup_{\alpha\in\operatorname{Src} f} f|_{\uparrow\{\alpha\}} = (\mathsf{FCD})\operatorname{Compl} f.$$

□

CONJECTURE 1111.

1°. $\operatorname{Compl}(\mathsf{RLD})_{\mathrm{in}}g = (\mathsf{RLD})_{\mathrm{in}}\operatorname{Compl} g$;
2°. $\operatorname{Compl}(\mathsf{RLD})_{\mathrm{out}}g = (\mathsf{RLD})_{\mathrm{out}}\operatorname{Compl} g$.

Note that the above Galois connection between funcoids and reloids is a Galois surjection.

PROPOSITION 1112. $(\mathsf{RLD})_{\mathrm{in}} g = \max\left\{ \frac{f \in \mathsf{RLD}}{(\mathsf{FCD}) f \sqsubseteq g} \right\} = \max\left\{ \frac{f \in \mathsf{RLD}}{(\mathsf{FCD}) f = g} \right\}$.

PROOF. By theorem 131 and proposition 323. □

9.4. Funcoidal reloids

DEFINITION 1113. I call *funcoidal* such a reloid ν that

$$\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \not\asymp \nu \Rightarrow \exists \mathcal{X}' \in \mathscr{F}(\operatorname{Base}(\mathcal{X})) \setminus \{\bot\}, \mathcal{Y}' \in \mathscr{F}(\operatorname{Base}(\mathcal{Y})) \setminus \{\bot\} : (\mathcal{X}' \sqsubseteq \mathcal{X} \wedge \mathcal{Y}' \sqsubseteq \mathcal{Y} \wedge \mathcal{X}' \times^{\mathsf{RLD}} \mathcal{Y}' \sqsubseteq \nu)$$

for every $\mathcal{X} \in \mathscr{F}(\operatorname{Src} \nu)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} \nu)$.

REMARK 1114. See theorem 1119 below for how they are bijectively related with funcoids (and thus named funcoidal).

PROPOSITION 1115. A reloid ν is funcoidal iff $x \times^{\mathsf{RLD}} y \not\asymp \nu \Rightarrow x \times^{\mathsf{RLD}} y \sqsubseteq \nu$ for every atomic filter objects x and y on respective sets.

PROOF.

$\Rightarrow$. $x \times^{\mathsf{RLD}} y \not\asymp \nu \Rightarrow \exists \mathcal{X}' \in \operatorname{atoms} x, \mathcal{Y}' \in \operatorname{atoms} y : \mathcal{X}' \times^{\mathsf{RLD}} \mathcal{Y}' \sqsubseteq \nu \Rightarrow x \times^{\mathsf{RLD}} y \sqsubseteq \nu$.

$\Leftarrow$.

$$\begin{aligned}
&\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \not\asymp \nu \Rightarrow \\
&\exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y} : x \times^{\mathsf{RLD}} y \not\asymp \nu \Rightarrow \\
&\exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y} : x \times^{\mathsf{RLD}} y \sqsubseteq \nu \Rightarrow \\
&\exists \mathcal{X}' \in \mathscr{F}(\operatorname{Base}(\mathcal{X})) \setminus \{\bot\}, \mathcal{Y}' \in \mathscr{F}(\operatorname{Base}(\mathcal{Y})) \setminus \{\bot\} : \\
&\qquad (\mathcal{X}' \sqsubseteq \mathcal{X} \wedge \mathcal{Y}' \sqsubseteq \mathcal{Y} \wedge \mathcal{X}' \times^{\mathsf{RLD}} \mathcal{Y}' \sqsubseteq \nu).
\end{aligned}$$

□

PROPOSITION 1116.

$$(\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f = \bigsqcup \left\{ \frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} \nu)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} \nu)}, a \times^{\mathsf{RLD}} b \not\asymp f} \right\}.$$

PROOF.

$$\begin{aligned}
(\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f &= \\
\bigsqcup \left\{ \frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} \nu)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} \nu)}, a \times^{\mathsf{FCD}} b \sqsubseteq (\mathsf{FCD}) f} \right\} &= \\
\bigsqcup \left\{ \frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} \nu)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} \nu)}, a \mathrel{[(\mathsf{FCD}) f]} b} \right\} &= \\
\bigsqcup \left\{ \frac{a \times^{\mathsf{RLD}} b}{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} \nu)}, b \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} \nu)}, a \times^{\mathsf{RLD}} b \not\asymp f} \right\}&.
\end{aligned}$$

□

DEFINITION 1117. I call $(\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f$ *funcoidization* of a reloid f.

LEMMA 1118. $(\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f$ is funcoidal for every reloid f.

PROOF. $x \times^{\mathsf{RLD}} y \not\asymp (\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f \Rightarrow x \times^{\mathsf{RLD}} y \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f$ for atomic filters x and y. □

THEOREM 1119. $(\mathsf{RLD})_{\mathrm{in}}$ is a bijection from $\mathsf{FCD}(A,B)$ to the set of funcoidal reloids from A to B. The reverse bijection is given by (FCD).

PROOF. Let $f \in \mathsf{FCD}(A,B)$. Prove that $(\mathsf{RLD})_{\mathrm{in}} f$ is funcoidal.

Really $(\mathsf{RLD})_{\mathrm{in}} f = (\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f$ and thus we can use the lemma stating that it is funcoidal.

It remains to prove $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f = f$ for a funcoidal reloid f. $((\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} g = g$ for every funcoid g is already proved above.)

$$\begin{aligned}(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f &=\\ \bigsqcup\left\{\frac{x \times^{\mathsf{RLD}} y}{x \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, y \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}, x \times^{\mathsf{RLD}} y \not\asymp f}\right\} &=\\ \bigsqcup\left\{\frac{p \in \operatorname{atoms}(x \times^{\mathsf{RLD}} y)}{x \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, y \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}, x \times^{\mathsf{RLD}} y \not\asymp f}\right\} &=\\ \bigsqcup\left\{\frac{p \in \operatorname{atoms}(x \times^{\mathsf{RLD}} y)}{x \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}, y \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f)}, x \times^{\mathsf{RLD}} y \sqsubseteq f}\right\} &=\\ \bigsqcup \operatorname{atoms} f = f.\end{aligned}$$

□

COROLLARY 1120. Funcoidal reloids are convex.

PROOF. Every $(\mathsf{RLD})_{\mathrm{in}} f$ is obviously convex. □

THEOREM 1121. $(\mathsf{RLD})_{\mathrm{in}}(g \circ f) = (\mathsf{RLD})_{\mathrm{in}} g \circ (\mathsf{RLD})_{\mathrm{in}} f$ for every composable funcoids f and g.

PROOF.

$$(\mathsf{RLD})_{\mathrm{in}} g \circ (\mathsf{RLD})_{\mathrm{in}} f = (\text{corollary } 1011) = \bigsqcup^{\mathsf{RLD}}\left\{\frac{G \circ F}{F \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f, G \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} g}\right\}$$

Let F be an atom of the poset $\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)$.

$$F \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f \Rightarrow \operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F \not\asymp (\mathsf{RLD})_{\mathrm{in}} f \Rightarrow \\ (\text{because } (\mathsf{RLD})_{\mathrm{in}} f \text{ is a funcoidal reloid}) \Rightarrow \\ \operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f$$

but $\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f \Rightarrow F \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f$ is obvious.

So

$$F \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow \operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f \Rightarrow \\ (\mathsf{FCD})(\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F) \sqsubseteq (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow \operatorname{dom} F \times^{\mathsf{FCD}} \operatorname{im} F \sqsubseteq f.$$

But

$$\operatorname{dom} F \times^{\mathsf{FCD}} \operatorname{im} F \sqsubseteq f \Rightarrow (\mathsf{RLD})_{\mathrm{in}}(\operatorname{dom} F \times^{\mathsf{FCD}} \operatorname{im} F) \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow \\ \operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f.$$

So $F \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow \operatorname{dom} F \times^{\mathsf{FCD}} \operatorname{im} F \sqsubseteq f$.

Let $F \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f$, $G \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} g$. Then $\operatorname{dom} F \times^{\mathsf{FCD}} \operatorname{im} F \sqsubseteq f$ and $\operatorname{dom} G \times^{\mathsf{FCD}} \operatorname{im} G \sqsubseteq g$. Provided that $\operatorname{im} F \not\asymp \operatorname{dom} G$, we have:

$$\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} G = (\operatorname{dom} G \times^{\mathsf{RLD}} \operatorname{im} G) \circ (\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F) =$$

$$\bigsqcup^{\mathsf{RLD}} \left\{ \frac{G' \circ F'}{F' \in \operatorname{atoms}(\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F), G' \in \operatorname{atoms}(\operatorname{dom} G \times^{\mathsf{RLD}} \operatorname{im} G)} \right\} \sqsubseteq (*)$$

$$\bigsqcup^{\mathsf{RLD}} \left\{ \frac{G' \circ F'}{\begin{array}{l} F' \in \operatorname{atoms}^{\mathsf{RLD}(\operatorname{Src} F, \operatorname{Dst} F)}, G' \in \operatorname{atoms}^{\mathsf{RLD}(\operatorname{Src} G, \operatorname{Dst} G)}, \\ F' \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f, G' \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} g \end{array}} \right\} =$$

$$\bigsqcup^{\mathsf{RLD}} \left\{ \frac{G' \circ F'}{F' \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f, G' \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} g} \right\} = (\mathsf{RLD})_{\mathrm{in}} g \circ (\mathsf{RLD})_{\mathrm{in}} f.$$

(*) $F' \in \operatorname{atoms}(\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F)$ and $\operatorname{dom} F \times^{\mathsf{FCD}} \operatorname{im} F \sqsubseteq f$ implies $\operatorname{dom} F' \times^{\mathsf{FCD}} \operatorname{im} F' \sqsubseteq f$; thus $\operatorname{dom} F' \times^{\mathsf{RLD}} \operatorname{im} F' \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f$ and thus $F' \sqsubseteq (\mathsf{RLD})_{\mathrm{in}} f$. Likewise for G and G'.

Thus $(\mathsf{RLD})_{\mathrm{in}} g \circ (\mathsf{RLD})_{\mathrm{in}} f \sqsupseteq \bigsqcup^{\mathsf{RLD}} \left\{ \frac{\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} G}{F \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f, G \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} g, \operatorname{im} F \not\asymp \operatorname{dom} G} \right\}$.

But

$$(\mathsf{RLD})_{\mathrm{in}} g \circ (\mathsf{RLD})_{\mathrm{in}} f \sqsubseteq \bigsqcup^{\mathsf{RLD}} \left\{ \frac{(\operatorname{dom} G \times^{\mathsf{RLD}} \operatorname{im} G) \circ (\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} F)}{F \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f, G \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} g} \right\} =$$

$$\bigsqcup^{\mathsf{RLD}} \left\{ \frac{\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} G}{F \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f, G \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} g, \operatorname{im} F \not\asymp \operatorname{dom} G} \right\}.$$

Thus

$$(\mathsf{RLD})_{\mathrm{in}} g \circ (\mathsf{RLD})_{\mathrm{in}} f =$$

$$\bigsqcup^{\mathsf{RLD}} \left\{ \frac{\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} G}{F \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} f, G \in \operatorname{atoms}(\mathsf{RLD})_{\mathrm{in}} g, \operatorname{im} F \not\asymp \operatorname{dom} G} \right\} =$$

$$\bigsqcup^{\mathsf{RLD}} \left\{ \frac{\operatorname{dom} F \times^{\mathsf{RLD}} \operatorname{im} G}{\begin{array}{l} F \in \operatorname{atoms}^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)}, G \in \operatorname{atoms}^{\mathsf{RLD}(\operatorname{Dst} f, \operatorname{Dst} g)}, \\ \operatorname{dom} F \times^{\mathsf{FCD}} \operatorname{im} F \sqsubseteq f, \operatorname{dom} G \times^{\mathsf{FCD}} \operatorname{im} G \sqsubseteq g, \operatorname{im} F \not\asymp \operatorname{dom} G \end{array}} \right\}.$$

But

$$(\mathsf{RLD})_{\mathrm{in}}(g \circ f) = \bigsqcup \left\{ \frac{a \times^{\mathsf{RLD}} c}{a \times^{\mathsf{FCD}} c \in \operatorname{atoms}(g \circ f)} \right\} = (\text{proposition } 910) =$$

$$\bigsqcup \left\{ \frac{a \times^{\mathsf{RLD}} c}{\begin{array}{l} a \in \mathscr{F}(\operatorname{Src} f), c \in \mathscr{F}(\operatorname{Dst} g), \\ \exists b \in \mathscr{F}(\operatorname{Dst} f) : (a \times^{\mathsf{FCD}} b \in \operatorname{atoms} f \wedge b \times^{\mathsf{FCD}} c \in \operatorname{atoms} g) \end{array}} \right\} =$$

$$\bigsqcup \left\{ \frac{a \times^{\mathsf{RLD}} c}{\begin{array}{l} a \in \mathscr{F}(\operatorname{Src} f), c \in \mathscr{F}(\operatorname{Dst} g), \\ \exists b_0, b_1 \in \mathscr{F}(\operatorname{Dst} f) : (a \times^{\mathsf{FCD}} b \in \operatorname{atoms} f \wedge b \times^{\mathsf{FCD}} c \in \operatorname{atoms} g \wedge b_0 \not\asymp b_1) \end{array}} \right\}.$$

Now it becomes obvious that $(\mathsf{RLD})_{\mathrm{in}} g \circ (\mathsf{RLD})_{\mathrm{in}} f = (\mathsf{RLD})_{\mathrm{in}}(g \circ f)$. □

9.5. Complete funcoids and reloids

For the proof below assume

$$\theta=\left(\bigsqcup_{x\in\operatorname{Src}f}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{RLD}}\langle f\rangle^*@\{x\})\mapsto\bigsqcup_{x\in\operatorname{Src}f}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{FCD}}\langle f\rangle^*@\{x\})\right)$$

(where f ranges the set of complete funcoids).

Lemma 1122. θ is a bijection from complete reloids into complete funcoids.

Proof. Theorems 931 and 1042. □

Lemma 1123. $(\mathsf{FCD})g=\theta g$ for every complete reloid g.

Proof. Really, $g=\bigsqcup_{x\in\operatorname{Src}f}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{RLD}}\langle f\rangle^*@\{x\})$ for a complete reloid g and thus

$$(\mathsf{FCD})g=\bigsqcup_{x\in\operatorname{Src}f}(\mathsf{FCD})(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{RLD}}\langle f\rangle^*@\{x\})=\bigsqcup_{x\in\operatorname{Src}f}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{FCD}}\langle f\rangle^*@\{x\})=\theta g.$$

□

Lemma 1124. $(\mathsf{RLD})_{\mathrm{out}}f=\theta^{-1}f$ for every complete funcoid f.

Proof. We have $f=\bigsqcup_{x\in\operatorname{Src}f}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{FCD}}\langle f\rangle^*@\{x\})$. We need to prove $(\mathsf{RLD})_{\mathrm{out}}f=\bigsqcup_{x\in\operatorname{Src}f}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{RLD}}\langle f\rangle^*@\{x\})$.

Really, $(\mathsf{RLD})_{\mathrm{out}}f\sqsupseteq\bigsqcup_{x\in\operatorname{Src}f}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{RLD}}\langle f\rangle^*@\{x\})$.

It remains to prove that $\bigsqcup_{x\in\operatorname{Src}f}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{RLD}}\langle f\rangle^*@\{x\})\sqsupseteq(\mathsf{RLD})_{\mathrm{out}}f$.

Let $L\in\operatorname{up}\bigsqcup_{x\in\operatorname{Src}f}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{RLD}}\langle f\rangle^*@\{x\})$. We will prove $L\in\operatorname{up}(\mathsf{RLD})_{\mathrm{out}}f$.

We have

$$L\in\bigcap_{x\in\operatorname{Src}f}\operatorname{up}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{RLD}}\langle f\rangle^*@\{x\}).$$

$\langle L\rangle^*\{x\}=G(x)$ for some $G(x)\in\operatorname{up}\langle f\rangle^*@\{x\}$ (because $L\in\operatorname{up}(\uparrow^{\operatorname{Src}f}\{x\}\times^{\mathsf{RLD}}\langle f\rangle^*@\{x\})$.

Thus $L=G\in\operatorname{up}f$ (because f is complete). Thus $L\in\operatorname{up}f$ and so $L\in\operatorname{up}(\mathsf{RLD})_{\mathrm{out}}f$.

□

Proposition 1125. (FCD) and $(\mathsf{RLD})_{\mathrm{out}}$ form mutually inverse bijections between complete reloids and complete funcoids.

Proof. From two last lemmas. □

Theorem 1126. The diagram at the figure 1 (with the "unnamed" arrow from $\mathsf{ComplRLD}(A,B)$ to $\mathscr{F}(B)^A$ *defined* as the inverse isomorphism of its opposite arrow) is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity (therefore "parallel" arrows are mutually inverse). The arrows preserve order.

Proof. It's proved above, that all morphisms (except the "unnamed" arrow, which is the inverse morphism by definition) depicted on the diagram are bijections and the depicted "opposite" morphisms are mutually inverse.

That arrows preserve order is obvious.

$$\begin{tikzcd}
& \mathscr{F}(B)^A \arrow[dl, shift left, "G\mapsto\bigsqcup\left\{\frac{\{\alpha\}\times^{\mathsf{FCD}}G(\alpha)}{\alpha\in A}\right\}"] \arrow[dr, shift left] & \\
\mathsf{ComplFCD}(A,B) \arrow[ur, shift left, "f\mapsto(\alpha\mapsto\langle f\rangle^*\{\alpha\})"] \arrow[rr, shift left, "(\mathsf{RLD})_{\mathrm{out}}"] & & \mathsf{ComplRLD}(A,B) \arrow[ll, shift left, "(\mathsf{FCD})"] \arrow[ul, shift left, "G\mapsto\bigsqcup\left\{\frac{\{\alpha\}\times^{\mathsf{RLD}}G(\alpha)}{\alpha\in A}\right\}"']
\end{tikzcd}$$

FIGURE 8

It remains to apply lemma 196 (taking into account that θ can be decomposed into $\left(G\mapsto\bigsqcup\left\{\frac{\{\alpha\}\times^{\mathsf{RLD}}G(\alpha)}{\alpha\in A}\right\}\right)^{-1}$ and $G\mapsto\bigsqcup\left\{\frac{\{\alpha\}\times^{\mathsf{FCD}}G(\alpha)}{\alpha\in A}\right\}$). □

THEOREM 1127. Composition of complete reloids is complete.

PROOF. Let f, g be complete reloids. Then $(\mathsf{FCD})(g\circ f)=(\mathsf{FCD})g\circ(\mathsf{FCD})f$. Thus (because $(\mathsf{FCD})(g\circ f)$ is a complete funcoid) we have $g\circ f=(\mathsf{RLD})_{\mathrm{out}}((\mathsf{FCD})g\circ(\mathsf{FCD})f)$, but $(\mathsf{FCD})g\circ(\mathsf{FCD})f$ is a complete funcoid, thus $g\circ f$ is a complete reloid. □

THEOREM 1128.

1°. $(\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{RLD})_{\mathrm{out}}f=(\mathsf{RLD})_{\mathrm{out}}(g\circ f)$ for composable complete funcoids f and g.
2°. $(\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{RLD})_{\mathrm{out}}f=(\mathsf{RLD})_{\mathrm{out}}(g\circ f)$ for composable co-complete funcoids f and g.

PROOF. Let f, g be composable complete funcoids.
$(\mathsf{FCD})((\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{RLD})_{\mathrm{out}}f)=(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}}f=g\circ f$.
Thus (taking into account that $(\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{RLD})_{\mathrm{out}}f$ is complete) we have $(\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{RLD})_{\mathrm{out}}f=(\mathsf{RLD})_{\mathrm{out}}(g\circ f)$.
For co-complete funcoids it's dual. □

PROPOSITION 1129. If f is a (co-)complete funcoid then $\operatorname{up}f$ is a filter.

PROOF. It is enough to consider the case if f is complete.
We need to prove that $\forall F,G\in\operatorname{up}f:F\sqcap G\in\operatorname{up}f$.
For every $F\in\mathbf{Rel}(\operatorname{Src}f,\operatorname{Dst}f)$ we have
$F\in\operatorname{up}f\Leftrightarrow F\sqsupseteq f\Leftrightarrow\langle F\rangle^*\{x\}\sqsupseteq\langle f\rangle^*\{x\}$.
Thus $F,G\in\operatorname{up}f\Rightarrow\langle F\rangle^*\{x\}\sqsupseteq\langle f\rangle^*\{x\}\wedge\langle G\rangle^*\{x\}\sqsupseteq\langle f\rangle^*\{x\}\Rightarrow\langle F\sqcap G\rangle^*\{x\}=\langle F\rangle^*\{x\}\sqcap\langle G\rangle^*\{x\}\sqsupseteq\langle f\rangle^*\{x\}\Rightarrow F\sqcap G\in\operatorname{up}f$.
That $\operatorname{up}f$ is nonempty and up-directed is obvious. □

COROLLARY 1130.

1°. If f is a (co-)complete funcoid then $\operatorname{up}f=\operatorname{up}(\mathsf{RLD})_{\mathrm{out}}f$.

2°. If f is a (co-)complete reloid then $\operatorname{up} f = \operatorname{up}(\mathsf{FCD})f$.

PROOF. By order isomorphism, it is enough to prove the first.

Because $\operatorname{up} f$ is a filter, by properties of generalized filter bases we have $F \in \operatorname{up}(\mathsf{RLD})_{\mathrm{out}} f \Leftrightarrow \exists g \in \operatorname{up} f : F \sqsupseteq g \Leftrightarrow F \in \operatorname{up} f$. □

9.6. Properties preserved by relationships

PROPOSITION 1131. $(\mathsf{FCD})f$ is reflexive iff f is reflexive (for every endoreloid f).

PROOF. f is reflexive $\Leftrightarrow 1^{\mathbf{Rel}}_{\operatorname{Ob} f} \sqsubseteq f \Leftrightarrow \forall F \in \operatorname{up} f : 1^{\mathbf{Rel}}_{\operatorname{Ob} f} \sqsubseteq F \Leftrightarrow 1^{\mathbf{Rel}}_{\operatorname{Ob} f} \sqsubseteq \bigsqcap^{\mathsf{FCD}} \operatorname{up} f \Leftrightarrow 1^{\mathbf{Rel}}_{\operatorname{Ob} f} \sqsubseteq (\mathsf{FCD})f \Leftrightarrow (\mathsf{FCD})f$ is reflexive. □

PROPOSITION 1132. $(\mathsf{RLD})_{\mathrm{out}} f$ is reflexive iff f is reflexive (for every endofuncoid f).

PROOF. f is reflexive $\Leftrightarrow 1^{\mathbf{Rel}}_{\operatorname{Ob} f} \sqsubseteq f \Leftrightarrow$ (corollary 925) $\Leftrightarrow \forall F \in \operatorname{up} f : 1^{\mathbf{Rel}}_{\operatorname{Ob} f} \sqsubseteq F \Leftrightarrow 1^{\mathbf{Rel}}_{\operatorname{Ob} f} \sqsubseteq \bigsqcap^{\mathsf{RLD}} \operatorname{up} f \Leftrightarrow 1^{\mathbf{Rel}}_{\operatorname{Ob} f} \sqsubseteq (\mathsf{RLD})_{\mathrm{out}} f \Leftrightarrow (\mathsf{RLD})_{\mathrm{out}} f$ is reflexive. □

PROPOSITION 1133. $(\mathsf{RLD})_{\mathrm{in}} f$ is reflexive iff f is reflexive (for every endofuncoid f).

PROOF. $(\mathsf{RLD})_{\mathrm{in}} f$ is reflexive iff $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f$ if reflexive iff f is reflexive. □

OBVIOUS 1134. (FCD), $(\mathsf{RLD})_{\mathrm{in}}$, and $(\mathsf{RLD})_{\mathrm{out}}$ preserve symmetry of the argument funcoid or reloid.

PROPOSITION 1135. $a \times^{\mathsf{RLD}}_F a = \bot$ for every nontrivial ultrafilter a.

PROOF. $a \times^{\mathsf{RLD}}_F a = (\mathsf{RLD})_{\mathrm{out}}(a \times^{\mathsf{FCD}} a) = \bigsqcap^{\mathsf{RLD}} \operatorname{up}(a \times^{\mathsf{FCD}} a) \sqsubseteq 1^{\mathsf{FCD}} \sqcap (\top^{\mathsf{FCD}} \setminus 1^{\mathsf{FCD}}) = \bot^{\mathsf{FCD}}$. □

EXAMPLE 1136. There exist filters $\mathcal{A}$ and $\mathcal{B}$ such that $(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}) \sqsubset \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$.

PROOF. Take $\mathcal{A} = \mathcal{B} = a$ for a nontrivial ultrafilter a. $a \times^{\mathsf{RLD}}_F a = \bot$. Thus $(\mathsf{FCD})(a \times^{\mathsf{RLD}}_F a) = \bot \sqsubset a \times^{\mathsf{FCD}} a$. □

CONJECTURE 1137. There exist filters $\mathcal{A}$ and $\mathcal{B}$ such that $(\mathsf{FCD})(\mathcal{A} \ltimes \mathcal{B}) \sqsubset \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$.

EXAMPLE 1138. There is such a non-symmetric reloid f that $(\mathsf{FCD})f$ is symmetric.

PROOF. Take $f = ((\mathsf{RLD})_{\mathrm{in}}(=)|_{\mathbb{R}}) \sqcap (\geq)_{\mathbb{R}}$. f is non-symmetric because $f \not\asymp (>)_{\mathbb{R}}$ but $f \asymp (<)_{\mathbb{R}}$. $(\mathsf{FCD})f = (=)|_{\mathbb{R}}$ because $(=)|_{\mathbb{R}} \sqsubseteq f \sqsubseteq (\mathsf{RLD})_{\mathrm{in}}(=)|_{\mathbb{R}}$. □

PROPOSITION 1139. If $(\mathsf{RLD})_{\mathrm{in}} f$ is symmetric then endofuncoid f is symmetric.

PROOF. Suppose $(\mathsf{RLD})_{\mathrm{in}} f$ is symmetric then $f = (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f$ is symmetric. □

CONJECTURE 1140. If $(\mathsf{RLD})_{\mathrm{out}} f$ is symmetric then endofuncoid f is symmetric.

PROPOSITION 1141. If f is a transitive endoreloid, then $(\mathsf{FCD})f$ is a transitive funcoid.

PROOF. $f = f \circ f$; $(\mathsf{FCD})f = (\mathsf{FCD})(f \circ f)$; $(\mathsf{FCD})f = (\mathsf{FCD})f \circ (\mathsf{FCD})f$. □

CONJECTURE 1142. There exists a non-transitive endoreloid f such that $(\mathsf{FCD})f$ is a transitive funcoid.

PROPOSITION 1143. $(\mathsf{RLD})_{\mathrm{in}} f$ is transitive iff f is transitive (for every endofuncoid f).

PROOF. $f = f \circ f \Rightarrow (\mathsf{RLD})_{\mathrm{in}} f = (\mathsf{RLD})_{\mathrm{in}}(f \circ f) \Leftrightarrow$ (theorem 1121) $\Leftrightarrow (\mathsf{RLD})_{\mathrm{in}} f = (\mathsf{RLD})_{\mathrm{in}} f \circ (\mathsf{RLD})_{\mathrm{in}} f \Rightarrow (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f \circ (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow f = f \circ f$. Thus $f = f \circ f \Leftrightarrow (\mathsf{RLD})_{\mathrm{in}} f \circ (\mathsf{RLD})_{\mathrm{in}} f$. □

CONJECTURE 1144.
1°. There exists such a transitive endofuncoid f, that $(\mathsf{RLD})_{\mathrm{out}} f$ is not a transitive reloid.
2°. There exists such a non-transitive endofuncoid f, that $(\mathsf{RLD})_{\mathrm{out}} f$ is transitive reloid.

9.7. Some sub-posets of funcoids and reloids

PROPOSITION 1145. The following are complete sub-meet-semilattices (that is subsets closed for arbitrary meets) of $\mathsf{RLD}(A, A)$ (for every set A):
1°. symmetric reloids on A;
2°. reflexive reloids on A;
3°. symmetric reflexive reloids on A;
4°. transitive reloids on A;
5°. symmetric reflexive transitive reloids (= reloids of equivalence = uniform spaces) on A.

PROOF. The first three items are obvious.

Fourth: Let S be a set of transitive reloids on A. That is $f \circ f \sqsubseteq f$ for every $f \in S$. Then $(\bigsqcap S) \circ (\bigsqcap S) \sqsubseteq f \circ f \sqsubseteq f$. Consequently $(\bigsqcap S) \circ (\bigsqcap S) \sqsubseteq \bigsqcap S$.

The last item follows from the previous. □

PROPOSITION 1146. The following are complete sub-meet-semilattices (that is subsets closed for arbitrary meets) of $\mathsf{FCD}(A, A)$ (for every set A):
1°. symmetric funcoids on A;
2°. reflexive funcoids on A;
3°. symmetric reflexive funcoids on A;
4°. transitive funcoids on A;
5°. symmetric reflexive transitive funcoids (= funcoids of equivalence = proximity spaces) on A.

PROOF. Analogous. □

Obvious corollaries:

COROLLARY 1147. The following are complete lattices (for every set A):
1°. symmetric reloids on A;
2°. reflexive reloids on A;
3°. symmetric reflexive reloids on A;
4°. transitive reloids on A;
5°. symmetric reflexive transitive reloids (= reloids of equivalence = uniform spaces) on A.

COROLLARY 1148. The following are complete lattices (for every set A):
1°. symmetric funcoids on A;
2°. reflexive funcoids on A;
3°. symmetric reflexive funcoids on A;
4°. transitive funcoids on A;
5°. symmetric reflexive transitive funcoids (= funcoids of equivalence = proximity spaces) on A.

The following conjecture was inspired by theorem 2.2 in [41]:

CONJECTURE 1149. Join of a set S on the lattice of transitive reloids is the join (on the lattice of reloids) of all compositions of finite sequences of elements of S.

The similar question can be asked about uniform spaces.
Does the same hold for funcoids?

9.8. Double filtrators

Below I show that it's possible to describe (FCD), $(\mathsf{RLD})_{\mathrm{out}}$, and $(\mathsf{RLD})_{\mathrm{in}}$ entirely in terms of filtrators (order). This seems not to lead to really interesting results but it's curious.

DEFINITION 1150. *Double filtrator* is a triple $(\mathfrak{A}, \mathfrak{B}, \mathfrak{Z})$ of posets such that $\mathfrak{Z}$ is a sub-poset of both $\mathfrak{A}$ and $\mathfrak{B}$.

In other words, a double filtrator $(\mathfrak{A}, \mathfrak{B}, \mathfrak{Z})$ is a triple such that both $(\mathfrak{A}, \mathfrak{Z})$ and $(\mathfrak{B}, \mathfrak{Z})$ are filtrators.

DEFINITION 1151. *Double filtrator of funcoids and reloids* is $(\mathsf{FCD}, \mathsf{RLD}, \mathbf{Rel})$.

DEFINITION 1152. $(\mathsf{FCD})f = \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} f$ for $f \in \mathfrak{B}$.

DEFINITION 1153. $(\mathsf{RLD})_{\mathrm{out}} f = \bigsqcap^{\mathfrak{B}} \operatorname{up}^{\mathfrak{Z}} f$ for $f \in \mathfrak{A}$.

DEFINITION 1154. If (FCD) is a lower adjoint, define $(\mathsf{RLD})_{\mathrm{in}}$ as the upper adjoint of (FCD).

9.8.1. Embedding of $\mathfrak{A}$ into $\mathfrak{B}$. In this section we will suppose that (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ form a Galois surjection, that is $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$ for every $f \in \mathfrak{A}$.
Then $(\mathsf{RLD})_{\mathrm{in}}$ is an order embedding from $\mathfrak{A}$ to $\mathfrak{B}$.

9.8.2. One more core part. I this section we will assume that (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ form a Galois surjection and equate $\mathfrak{A}$ with its image by $(\mathsf{RLD})_{\mathrm{in}}$ in $\mathfrak{B}$. We will also assume $(\mathfrak{A}, \mathfrak{Z})$ being a filtered filtrator.

PROPOSITION 1155. $(\mathsf{FCD})f = \operatorname{Cor}^{\mathfrak{A}} f$ for every $f \in \mathfrak{B}$.

PROOF. $\operatorname{Cor}^{\mathfrak{A}} f = \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{A}} f \sqsubseteq \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} f = (\mathsf{FCD})f$. But for every $g \in \operatorname{up}^{\mathfrak{A}} f$ we have $g = \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} g \sqsupseteq \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} f$, thus $\bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{A}} f \sqsupseteq \bigsqcap^{\mathfrak{A}} \operatorname{up}^{\mathfrak{Z}} f$. □

EXAMPLE 1156. $(\mathsf{FCD})f \neq \operatorname{Cor}'^{\mathfrak{A}} f$ for the double filtrator of funcoids and reloids.

PROOF. Consider a nontrivial ultrafiler a and the reloid $f = \mathrm{id}^{\mathsf{RLD}}_a$. $\operatorname{Cor}'^{\mathfrak{A}} f = \operatorname{Cor}'^{\mathsf{FCD}} \mathrm{id}^{\mathsf{RLD}}_a = \bigsqcup^{\mathsf{FCD}} \operatorname{down}^{\mathsf{FCD}} \mathrm{id}^{\mathsf{RLD}}_a = \bigsqcup^{\mathsf{FCD}} \emptyset = \bot^{\mathsf{FCD}} \neq a \times^{\mathsf{FCD}} a = (\mathsf{FCD})\, \mathrm{id}^{\mathsf{RLD}}_a$. □

I leave to a reader's exercise to apply the above theory to complete funcoids and reloids.

CHAPTER 10

On distributivity of composition with a principal reloid

10.1. Decomposition of composition of binary relations

REMARK 1157. Sorry for an unfortunate choice of terminology: "composition" and "decomposition" are unrelated.

The idea of the proof below is that composition of binary relations can be decomposed into two operations: $\otimes$ and dom:

$$g \otimes f = \left\{ \frac{((x,z),y)}{x \mathrel{f} y \wedge y \mathrel{g} z} \right\}.$$

Composition of binary relations can be decomposed: $g \circ f = \operatorname{dom}(g \otimes f)$.

It can be decomposed even further: $g \otimes f = \Theta_0 f \cap \Theta_1 g$ where

$$\Theta_0 f = \left\{ \frac{((x,z),y)}{x \mathrel{f} y, z \in \mho} \right\} \quad \text{and} \quad \Theta_1 f = \left\{ \frac{((x,z),y)}{y \mathrel{f} z, x \in \mho} \right\}.$$

(Here $\mho$ is the Grothendieck universe.)

Now we will do a similar trick with reloids.

10.2. Decomposition of composition of reloids

A similar thing for reloids:

In this chapter we will equate reloids with filters on cartesian products of sets.

For composable reloids f and g we have

$$g \circ f = \\ \bigsqcap^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} g)} \left\{ \frac{G \circ F}{F \in \operatorname{GR} f, G \in \operatorname{GR} g} \right\} = \\ \bigsqcap^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} g)} \left\{ \frac{\operatorname{dom}(G \otimes F)}{F \in \operatorname{GR} f, G \in \operatorname{GR} g} \right\}.$$

LEMMA 1158. $\left\{ \frac{G \otimes F}{F \in \operatorname{GR} f, G \in \operatorname{GR} g} \right\}$ is a filter base.

PROOF. Let $P, Q \in \left\{ \frac{G \otimes F}{F \in \operatorname{GR} f, G \in \operatorname{GR} g} \right\}$. Then $P = G_0 \otimes F_0$, $Q = G_1 \otimes F_1$ for some $F_0, F_1 \in f$, $G_0, G_1 \in g$. Then $F_0 \cap F_1 \in \operatorname{up} f$, $G_0 \cap G_1 \in \operatorname{up} g$ and thus

$$P \cap Q \supseteq (F_0 \cap F_1) \otimes (G_0 \cap G_1) \in \left\{ \frac{G \otimes F}{F \in \operatorname{GR} f, G \in \operatorname{GR} g} \right\}.$$

□

COROLLARY 1159. $\left\{ \frac{\uparrow^{\mathscr{F}(\operatorname{Src} f \times \operatorname{Dst} g)} (G \otimes F)}{F \in \operatorname{GR} f, G \in \operatorname{GR} g} \right\}$ is a generalized filter base.

PROPOSITION 1160. $g \circ f = \operatorname{dom} \bigsqcap^{\mathscr{F}(\operatorname{Src} f \times \operatorname{Dst} g)} \left\{ \frac{G \otimes F}{F \in \operatorname{GR} f, G \in \operatorname{GR} g} \right\}$.

PROOF. $\uparrow^{\mathscr{F}(\mathrm{Src}\, f\times\mathrm{Dst}\, g)}\mathrm{dom}(G\otimes F)\sqsupseteq\mathrm{dom}\bigsqcap^{\mathscr{F}(\mathrm{Src}\, f\times\mathrm{Dst}\, g)}\left\{\frac{G\otimes F}{F\in\mathrm{GR}\, f,G\in\mathrm{GR}\, g}\right\}$. Thus

$$g\circ f\sqsupseteq\mathrm{dom}\bigsqcap^{\mathscr{F}(\mathrm{Src}\, f\times\mathrm{Dst}\, g)}\left\{\frac{G\otimes F}{F\in\mathrm{GR}\, f,G\in\mathrm{GR}\, g}\right\}.$$

Let $X\in\mathrm{up}\,\mathrm{dom}\bigsqcap^{\mathscr{F}(\mathrm{Src}\, f\times\mathrm{Dst}\, g)}\left\{\frac{G\otimes F}{F\in\mathrm{up}\, f,G\in\mathrm{up}\, g}\right\}$. Then there exist Y such that

$$X\times Y\in\mathrm{up}\bigsqcap^{\mathscr{F}(\mathrm{Src}\, f\times\mathrm{Dst}\, g)}\left\{\frac{G\otimes F}{F\in\mathrm{up}\, f,G\in\mathrm{up}\, g}\right\}.$$

So because it is a generalized filter base $X\times Y\supseteq G\otimes F$ for some $F\in\mathrm{up}\, f$, $G\in\mathrm{up}\, g$. Thus $X\in\mathrm{up}\,\mathrm{dom}(G\otimes F)$. $X\in\mathrm{up}(g\circ f)$. □

We can define $g\otimes f$ for reloids f, g:

$$g\otimes f=\left\{\frac{G\otimes F}{F\in\mathrm{GR}\, f,G\in\mathrm{GR}\, g}\right\}.$$

Then

$$g\circ f=\bigsqcap^{\mathscr{F}(\mathrm{Src}\, f\times\mathrm{Dst}\, g)}\langle\mathrm{dom}\rangle^*(g\otimes f)=\mathrm{dom}\left\langle\uparrow^{\mathsf{RLD}(\mathrm{Src}\, f\times\mathrm{Dst}\, g,\mho)}\right\rangle^*(g\otimes f).$$

10.3. Lemmas for the main result

LEMMA 1161. $(g\otimes f)\cap(h\otimes f)=(g\cap h)\otimes f$ for binary relations f, g, h.

PROOF.

$$(g\cap h)\otimes f=\Theta_0 f\cap\Theta_1(g\cap h)=\Theta_0 f\cap(\Theta_1 g\cap\Theta_1 h)=\\(\Theta_0 f\cap\Theta_1 g)\cap(\Theta_0 f\cap\Theta_1 h)=(g\otimes f)\cap(h\otimes f).$$

□

LEMMA 1162. Let $F=\uparrow^{\mathsf{RLD}} f$ be a principal reloid (for a **Rel**-morphism f), T be a set of reloids from $\mathrm{Dst}\, F$ to a set V.

$$\bigsqcap_{G\in\mathrm{up}\bigsqcup T}^{\mathsf{RLD}(\mathrm{Src}\, f\times V,\mho)}(G\otimes f)=\bigsqcup_{G\in T}\bigsqcap^{\mathsf{RLD}(\mathrm{Src}\, f\times V,\mho)}(G\otimes F).$$

PROOF. $\bigsqcap_{G\in\mathrm{up}\bigsqcup T}^{\mathsf{RLD}(\mathrm{Src}\, f\times V,\mho)}(G\otimes f)\sqsupseteq\bigsqcup_{G\in T}\bigsqcap^{\mathsf{RLD}(\mathrm{Src}\, f\times V,\mho)}(G\otimes F)$ is obvious.

Let $K\in\mathrm{up}\bigsqcup_{G\in T}\bigsqcap^{\mathsf{RLD}(\mathrm{Src}\, f\times V,\mho)}(G\otimes F)$. Then for each $G\in T$

$$K\in\mathrm{up}\bigsqcap^{\mathsf{RLD}(\mathrm{Src}\, f\times V,\mho)}(G\otimes F);$$

$K\in\mathrm{up}\bigsqcap^{\mathsf{RLD}(\mathrm{Src}\, f\times V,\mho)}\left\{\frac{\Gamma\otimes f}{\Gamma\in G}\right\}$. Then $K\in\left\{\frac{\Gamma\otimes f}{\Gamma\in G}\right\}$ by properties of generalized filter bases.

$K\in\left\{\frac{(\Gamma_0\cap\dots\cap\Gamma_n)\otimes f}{n\in\mathbb{N},\Gamma_i\in G}\right\}=\left\{\frac{(\Gamma_0\otimes f)\cap\dots\cap(\Gamma_n\otimes f)}{n\in\mathbb{N},\Gamma_i\in G}\right\}$.

$\forall G\in T:K\supseteq(\Gamma_{G,0}\otimes f)\cap\dots\cap(\Gamma_{G,n}\otimes f)$ for some $n\in\mathbb{N}$, $\Gamma_{G,i}\in G$.

$K\supseteq\left\{\frac{(\Gamma_0\otimes f)\cap\dots\cap(\Gamma_n\otimes f)}{n\in\mathbb{N},\Gamma_i\in G}\right\}$ where $\Gamma_i=\bigcup_{g\in G}\Gamma_{g,i}\in\mathrm{up}\bigsqcup T$.

$K \in \left\{ \frac{(\Gamma_0 \otimes f) \cap \cdots \cap (\Gamma_n \otimes f)}{n \in \mathbb{N}} \right\}$. So

$$K \in \left\{ \frac{(\Gamma'_0 \otimes f) \cap \cdots \cap (\Gamma'_n \otimes f)}{n \in \mathbb{N}, \Gamma'_i \in \operatorname{up} \bigsqcup T} \right\} = \left\{ \frac{(\Gamma'_0 \cap \cdots \cap \Gamma'_n) \otimes f}{n \in \mathbb{N}, \Gamma'_i \in \operatorname{up} \bigsqcup T} \right\} = \operatorname{up} \prod^{\mathsf{RLD}(\operatorname{Src} f \times V, \mho)} \left\{ \frac{G \otimes f}{G \in \operatorname{up} \bigsqcup T} \right\}.$$

□

10.4. Proof of the main result

Let's prove a special case of conjecture 1058:

THEOREM 1163. $(\bigsqcup T) \circ F = \bigsqcup \left\{ \frac{G \circ F}{G \in T} \right\}$ for every principal reloid $F = \uparrow^{\mathsf{RLD}} f$ (for a **Rel**-morphism f) and a set T of reloids from $\operatorname{Dst} F$ to some set V. (In other words principal reloids are co-metacomplete and thus also metacomplete by duality.)

PROOF.

$$\left(\bigsqcup T\right) \circ F = \prod^{\mathsf{RLD}(\operatorname{Src} f, V)} \langle \operatorname{dom} \rangle^* \left(\left(\bigsqcup T\right) \otimes F\right) = \operatorname{dom} \prod^{\mathsf{RLD}(\operatorname{Src} f \times V, \mho)} \left(\left(\bigsqcup T\right) \otimes F\right) = \operatorname{dom} \prod^{\mathsf{RLD}(\operatorname{Src} f \times V, \mho)}_{G \in \operatorname{up} \bigsqcup T} (G \otimes f);$$

$$\bigsqcup_{G \in T} (G \circ F) = \bigsqcup_{G \in T} \prod^{\mathsf{RLD}(\operatorname{Src} f, V)} \langle \operatorname{dom} \rangle^* (G \otimes F) = \bigsqcup_{G \in T} \operatorname{dom} \prod^{\mathsf{RLD}(\operatorname{Src} f \times V, \mho)} (G \otimes F) = \operatorname{dom} \bigsqcup_{G \in T} \prod^{\mathsf{RLD}(\operatorname{Src} f \times V, \mho)} (G \otimes F).$$

It's enough to prove

$$\prod^{\mathsf{RLD}(\operatorname{Src} f \times V, \mho)}_{G \in \operatorname{up} \bigsqcup T} (G \otimes f) = \bigsqcup_{G \in T} \prod^{\mathsf{RLD}(\operatorname{Src} f \times V, \mho)} (G \otimes F)$$

but this is the statement of the lemma. □

10.5. Embedding reloids into funcoids

DEFINITION 1164. Let f be a reloid. The funcoid

$$\rho f = \mathsf{FCD}(\mathscr{P}(\operatorname{Src} f \times \operatorname{Src} f), \mathscr{P}(\operatorname{Dst} f \times \operatorname{Dst} f))$$

is defined by the formulas:

$$\langle \rho f \rangle x = f \circ x \quad \text{and} \quad \left\langle \rho f^{-1} \right\rangle y = f^{-1} \circ y$$

where x are endoreloids on $\operatorname{Src} f$ and y are endoreloids on $\operatorname{Dst} f$.

PROPOSITION 1165. It is really a funcoid (if we equate reloids x and y with corresponding filters on Cartesian products of sets).

PROOF. $y \not\asymp \langle \rho f \rangle x \Leftrightarrow y \not\asymp f \circ x \Leftrightarrow f^{-1} \circ y \not\asymp x \Leftrightarrow \left\langle \rho f^{-1} \right\rangle y \not\asymp x$. □

COROLLARY 1166. $(\rho f)^{-1} = \rho f^{-1}$.

DEFINITION 1167. It can be continued to arbitrary funcoids x having destination $\operatorname{Src} f$ by the formula $\langle \rho^* f \rangle x = \langle \rho f \rangle \operatorname{id}_{\operatorname{Src} f} \circ x = f \circ x$.

PROPOSITION 1168. ρ is an injection.

PROOF. Consider $x = \operatorname{id}_{\operatorname{Src} f}$. □

PROPOSITION 1169. $\rho(g \circ f) = (\rho g) \circ (\rho f)$.

PROOF. $\langle \rho(g \circ f) \rangle x = g \circ f \circ x = \langle \rho g \rangle \langle \rho f \rangle x = (\langle \rho g \rangle \circ \langle \rho f \rangle) x$. Thus $\langle \rho(g \circ f) \rangle = \langle \rho g \rangle \circ \langle \rho f \rangle = \langle (\rho g) \circ (\rho f) \rangle$ and so $\rho(g \circ f) = (\rho g) \circ (\rho f)$. □

THEOREM 1170. $\rho \bigsqcup F = \bigsqcup \langle \rho \rangle^* F$ for a set F of reloids.

PROOF. It's enough to prove $\langle \rho \bigsqcup F \rangle^* X = \left\langle \bigsqcup \langle \rho \rangle^* F \right\rangle^* X$ for a set X.
Really,

$$\begin{aligned}
\left\langle \rho \bigsqcup F \right\rangle^* X &= \\
\left\langle \rho \bigsqcup F \right\rangle \uparrow X &= \\
\bigsqcup F \circ \uparrow X &= \\
\bigsqcup \left\{ \frac{f \circ \uparrow X}{f \in F} \right\} &= \\
\bigsqcup \left\{ \frac{\langle \rho f \rangle \uparrow X}{f \in F} \right\} &= \\
\left\langle \bigsqcup \left\{ \frac{\rho f}{f \in F} \right\} \right\rangle X &= \\
\left\langle \bigsqcup \langle \rho \rangle^* F \right\rangle^* X.&
\end{aligned}$$

□

CONJECTURE 1171. $\rho \bigsqcap F = \bigsqcap \langle \rho \rangle^* F$ for a set F of reloids.

PROPOSITION 1172. $\rho 1_A^{\mathsf{RLD}} = 1_{\mathscr{P}(A \times A)}^{\mathsf{FCD}}$.

PROOF. $\left\langle \rho 1_A^{\mathsf{RLD}} \right\rangle x = 1_A^{\mathsf{RLD}} \circ x = x = \left\langle 1_{\mathscr{P}(A \times A)}^{\mathsf{FCD}} \right\rangle x$. □

We can try to develop further theory by applying embedding of reloids into funcoids for researching of properties of reloids.

THEOREM 1173. Reloid f is monovalued iff funcoid ρf is monovalued.

Proof.

$$
\begin{aligned}
\rho f \text{ is monovalued} &\Leftrightarrow \\
(\rho f) \circ (\rho f)^{-1} \sqsubseteq 1_{\operatorname{Dst} \rho f} &\Leftrightarrow \\
\rho (f \circ f^{-1}) \sqsubseteq 1_{\operatorname{Dst} \rho f} &\Leftrightarrow \\
\rho (f \circ f^{-1}) \sqsubseteq 1^{\mathsf{FCD}}_{\mathscr{P}(\operatorname{Dst} f \times \operatorname{Dst} f)} &\Leftrightarrow \\
\rho (f \circ f^{-1}) \sqsubseteq \rho 1^{\mathsf{RLD}}_{\operatorname{Dst} f} &\Leftrightarrow \\
f \circ f^{-1} \sqsubseteq 1^{\mathsf{RLD}}_{\operatorname{Dst} f} &\Leftrightarrow \\
f \text{ is monovalued.} &
\end{aligned}
$$

□

CHAPTER 11

Continuous morphisms

This chapter uses the apparatus from the section "Partially ordered dagger categories".

11.1. Traditional definitions of continuity

In this section we will show that having a funcoid or reloid $\uparrow f$ corresponding to a function f we can express continuity of it by the formula $\uparrow f \circ \mu \sqsubseteq \nu \circ \uparrow f$ (or similar formulas) where μ and ν are some spaces.

11.1.1. Pretopology. Let (A, cl_A) and (B, cl_B) be preclosure spaces. Then by definition a function $f : A \to B$ is continuous iff $f\,\mathrm{cl}_A(X) \subseteq \mathrm{cl}_B(fX)$ for every $X \in \mathscr{P}A$. Let now μ and ν be endofuncoids corresponding correspondingly to cl_A and cl_B. Then the condition for continuity can be rewritten as

$$\uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)} f \circ \mu \sqsubseteq \nu \circ \uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)} f.$$

11.1.2. Proximity spaces. Let μ and ν be proximity spaces (which I consider a special case of endofuncoids). By definition a **Set**-morphism f is a proximity-continuous map from μ to ν iff

$$\forall X, Y \in \mathscr{T}(\mathrm{Ob}\,\mu) : (X\ [\mu]^*\ Y \Rightarrow \langle f \rangle^* X\ [\nu]^*\ \langle f \rangle^* Y).$$

Equivalently transforming this formula we get

$$\forall X, Y \in \mathscr{T}(\mathrm{Ob}\,\mu) : (X\ [\mu]^*\ Y \Rightarrow \langle f \rangle \uparrow X\ [\nu]\ \langle f \rangle \uparrow Y);$$
$$\forall X, Y \in \mathscr{T}(\mathrm{Ob}\,\mu) : (X\ [\mu]^*\ Y \Rightarrow \uparrow X\ \left[f^{-1} \circ \nu \circ f\right] \uparrow Y);$$
$$\forall X, Y \in \mathscr{T}(\mathrm{Ob}\,\mu) : (X\ [\mu]^*\ Y \Rightarrow X\ \left[f^{-1} \circ \nu \circ f\right]^*\ Y);$$
$$\mu \sqsubseteq f^{-1} \circ \nu \circ f.$$

So a function f is proximity continuous iff $\mu \sqsubseteq f^{-1} \circ \nu \circ f$.

11.1.3. Uniform spaces. Uniform spaces are a special case of endoreloids.

Let μ and ν be uniform spaces. By definition a **Set**-morphism f is a uniformly continuous map from μ to ν iff

$$\forall \varepsilon \in \mathrm{up}\,\nu \exists \delta \in \mathrm{up}\,\nu \forall (x, y) \in \delta : (fx, fy) \in \varepsilon.$$

Equivalently transforming this formula we get:

$$\forall \epsilon \in \mathrm{up}\,\nu \exists \delta \in \mathrm{up}\,\mu \forall (x, y) \in \delta : \{(fx, fy)\} \subseteq \epsilon;$$
$$\forall \epsilon \in \mathrm{up}\,\nu \exists \delta \in \mathrm{up}\,\mu \forall (x, y) \in \delta : f \circ \{(x, y)\} \circ f^{-1} \subseteq \epsilon;$$
$$\forall \epsilon \in \mathrm{up}\,\nu \exists \delta \in \mathrm{up}\,\mu : f \circ \delta \circ f^{-1} \subseteq \epsilon;$$
$$\forall \epsilon \in \mathrm{up}\,\nu : \uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)} f \circ \mu \circ (\uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)} f)^{-1} \sqsubseteq \uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)} \epsilon;$$
$$\uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)} f \circ \mu \circ (\uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu)} f)^{-1} \sqsubseteq \nu.$$

So a function f is uniformly continuous iff $f \circ \mu \circ f^{-1} \sqsubseteq \nu$.

11.2. Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν be endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in \mathrm{C}(\mu, \nu) \Leftrightarrow f \in \mathrm{Hom}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu) \wedge f \circ \mu \sqsubseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$f \in \mathrm{C}'(\mu, \nu) \Leftrightarrow f \in \mathrm{Hom}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu) \wedge \mu \sqsubseteq f^\dagger \circ \nu \circ f;$$

$$f \in \mathrm{C}''(\mu, \nu) \Leftrightarrow f \in \mathrm{Hom}(\mathrm{Ob}\,\mu, \mathrm{Ob}\,\nu) \wedge f \circ \mu \circ f^\dagger \sqsubseteq \nu.$$

REMARK 1174. In the examples (above) about funcoids and reloids the "dagger functor" is the reverse of a funcoid or reloid, that is $f^\dagger = f^{-1}$.

PROPOSITION 1175. Every of these three definitions of continuity forms a wide sub-precategory (wide subcategory if the original precategory is a category).

PROOF.

C. Let $f \in \mathrm{C}(\mu, \nu)$, $g \in \mathrm{C}(\nu, \pi)$. Then $f \circ \mu \sqsubseteq \nu \circ f$, $g \circ \nu \sqsubseteq \pi \circ g$, $g \circ f \circ \mu \sqsubseteq g \circ \nu \circ f \sqsubseteq \pi \circ g \circ f$. So $g \circ f \in \mathrm{C}(\mu, \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}(\mu, \mu)$ is obvious.

C′. Let $f \in \mathrm{C}'(\mu, \nu)$, $g \in \mathrm{C}'(\nu, \pi)$. Then $\mu \sqsubseteq f^\dagger \circ \nu \circ f$, $\nu \sqsubseteq g^\dagger \circ \pi \circ g$;

$$\mu \sqsubseteq f^\dagger \circ g^\dagger \circ \pi \circ g \circ f; \quad \mu \sqsubseteq (g \circ f)^\dagger \circ \pi \circ (g \circ f).$$

So $g \circ f \in \mathrm{C}'(\mu, \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}'(\mu, \mu)$ is obvious.

C″. Let $f \in \mathrm{C}''(\mu, \nu)$, $g \in \mathrm{C}''(\nu, \pi)$. Then $f \circ \mu \circ f^\dagger \sqsubseteq \nu$, $g \circ \nu \circ g^\dagger \sqsubseteq \pi$;

$$g \circ f \circ \mu \circ f^\dagger \circ g^\dagger \sqsubseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^\dagger \sqsubseteq \pi.$$

So $g \circ f \in \mathrm{C}''(\mu, \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}''(\mu, \mu)$ is obvious.

□

PROPOSITION 1176. For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}'(\mu, \nu) \Rightarrow f \in \mathrm{C}(\mu, \nu) \Rightarrow f \in \mathrm{C}''(\mu, \nu).$$

PROOF. Let $f \in \mathrm{C}'(\mu, \nu)$. Then $\mu \sqsubseteq f^\dagger \circ \nu \circ f$;

$$f \circ \mu \sqsubseteq f \circ f^\dagger \circ \nu \circ f \sqsubseteq 1_{\mathrm{Dst}\,f} \circ \nu \circ f = \nu \circ f; \quad f \in \mathrm{C}(\mu, \nu).$$

Let $f \in \mathrm{C}(\mu, \nu)$. Then $f \circ \mu \sqsubseteq \nu \circ f$;

$$f \circ \mu \circ f^\dagger \sqsubseteq \nu \circ f \circ f^\dagger \sqsubseteq \nu \circ 1_{\mathrm{Dst}\,f} = \nu; \quad f \in \mathrm{C}''(\mu, \nu).$$

□

PROPOSITION 1177. For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in \mathrm{C}''(\mu, \nu) \Rightarrow f \in \mathrm{C}(\mu, \nu) \Rightarrow f \in \mathrm{C}'(\mu, \nu).$$

PROOF. Let $f \in \mathrm{C}''(\mu, \nu)$. Then $f \circ \mu \circ f^\dagger \sqsubseteq \nu$; $f \circ \mu \circ f^\dagger \circ f \sqsubseteq \nu \circ f$; $f \circ \mu \circ 1_{\mathrm{Src}\,f} \sqsubseteq \nu \circ f$; $f \circ \mu \sqsubseteq \nu \circ f$; $f \in \mathrm{C}(\mu, \nu)$.

Let $f \in \mathrm{C}(\mu, \nu)$. Then $f \circ \mu \sqsubseteq \nu \circ f$; $f^\dagger \circ f \circ \mu \sqsubseteq f^\dagger \circ \nu \circ f$; $1_{\mathrm{Src}\,\mu} \circ \mu \sqsubseteq f^\dagger \circ \nu \circ f$; $\mu \sqsubseteq f^\dagger \circ \nu \circ f$; $f \in \mathrm{C}'(\mu, \nu)$. □

For entirely defined monovalued morphisms our three definitions of continuity coincide:

THEOREM 1178. If f is a monovalued and entirely defined morphism of a partially ordered dagger precategory then

$$f \in \mathrm{C}'(\mu,\nu) \Leftrightarrow f \in \mathrm{C}(\mu,\nu) \Leftrightarrow f \in \mathrm{C}''(\mu,\nu).$$

PROOF. From two previous propositions. □

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is proximity-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

THEOREM 1179. If an entirely defined morphism of the category of reloids $f \in \mathrm{C}''(\mu,\nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\mathsf{FCD})f \in \mathrm{C}'((\mathsf{FCD})\mu, (\mathsf{FCD})\nu)$.

EXERCISE 1180. I leave a simple exercise for the reader to prove the last theorem.

THEOREM 1181. Let μ and ν be endomorphisms of some partially ordered dagger precategory and $f \in \operatorname{Hom}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$ be a monovalued, entirely defined morphism. Then

$$f \in \mathrm{C}(\mu,\nu) \Leftrightarrow f \in \mathrm{C}(\mu^\dagger,\nu^\dagger).$$

PROOF. $f\circ\mu \sqsubseteq \nu\circ f \Leftrightarrow \mu \sqsubseteq f^\dagger\circ\nu\circ f \Rightarrow \mu\circ f^\dagger \sqsubseteq f^\dagger\circ\nu\circ f\circ f^\dagger \Rightarrow \mu\circ f^\dagger \sqsubseteq f^\dagger\circ\nu \Leftrightarrow f\circ\mu^\dagger \sqsubseteq \nu^\dagger\circ f \Rightarrow f^\dagger\circ f\circ\mu^\dagger \sqsubseteq f^\dagger\circ\nu^\dagger\circ f \Rightarrow \mu^\dagger \sqsubseteq f^\dagger\circ\nu^\dagger\circ f \Leftrightarrow \mu \sqsubseteq f^\dagger\circ\nu\circ f$.

Thus $f\circ\mu \sqsubseteq \nu\circ f \Leftrightarrow \mu \Leftrightarrow f\circ\mu^\dagger \sqsubseteq \nu^\dagger\circ f$. □

11.3. Continuity for topological spaces

PROPOSITION 1182. The following are pairwise equivalent for funcoids μ, ν and a monovalued, entirely defined morphism $f \in \operatorname{Hom}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$:

1°. $\forall A \in \mathscr{T}\operatorname{Ob}\mu, B \in \operatorname{up}\langle\nu\rangle\langle f\rangle^* A : \langle f^{-1}\rangle^* B \in \operatorname{up}\langle\mu\rangle^* A$.
2°. $f \in \mathrm{C}(\mu,\nu)$.
3°. $f \in \mathrm{C}(\mu^{-1},\nu^{-1})$.

PROOF.

2°⇔3°. By general $f\circ\mu \sqsubseteq \nu\circ f \Leftrightarrow f\circ\mu^\dagger \sqsubseteq \nu^\dagger\circ f$ formula above.

1°⇔2°. 1° is equivalent to $\left\langle\langle f^{-1}\rangle^*\right\rangle^* \operatorname{up}\langle\nu\rangle\langle f\rangle^* A \subseteq \operatorname{up}\langle\mu\rangle^* A$ equivalent to $\langle\nu\rangle\langle f\rangle^* A \sqsupseteq \langle f\rangle\langle\mu\rangle^* A$ (used "Orderings of filters" chapter).

□

COROLLARY 1183. The following are pairwise equivalent for topological spaces μ, ν and a monovalued, entirely defined morphism $f \in \operatorname{Hom}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$:

1°. $\forall x \in \operatorname{Ob}\mu, B \in \operatorname{up}\langle\nu\rangle\langle f\rangle^*\{x\} : \langle f^{-1}\rangle^* B \in \operatorname{up}\langle\mu\rangle^*\{x\}$.
2°. Preimages (by f) of open sets are open.
3°. $f \in \mathrm{C}(\mu,\nu)$ that is $\langle f\rangle\langle\mu\rangle^*\{x\} \sqsubseteq \langle\nu\rangle\langle f\rangle^*\{x\}$ for every $x \in \operatorname{Ob}\mu$.
4°. $f \in \mathrm{C}(\mu^{-1},\nu^{-1})$ that is $\langle f\rangle\langle\mu^{-1}\rangle^* A \sqsubseteq \langle\nu^{-1}\rangle\langle f\rangle^* A$ for every $A \in \mathscr{T}\operatorname{Ob}\mu$.

PROOF. 2° from the previous proposition is equivalent to $\langle f\rangle\langle\mu\rangle^*\{x\} \sqsubseteq \langle\nu\rangle\langle f\rangle^*\{x\}$ equivalent to $\left\langle\langle f^{-1}\rangle^*\right\rangle^* \operatorname{up}\langle\nu\rangle\langle f\rangle^*\{x\} \subseteq \operatorname{up}\langle\mu\rangle^*\{x\}$ for every $x \in \operatorname{Ob}\mu$, equivalent to 1° (used "Orderings of filters" chapter).

It remains to prove 3°⇔2°.

3°⇒2°. Let B be an open set in ν. For every $x \in \langle f^{-1}\rangle^* B$ we have $f(x) \in B$ that is B is a neighborhood of $f(x)$, thus $\langle f^{-1}\rangle^* B$ is a neighborhood of x. We have proved that $\langle f^{-1}\rangle^* B$ is open.

2°⇒3°. Let B be a neighborhood of $f(x)$. Then there is an open neighborhood $B' \subseteq B$ of $f(x)$. $\langle f^{-1}\rangle^* B'$ is open and thus is a neighborhood of x ($x \in \langle f^{-1}\rangle^* B'$ because $f(x) \in B'$). Consequently $\langle f^{-1}\rangle^* B$ is a neighborhood of x.

Alternative proof of 2°⇔4°: http://math.stackexchange.com/a/1855782/4876

□

11.4. $\mathrm{C}(\mu \circ \mu^{-1}, \nu \circ \nu^{-1})$

PROPOSITION 1184. $f \in \mathrm{C}(\mu, \nu) \Rightarrow f \in \mathrm{C}''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1})$ for endofuncoids μ, ν and monovalued funcoid $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$.

PROOF. Let $f \in \mathrm{C}(\mu, \nu)$.
$X \; [f \circ \mu \circ \mu^{-1} \circ f^{-1}]^* \; Z \Leftrightarrow \exists p \in \operatorname{atoms}^{\mathscr{F}} : \left(X \; [\mu^{-1} \circ f^{-1}]^* \; p \wedge p \; [f \circ \mu]^* \; Z\right) \Leftrightarrow \exists p \in \operatorname{atoms}^{\mathscr{F}} : (p \; [f \circ \mu]^* \; X \wedge p \; [f \circ \mu]^* \; Z) \Rightarrow \exists p \in \operatorname{atoms}^{\mathscr{F}} : (p \; [\nu \circ f]^* \; X \wedge p \; [\nu \circ f]^* \; Z) \Leftrightarrow \exists p \in \operatorname{atoms}^{\mathscr{F}} : \left(\langle f\rangle^* p \; [\nu]^* \; X \wedge \langle f\rangle^* p \; [\nu]^* \; Z\right) \Rightarrow X \; [\nu \circ \nu^{-1}]^* \; Z$ (taken into account monovaluedness of f and thus that $\langle f\rangle^* p$ is atomic or least). Thus $f \circ \mu \circ \mu^{-1} \circ f^{-1} \sqsubseteq \nu \circ \nu^{-1}$ that is $f \in \mathrm{C}''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1})$. □

PROPOSITION 1185. $f \in \mathrm{C}''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1}) \Rightarrow f \in \mathrm{C}''(\mu, \nu)$ for complete endofuncoids μ, ν and principal funcoid $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, provided that μ is reflexive, and ν is T_1-separable.

PROOF. $f \in \mathrm{C}''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1}) \Leftrightarrow f \circ \mu \circ \mu^{-1} \circ f^{-1} \sqsubseteq \nu \circ \nu^{-1} \Rightarrow$ (reflexivity of μ) $\Rightarrow f \circ \mu \circ f^{-1} \sqsubseteq \nu \circ \nu^{-1} \Leftrightarrow f \circ \mu^{-1} \circ f^{-1} \sqsubseteq \nu \circ \nu^{-1} \Rightarrow \langle f \circ \mu^{-1} \circ f^{-1}\rangle^* X \sqsubseteq \langle \nu\rangle^* \langle \nu^{-1}\rangle^* X \Rightarrow \operatorname{Cor}\langle f \circ \mu^{-1} \circ f^{-1}\rangle^* X \sqsubseteq \operatorname{Cor}\langle \nu\rangle^* \langle \nu^{-1}\rangle^* X \Leftrightarrow \langle f \circ \mu^{-1} \circ f^{-1}\rangle^* X \sqsubseteq \operatorname{Cor}\langle \nu\rangle^* \langle \nu^{-1}\rangle^* X \Rightarrow$ (T_1-separability) $\Rightarrow \langle f \circ \mu^{-1} \circ f^{-1}\rangle^* X \sqsubseteq \langle \nu^{-1}\rangle^* X$ for any typed set X on $\operatorname{Ob}\nu$. Thus $f \in \mathrm{C}''(\mu \circ \mu^{-1}, \nu \circ \nu^{-1}) \Rightarrow f \circ \mu^{-1} \circ f^{-1} \sqsubseteq \nu^{-1} \Leftrightarrow f \circ \mu \circ f^{-1} \sqsubseteq \nu \Leftrightarrow f \in \mathrm{C}''(\mu, \nu)$. □

THEOREM 1186. $f \in \mathrm{C}(\mu \circ \mu^{-1}, \nu \circ \nu^{-1}) \Leftrightarrow f \in \mathrm{C}(\mu, \nu)$ for complete endofuncoids μ, ν and principal monovalued and entirely defined funcoid $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, provided that μ is reflexive, and ν is T_1-separable.

PROOF. Two above propositions and theorem 1178. □

11.5. Continuity of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids on some set regarding the composition.) Consider also some lattice (*lattice of objects*). (For example take the lattice of set theoretic filters.)

We will map every object A to so called *restricted identity* element I_A of the semigroup (for example restricted identity funcoid or restricted identity reloid). For identity elements we will require

1°. $I_A \circ I_B = I_{A \sqcap B}$;
2°. $f \circ I_A \sqsubseteq f$; $I_A \circ f \sqsubseteq f$.

In the case when our semigroup is "dagger" (that is is a dagger precategory) we will require also $(I_A)^\dagger = I_A$.

We can define restricting an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define *rectangular restricting* an element f of our semigroup to objects A and B as $I_B \circ f \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for funcoids and for reloids):

$$f \sqcap (A \times B) = I_B \circ f \circ I_A.$$

Square restricting of an element f to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ f \circ I_A$ (or by the formula $f \sqcap (A \times A)$).

THEOREM 1187. For every elements f, μ, ν of our semigroup and an object A

1°. $f \in \mathrm{C}(\mu, \nu) \Rightarrow f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A, \nu)$;
2°. $f \in \mathrm{C}'(\mu, \nu) \Rightarrow f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A, \nu)$;
3°. $f \in \mathrm{C}''(\mu, \nu) \Rightarrow f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A, \nu)$.

(Two last items are true for the case when our semigroup is dagger.)

PROOF.

1°.

$$\begin{aligned} f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A, \nu) &\Leftrightarrow \\ f|_A \circ I_A \circ \mu \circ I_A \sqsubseteq \nu \circ f|_A &\Leftrightarrow \\ f \circ I_A \circ I_A \circ \mu \circ I_A \sqsubseteq \nu \circ f|_A &\Leftrightarrow \\ f \circ I_A \circ \mu \circ I_A \sqsubseteq \nu \circ f \circ I_A &\Leftarrow \\ f \circ I_A \circ \mu \sqsubseteq \nu \circ f &\Leftarrow \\ f \circ \mu \sqsubseteq \nu \circ f &\Leftrightarrow \\ f \in \mathrm{C}(\mu, \nu). \end{aligned}$$

2°.

$$\begin{aligned} f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A, \nu) &\Leftrightarrow \\ I_A \circ \mu \circ I_A \sqsubseteq (f|_A)^\dagger \circ \nu \circ f|_A &\Leftarrow \\ I_A \circ \mu \circ I_A \sqsubseteq (f \circ I_A)^\dagger \circ \nu \circ f \circ I_A &\Leftrightarrow \\ I_A \circ \mu \circ I_A \sqsubseteq I_A \circ f^\dagger \circ \nu \circ f \circ I_A &\Leftarrow \\ \mu \sqsubseteq f^\dagger \circ \nu \circ f &\Leftrightarrow \\ f \in \mathrm{C}'(\mu, \nu). \end{aligned}$$

3°.

$$\begin{aligned} f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A, \nu) &\Leftrightarrow \\ f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^\dagger \sqsubseteq \nu &\Leftrightarrow \\ f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^\dagger \sqsubseteq \nu &\Leftrightarrow \\ f \circ I_A \circ \mu \circ I_A \circ f^\dagger \sqsubseteq \nu &\Leftarrow \\ f \circ \mu \circ f^\dagger \sqsubseteq \nu &\Leftrightarrow \\ f \in \mathrm{C}''(\mu, \nu). \end{aligned}$$

□

CHAPTER 12

Connectedness regarding funcoids and reloids

12.1. Some lemmas

LEMMA 1188. Let U be a set, $A, B \in \mathscr{T}U$ be typed sets, f be an endo-funcoid on U. If $\neg(A\ [f]^*\ B) \wedge A \sqcup B \in \operatorname{up}(\operatorname{dom} f \sqcup \operatorname{im} f)$ then f is closed on A.

PROOF. Let $A \sqcup B \in \operatorname{up}(\operatorname{dom} f \sqcup \operatorname{im} f)$.

$$\begin{aligned}
\neg(A\ [f]^*\ B) &\Leftrightarrow \\
B \sqcap \langle f\rangle^* A = \bot &\Rightarrow \\
(\operatorname{dom} f \sqcup \operatorname{im} f) \sqcap B \sqcap \langle f\rangle^* A = \bot &\Rightarrow \\
((\operatorname{dom} f \sqcup \operatorname{im} f) \setminus A) \sqcap \langle f\rangle^* A = \bot &\Leftrightarrow \\
\langle f\rangle^* A \sqsubseteq A.&
\end{aligned}$$

□

COROLLARY 1189. If $\neg(A\ [f]^*\ B) \wedge A \sqcup B \in \operatorname{up}(\operatorname{dom} f \sqcup \operatorname{im} f)$ then f is closed on $A \setminus B$ for a funcoid $f \in \mathsf{FCD}(U, U)$ for every sets U and typed sets $A, B \in \mathscr{T}U$.

PROOF. Let $\neg(A\ [f]^*\ B) \wedge A \sqcup B \in \operatorname{up}(\operatorname{dom} f \sqcup \operatorname{im} f)$. Then

$$\neg((A \setminus B)\ [f]^*\ B) \wedge (A \setminus B) \sqcup B \in \operatorname{up}(\operatorname{dom} f \sqcup \operatorname{im} f).$$

□

LEMMA 1190. If $\neg(A\ [f]^*\ B) \wedge A \sqcup B \in \operatorname{up}(\operatorname{dom} f \sqcup \operatorname{im} f)$ then $\neg(A\ [f^n]^*\ B)$ for every whole positive n.

PROOF. Let $\neg(A\ [f]^*\ B) \wedge A \sqcup B \in \operatorname{up}(\operatorname{dom} f \sqcup \operatorname{im} f)$. From the above lemma $\langle f\rangle^* A \sqsubseteq A$. $B \sqcap \langle f\rangle A = \bot$, consequently $\langle f\rangle^* A \sqsubseteq A \setminus B$. Because (by the above corollary) f is closed on $A \setminus B$, then $\langle f\rangle\langle f\rangle A \sqsubseteq A \setminus B$, $\langle f\rangle\langle f\rangle\langle f\rangle A \sqsubseteq A \setminus B$, etc. So $\langle f^n\rangle A \sqsubseteq A \setminus B$, $B \asymp \langle f^n\rangle A$, $\neg(A\ [f^n]^*\ B)$. □

12.2. Endomorphism series

DEFINITION 1191. $S_1(\mu) = \mu \sqcup \mu^2 \sqcup \mu^3 \sqcup \dots$ for an endomorphism μ of a precategory with countable join of morphisms (that is join defined for every countable set of morphisms).

DEFINITION 1192. $S(\mu) = \mu^0 \sqcup S_1(\mu) = \mu^0 \sqcup \mu \sqcup \mu^2 \sqcup \mu^3 \sqcup \dots$where $\mu^0 = 1_{\operatorname{Ob} \mu}$ (identity morphism for the object $\operatorname{Ob} \mu$) where $\operatorname{Ob} \mu$ is the object of endomorphism μ for an endomorphism μ of a category with countable join of morphisms.

I call S_1 and S *endomorphism series*.

PROPOSITION 1193. The relation $S(\mu)$ is transitive for the category **Rel**.

PROOF.

$$S(\mu) \circ S(\mu) = \mu^0 \sqcup S(\mu) \sqcup \mu \circ S(\mu) \sqcup \mu^2 \circ S(\mu) \sqcup \dots = (\mu^0 \sqcup \mu^1 \sqcup \mu^2 \sqcup \dots) \sqcup (\mu^1 \sqcup \mu^2 \sqcup \mu^3 \sqcup \dots) \sqcup (\mu^2 \sqcup \mu^3 \sqcup \mu^4 \sqcup \dots) = \mu^0 \sqcup \mu^1 \sqcup \mu^2 \sqcup \dots = S(\mu).$$

□

12.3. Connectedness regarding binary relations

Before going to research connectedness for funcoids and reloids we will excurse into the basic special case of connectedness regarding binary relations on a set $\mho$.

This is commonly studied in "graph theory" courses. *Digraph* as commonly defined is essentially the same as an endomorphism of the category **Rel**.

DEFINITION 1194. A set A is called *(strongly) connected* regarding a binary relation μ on U when

$$\forall X, Y \in \mathscr{P}U \setminus \{\emptyset\} : (X \cup Y = A \Rightarrow X \; [\mu]^* \; Y).$$

DEFINITION 1195. A typed set A of type U is called *(strongly) connected* regarding a **Rel**-endomorphism μ on U when

$$\forall X, Y \in \mathscr{T}(\operatorname{Ob}\mu) \setminus \{\bot^{\mathscr{T}(\operatorname{Ob}\mu)}\} : (X \sqcup Y = A \Rightarrow X \; [\mu]^* \; Y).$$

OBVIOUS 1196. A typed set A is connected regarding **Rel**-endomorphism μ on its type iff $\operatorname{GR} A$ is connected regarding $\operatorname{GR}\mu$.

Let $\mho$ be a set.

DEFINITION 1197. *Path* between two elements $a, b \in \mho$ in a set $A \subseteq \mho$ through binary relation μ is the finite sequence $x_0 \dots x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i \; (\mu \cap A \times A) \; x_{i+1}$ for every $i = 0, \dots, n-1$. n is called path *length*.

PROPOSITION 1198. There exists path between every element $a \in \mho$ and that element itself.

PROOF. It is the path consisting of one vertex (of length 0). □

PROPOSITION 1199. There is a path from element a to element b in a set A through a binary relation μ iff $a \; (S(\mu \cap A \times A)) \; b$ (that is $(a, b) \in S(\mu \cap A \times A)$).

PROOF.

$\Rightarrow$. If a path from a to b exists, then $\{b\} \subseteq \langle(\mu \cap A \times A)^n\rangle^* \{a\}$ where n is the path length. Consequently $\{b\} \subseteq \langle S(\mu \cap A \times A)\rangle^* \{a\}$; $a \; (S(\mu \cap A \times A)) \; b$.

$\Leftarrow$. If $a \; (S(\mu \cap A \times A)) \; b$ then there exists $n \in \mathbb{N}$ such that $a \; (\mu \cap A \times A)^n \; b$. By definition of composition of binary relations this means that there exist finite sequence $x_0 \dots x_n$ where $x_0 = a$, $x_n = b$ for $n \in N$ and $x_i \; (\mu \cap A \times A) \; x_{i+1}$ for every $i = 0, \dots, n-1$. That is there is a path from a to b.

□

PROPOSITION 1200. There is a path from element a to element b in a set A through a binary relation μ iff $a \; (S_1(\mu \cap A \times A)) \; b$ (that is $(a, b) \in S_1(\mu \cap A \times A)$).

PROOF. Similar to the previous proof. □

THEOREM 1201. The following statements are equivalent for a binary relation μ and a set A:

1°. For every $a, b \in A$ there is a nonzero-length path between a and b in A through μ.

2°. $S_1(\mu \cap (A \times A)) \supseteq A \times A$.
3°. $S_1(\mu \cap (A \times A)) = A \times A$.
4°. A is connected regarding μ.

PROOF.

1°⇒2°. Let for every $a, b \in A$ there is a nonzero-length path between a and b in A through μ. Then $a\ (S_1(\mu \cap A \times A))\ b$ for every $a, b \in A$. It is possible only when $S_1(\mu \cap (A \times A)) \supseteq A \times A$.

3°⇒1°. For every two vertices a and b we have $a\ (S_1(\mu \cap A \times A))\ b$. So (by the previous) for every two vertices a and b there exists a nonzero-length path from a to b.

3°⇒4°. Suppose $\neg(X\ [\mu \cap (A \times A)]^*\ Y)$ for some $X, Y \in \mathscr{P}\mho \setminus \{\emptyset\}$ such that $X \cup Y = A$. Then by a lemma $\neg(X\ [(\mu \cap (A \times A))^n]^*\ Y)$ for every $n \in \mathbb{Z}_+$. Consequently $\neg(X\ [S_1(\mu \cap (A \times A))]^*\ Y)$. So $S_1(\mu \cap (A \times A)) \neq A \times A$.

4°⇒3°. If $\langle S_1(\mu \cap (A \times A))\rangle^*\{v\} = A$ for every vertex v then $S_1(\mu \cap (A \times A)) = A \times A$. Consider the remaining case when $V \overset{\text{def}}{=} \langle S_1(\mu \cap (A \times A))\rangle^*\{v\} \subset A$ for some vertex v. Let $W = A \setminus V$. If $\operatorname{card} A = 1$ then $S_1(\mu \cap (A \times A)) \supseteq \mathrm{id}_A = A \times A$; otherwise $W \neq \emptyset$. Then $V \cup W = A$ and so $V\ [\mu]^*\ W$ what is equivalent to $V\ [\mu \cap (A \times A)]^*\ W$ that is $\langle \mu \cap (A \times A)\rangle^* V \cap W \neq \emptyset$. This is impossible because

$$\langle \mu \cap (A \times A)\rangle^* V = \langle \mu \cap (A \times A)\rangle^* \langle S_1(\mu \cap (A \times A))\rangle^* V = \\ \left\langle (\mu \cap (A \times A))^2 \cup (\mu \cap (A \times A))^3 \cup \cdots \cup \right\rangle^* V \subseteq \langle S_1(\mu \cap (A \times A))\rangle^* V = V.$$

2°⇒3°. Because $S_1(\mu \cap (A \times A)) \subseteq A \times A$.

□

COROLLARY 1202. A set A is connected regarding a binary relation μ iff it is connected regarding $\mu \cap (A \times A)$.

DEFINITION 1203. A *connected component* of a set A regarding a binary relation F is a maximal connected subset of A.

THEOREM 1204. The set A is partitioned into connected components (regarding every binary relation F).

PROOF. Consider the binary relation $a \sim b \Leftrightarrow a\ (S(F))\ b \wedge b\ (S(F))\ a$. $\sim$ is a symmetric, reflexive, and transitive relation. So all points of A are partitioned into a collection of sets Q. Obviously each component is (strongly) connected. If a set $R \subseteq A$ is greater than one of that connected components A then it contains a point $b \in B$ where B is some other connected component. Consequently R is disconnected. □

PROPOSITION 1205. A set is connected (regarding a binary relation) iff it has one connected component.

PROOF. Direct implication is obvious. Reverse is proved by contradiction. □

12.4. Connectedness regarding funcoids and reloids

DEFINITION 1206. *Connectivity reloid* $S_1^*(\mu) = \bigsqcap^{\mathsf{RLD}}_{M \in \operatorname{up} \mu} S_1(M)$ for an endoreloid μ.

DEFINITION 1207. $S^*(\mu)$ for an endoreloid μ is defined as follows:

$$S^*(\mu) = \bigsqcap^{\mathsf{RLD}}_{M \in \operatorname{up} \mu} S(M).$$

Do not mess the word *connectivity* with the word *connectedness* which means being connected.[1]

PROPOSITION 1208. $S^*(\mu) = 1^{\mathsf{RLD}}_{\operatorname{Ob}\mu} \sqcup S_1^*(\mu)$ for every endoreloid μ.

PROOF. By the proposition 610. □

PROPOSITION 1209. $S^*(\mu) = S(\mu)$ and $S_1^*(\mu) = S_1(\mu)$ if μ is a principal reloid.

PROOF. $S^*(\mu) = \bigsqcap\{S(\mu)\} = S(\mu)$; $S_1^*(\mu) = \bigsqcap\{S_1(\mu)\} = S_1(\mu)$. □

DEFINITION 1210. A filter $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$ is called *connected* regarding an endoreloid μ when $S_1^*(\mu \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})) \sqsupseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

OBVIOUS 1211. A filter $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$ is connected regarding an endoreloid μ iff $S_1^*(\mu \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

DEFINITION 1212. A filter $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$ is called *connected* regarding an endofuncoid μ when

$$\forall \mathcal{X}, \mathcal{Y} \in \mathscr{F}(\operatorname{Ob}\mu) \setminus \{\bot^{\mathscr{F}(\operatorname{Ob}\mu)}\} : (\mathcal{X} \sqcup \mathcal{Y} = \mathcal{A} \Rightarrow \mathcal{X} \mathrel{[\mu]} \mathcal{Y}).$$

PROPOSITION 1213. Let A be a typed set of type U. The filter $\uparrow A$ is connected regarding an endofuncoid μ on U iff

$$\forall X, Y \in \mathscr{T}(\operatorname{Ob}\mu) \setminus \{\bot^{\mathscr{T}(\operatorname{Ob}\mu)}\} : (X \sqcup Y = A \Rightarrow X \mathrel{[\mu]^*} Y).$$

PROOF.

$\Rightarrow$. Obvious.

$\Leftarrow$. It follows from co-separability of filters.

□

THEOREM 1214. The following are equivalent for every typed set A of type U and **Rel**-endomorphism μ on a set U:

1°. A is connected regarding μ.
2°. $\uparrow A$ is connected regarding $\uparrow^{\mathsf{RLD}} \mu$.
3°. $\uparrow A$ is connected regarding $\uparrow^{\mathsf{FCD}} \mu$.

PROOF.

1°⇔2°.

$$S_1^*(\uparrow^{\mathsf{RLD}} \mu \sqcap (A \times^{\mathsf{RLD}} A)) = \\ S_1^*(\uparrow^{\mathsf{RLD}} (\mu \sqcap (A \times A))) = \\ \uparrow^{\mathsf{RLD}} S_1(\mu \sqcap (A \times A)).$$

So

$$S_1^*(\uparrow^{\mathsf{RLD}} \mu \sqcap (A \times^{\mathsf{RLD}} A)) \sqsupseteq A \times^{\mathsf{RLD}} A \Leftrightarrow \\ \uparrow^{\mathsf{RLD}} S_1(\mu \sqcap (A \times A)) \sqsupseteq \uparrow^{\mathsf{RLD}} (A \times A) = A \times^{\mathsf{RLD}} A.$$

1°⇔3°. It follows from the previous proposition.

□

Next is conjectured a statement more strong than the above theorem:

CONJECTURE 1215. Let $\mathcal{A}$ be a filter on a set U and F be a **Rel**-endomorphism on U.

$\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{FCD}} F$ iff $\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{RLD}} F$.

[1] In some math literature these two words are used interchangeably.

OBVIOUS 1216. A filter $\mathcal{A}$ is connected regarding a reloid μ iff it is connected regarding the reloid $\mu \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})$.

OBVIOUS 1217. A filter $\mathcal{A}$ is connected regarding a funcoid μ iff it is connected regarding the funcoid $\mu \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{A})$.

THEOREM 1218. A filter $\mathcal{A}$ is connected regarding a reloid f iff $\mathcal{A}$ is connected regarding every $F \in \langle \uparrow^{\mathsf{RLD}} \rangle^* \operatorname{up} f$.

PROOF.

$\Rightarrow$. Obvious.

$\Leftarrow$. $\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{RLD}} F$ iff $S_1(F) = F^1 \sqcup F^2 \sqcup \dots \in \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})$.

$$S_1^*(f) = \textstyle\bigsqcap^{\mathsf{RLD}}_{F \in \operatorname{up} f} S_1(F) \sqsupseteq \bigsqcap_{F \in \operatorname{up} f} (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}.$$

□

CONJECTURE 1219. A filter $\mathcal{A}$ is connected regarding a funcoid f iff $\mathcal{A}$ is connected regarding every $F \in \langle \uparrow^{\mathsf{FCD}} \rangle^* \operatorname{up} f$.

The above conjecture is open even for the case when $\mathcal{A}$ is a principal filter.

CONJECTURE 1220. A filter $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding the funcoid $(\mathsf{FCD})f$.

The above conjecture is true in the special case of principal filters:

PROPOSITION 1221. A filter $\uparrow A$ (for a typed set A) is connected regarding an endoreloid f on the suitable object iff it is connected regarding the endofuncoid $(\mathsf{FCD})f$.

PROOF. $\uparrow A$ is connected regarding a reloid f iff A is connected regarding every $F \in \operatorname{up} f$ that is when (taken into account that connectedness for $\uparrow^{\mathsf{RLD}} F$ is the same as connectedness of $\uparrow^{\mathsf{FCD}} F$)

$$\begin{aligned}
&\forall F \in \operatorname{up} f \forall \mathcal{X}, \mathcal{Y} \in \mathscr{F}(\operatorname{Ob} f) \setminus \{\bot^{\mathscr{F}(\operatorname{Ob} f)}\} : (\mathcal{X} \sqcup \mathcal{Y} = \uparrow A \Rightarrow \mathcal{X} \left[\uparrow^{\mathsf{FCD}} F\right] \mathcal{Y}) \Leftrightarrow \\
&\forall \mathcal{X}, \mathcal{Y} \in \mathscr{F}(\operatorname{Ob} f) \setminus \{\bot^{\mathscr{F}(\operatorname{Ob} f)}\} \forall F \in \operatorname{up} f : (\mathcal{X} \sqcup \mathcal{Y} = \uparrow A \Rightarrow \mathcal{X} \left[\uparrow^{\mathsf{FCD}} F\right] \mathcal{Y}) \Leftrightarrow \\
&\forall \mathcal{X}, \mathcal{Y} \in \mathscr{F}(\operatorname{Ob} f) \setminus \{\bot^{\mathscr{F}(\operatorname{Ob} f)}\} (\mathcal{X} \sqcup \mathcal{Y} = \uparrow A \Rightarrow \forall F \in \operatorname{up} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}} F\right] \mathcal{Y}) \Leftrightarrow \\
&\qquad \forall \mathcal{X}, \mathcal{Y} \in \mathscr{F}(\operatorname{Ob} f) \setminus \{\bot^{\mathscr{F}(\operatorname{Ob} f)}\} (\mathcal{X} \sqcup \mathcal{Y} = \uparrow A \Rightarrow \mathcal{X} \left[(\mathsf{FCD})f\right] \mathcal{Y})
\end{aligned}$$

that is when the set $\uparrow A$ is connected regarding the funcoid $(\mathsf{FCD})f$. □

CONJECTURE 1222. A set A is connected regarding an endofuncoid μ iff for every $a, b \in A$ there exists a totally ordered set $P \subseteq A$ such that $\min P = a$, $\max P = b$ and

$$\forall q \in P \setminus \{b\} : \left\{ \frac{x \in P}{x \leq q} \right\} [\mu]^* \left\{ \frac{x \in P}{x > q} \right\}.$$

Weaker condition:

$$\forall q \in P \setminus \{b\} : \left\{ \frac{x \in P}{x \leq q} \right\} [\mu]^* \left\{ \frac{x \in P}{x > q} \right\} \vee \forall q \in P \setminus \{a\} : \left\{ \frac{x \in P}{x < q} \right\} [\mu]^* \left\{ \frac{x \in P}{x \geq q} \right\}.$$

12.5. Algebraic properties of S and S^*

THEOREM 1223. $S^*(S^*(f)) = S^*(f)$ for every endoreloid f.

PROOF.

$$S^*(S^*(f)) = \prod^{\mathsf{RLD}}_{R\in\operatorname{up} S^*(f)} S(R) \sqsubseteq \prod^{\mathsf{RLD}}_{R\in\left\{\frac{S(F)}{F\in\operatorname{up} f}\right\}} S(R) = \prod^{\mathsf{RLD}}_{R\in\operatorname{up} f} S(S(R)) = \prod^{\mathsf{RLD}}_{R\in\operatorname{up} f} S(R) = S^*(f).$$

So $S^*(S^*(f)) \sqsubseteq S^*(f)$. That $S^*(S^*(f)) \sqsupseteq S^*(f)$ is obvious. □

COROLLARY 1224. $S^*(S(f)) = S(S^*(f)) = S^*(f)$ for every endoreloid f.

PROOF. Obviously $S^*(S(f)) \sqsupseteq S^*(f)$ and $S(S^*(f)) \sqsupseteq S^*(f)$.
But $S^*(S(f)) \sqsubseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \sqsubseteq S^*(S^*(f)) = S^*(f)$. □

CONJECTURE 1225. $S(S(f)) = S(f)$ for

1°. every endoreloid f;
2°. every endofuncoid f.

CONJECTURE 1226. $S(f) \circ S(f) = S(f)$ for every endoreloid f.

THEOREM 1227. $S^*(f) \circ S^*(f) = S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$ for every endoreloid f.

PROOF. [2]

It is enough to prove $S^*(f) \circ S^*(f) = S^*(f)$ because $S^*(f) \sqsubseteq S(f) \circ S^*(f) \sqsubseteq S^*(f) \circ S^*(f)$ and likewise for $S^*(f) \circ S(f)$.

$S^*(\mu) \circ S^*(\mu) = \prod^{\mathsf{RLD}}_{F\in\operatorname{up} S^*(\mu)}(F \circ F) = \text{(see below)} = \prod^{\mathsf{RLD}}_{X\in\operatorname{up}\mu}(S(X) \circ S(X)) = \prod^{\mathsf{RLD}}_{X\in\operatorname{up}\mu} S(X) = S^*(\mu)$.

$F \in \operatorname{up} S^*(\mu) \Leftrightarrow F \in \operatorname{up} \prod^{\mathscr{F}}_{F\in\operatorname{up}\mu} S(F) \Rightarrow$ (by properties of filter bases) $\Rightarrow \exists X \in \operatorname{up}\mu : F \sqsupseteq S(X) \Rightarrow \exists X \in \operatorname{up}\mu : F \circ F \sqsupseteq S(X) \circ S(X)$ thus

$$\prod^{\mathsf{RLD}}_{F\in\operatorname{up} S^*(\mu)} F \circ F \sqsupseteq \prod^{\mathsf{RLD}}_{X\in\operatorname{up}\mu} (S(X) \circ S(X));$$

$X \in \operatorname{up}\mu \Rightarrow S(X) \in \operatorname{up} S^*(\mu) \Rightarrow \exists F \in \operatorname{up} S^*(\mu) : S(X) \circ S(X) \sqsupseteq F \circ F$ thus

$$\prod^{\mathsf{RLD}}_{F\in\operatorname{up} S^*(\mu)} F \circ F \sqsubseteq \prod^{\mathsf{RLD}}_{X\in\operatorname{up}\mu} (S(X) \circ S(X)).$$

□

CONJECTURE 1228. $S(f) \circ S(f) = S(f)$ for every endofuncoid f.

12.6. Irreflexive reloids

DEFINITION 1229. Endoreloid f is irreflexive iff $f \asymp 1^{\operatorname{Ob} f}$.

PROPOSITION 1230. Endoreloid f is irreflexive iff $f \sqsubseteq \top \setminus 1$.

PROOF. By theorem 604. □

OBVIOUS 1231. $f \setminus 1$ is an irreflexive endoreloid if f is an endoreloid.

PROPOSITION 1232. $S(f) = S(f \sqcup 1)$ if f is an endoreloid, endofuncoid, or endorelation.

[2] Can be more succintly proved considering $\mu \mapsto S^*(\mu)$ as a pointfree funcoid?

PROOF. First prove $(f \sqcup 1)^n = 1 \sqcup f \sqcup \ldots \sqcup f^n$ for $n \in \mathbb{N}$. For $n = 0$ it's obvious. By induction we have

$$\begin{aligned} (f \sqcup 1)^{n+1} &= \\ (f \sqcup 1)^n \circ (f \sqcup 1) &= \\ (1 \sqcup f \sqcup \cdots \sqcup f^n) \circ (f \sqcup 1) &= \\ (f \sqcup f^2 \sqcup \cdots \sqcup f^{n+1}) \sqcup (1 \sqcup f \sqcup \cdots \sqcup f^n) &= \\ 1 \sqcup f \sqcup \cdots \sqcup f^{n+1}&. \end{aligned}$$

So $S(f \sqcup 1) = 1 \sqcup (1 \sqcup f) \sqcup (1 \sqcup f \sqcup f^2) \sqcup \ldots = 1 \sqcup f \sqcup f^2 \sqcup \ldots = S(f)$. □

COROLLARY 1233. $S(f) = S(f \sqcup 1) = S(f \setminus 1)$ if f is an endoreloid (or just an endorelation).

PROOF. $S(f \setminus 1) = S((f \setminus 1) \sqcup 1) \sqsupseteq S(f)$. But $S(f \setminus 1) \sqsubseteq S(f)$ is obvious. So $S(f \setminus 1) = S(f)$. □

12.7. Micronization

"Micronization" was a thoroughly wrong idea with several errors in the proofs. This section is removed from the book.

CHAPTER 13

Total boundness of reloids

13.1. Thick binary relations

DEFINITION 1234. I will call *α-thick* and denote $\operatorname{thick}_\alpha(E)$ a **Rel**-endomorphism E when there exists a finite cover S of $\operatorname{Ob} E$ such that $\forall A \in S : A \times A \subseteq \operatorname{GR} E$.

DEFINITION 1235. $\operatorname{CS}(S) = \bigcup \left\{ \frac{A \times A}{A \in S} \right\}$ for a collection S of sets.

REMARK 1236. CS means "Cartesian squares".

OBVIOUS 1237. A **Rel**-endomorphism is α-thick iff there exists a finite cover S of $\operatorname{Ob} E$ such that $\operatorname{CS}(S) \subseteq \operatorname{GR} E$.

DEFINITION 1238. I will call *β-thick* and denote $\operatorname{thick}_\beta(E)$ a **Rel**-endomorphism E when there exists a finite set B such that $\langle \operatorname{GR} E \rangle^* B = \operatorname{Ob} E$.

PROPOSITION 1239. $\operatorname{thick}_\alpha(E) \Rightarrow \operatorname{thick}_\beta(E)$.

PROOF. Let $\operatorname{thick}_\alpha(E)$. Then there exists a finite cover S of the set $\operatorname{Ob} E$ such that $\forall A \in S : A \times A \subseteq \operatorname{GR} E$. Without loss of generality assume $A \neq \emptyset$ for every $A \in S$. So $A \subseteq \langle \operatorname{GR} E \rangle^* \{x_A\}$ for some x_A for every $A \in S$. So

$$\langle \operatorname{GR} E \rangle^* \left\{ \frac{x_A}{A \in S} \right\} = \bigcup \left\{ \frac{\langle \operatorname{GR} E \rangle^* \{x_A\}}{A \in S} \right\} = \operatorname{Ob} E$$

and thus E is β-thick. □

OBVIOUS 1240. Let X be a set, A and B be **Rel**-endomorphisms on X and $B \sqsupseteq A$. Then:

- $\operatorname{thick}_\alpha(A) \Rightarrow \operatorname{thick}_\alpha(B)$;
- $\operatorname{thick}_\beta(A) \Rightarrow \operatorname{thick}_\beta(B)$.

EXAMPLE 1241. There is a β-thick Rel-morphism which is not α-thick.

PROOF. Consider the **Rel**-morphism on $[0;1]$ with the graph on figure 1:

$$\Gamma = \left\{ \frac{(x,x)}{x \in [0;1]} \right\} \cup \left\{ \frac{(x,0)}{x \in [0;1]} \right\} \cup \left\{ \frac{(0,x)}{x \in [0;1]} \right\}.$$

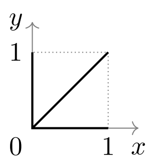

FIGURE 9. Thickness counterexample graph

Γ is β-thick because $\langle \Gamma \rangle^* \{0\} = [0;1]$.

To prove that Γ is not α-thick it's enough to prove that every set A such that $A \times A \subseteq \Gamma$ is finite.

Suppose for the contrary that A is infinite. Then A contains more than one non-zero points y, z ($y \neq z$). Without loss of generality $y < z$. So we have that (y, z) is not of the form (y, y) nor $(0, y)$ nor $(y, 0)$. Therefore $A \times A$ isn't a subset of Γ. □

13.2. Totally bounded endoreloids

The below is a straightforward generalization of the customary definition of totally bounded sets on uniform spaces (it's proved below that for uniform spaces the below definitions are equivalent).

DEFINITION 1242. An endoreloid f is *α-totally bounded* ($\operatorname{totBound}_\alpha(f)$) if every $E \in \operatorname{up} f$ is α-thick.

DEFINITION 1243. An endoreloid f is *β-totally bounded* ($\operatorname{totBound}_\beta(f)$) if every $E \in \operatorname{up} f$ is β-thick.

REMARK 1244. We could rewrite the above definitions in a more algebraic way like $\operatorname{up} f \subseteq \operatorname{thick}_\alpha$ (with $\operatorname{thick}_\alpha$ would be defined as a set rather than as a predicate), but we don't really need this simplification.

PROPOSITION 1245. If an endoreloid is α-totally bounded then it is β-totally bounded.

PROOF. Because $\operatorname{thick}_\alpha(E) \Rightarrow \operatorname{thick}_\beta(E)$. □

PROPOSITION 1246. If an endoreloid f is reflexive and $\operatorname{Ob} f$ is finite then f is both α-totally bounded and β-totally bounded.

PROOF. It enough to prove that f is α-totally bounded. Really, every $E \in \operatorname{up} f$ is reflexive. Thus $\{x\} \times \{x\} \subseteq \operatorname{GR} E$ for $x \in \operatorname{Ob} f$ and thus $\left\{ \frac{\{x\}}{x \in \operatorname{Ob} f} \right\}$ is a sought for finite cover of $\operatorname{Ob} f$. □

OBVIOUS 1247.

- A principal endoreloid induced by a **Rel**-morphism E is α-totally bounded iff E is α-thick.
- A principal endoreloid induced by a **Rel**-morphism E is β-totally bounded iff E is β-thick.

EXAMPLE 1248. There is a β-totally bounded endoreloid which is not α-totally bounded.

PROOF. It follows from the example above and properties of principal endoreloids. □

13.3. Special case of uniform spaces

Remember that *uniform space* is essentially the same as symmetric, reflexive and transitive endoreloid.

THEOREM 1249. Let f be such an endoreloid that $f \circ f^{-1} \sqsubseteq f$. Then f is α-totally bounded iff it is β-totally bounded.

PROOF.

$\Rightarrow$. Proved above.

$\Leftarrow$. For every $\epsilon \in \operatorname{up} f$ we have that $\langle \operatorname{GR} \epsilon \rangle^* \{c_0\}, \ldots, \langle \operatorname{GR} \epsilon \rangle^* \{c_n\}$ covers the space. $\langle \operatorname{GR} \epsilon \rangle^* \{c_i\} \times \langle \operatorname{GR} \epsilon \rangle^* \{c_i\} \subseteq \operatorname{GR}(\epsilon \circ \epsilon^{-1})$ because for $x \in \langle \operatorname{GR} \epsilon \rangle^* \{c_i\}$ (the same as $c_i \in \langle \operatorname{GR} \epsilon \rangle^* \{x\}$) we have

$$\left\langle \langle \operatorname{GR} \epsilon \rangle^* \{c_i\} \times \langle \operatorname{GR} \epsilon \rangle^* \{c_i\} \right\rangle^* \{x\} = \langle \operatorname{GR} \epsilon \rangle^* \{c_i\} \subseteq \langle \operatorname{GR} \epsilon \rangle^* \left\langle \operatorname{GR} \epsilon^{-1} \right\rangle^* \{x\} = \left\langle \operatorname{GR}(\epsilon \circ \epsilon^{-1}) \right\rangle^* \{x\}.$$

For every $\epsilon' \in \operatorname{up} f$ exists $\epsilon \in \operatorname{up} f$ such that $\epsilon \circ \epsilon^{-1} \sqsubseteq \epsilon'$ because $f \circ f^{-1} \sqsubseteq f$. Thus for every ϵ' we have $\langle \operatorname{GR} \epsilon \rangle^* \{c_i\} \times \langle \operatorname{GR} \epsilon \rangle^* \{c_i\} \subseteq \operatorname{GR} \epsilon'$ and so $\langle \operatorname{GR} \epsilon \rangle^* \{c_0\}, \ldots, \langle \operatorname{GR} \epsilon \rangle^* \{c_n\}$ is a sought for finite cover.

□

Corollary 1250. A uniform space is α-totally bounded iff it is β-totally bounded.

Proof. From the theorem and the definition of uniform spaces. □

Thus we can say about just *totally bounded* uniform spaces (without specifying whether it is α or β).

13.4. Relationships with other properties

Theorem 1251. Let μ and ν be endoreloids. Let f be a principal $\mathrm{C}'(\mu, \nu)$ continuous, monovalued, surjective reloid. Then if μ is β-totally bounded then ν is also β-totally bounded.

Proof. Let φ be the monovalued, surjective function, which induces the reloid f.

We have $\mu \sqsubseteq f^{-1} \circ \nu \circ f$.

Let $F \in \operatorname{up} \nu$. Then there exists $E \in \operatorname{up} \mu$ such that $E \subseteq \varphi^{-1} \circ F \circ \varphi$.

Since μ is β-totally bounded, there exists a finite typed subset A of $\operatorname{Ob} \mu$ such that $\langle \operatorname{GR} E \rangle^* A = \operatorname{Ob} \mu$.

We claim $\langle \operatorname{GR} F \rangle^* \langle \varphi \rangle^* A = \operatorname{Ob} \nu$.

Indeed let $y \in \operatorname{Ob} \nu$ be an arbitrary point. Since φ is surjective, there exists $x \in \operatorname{Ob} \mu$ such that $\varphi x = y$. Since $\langle \operatorname{GR} E \rangle^* A = \operatorname{Ob} \mu$ there exists $a \in A$ such that $a \ (\operatorname{GR} E) \ x$ and thus $a \ (\varphi^{-1} \circ F \circ \varphi) \ x$. So $(\varphi a, y) = (\varphi a, \varphi x) \in \operatorname{GR} F$. Therefore $y \in \langle \operatorname{GR} F \rangle^* \langle \varphi \rangle^* A$. □

Theorem 1252. Let μ and ν be endoreloids. Let f be a principal $\mathrm{C}''(\mu, \nu)$ continuous, surjective reloid. Then if μ is α-totally bounded then ν is also α-totally bounded.

Proof. Let φ be the surjective binary relation which induces the reloid f.

We have $f \circ \mu \circ f^{-1} \sqsubseteq \nu$.

Let $F \in \operatorname{up} \nu$. Then there exists $E \in \operatorname{up} \mu$ such that $\varphi \circ E \circ \varphi^{-1} \subseteq F$.

There exists a finite cover S of $\operatorname{Ob} \mu$ such that $\bigcup \left\{ \frac{A \times A}{A \in S} \right\} \subseteq \operatorname{GR} E$.

Thus $\varphi \circ \left(\bigcup \left\{ \frac{A \times A}{A \in S} \right\} \right) \circ \varphi^{-1} \subseteq \operatorname{GR} F$ that is $\bigcup \left\{ \frac{\langle \varphi \rangle^* A \times \langle \varphi \rangle^* A}{A \in S} \right\} \subseteq \operatorname{GR} F$.

It remains to prove that $\left\{ \frac{\langle \varphi \rangle^* A}{A \in S} \right\}$ is a cover of $\operatorname{Ob} \nu$. It is true because φ is a surjection and S is a cover of $\operatorname{Ob} \mu$. □

A stronger statement (principality requirement removed):

Conjecture 1253. The image of a uniformly continuous entirely defined monovalued surjective reloid from a (α-, β-)totally bounded endoreloid is also (α-, β-)totally bounded.

Can we remove the requirement to be entirely defined from the above conjecture?

QUESTION 1254. Under which conditions it's true that join of (α-, β-) totally bounded reloids is also totally bounded?

13.5. Additional predicates

We may consider also the following predicates expressing different kinds of what is intuitively is understood as boundness. Their usefulness is unclear, but I present them for completeness.

- $\operatorname{totBound}_\alpha(f)$
- $\operatorname{totBound}_\beta(f)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^n)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\beta(E^n)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^0 \sqcup \ldots \sqcup E^n)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\beta(E^0 \sqcup \ldots \sqcup E^n)$
- $\exists n \in \mathbb{N} : \operatorname{totBound}_\alpha(f^n)$
- $\exists n \in \mathbb{N} : \operatorname{totBound}_\beta(f^n)$
- $\exists n \in \mathbb{N} : \operatorname{totBound}_\alpha(f^0 \sqcup \ldots \sqcup f^n)$
- $\exists n \in \mathbb{N} : \operatorname{totBound}_\beta(f^0 \sqcup \ldots \sqcup f^n)$
- $\operatorname{totBound}_\alpha(S(f))$
- $\operatorname{totBound}_\beta(S(f))$

Some of the above defined predicates are equivalent:

PROPOSITION 1255.

- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^n) \Leftrightarrow \exists n \in \mathbb{N} : \operatorname{totBound}_\alpha(f^n)$.
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\beta(E^n) \Leftrightarrow \exists n \in \mathbb{N} : \operatorname{totBound}_\beta(f^n)$.

PROOF. Because for every $E \in \operatorname{up} f$ some $F \in \operatorname{up} f^n$ is a subset of E^n, we have

$$\forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^n) \Leftrightarrow \forall F \in \operatorname{up} f^n : \operatorname{thick}_\alpha(F)$$

and likewise for $\operatorname{thick}_\beta$. □

PROPOSITION 1256.

- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\alpha(E^0 \sqcup \ldots \sqcup E^n) \Leftrightarrow \exists n \in \mathbb{N} : \operatorname{totBound}_\alpha(f^0 \sqcup \ldots \sqcup f^n)$
- $\exists n \in \mathbb{N} \forall E \in \operatorname{up} f : \operatorname{thick}_\beta(E^0 \sqcup \ldots \sqcup E^n) \Leftrightarrow \exists n \in \mathbb{N} : \operatorname{totBound}_\beta(f^0 \sqcup \ldots \sqcup f^n)$

PROOF. It's enough to prove

$$\forall E \in \operatorname{up} f \exists F \in \operatorname{up}(f^0 \sqcup \cdots \sqcup f^n) : F \sqsubseteq E^0 \sqcup \ldots \sqcup E^n \text{ and} \tag{15}$$

$$\forall F \in \operatorname{up}(f^0 \sqcup \cdots \sqcup f^n) \exists E \in \operatorname{up} f : E^0 \sqcup \ldots \sqcup E^n \sqsubseteq F. \tag{16}$$

For the formula (15) take $F = E^0 \sqcup \cdots \sqcup E^n$.

Let's prove (16). Let $F \in \operatorname{up}(f^0 \sqcup \cdots \sqcup f^n)$. Using the fact that $F \in \operatorname{up} f^i$ take $E_i \in \operatorname{up} f$ for $i = 0, \ldots, n$ such that $E_i^i \sqsubseteq F$ (exercise 1007 and properties of generalized filter bases) and then $E = E_0 \sqcap \cdots \sqcap E_n \in \operatorname{up} f$. We have $E^0 \sqcup \ldots \sqcup E^n \sqsubseteq F$. □

PROPOSITION 1257. All predicates in the above list are pairwise equivalent in the case if f is a uniform space.

PROOF. Because $f \circ f = f$ and thus $f^n = f^0 \sqcup \cdots \sqcup f^n = S(f) = f$. □

CHAPTER 14

Orderings of filters in terms of reloids

Whilst the other chapters of this book use filters to research funcoids and reloids, here the opposite thing is discussed, the theory of reloids is used to describe properties of filters.

In this chapter the word *filter* is used to denote a filter on a set (not on an arbitrary poset) only.

14.1. Ordering of filters

Below I will define some categories having filters (with possibly different bases) as their objects and some relations having two filters (with possibly different bases) as arguments induced by these categories (defined as existence of a morphism between these two filters).

THEOREM 1258. $\operatorname{card} a = \operatorname{card} U$ for every ultrafilter a on U if U is infinite.

PROOF. Let $f(X) = X$ if $X \in a$ and $f(X) = U \setminus X$ if $X \notin a$. Obviously f is a surjection from U to a.

Every $X \in a$ appears as a value of f exactly twice, as $f(X)$ and $f(U \setminus X)$. So $\operatorname{card} a = (\operatorname{card} U)/2 = \operatorname{card} U$. □

COROLLARY 1259. Cardinality of every two ultrafilters on a set U is the same.

PROOF. For infinite U it follows from the theorem. For finite case it is obvious. □

PROPOSITION 1260. $\langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A} = \left\{\frac{C \in \mathscr{P}(\operatorname{Dst} f)}{\langle f^{-1}\rangle^* C \in \mathcal{A}}\right\}$ for every **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$. (Here a funcoid is considered as a pair of functions $\mathfrak{F}(\operatorname{Base}(\mathcal{A})) \to \mathfrak{F}(\operatorname{Base}(\mathcal{B}))$, $\mathfrak{F}(\operatorname{Base}(\mathcal{B})) \to \mathfrak{F}(\operatorname{Base}(\mathcal{A}))$ rather than as a pair of functions $\mathscr{F}(\operatorname{Base}(\mathcal{A})) \to \mathscr{F}(\operatorname{Base}(\mathcal{B}))$, $\mathscr{F}(\operatorname{Base}(\mathcal{B})) \to \mathscr{F}(\operatorname{Base}(\mathcal{A}))$.)

PROOF. For every set $C \in \mathscr{P}\operatorname{Base}(\mathcal{B})$ we have

$$\begin{aligned}\langle f^{-1}\rangle^* C \in \mathcal{A} &\Rightarrow\\ \exists K \in \mathcal{A} : \langle f^{-1}\rangle^* C = K &\Rightarrow\\ \exists K \in \mathcal{A} : \langle f\rangle^* \langle f^{-1}\rangle^* C = \langle f\rangle^* K &\Rightarrow\\ \exists K \in \mathcal{A} : C \supseteq \langle f\rangle^* K &\Leftrightarrow\\ \exists K \in \mathcal{A} : C \in \langle\uparrow^{\mathsf{FCD}} f\rangle^* K &\Rightarrow\\ C \in \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}.\end{aligned}$$

So $C \in \left\{\frac{C \in \mathscr{P}(\operatorname{Dst} f)}{\langle f^{-1}\rangle^* C \in \mathcal{A}}\right\} \Rightarrow C \in \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}$.

Let now $C \in \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}$. Then $\uparrow \langle f^{-1}\rangle^* C \sqsupseteq \langle\uparrow^{\mathsf{FCD}} f^{-1}\rangle\langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A} \sqsupseteq \mathcal{A}$ and thus $\langle f^{-1}\rangle^* C \in \mathcal{A}$. □

Below I'll define some directed multigraphs. By an abuse of notation, I will denote these multigraphs the same as (below defined) categories based on some

of these directed multigraphs with added composition of morphisms (of directed multigraphs edges). As such I will call vertices of these multigraphs objects and edges morphisms.

DEFINITION 1261. I will denote $\mathbf{GreFunc}_1$ the multigraph whose objects are filters and whose morphisms between objects $\mathcal{A}$ and $\mathcal{B}$ are $\mathbf{Set}$-morphisms from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}$.

DEFINITION 1262. I will denote $\mathbf{GreFunc}_2$ the multigraph whose objects are filters and whose morphisms between objects $\mathcal{A}$ and $\mathcal{B}$ are $\mathbf{Set}$-morphisms from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}$.

DEFINITION 1263. Let $\mathcal{A}$ be a filter on a set X and $\mathcal{B}$ be a filter on a set Y. $\mathcal{A} \geq_1 \mathcal{B}$ iff $\operatorname{Hom}_{\mathbf{GreFunc}_1}(\mathcal{A}, \mathcal{B})$ is not empty.

DEFINITION 1264. Let $\mathcal{A}$ be a filter on a set X and $\mathcal{B}$ be a filter on a set Y. $\mathcal{A} \geq_2 \mathcal{B}$ iff $\operatorname{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B})$ is not empty.

PROPOSITION 1265.

1°. $f \in \operatorname{Hom}_{\mathbf{GreFunc}_1}(\mathcal{A}, \mathcal{B})$ iff f is a $\mathbf{Set}$-morphism from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that
$$C \in \mathcal{B} \Leftarrow \langle f^{-1}\rangle^* C \in \mathcal{A}$$
for every $C \in \mathscr{P}\operatorname{Base}(\mathcal{B})$.

2°. $f \in \operatorname{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B})$ iff f is a $\mathbf{Set}$-morphism from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that
$$C \in \mathcal{B} \Leftrightarrow \langle f^{-1}\rangle^* C \in \mathcal{A}$$
for every $C \in \mathscr{P}\operatorname{Base}(\mathcal{B})$.

PROOF.

1°.
$$f \in \operatorname{Hom}_{\mathbf{GreFunc}_1}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{B} \sqsubseteq \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A} \Leftrightarrow \forall C \in \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A} : C \in \mathcal{B} \Leftrightarrow \forall C \in \mathscr{P}\operatorname{Base}(\mathcal{B}) : (\langle f^{-1}\rangle^* C \in \mathcal{A} \Rightarrow C \in \mathcal{B}).$$

2°.
$$f \in \operatorname{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{B} = \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A} \Leftrightarrow \forall C : (C \in \mathcal{B} \Leftrightarrow C \in \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}) \Leftrightarrow \forall C \in \mathscr{P}\operatorname{Base}(\mathcal{B}) : (C \in \mathcal{B} \Leftrightarrow C \in \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}) \Leftrightarrow \forall C \in \mathscr{P}\operatorname{Base}(\mathcal{B}) : (\langle f^{-1}\rangle^* C \in \mathcal{A} \Leftrightarrow C \in \mathcal{B}).$$

□

DEFINITION 1266. The directed multigraph $\mathbf{FuncBij}$ is the directed multigraph got from $\mathbf{GreFunc}_2$ by restricting to only bijective morphisms.

DEFINITION 1267. A filter $\mathcal{A}$ is *directly isomorphic* to a filter $\mathcal{B}$ iff there is a morphism $f \in \operatorname{Hom}_{\mathbf{FuncBij}}(\mathcal{A}, \mathcal{B})$.

OBVIOUS 1268. $f \in \operatorname{Hom}_{\mathbf{GreFunc}_1}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{B} \sqsubseteq \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}$ for every $\mathbf{Set}$-morphism from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$.

OBVIOUS 1269. $f \in \operatorname{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B}) \Leftrightarrow \mathcal{B} = \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}$ for every $\mathbf{Set}$-morphism from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$.

COROLLARY 1270. $\mathcal{A} \geq_1 \mathcal{B}$ iff it exists a $\mathbf{Set}$-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}$.

COROLLARY 1271. $\mathcal{A} \geq_2 \mathcal{B}$ iff it exists a $\mathbf{Set}$-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A}$.

PROPOSITION 1272. For a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ the following are equivalent:

1°. $\mathcal{B} = \left\{ \frac{C \in \mathscr{P} \operatorname{Base}(\mathcal{B})}{\langle f^{-1} \rangle^* C \in \mathcal{A}} \right\}$.
2°. $\forall C \in \operatorname{Base}(\mathcal{B}) : (C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A})$.
3°. $\forall C \in \operatorname{Base}(\mathcal{A}) : (C \in \langle f \rangle^* \mathcal{B} \Leftrightarrow C \in \mathcal{A})$.
4°. $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is a bijection from $\mathcal{A}$ to $\mathcal{B}$.
5°. $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is a function onto $\mathcal{B}$.
6°. $\mathcal{B} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$.
7°. $f \in \operatorname{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B})$.
8°. $f \in \operatorname{Hom}_{\mathbf{FuncBij}}(\mathcal{A}, \mathcal{B})$.

PROOF.

1°⇔2°.
$$\mathcal{B} = \left\{ \frac{C \in \mathscr{P} \operatorname{Base}(\mathcal{B})}{\langle f^{-1} \rangle^* C \in \mathcal{A}} \right\} \Leftrightarrow \forall C \in \mathscr{P} \operatorname{Base}(\mathcal{B}) : (C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A}).$$

2°⇔3°. Because f is a bijection.

2°⇒5°. For every $C \in \mathcal{B}$ we have $\langle f^{-1} \rangle^* C \in \mathcal{A}$ and thus $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}} \langle \uparrow^{\mathsf{FCD}} f^{-1} \rangle C = \langle f \rangle^* \langle f^{-1} \rangle^* C = C$. Thus $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is onto $\mathcal{B}$.

4°⇒5°. Obvious.

5°⇒4°. We need to prove only that $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is an injection. But this follows from the fact that f is a bijection.

4°⇒3°. We have $\forall C \in \operatorname{Base}(\mathcal{A}) : ((\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}) C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A})$ and consequently $\forall C \in \operatorname{Base}(\mathcal{A}) : (\langle f \rangle^* C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A})$.

6°⇔1°. From the last corollary.

1°⇔7°. Obvious.

7°⇔8°. Obvious.

□

COROLLARY 1273. The following are equivalent for every filters $\mathcal{A}$ and $\mathcal{B}$:

1°. $\mathcal{A}$ is directly isomorphic to $\mathcal{B}$.
2°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P} \operatorname{Base}(\mathcal{B})$
$$C \in \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle^* C \in \mathcal{A}.$$
3°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P} \operatorname{Base}(\mathcal{B})$
$$\langle f \rangle^* C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A}.$$
4°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is a bijection from $\mathcal{A}$ to $\mathcal{B}$.
5°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\langle \uparrow^{\mathsf{FCD}} f \rangle|_{\mathcal{A}}$ is a function onto $\mathcal{B}$.
6°. There is a bijective **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$.
7°. There is a bijective morphism $f \in \operatorname{Hom}_{\mathbf{GreFunc}_2}(\mathcal{A}, \mathcal{B})$.
8°. There is a bijective morphism $f \in \operatorname{Hom}_{\mathbf{FuncBij}}(\mathcal{A}, \mathcal{B})$.

PROPOSITION 1274. $\mathbf{GreFunc}_1$ and $\mathbf{GreFunc}_2$ with function composition are categories.

PROOF. Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be morphisms of $\mathbf{GreFunc}_1$. Then $\mathcal{B} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$ and $\mathcal{C} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} g \rangle \mathcal{B}$. So
$$\langle \uparrow^{\mathsf{FCD}} (g \circ f) \rangle \mathcal{A} = \langle \uparrow^{\mathsf{FCD}} g \rangle \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A} \sqsupseteq \langle \uparrow^{\mathsf{FCD}} g \rangle \mathcal{B} \sqsupseteq \mathcal{C}.$$

Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_1$. Associativity law is evident. $\mathrm{id}_{\mathrm{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_1$ for every filter $\mathcal{A}$.

Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be morphisms of $\mathbf{GreFunc}_2$. Then $\mathcal{B} = \left\langle \uparrow^{\mathsf{FCD}} f \right\rangle \mathcal{A}$ and $\mathcal{C} = \left\langle \uparrow^{\mathsf{FCD}} g \right\rangle \mathcal{B}$. So

$$\left\langle \uparrow^{\mathsf{FCD}} (g \circ f) \right\rangle \mathcal{A} = \left\langle \uparrow^{\mathsf{FCD}} g \right\rangle \left\langle \uparrow^{\mathsf{FCD}} f \right\rangle \mathcal{A} = \left\langle \uparrow^{\mathsf{FCD}} g \right\rangle \mathcal{B} = \mathcal{C}.$$

Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_2$. Associativity law is evident. $\mathrm{id}_{\mathrm{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_2$ for every filter $\mathcal{A}$. □

Corollary 1275. $\leq_1$ and $\leq_2$ are preorders.

Theorem 1276. **FuncBij** is a groupoid.

Proof. First let's prove it is a category. Let $f : \mathcal{A} \to \mathcal{B}$ and $g : \mathcal{B} \to \mathcal{C}$ be morphisms of **FuncBij**. Then $f : \mathrm{Base}(\mathcal{A}) \to \mathrm{Base}(\mathcal{B})$ and $g : \mathrm{Base}(\mathcal{B}) \to \mathrm{Base}(\mathcal{C})$ are bijections and $\mathcal{B} = \left\langle \uparrow^{\mathsf{FCD}} f \right\rangle \mathcal{A}$ and $\mathcal{C} = \left\langle \uparrow^{\mathsf{FCD}} g \right\rangle \mathcal{B}$. Thus $g \circ f : \mathrm{Base}(\mathcal{A}) \to \mathrm{Base}(\mathcal{C})$ is a bijection and $\mathcal{C} = \left\langle \uparrow^{\mathsf{FCD}} (g \circ f) \right\rangle \mathcal{A}$. Thus $g \circ f$ is a morphism of **FuncBij**. $\mathrm{id}_{\mathrm{Base}(\mathcal{A})}$ is the identity morphism of **FuncBij** for every filter $\mathcal{A}$. Thus it is a category.

It remains to prove only that every morphism $f \in \mathrm{Hom}_{\mathbf{FuncBij}}(\mathcal{A}, \mathcal{B})$ has a reverse (for every filters $\mathcal{A}$, $\mathcal{B}$). We have f is a bijection $\mathrm{Base}(\mathcal{A}) \to \mathrm{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P}\,\mathrm{Base}(\mathcal{A})$

$$\langle f \rangle^* C \in \mathcal{B} \Leftrightarrow C \in \mathcal{A}.$$

Then $f^{-1} : \mathrm{Base}(\mathcal{B}) \to \mathrm{Base}(\mathcal{A})$ is a bijection such that for every $C \in \mathscr{P}\,\mathrm{Base}(\mathcal{B})$

$$\left\langle f^{-1} \right\rangle^* C \in \mathcal{A} \Leftrightarrow C \in \mathcal{B}.$$

Thus $f^{-1} \in \mathrm{Hom}_{\mathbf{FuncBij}}(\mathcal{B}, \mathcal{A})$. □

Corollary 1277. Being directly isomorphic is an equivalence relation.

Rudin-Keisler order of ultrafilters is considered in such a book as [40].

Obvious 1278. For the case of ultrafilters being directly isomorphic is the same as being Rudin-Keisler equivalent.

Definition 1279. A filter $\mathcal{A}$ is *isomorphic* to a filter $\mathcal{B}$ iff there exist sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$.

Obvious 1280. Equivalent filters are isomorphic.

Theorem 1281. Being isomorphic (for small filters) is an equivalence relation.

Proof.

Reflexivity. Because every filter is directly isomorphic to itself.

Symmetry. If filter $\mathcal{A}$ is isomorphic to $\mathcal{B}$ then there exist sets $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$ and thus $\mathcal{B} \div B$ is directly isomorphic to $\mathcal{A} \div A$. So $\mathcal{B}$ is isomorphic to $\mathcal{A}$.

Transitivity. Let $\mathcal{A}$ be isomorphic to $\mathcal{B}$ and $\mathcal{B}$ be isomorphic to $\mathcal{C}$. Then exist $A \in \mathcal{A}$, $B_1 \in \mathcal{B}$, $B_2 \in \mathcal{B}$, $C \in \mathcal{C}$ such that there are bijections $f : A \to B_1$ and $g : B_2 \to C$ such that

$$\forall X \in \mathscr{P} A : \left(X \in \mathcal{B} \Leftrightarrow \left\langle f^{-1} \right\rangle^* X \in \mathcal{A}\right) \quad \text{and} \quad \forall X \in \mathscr{P} B_1 : \left(X \in \mathcal{A} \Leftrightarrow \langle f \rangle^* X \in \mathcal{B}\right)$$

and also $\forall X \in \mathscr{P} B_2 : (X \in \mathcal{B} \Leftrightarrow \langle g \rangle^* X \in \mathcal{C})$.

So $g \circ f$ is a bijection from $\left\langle f^{-1} \right\rangle^* (B_1 \cap B_2) \in \mathcal{A}$ to $\langle g \rangle^* (B_1 \cap B_2) \in \mathcal{C}$ such that

$$X \in \mathcal{A} \Leftrightarrow \langle f \rangle^* X \in \mathcal{B} \Leftrightarrow \langle g \rangle^* \langle f \rangle^* X \in \mathcal{C} \Leftrightarrow \langle g \circ f \rangle^* X \in \mathcal{C}.$$

Thus $g \circ f$ establishes a bijection which proves that $\mathcal{A}$ is isomorphic to $\mathcal{C}$.

□

LEMMA 1282. Let $\operatorname{card} X = \operatorname{card} Y$, u be an ultrafilter on X and v be an ultrafilter on Y; let $A \in u$ and $B \in v$. Let $u \div A$ and $v \div B$ be directly isomorphic. Then if $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ we have u and v directly isomorphic.

PROOF. Arbitrary extend the bijection witnessing being directly isomorphic to the sets $X \setminus A$ and $X \setminus B$. □

THEOREM 1283. If $\operatorname{card} X = \operatorname{card} Y$ then being isomorphic and being directly isomorphic are the same for ultrafilters u on X and v on Y.

PROOF. That if two filters are isomorphic then they are directly isomorphic is obvious.

Let ultrafilters u and v be isomorphic that is there is a bijection $f : A \to B$ where $A \in u$, $B \in v$ witnessing isomorphism of u and v.

If one of the filters u or v is a trivial ultrafilter then the other is also a trivial ultrafilter and as it is easy to show they are directly isomorphic. So we can assume u and v are not trivial ultrafilters.

If $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ our statement follows from the last lemma.

Now assume without loss of generality $\operatorname{card}(X \setminus A) < \operatorname{card}(Y \setminus B)$.

$\operatorname{card} B = \operatorname{card} Y$ because otherwise $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$.

It is easy to show that there exists $B' \supset B$ such that $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B')$ and $\operatorname{card} B' = \operatorname{card} B$.

We will find a bijection g from B to B' which witnesses direct isomorphism of v to v itself. Then the composition $g \circ f$ witnesses a direct isomorphism of $u \div A$ and $v \div B'$ and by the lemma u and v are directly isomorphic.

Let $D = B' \setminus B$. We have $D \notin v$.

There exists a set $E \subseteq B$ such that $\operatorname{card} E \geq \operatorname{card} D$ and $E \notin v$.

We have $\operatorname{card} E = \operatorname{card}(D \cup E)$ and thus there exists a bijection $h : E \to D \cup E$.

Let

$$g(x) = \begin{cases} x & \text{if } x \in B \setminus E; \\ h(x) & \text{if } x \in E. \end{cases}$$

$g|_{B \setminus E}$ and $g|_E$ are bijections.

$\operatorname{im}(g|_{B \setminus E}) = B \setminus E$; $\operatorname{im}(g|_E) = \operatorname{im} h = D \cup E$;

$$(D \cup E) \cap (B \setminus E) = (D \cap (B \setminus E)) \cup (E \cap (B \setminus E)) = \emptyset \cup \emptyset = \emptyset.$$

Thus g is a bijection from B to $(B \setminus E) \cup (D \cup E) = B \cup D = B'$.

To finish the proof it's enough to show that $\langle g \rangle^* v = v$. Indeed it follows from $B \setminus E \in v$. □

PROPOSITION 1284.

1°. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $\mathcal{A} \geq_2 \mathcal{B}$ iff $\mathcal{A} \div A \geq_2 \mathcal{B} \div B$.
2°. For every $A \in \mathcal{A}$ and $B \in \mathcal{B}$ we have $\mathcal{A} \geq_1 \mathcal{B}$ iff $\mathcal{A} \div A \geq_1 \mathcal{B} \div B$.

PROOF.

1°. $\mathcal{A} \geq_2 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$.

2°. $\mathcal{A} \geq_1 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} \subseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$.

□

PROPOSITION 1285. For ultrafilters $\geq_2$ is the same as Rudin-Keisler ordering (as defined in [40]).

PROOF. $x \geq_2 y$ iff there exist sets $A \in x$ and $B \in y$ and a bijective **Set**-morphism $f : X \to Y$ such that

$$y \div B = \left\{ \frac{C \in \mathscr{P}Y}{\langle f^{-1} \rangle^* C \in x \div A} \right\}$$

that is when $C \in y \div B \Leftrightarrow \langle f^{-1} \rangle^* C \in x \div A$ what is equivalent to $C \in y \Leftrightarrow \langle f^{-1} \rangle^* C \in x$ what is the definition of Rudin-Keisler ordering. □

REMARK 1286. The relation of being isomorphic for ultrafilters is traditionally called *Rudin-Keisler equivalence*.

OBVIOUS 1287. $(\geq_1) \supseteq (\geq_2)$.

DEFINITION 1288. Let Q and R be binary relations on the set of (small) filters. I will denote $\mathbf{MonRld}_{Q,R}$ the directed multigraph with objects being filters and morphisms such monovalued reloids f that $(\operatorname{dom} f)\ Q\ \mathcal{A}$ and $(\operatorname{im} f)\ R\ \mathcal{B}$.

I will also denote $\mathbf{CoMonRld}_{Q,R}$ the directed multigraph with objects being filters and morphisms such injective reloids f that $(\operatorname{im} f)\ Q\ \mathcal{A}$ and $(\operatorname{dom} f)\ R\ \mathcal{B}$. These are essentially the duals.

Some of these directed multigraphs are categories with reloid composition (see below). By abuse of notation I will denote these categories the same as these directed multigraphs.

LEMMA 1289. $\mathbf{CoMonRld}_{Q,R} \neq \emptyset \Leftrightarrow \mathbf{MonRld}_{Q,R} \neq \emptyset$.

PROOF. $f \in \mathbf{CoMonRld}_{Q,R} \Leftrightarrow (\operatorname{im} f)\ Q\ \mathcal{A} \wedge (\operatorname{dom} f)\ R\ \mathcal{B} \Leftrightarrow (\operatorname{dom} f^{-1})\ Q\ \mathcal{A} \wedge (\operatorname{im} f^{-1})\ R\ \mathcal{B} \Leftrightarrow f^{-1} \in \mathbf{MonRld}_{Q,R}$ for every monovalued reloid f (or what is the same, injective reloid f^{-1}). □

THEOREM 1290. For every filters $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1°. $\mathcal{A} \geq_1 \mathcal{B}$.
2°. $\operatorname{Hom}_{\mathbf{MonRld}_{=,\sqsupseteq}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
3°. $\operatorname{Hom}_{\mathbf{MonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
4°. $\operatorname{Hom}_{\mathbf{MonRld}_{\sqsubseteq,=}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
5°. $\operatorname{Hom}_{\mathbf{CoMonRld}_{=,\sqsupseteq}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
6°. $\operatorname{Hom}_{\mathbf{CoMonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
7°. $\operatorname{Hom}_{\mathbf{CoMonRld}_{\sqsubseteq,=}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.

PROOF.

1°⇒2°. There exists a **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. We have

$$\operatorname{dom}(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \mathcal{A} \sqcap \top(\operatorname{Base}(\mathcal{A})) = \mathcal{A}$$

and

$$\operatorname{im}(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im}(\mathsf{FCD})(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im}(\uparrow^{\mathsf{FCD}} f)|_{\mathcal{A}} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A} \sqsupseteq \mathcal{B}.$$

Thus $(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}}$ is a monovalued reloid such that $\operatorname{dom}(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \mathcal{A}$ and $\operatorname{im}(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} \sqsupseteq \mathcal{B}$.

2°⇒3°, 4°⇒3°, 5°⇒6°, 7°⇒6°. Obvious.

3°⇒1°. We have $\mathcal{B} \sqsubseteq \langle (\mathsf{FCD}) f \rangle \mathcal{A}$ for a monovalued reloid $f \in \mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{B}))$. Then there exists a **Set**-morphism $F : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} F \rangle \mathcal{A}$ that is $\mathcal{A} \geq_1 \mathcal{B}$.

6°⇒7°. Let f be an injective reloid such that $\operatorname{im} f \sqsubseteq \mathcal{A}$ and $\operatorname{dom} f \sqsupseteq \mathcal{B}$. Then $\operatorname{im} f|_{\mathcal{B}} \sqsubseteq \mathcal{A}$ and $\operatorname{dom} f|_{\mathcal{B}} = \mathcal{B}$. So $f|_{\mathcal{B}} \in \operatorname{Hom}_{\mathbf{CoMonRld}_{\sqsubseteq,=}}(\mathcal{A}, \mathcal{B})$.

2°⇔5°, 3°⇔6°, 4°⇔7°. By the lemma.

□

THEOREM 1291. For every filters $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1°. $\mathcal{A} \geq_2 \mathcal{B}$.
2°. $\operatorname{Hom}_{\mathbf{MonRld}_{=,=}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.
3°. $\operatorname{Hom}_{\mathbf{CoMonRld}_{=,=}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.

PROOF.

1°⇒2°. Let $\mathcal{A} \geq_2 \mathcal{B}$ that is $\mathcal{B} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$ for some **Set**-morphism $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$. Then $\operatorname{dom}(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \mathcal{A}$ and

$$\operatorname{im}(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im}(\mathsf{FCD})(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im}(\uparrow^{\mathsf{FCD}} f)|_{\mathcal{A}} = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A} = \mathcal{B}.$$

So $(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}}$ is a sought for reloid.

2°⇒1°. There exists a monovalued reloid f with domain $\mathcal{A}$ such that $\langle (\mathsf{FCD}) f \rangle \mathcal{A} = \mathcal{B}$. By corollary 1327 below, there exists a **Set**-morphism $F : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $f = (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}$. Thus

$$\langle \uparrow^{\mathsf{FCD}} F \rangle \mathcal{A} = \operatorname{im}(\uparrow^{\mathsf{FCD}} F)|_{\mathcal{A}} = \operatorname{im}(\mathsf{FCD})(\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}} = \operatorname{im}(\mathsf{FCD}) f = \operatorname{im} f = \mathcal{B}.$$

Thus $\mathcal{A} \geq_2 \mathcal{B}$ is testified by the morphism F.

2°⇔3°. By the lemma.

□

THEOREM 1292. The following are categories (with reloid composition):

1°. $\mathbf{MonRld}_{\sqsubseteq,\sqsupseteq}$;
2°. $\mathbf{MonRld}_{\sqsubseteq,=}$;
3°. $\mathbf{MonRld}_{=,=}$;
4°. $\mathbf{CoMonRld}_{\sqsubseteq,\sqsupseteq}$;
5°. $\mathbf{CoMonRld}_{\sqsubseteq,=}$;
6°. $\mathbf{CoMonRld}_{=,=}$.

PROOF. We will prove only the first three. The rest follow from duality. We need to prove only that composition of morphisms is a morphism, because associativity and existence of identity morphism are evident. We have:

1°. Let $f \in \operatorname{Hom}_{\mathbf{MonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{A}, \mathcal{B})$, $g \in \operatorname{Hom}_{\mathbf{MonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{B}, \mathcal{C})$. Then $\operatorname{dom} f \sqsubseteq \mathcal{A}$, $\operatorname{im} f \sqsupseteq \mathcal{B}$, $\operatorname{dom} g \sqsubseteq \mathcal{B}$, $\operatorname{im} g \sqsupseteq \mathcal{C}$. So $\operatorname{dom}(g \circ f) \sqsubseteq \mathcal{A}$, $\operatorname{im}(g \circ f) \sqsupseteq \mathcal{C}$ that is $g \circ f \in \operatorname{Hom}_{\mathbf{MonRld}_{\sqsubseteq,\sqsupseteq}}(\mathcal{A}, \mathcal{C})$.

2°. Let $f \in \operatorname{Hom}_{\mathbf{MonRld}_{\sqsubseteq,=}}(\mathcal{A}, \mathcal{B})$, $g \in \operatorname{Hom}_{\mathbf{MonRld}_{\sqsubseteq,=}}(\mathcal{B}, \mathcal{C})$. Then $\operatorname{dom} f \sqsubseteq \mathcal{A}$, $\operatorname{im} f = \mathcal{B}$, $\operatorname{dom} g \sqsubseteq \mathcal{B}$, $\operatorname{im} g = \mathcal{C}$. So $\operatorname{dom}(g \circ f) \sqsubseteq \mathcal{A}$, $\operatorname{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \operatorname{Hom}_{\mathbf{MonRld}_{\sqsubseteq,=}}(\mathcal{A}, \mathcal{C})$.

3°. Let $f \in \operatorname{Hom}_{\mathbf{MonRld}_{=,=}}(\mathcal{A}, \mathcal{B})$, $g \in \operatorname{Hom}_{\mathbf{MonRld}_{=,=}}(\mathcal{B}, \mathcal{C})$. Then $\operatorname{dom} f = \mathcal{A}$, $\operatorname{im} f = \mathcal{B}$, $\operatorname{dom} g = \mathcal{B}$, $\operatorname{im} g = \mathcal{C}$. So $\operatorname{dom}(g \circ f) = \mathcal{A}$, $\operatorname{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \operatorname{Hom}_{\mathbf{MonRld}_{=,=}}(\mathcal{A}, \mathcal{C})$.

□

DEFINITION 1293. Let **BijRld** be the groupoid of all bijections of the category of reloid triples. Its objects are filters and its morphisms from a filter $\mathcal{A}$ to filter $\mathcal{B}$ are monovalued injective reloids f such that $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{im} f = \mathcal{B}$.

THEOREM 1294. Filters $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff $\operatorname{Hom}_{\mathbf{BijRld}}(\mathcal{A}, \mathcal{B}) \neq \emptyset$.

PROOF.

$\Rightarrow$. Let $\mathcal{A}$ and $\mathcal{B}$ be isomorphic. Then there are sets $A \in \mathcal{A}$, $B \in \mathcal{B}$ and a bijective **Set**-morphism $F : A \to B$ such that $\langle F \rangle^* : \mathscr{P}A \cap \mathcal{A} \to \mathscr{P}B \cap \mathcal{B}$ is a bijection.

Obviously $f = (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}$ is monovalued and injective.

$$\operatorname{im} f = \bigsqcap^{\mathfrak{F}} \left\{ \frac{\operatorname{im} G}{G \in \operatorname{up}(\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}} \right\} = \bigsqcap^{\mathfrak{F}} \left\{ \frac{\operatorname{im}(H \cap F|_X)}{H \in \operatorname{up}(\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}, X \in \mathcal{A}} \right\} = \bigsqcap^{\mathfrak{F}} \left\{ \frac{\operatorname{im} F|_P}{P \in \mathcal{A}} \right\} = \bigsqcap^{\mathfrak{F}} \left\{ \frac{\langle F \rangle^* P}{P \in \mathcal{A}} \right\} = \bigsqcap^{\mathfrak{F}} \left\{ \frac{\langle F \rangle^* P}{P \in \mathscr{P}A \cap \mathcal{A}} \right\} = \bigsqcap^{\mathfrak{F}} (\mathscr{P}B \cap \mathcal{B}) = \bigsqcap^{\mathfrak{F}} \mathcal{B} = \mathcal{B}.$$

Thus $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{im} f = \mathcal{B}$.

$\Leftarrow$. Let f be a monovalued injective reloid such that $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{im} f = \mathcal{B}$. Then there exist a function F' and an injective binary relation F'' such that $F', F'' \in f$. Thus $F = F' \cap F''$ is an injection such that $F \in f$. The function F is a bijection from $A = \operatorname{dom} F$ to $B = \operatorname{im} F$. The function $\langle F \rangle^*$ is an injection on $\mathscr{P}A \cap \mathcal{A}$ (and moreover on $\mathscr{P}A$). It's simple to show that $\forall X \in \mathscr{P}A \cap \mathcal{A} : \langle F \rangle^* X \in \mathscr{P}B \cap \mathcal{B}$ and similarly

$$\forall Y \in \mathscr{P}B \cap \mathcal{B} : (\langle F \rangle^*)^{-1} Y = \langle F^{-1} \rangle^* Y \in \mathscr{P}A \cap \mathcal{A}.$$

Thus $\langle F \rangle^*|_{\mathscr{P}A \cap \mathcal{A}}$ is a bijection $\mathscr{P}A \cap \mathcal{A} \to \mathscr{P}B \cap \mathcal{B}$. So filters $\mathcal{A}$ and $\mathcal{B}$ are isomorphic.

□

PROPOSITION 1295. $(\geq_1) = (\sqsupseteq) \circ (\geq_2)$ (when we limit to small filters).

PROOF. $\mathcal{A} \geq_1 \mathcal{B}$ iff exists a function $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. But $\mathcal{B} \sqsubseteq \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$ is equivalent to $\exists \mathcal{B}' \in \mathscr{F} : (\mathcal{B}' \sqsupseteq \mathcal{B} \wedge \mathcal{B}' = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A})$. So $\mathcal{A} \geq_1 \mathcal{B}$ is equivalent to existence of $\mathcal{B}' \in \mathscr{F}$ such that $\mathcal{B}' \sqsupseteq \mathcal{B}$ and existence of a function $f : \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B}' = \langle \uparrow^{\mathsf{FCD}} f \rangle \mathcal{A}$. This is equivalent to $\mathcal{A}\ ((\sqsupseteq) \circ (\geq_2))\ \mathcal{B}$. □

PROPOSITION 1296. If a and b are ultrafilters then $b \geq_1 a \Leftrightarrow b \geq_2 a$.

PROOF. We need to prove only $b \geq_1 a \Rightarrow b \geq_2 a$. If $b \geq_1 a$ then there exists a monovalued reloid $f : \operatorname{Base}(b) \to \operatorname{Base}(a)$ such that $\operatorname{dom} f = b$ and $\operatorname{im} f \sqsupseteq a$. Then $\operatorname{im} f = \operatorname{im}(\mathsf{FCD}) f \in \{\bot^{\mathscr{F}(\operatorname{Base}(a))}\} \cup \operatorname{atoms}^{\mathscr{F}(\operatorname{Base}(a))}$ because $(\mathsf{FCD}) f$ is a monovalued funcoid. So $\operatorname{im} f = a$ (taken into account $\operatorname{im} f \neq \bot^{\mathscr{F}(\operatorname{Base}(a))}$) and thus $b \geq_2 a$. □

COROLLARY 1297. For atomic filters $\geq_1$ is the same as $\geq_2$.

Thus I will write simply $\geq$ for atomic filters.

14.1.1. Existence of no more than one monovalued injective reloid for a given pair of ultrafilters.

14.1.1.1. *The lemmas.* The lemmas in this section were provided to me by Robert Martin Solovay in [39]. They are based on Wistar Comfort's work.

In this section we will assume μ is an ultrafilter on a set I and function $f : I \to I$ has the property $X \in \mu \Leftrightarrow \langle f^{-1} \rangle^* X \in \mu$.

Lemma 1298. If $X \in \mu$ then $X \cap \langle f \rangle^* X \in \mu$.

Proof. If $\langle f \rangle^* X \notin \mu$ then $X \subseteq \langle f^{-1} \rangle^* \langle f \rangle^* X \notin \mu$ and so $X \notin \mu$. Thus $X \in \mu \wedge \langle f \rangle^* X \in \mu$ and consequently $X \cap \langle f \rangle^* X \in \mu$. □

We will say that x is *periodic* when $f^n(x) = x$ for some positive integer x. The least such n is called *the period* of x.

Let's define $x \sim y$ iff there exist $i, j \in \mathbb{N}$ such that $f^i(x) = f^j(y)$. Trivially it is an equivalence relation. If x and y are periodic, then $x \sim y$ iff exists $n \in \mathbb{N}$ such that $f^n(y) = x$.

Let $A = \left\{ \frac{x \in I}{x \text{ is periodic with period} > 1} \right\}$.

We will show $A \notin \mu$. Let's assume $A \in \mu$.

Let a set $D \subseteq A$ contains (by the axiom of choice) exactly one element from each equivalence class of A defined by the relation $\sim$.

Let α be a function $A \to \mathbb{N}$ defined as follows. Let $x \in A$. Let y be the unique element of D such that $x \sim y$. Let $\alpha(x)$ be the least $n \in \mathbb{N}$ such that $f^n(y) = x$.

Let $B_0 = \left\{ \frac{x \in A}{\alpha(x) \text{ is even}} \right\}$ and $B_1 = \left\{ \frac{x \in A}{\alpha(x) \text{ is odd}} \right\}$.

Let $B_2 = \left\{ \frac{x \in A}{\alpha(x) = 0} \right\}$.

Lemma 1299. $B_0 \cap \langle f \rangle^* B_0 \subseteq B_2$.

Proof. If $x \in B_0 \cap \langle f \rangle^* B_0$ then for a minimal even n and $x = f(x')$ where $f^m(y') = x'$ for a minimal even m. Thus $f^n(y) = f(x')$ thus y and x' laying in the same equivalence class and thus $y = y'$. So we have $f^n(y) = f^{m+1}(y)$. Thus $n \leq m + 1$ by minimality.

x' lies on an orbit and thus $x' = f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit; $f^m(y) = f^{-1}(x)$ and thus $x' = f^{n-1}(y)$ thus $n - 1 \geq m$ by minimality or $n = 0$.

Thus $n = m + 1$ what is impossible for even n and m. We have a contradiction what proves $B_0 \cap \langle f \rangle^* B_0 \subseteq \emptyset$.

Remained the case $n = 0$, then $x = f^0(y)$ and thus $\alpha(x) = 0$. □

Lemma 1300. $B_1 \cap \langle f \rangle^* B_1 = \emptyset$.

Proof. Let $x \in B_1 \cap \langle f \rangle^* B_1$. Then $f^n(y) = x$ for an odd n and $x = f(x')$ where $f^m(y') = x'$ for an odd m. Thus $f^n(y) = f(x')$ thus y and x' laying in the same equivalence class and thus $y = y'$. So we have $f^n(y) = f^{m+1}(y)$. Thus $n \leq m + 1$ by minimality.

x' lies on an orbit and thus $x' = f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit;

$f^m(y) = f^{-1}(x)$ and thus $x' = f^{n-1}(y)$ thus $n - 1 \geq m$ by minimality ($n = 0$ is impossible because n is odd).

Thus $n = m + 1$ what is impossible for odd n and m. We have a contradiction what proves $B_1 \cap \langle f \rangle^* B_1 = \emptyset$. □

Lemma 1301. $B_2 \cap \langle f \rangle^* B_2 = \emptyset$.

Proof. Let $x \in B_2 \cap \langle f \rangle^* B_2$. Then $x = y$ and $x' = y$ where $x = f(x')$. Thus $x = f(x)$ and so $x \notin A$ what is impossible. □

LEMMA 1302. $A \notin \mu$.

PROOF. Suppose $A \in \mu$.
Since $A \in \mu$ we have $B_0 \in \mu$ or $B_1 \in \mu$.
So either $B_0 \cap \langle f \rangle^* B_0 \subseteq B_2$ or $B_1 \cap \langle f \rangle^* B_1 \subseteq B_2$. As such by the lemma 1298 we have $B_2 \in \mu$. This is incompatible with $B_2 \cap \langle f \rangle^* B_2 = \emptyset$. So we got a contradiction. □

Let C be the set of points x which are not periodic but $f^n(x)$ is periodic for some positive n.

LEMMA 1303. $C \notin \mu$.

PROOF. Let β be a function $C \to \mathbb{N}$ such that $\beta(x)$ is the least $n \in \mathbb{N}$ such that $f^n(x)$ is periodic.
Let $C_0 = \left\{ \frac{x \in C}{\beta(x) \text{ is even}} \right\}$ and $C_1 = \left\{ \frac{x \in C}{\beta(x) \text{ is odd}} \right\}$.
Obviously $C_j \cap \langle f \rangle^* C_j = \emptyset$ for $j = 0, 1$. Hence by lemma 1298 we have $C_0, C_1 \notin \mu$ and thus $C = C_0 \cup C_1 \notin \mu$. □

Let E be the set of $x \in I$ such that for no $n \in \mathbb{N}$ we have $f^n(x)$ periodic.

LEMMA 1304. Let $x, y \in E$ be such that $f^i(x) = f^j(y)$ and $f^{i'}(x) = f^{j'}(y)$ for some $i, j, i', j' \in \mathbb{N}$. Then $i - j = i' - j'$.

PROOF. $i \mapsto f^i(x)$ is a bijection.
So $y = f^{i-j}(y)$ and $y = f^{i'-j'}(y)$. Thus $f^{i-j}(y) = f^{i'-j'}(y)$ and so $i - j = i' - j'$. □

LEMMA 1305. $E \notin \mu$.

PROOF. Let $D' \subseteq E$ be a subset of E with exactly one element from each equivalence class of the relation $\sim$ on E.
Define the function $\gamma : E \to \mathbb{Z}$ as follows. Let $x \in E$. Let y be the unique element of D' such that $x \sim y$. Choose $i, j \in \mathbb{N}$ such that $f^i(y) = f^j(x)$. Let $\gamma(x) = i - j$. By the last lemma, γ is well-defined.
It is clear that if $x \in E$ then $f(x) \in E$ and moreover $\gamma(f(x)) = \gamma(x) + 1$.
Let $E_0 = \left\{ \frac{x \in E}{\gamma(x) \text{ is even}} \right\}$ and $E_1 = \left\{ \frac{x \in E}{\gamma(x) \text{ is odd}} \right\}$.
We have $E_0 \cap \langle f \rangle^* E_0 = \emptyset \notin \mu$ and hence $E_0 \notin \mu$.
Similarly $E_1 \notin \mu$.
Thus $E = E_0 \cup E_1 \notin \mu$. □

LEMMA 1306. f is the identity function on a set in μ.

PROOF. We have shown $A, C, E \notin \mu$. But the points which lie in none of these sets are exactly points periodic with period 1 that is fixed points of f. Thus the set of fixed points of f belongs to the filter μ. □

14.1.1.2. *The main theorem and its consequences.*

THEOREM 1307. For every ultrafilter a the morphism $(a, a, \mathrm{id}_a^{\mathsf{FCD}})$ is the only

1°. monovalued morphism of the category of reloid triples from a to a;
2°. injective morphism of the category of reloid triples from a to a;
3°. bijective morphism of the category of reloid triples from a to a.

PROOF. We will prove only 1° because the rest follow from it.
Let f be a monovalued morphism of reloid triples from a to a. Then it exists a **Set**-morphism F such that $F \in f$. Trivially $\langle \uparrow^{\mathsf{FCD}} F \rangle a \sqsupseteq a$ and thus $\langle F \rangle^* A \in a$ for every $A \in a$. Thus by the lemma we have that F is the identity function on a set in a and so obviously f is an identity. □

Corollary 1308. For every two atomic filters (with possibly different bases) $\mathcal{A}$ and $\mathcal{B}$ there exists at most one bijective reloid triple from $\mathcal{A}$ to $\mathcal{B}$.

Proof. Suppose that f and g are two different bijective reloids from $\mathcal{A}$ to $\mathcal{B}$. Then $g^{-1}\circ f$ is not the identity reloid (otherwise $g^{-1}\circ f = \mathrm{id}^{\mathsf{RLD}}_{\operatorname{dom} f}$ and so $f = g$ because f and g are isomorphisms). But $g^{-1}\circ f$ is a bijective reloid (as a composition of bijective reloids) from $\mathcal{A}$ to $\mathcal{A}$ what is impossible. □

14.2. Rudin-Keisler equivalence and Rudin-Keisler order

Theorem 1309. Atomic filters a and b (with possibly different bases) are isomorphic iff $a \geq b \wedge b \geq a$.

Proof. Let $a \geq b \wedge b \geq a$. Then there are a monovalued reloids f and g such that $\operatorname{dom} f = a$ and $\operatorname{im} f = b$ and $\operatorname{dom} g = b$ and $\operatorname{im} g = a$. Thus $g \circ f$ and $f \circ g$ are monovalued morphisms from a to a and from b to b. By the above we have $g \circ f = \mathrm{id}^{\mathsf{RLD}}_a$ and $f \circ g = \mathrm{id}^{\mathsf{RLD}}_b$ so $g = f^{-1}$ and $f^{-1}\circ f = \mathrm{id}^{\mathsf{RLD}}_a$ and $f \circ f^{-1} = \mathrm{id}^{\mathsf{RLD}}_b$. Thus f is an injective monovalued reloid from a to b and thus a and b are isomorphic. □

The last theorem cannot be generalized from atomic filters to arbitrary filters, as it's shown by the following example:

Example 1310. $\mathcal{A} \geq_1 \mathcal{B} \wedge \mathcal{B} \geq_1 \mathcal{A}$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ for some filters $\mathcal{A}$ and $\mathcal{B}$.

Proof. Consider $\mathcal{A} = \uparrow^{\mathbb{R}} [0;1]$ and $\mathcal{B} = \bigsqcap\left\{ \frac{\uparrow^{\mathbb{R}}[0;1+\epsilon[}{\epsilon>0} \right\}$. Then the function $f = \lambda x \in \mathbb{R} : x/2$ witnesses both inequalities $\mathcal{A} \geq_1 \mathcal{B}$ and $\mathcal{B} \geq_1 \mathcal{A}$. But these filters cannot be isomorphic because only one of them is principal. □

Lemma 1311. Let f_0 and f_1 be **Set**-morphisms. Let $f(x,y) = (f_0 x, f_1 y)$ for a function f. Then

$$\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f_0 \times \operatorname{Src} f_1, \operatorname{Dst} f_0 \times \operatorname{Dst} f_1)} f \right\rangle (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \langle \uparrow^{\mathsf{FCD}} f_0 \rangle \mathcal{A} \times^{\mathsf{RLD}} \langle \uparrow^{\mathsf{FCD}} f_1 \rangle \mathcal{B}.$$

Proof.

$$\begin{aligned}
&\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f_0 \times \operatorname{Src} f_1, \operatorname{Dst} f_0 \times \operatorname{Dst} f_1)} f \right\rangle (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \\
&\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f_0 \times \operatorname{Src} f_1, \operatorname{Dst} f_0 \times \operatorname{Dst} f_1)} f \right\rangle \bigsqcap\left\{ \frac{\uparrow^{\operatorname{Src} f_0 \times \operatorname{Src} f_1} (A \times B)}{A \in \mathcal{A}, B \in \mathcal{B}} \right\} = \\
&\bigsqcap\left\{ \frac{\uparrow^{\operatorname{Dst} f_0 \times \operatorname{Dst} f_1} \langle f \rangle^* (A \times B)}{A \in \mathcal{A}, B \in \mathcal{B}} \right\} = \\
&\bigsqcap\left\{ \frac{\uparrow^{\operatorname{Dst} f_0 \times \operatorname{Dst} f_1} (\langle f_0 \rangle^* A \times \langle f_1 \rangle^* B)}{A \in \mathcal{A}, B \in \mathcal{B}} \right\} = \\
&\bigsqcap\left\{ \frac{\uparrow^{\operatorname{Dst} f_0} \langle f_0 \rangle^* A \times \uparrow^{\operatorname{Dst} f_1} \langle f_1 \rangle^* B)}{A \in \mathcal{A}, B \in \mathcal{B}} \right\} = \text{ (theorem 891)} \\
&\bigsqcap\left\{ \frac{\uparrow^{\operatorname{Dst} f_0} \langle f_0 \rangle^* A}{A \in \mathcal{A}} \right\} \times^{\mathsf{RLD}} \bigsqcap\left\{ \frac{\uparrow^{\operatorname{Dst} f_1} \langle f_1 \rangle^* B}{B \in \mathcal{B}} \right\} = \\
&\langle \uparrow^{\mathsf{FCD}} f_0 \rangle \mathcal{A} \times^{\mathsf{RLD}} \langle \uparrow^{\mathsf{FCD}} f_1 \rangle \mathcal{B}.
\end{aligned}$$

□

Theorem 1312. Let f be a monovalued reloid. Then $\operatorname{GR} f$ is isomorphic to the filter $\operatorname{dom} f$.

PROOF. Let f be a monovalued reloid. There exists a function $F \in \operatorname{GR} f$. Consider the bijective function $p = \lambda x \in \operatorname{dom} F : (x, Fx)$.

$\langle p \rangle^* \operatorname{dom} F = F$ and consequently

$$\langle p \rangle \operatorname{dom} f = \bigsqcap_{K \in \operatorname{up} f}^{\mathsf{RLD}} \langle p \rangle^* \operatorname{dom} K = \bigsqcap_{K \in \operatorname{up} f}^{\mathsf{RLD}} \langle p \rangle^* \operatorname{dom}(K \cap F) = \bigsqcap_{K \in \operatorname{up} f}^{\mathsf{RLD}} (K \cap F) = \bigsqcap_{K \in \operatorname{up} f}^{\mathsf{RLD}} K = f.$$

Thus p witnesses that f is isomorphic to the filter $\operatorname{dom} f$. □

COROLLARY 1313. The graph of a monovalued reloid with atomic domain is atomic.

COROLLARY 1314. $\operatorname{id}^{\mathsf{RLD}}_{\mathcal{A}}$ is isomorphic to $\mathcal{A}$ for every filter $\mathcal{A}$.

THEOREM 1315. There are atomic filters incomparable by Rudin-Keisler order. (Elements a and b are *incomparable* when $a \not\sqsubseteq b \wedge b \not\sqsubseteq a$.)

PROOF. See [13]. □

THEOREM 1316. $\geq_1$ and $\geq_2$ are different relations.

PROOF. Consider a is an arbitrary non-empty filter. Then $a \geq_1 \perp^{\mathscr{F}(\operatorname{Base}(a))}$ but not $a \geq_2 \perp^{\mathscr{F}(\operatorname{Base}(a))}$. □

PROPOSITION 1317. If $a \geq_2 b$ where a is an ultrafilter then b is also an ultrafilter.

PROOF. $b = \left\langle \uparrow^{\mathsf{FCD}} f \right\rangle a$ for some $f : \operatorname{Base}(a) \to \operatorname{Base}(b)$. So b is an ultrafilter since f is monovalued. □

COROLLARY 1318. If $a \geq_1 b$ where a is an ultrafilter then b is also an ultrafilter or $\perp^{\mathscr{F}(\operatorname{Base}(a))}$.

PROOF. $b \sqsubseteq \left\langle \uparrow^{\mathsf{FCD}} f \right\rangle a$ for some $f : \operatorname{Base}(a) \to \operatorname{Base}(b)$. Therefore $b' = \left\langle \uparrow^{\mathsf{FCD}} f \right\rangle a$ is an ultrafilter. From this our statement follows. □

PROPOSITION 1319. Principal filters, generated by sets of the same cardinality, are isomorphic.

PROOF. Let A and B be sets of the same cardinality. Then there are a bijection f from A to B. We have $\langle f \rangle^* A = B$ and thus A and B are isomorphic. □

PROPOSITION 1320. If a filter is isomorphic to a principal filter, then it is also a principal filter induced by a set with the same cardinality.

PROOF. Let A be a principal filter and B is a filter isomorphic to A. Then there are sets $X \in A$ and $Y \in B$ such that there are a bijection $f : X \to Y$ such that $\langle f \rangle^* A = B$.

So $\min B$ exists and $\min B = \langle f \rangle^* \min A$ and thus B is a principal filter (of the same cardinality as A). □

Proposition 1321. A filter isomorphic to a non-trivial ultrafilter is a non-trivial ultrafilter.

Proof. Let a be a non-trivial ultrafilter and a be isomorphic to b. Then $a \geq_2 b$ and thus b is an ultrafilter. The filter b cannot be trivial because otherwise a would be also trivial. □

Theorem 1322. For an infinite set U there exist $2^{2^{\operatorname{card} U}}$ equivalence classes of isomorphic ultrafilters.

Proof. The number of bijections between any two given subsets of U is no more than $(\operatorname{card} U)^{\operatorname{card} U} = 2^{\operatorname{card} U}$. The number of bijections between all pairs of subsets of U is no more than $2^{\operatorname{card} U} \cdot 2^{\operatorname{card} U} = 2^{\operatorname{card} U}$. Therefore each isomorphism class contains at most $2^{\operatorname{card} U}$ ultrafilters. But there are $2^{2^{\operatorname{card} U}}$ ultrafilters. So there are $2^{2^{\operatorname{card} U}}$ classes. □

Remark 1323. One of the above mentioned equivalence classes contains trivial ultrafilters.

Corollary 1324. There exist non-isomorphic nontrivial ultrafilters on any infinite set.

14.3. Consequences

Theorem 1325. The graph of reloid $\mathcal{F} \times^{\mathsf{RLD}} \uparrow^A \{a\}$ is isomorphic to the filter $\mathcal{F}$ for every set A and $a \in A$.

Proof. From 1312. □

Theorem 1326. If f, g are reloids, $f \sqsubseteq g$ and g is monovalued then $g|_{\operatorname{dom} f} = f$.

Proof. It's simple to show that $f = \bigsqcup \left\{ \frac{f|_a}{a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}} \right\}$ (use the fact that $k \sqsubseteq f|_a$ for some $a \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Src} f)}$ for every $k \in \operatorname{atoms} f$ and the fact that $\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)$ is atomistic).

Suppose that $g|_{\operatorname{dom} f} \neq f$. Then there exists $a \in \operatorname{atoms} \operatorname{dom} f$ such that $g|_a \neq f|_a$.

Obviously $g|_a \sqsupseteq f|_a$.

If $g|_a \sqsupset f|_a$ then $g|_a$ is not atomic (because $f|_a \neq \bot^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)}$) what contradicts to a theorem above. So $g|_a = f|_a$ what is a contradiction and thus $g|_{\operatorname{dom} f} = f$. □

Corollary 1327. Every monovalued reloid is a restricted principal monovalued reloid.

Proof. Let f be a monovalued reloid. Then there exists a function $F \in \operatorname{GR} f$. So we have

$$(\uparrow^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)} F)|_{\operatorname{dom} f} = f.$$

□

Corollary 1328. Every monovalued injective reloid is a restricted injective monovalued principal reloid.

Proof. Let f be a monovalued injective reloid. There exists a function F such that $f = (\uparrow^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)} F)|_{\operatorname{dom} f}$. Also there exists an injection $G \in \operatorname{up} f$.

Thus

$$f = f \sqcap (\uparrow^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)} G)|_{\operatorname{dom} f} = \\ (\uparrow^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)} F)|_{\operatorname{dom} f} \sqcap (\uparrow^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)} G)|_{\operatorname{dom} f} = \\ (\uparrow^{\mathsf{RLD}(\operatorname{Src} f, \operatorname{Dst} f)} (F \sqcap G))|_{\operatorname{dom} f}.$$

Obviously $F \sqcap G$ is an injection. □

THEOREM 1329. If a reloid f is monovalued and $\operatorname{dom} f$ is an principal filter then f is principal.

PROOF. f is a restricted principal monovalued reloid. Thus $f = F|_{\operatorname{dom} f}$ where F is a principal monovalued reloid. Thus f is principal. □

LEMMA 1330. If a filter $\mathcal{A}$ is isomorphic to a filter $\mathcal{B}$ then if X is a typed set then there exists a typed set Y such that $\uparrow^{\operatorname{Base}(\mathcal{A})} X \sqcap \mathcal{A}$ is a filter isomorphic to $\uparrow^{\operatorname{Base}(\mathcal{B})} Y \sqcap \mathcal{B}$.

PROOF. Let f be a monovalued injective reloid such that $\operatorname{dom} f = \mathcal{A}$, $\operatorname{im} f = \mathcal{B}$. By proposition 629 we have: $\uparrow^{\operatorname{Base}(\mathcal{A})} X \sqcap \mathcal{A} = \mathcal{X}$ where $\mathcal{X}$ is a filter complementive to $\mathcal{A}$. Let $\mathcal{Y} = \mathcal{A} \setminus \mathcal{X}$.

$\langle(\mathsf{FCD})f\rangle\mathcal{X} \sqcap \langle(\mathsf{FCD})f\rangle\mathcal{Y} = \langle(\mathsf{FCD})f\rangle(\mathcal{X} \sqcap \mathcal{Y}) = \bot$ by injectivity of f.

$\langle(\mathsf{FCD})f\rangle\mathcal{X} \sqcup \langle(\mathsf{FCD})f\rangle\mathcal{Y} = \langle(\mathsf{FCD})f\rangle(\mathcal{X} \sqcup \mathcal{Y}) = \langle(\mathsf{FCD})f\rangle\mathcal{A} = \mathcal{B}$. So $\langle(\mathsf{FCD})f\rangle\mathcal{X}$ is a filter complementive to $\mathcal{B}$. So by proposition 629 there exists a set Y such that $\langle(\mathsf{FCD})f\rangle\mathcal{X} = \uparrow Y \sqcap \mathcal{B}$.

$f|_{\mathcal{X}}$ is obviously a monovalued injective reloid with $\operatorname{dom}(f|_{\mathcal{X}}) = \uparrow X \sqcap \mathcal{A}$ and $\operatorname{im}(f|_{\mathcal{X}}) = \uparrow Y \sqcap \mathcal{B}$. So $\uparrow X \sqcap \mathcal{A}$ is isomorphic to $\uparrow Y \sqcap \mathcal{B}$. □

EXAMPLE 1331. $\mathcal{A} \geq_2 \mathcal{B} \wedge \mathcal{B} \geq_2 \mathcal{A}$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ for some filters $\mathcal{A}$ and $\mathcal{B}$.

PROOF. (proof idea by ANDREAS BLASS, rewritten using reloids by me)

Let u_n, h_n with n ranging over the set $\mathbb{Z}$ be sequences of ultrafilters on $\mathbb{N}$ and functions $\mathbb{N} \to \mathbb{N}$ such that $\langle\uparrow^{\mathsf{FCD}(\mathbb{N},\mathbb{N})} h_n\rangle u_{n+1} = u_n$ and u_n are pairwise non-isomorphic. (See [6] for a proof that such ultrafilters and functions exist.)

$\mathcal{A} \stackrel{\text{def}}{=} \bigsqcup_{n\in\mathbb{Z}}(\uparrow^{\mathbb{Z}} \{n\} \times^{\mathsf{RLD}} u_{2n+1})$; $\mathcal{B} \stackrel{\text{def}}{=} \bigsqcup_{n\in\mathbb{Z}}(\uparrow^{\mathbb{Z}} \{n\} \times^{\mathsf{RLD}} u_{2n})$.

Let the **Set**-morphisms $f, g : \mathbb{Z} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$ be defined by the formulas $f(n, x) = (n, h_{2n}x)$ and $g(n, x) = (n - 1, h_{2n-1}x)$.

Using the fact that every function induces a complete funcoid and a lemma above we get:

$$\begin{aligned} \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{A} = \\ \bigsqcup\langle\langle\uparrow^{\mathsf{FCD}} f\rangle\rangle^{*}\left\{\frac{\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n+1}}{n\in\mathbb{Z}}\right\} = \\ \bigsqcup\left\{\frac{\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n}}{n\in\mathbb{Z}}\right\} = \\ \mathcal{B}. \end{aligned}$$

$$\begin{aligned} \langle\uparrow^{\mathsf{FCD}} g\rangle\mathcal{B} = \\ \bigsqcup\langle\langle\uparrow^{\mathsf{FCD}} g\rangle\rangle^{*}\left\{\frac{\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n}}{n\in\mathbb{Z}}\right\} = \\ \bigsqcup\left\{\frac{\uparrow^{\mathbb{Z}}\{n-1\}\times^{\mathsf{RLD}}u_{2n-1}}{n\in\mathbb{Z}}\right\} = \\ \bigsqcup\left\{\frac{\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n+1}}{n\in\mathbb{Z}}\right\} = \\ \mathcal{A}. \end{aligned}$$

It remains to show that $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic.

Let $X\in\operatorname{up}(\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n+1})$ for some $n\in\mathbb{Z}$. Then if $\uparrow^{\mathbb{Z}\times\mathbb{N}}X\sqcap\mathcal{A}$ is an ultrafilter we have $\uparrow^{\mathbb{Z}\times\mathbb{N}}X\sqcap\mathcal{A}=\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n+1}$ and thus by the theorem 1325 is isomorphic to u_{2n+1}.

If $X\notin\operatorname{up}(\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n+1})$ for every $n\in\mathbb{Z}$ then $(\mathbb{Z}\times\mathbb{N})\setminus X\in\operatorname{up}(\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n+1})$ and thus $(\mathbb{Z}\times\mathbb{N})\setminus X\in\operatorname{up}\mathcal{A}$ and thus $\uparrow^{\mathbb{Z}\times\mathbb{N}}X\sqcap\mathcal{A}=\bot^{\mathbb{Z}\times\mathbb{N}}$.

We have also

$$(\uparrow^{\mathbb{Z}}\{0\}\times^{\mathsf{RLD}}\mathbb{N})\sqcap\mathcal{B}=(\uparrow^{\mathbb{Z}}\{0\}\times^{\mathsf{RLD}}\mathbb{N})\sqcap\bigsqcup\left\{\frac{\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n}}{n\in\mathbb{Z}}\right\}=$$
$$\bigsqcup\left\{\frac{(\uparrow^{\mathbb{Z}}\{0\}\times^{\mathsf{RLD}}\mathbb{N})\sqcap(\uparrow^{\mathbb{Z}}\{n\}\times^{\mathsf{RLD}}u_{2n})}{n\in\mathbb{Z}}\right\}=\uparrow^{\mathbb{Z}}\{0\}\times^{\mathsf{RLD}}u_0\text{ (an ultrafilter).}$$

Thus every ultrafilter generated as intersecting $\mathcal{A}$ with a principal filter $\uparrow^{\mathbb{Z}\times\mathbb{N}}X$ is isomorphic to some u_{2n+1} and thus is not isomorphic to u_0. By the lemma it follows that $\mathcal{A}$ and $\mathcal{B}$ are non-isomorphic. □

14.3.1. Metamonovalued reloids.

PROPOSITION 1332. $(\bigcap G)\circ f=\bigcap_{g\in G}(g\circ f)$ for every function f and a set G of binary relations.

PROOF.

$$\begin{aligned} (x,z)\in\left(\bigcap G\right)\circ f\Leftrightarrow \\ \exists y:(fx=y\wedge(y,z)\in\bigcap G)\Leftrightarrow \\ (fx,z)\in\bigcap G\Leftrightarrow \\ \forall g\in G:(fx,z)\in g\Leftrightarrow \\ \forall g\in G\exists y:(fx=y\wedge(y,z)\in g)\Leftrightarrow \\ \forall g\in G:(x,z)\in g\circ f\Leftrightarrow \\ (x,z)\in\bigcap_{g\in G}(g\circ f). \end{aligned}$$

□

LEMMA 1333. $(\bigsqcap G) \circ f = \bigsqcap_{g \in G}(g \circ f)$ if f is a monovalued principal reloid and G is a set of reloids (with matching sources and destinations).

PROOF. Let $f = \uparrow^{\mathsf{RLD}} \varphi$ for some monovalued **Rel**-morphism φ.

$(\bigsqcap G) \circ f = \bigsqcap^{\mathsf{RLD}}_{g \in \operatorname{up} \bigsqcap G}(g \circ \varphi)$;

$$\operatorname{up} \bigsqcap_{g \in G}(g \circ f) =$$

$$\operatorname{up} \bigsqcap_{g \in G} \bigsqcap^{\mathsf{RLD}}_{\Gamma \in \operatorname{up} g}(\Gamma \circ \varphi) =$$

$$\operatorname{up} \bigsqcap \bigcup_{g \in G} \left\{ \frac{\uparrow^{\mathsf{RLD}} (\Gamma \circ \varphi)}{\Gamma \in \operatorname{up} g} \right\} =$$

$$\operatorname{up} \bigsqcap^{\mathsf{RLD}}_{\Gamma \in \operatorname{up} \bigsqcap G}(\Gamma \circ \varphi) =$$

$$\operatorname{up} \bigsqcap \left\{ \frac{(\Gamma_0 \circ \varphi) \sqcap \dots \sqcap (\Gamma_n \circ \varphi)}{\Gamma_i \in \operatorname{up} \bigsqcap G \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} = \quad \text{(proposition above)}$$

$$\operatorname{up} \bigsqcap \left\{ \frac{(\Gamma_0 \sqcap \dots \sqcap \Gamma_n) \circ \varphi}{\Gamma_i \in \operatorname{up} \bigsqcap G \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}} \right\} =$$

$$\operatorname{up} \bigsqcap \left\{ \frac{\Gamma \circ \varphi}{\Gamma \in \operatorname{up} \bigsqcap G} \right\}.$$

Thus $(\bigsqcap G) \circ f = \bigsqcap_{g \in G}(g \circ f)$. □

THEOREM 1334.

1°. Monovalued reloids are metamonovalued.
2°. Injective reloids are metainjective.

PROOF. We will prove only the first, as the second is dual.

Let G be a set of reloids and f be a monovalued reloid.

Let f' be a principal monovalued continuation of f (so that $f = f'|_{\operatorname{dom} f}$).

By the lemma $(\bigsqcap G) \circ f' = \bigsqcap_{g \in G}(g \circ f')$. Restricting this equality to $\operatorname{dom} f$ we get: $(\bigsqcap G) \circ f = \bigsqcap_{g \in G}(g \circ f)$. □

CONJECTURE 1335. Every metamonovalued reloid is monovalued.

CHAPTER 15

Counter-examples about funcoids and reloids

For further examples we will use the filter defined by the formula

$$\Delta = \bigsqcap^{\mathscr{F}(\mathbb{R})} \left\{ \frac{]-\epsilon;\epsilon[}{\epsilon \in \mathbb{R}, \epsilon > 0} \right\}.$$

I will denote $\Omega(A)$ the Fréchet filter on a set A.

EXAMPLE 1336. There exist a funcoid f and a set S of funcoids such that $f \sqcap \bigsqcup S \neq \bigsqcup \langle f \sqcap \rangle^* S$.

PROOF. Let $f = \Delta \times^{\mathsf{FCD}} \uparrow^{\mathscr{F}(\mathbb{R})} \{0\}$ and $S = \left\{ \frac{\uparrow^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}(]\epsilon;+\infty[\times\{0\})}{\epsilon\in\mathbb{R},\epsilon>0} \right\}$. Then

$$f \sqcap \bigsqcup S = (\Delta \times^{\mathsf{FCD}} \uparrow^{\mathscr{F}(\mathbb{R})} \{0\}) \sqcap \uparrow^{\mathsf{FCD}(\mathbb{R},\mathbb{R})} (]0;+\infty[\times\{0\}) = (\Delta \sqcap \uparrow^{\mathscr{F}(\mathbb{R})}]0;+\infty[) \times^{\mathsf{FCD}} \uparrow^{\mathscr{F}(\mathbb{R})} \{0\} \neq \bot^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}$$

while $\bigsqcup \langle f \sqcap \rangle^* S = \bigsqcup \{ \bot^{\mathsf{FCD}(\mathbb{R},\mathbb{R})} \} = \bot^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}$. □

EXAMPLE 1337. There exist a set R of funcoids and a funcoid f such that $f \circ \bigsqcup R \neq \bigsqcup \langle f \circ \rangle^* R$.

PROOF. Let $f = \Delta \times^{\mathsf{FCD}} \uparrow^{\mathscr{F}(\mathbb{R})} \{0\}$, $R = \left\{ \frac{\uparrow^{\mathbb{R}}\{0\}\times^{\mathsf{FCD}}\uparrow^{\mathbb{R}}]\epsilon;+\infty[}{\epsilon\in\mathbb{R},\epsilon>0} \right\}$.

We have $\bigsqcup R = \uparrow^{\mathbb{R}} \{0\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{R}}]0;+\infty[$; $f \circ \bigsqcup R = \uparrow^{\mathsf{FCD}(\mathbb{R},\mathbb{R})} (\{0\} \times \{0\}) \neq \bot^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}$ and $\bigsqcup \langle f \circ \rangle^* R = \bigsqcup \{ \bot^{\mathsf{FCD}(\mathbb{R},\mathbb{R})} \} = \bot^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}$. □

EXAMPLE 1338. There exist a set R of reloids and a reloid f such that $f \circ \bigsqcup R \neq \bigsqcup \langle f \circ \rangle^* R$.

PROOF. Let $f = \Delta \times^{\mathsf{RLD}} \uparrow^{\mathscr{F}(\mathbb{R})} \{0\}$, $R = \left\{ \frac{\uparrow^{\mathbb{R}}\{0\}\times^{\mathsf{RLD}}\uparrow^{\mathbb{R}}]\epsilon;+\infty[}{\epsilon\in\mathbb{R},\epsilon>0} \right\}$.

We have $\bigsqcup R = \uparrow^{\mathbb{R}} \{0\} \times^{\mathsf{RLD}} \uparrow^{\mathbb{R}}]0;+\infty[$; $f \circ \bigsqcup R = \uparrow^{\mathsf{RLD}(\mathbb{R},\mathbb{R})} (\{0\} \times \{0\}) \neq \bot^{\mathsf{RLD}(\mathbb{R},\mathbb{R})}$ and $\bigsqcup \langle f \circ \rangle^* R = \bigsqcup \{ \bot^{\mathsf{RLD}(\mathbb{R},\mathbb{R})} \} = \bot^{\mathsf{RLD}(\mathbb{R},\mathbb{R})}$. □

EXAMPLE 1339. There exist a set R of funcoids and filters $\mathcal{X}$ and $\mathcal{Y}$ such that

1°. $\mathcal{X} \left[\bigsqcup R \right] \mathcal{Y} \wedge \nexists f \in R : \mathcal{X} \left[f \right] \mathcal{Y}$;

2°. $\langle \bigsqcup R \rangle \mathcal{X} \sqsupset \bigsqcup \left\{ \frac{\langle f \rangle \mathcal{X}}{f \in R} \right\}$.

PROOF.

1°. Take $\mathcal{X} = \Delta$ and $\mathcal{Y} = \top^{\mathscr{F}(\mathbb{R})}$, $R = \left\{ \frac{\uparrow^{\mathsf{FCD}(\mathbb{R},\mathbb{R})}(]\epsilon;+\infty[\times\mathbb{R})}{\epsilon\in\mathbb{R},\epsilon>0} \right\}$. Then $\bigsqcup R = \uparrow^{\mathsf{FCD}(\mathbb{R},\mathbb{R})} (]0;+\infty[\times\mathbb{R})$. So $\mathcal{X} \left[\bigsqcup R \right] \mathcal{Y}$ and $\forall f \in R : \neg (\mathcal{X} \left[f \right] \mathcal{Y})$.

2°. With the same $\mathcal{X}$ and R we have $\langle \bigsqcup R \rangle \mathcal{X} = \top^{\mathscr{F}(\mathbb{R})}$ and $\langle f \rangle \mathcal{X} = \bot^{\mathscr{F}(\mathbb{R})}$ for every $f \in R$, thus $\bigsqcup \left\{ \frac{\langle f \rangle \mathcal{X}}{f \in R} \right\} = \bot^{\mathscr{F}(\mathbb{R})}$.

□

EXAMPLE 1340. $\bigsqcup_{\mathcal{B}\in T} (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \neq \mathcal{A} \times^{\mathsf{RLD}} \bigsqcup T$ for some filter $\mathcal{A}$ and set of filters T (with a common base).

Proof. Take $\mathbb{R}_+ = \left\{ \frac{x\in\mathbb{R}}{x>0} \right\}$, $\mathcal{A} = \Delta$, $T = \left\{ \frac{\uparrow\{x\}}{x\in\mathbb{R}_+} \right\}$ where $\uparrow = \uparrow^{\mathbb{R}}$.

$\bigsqcup T = \uparrow \mathbb{R}_+$; $\mathcal{A} \times^{\mathsf{RLD}} \bigsqcup T = \Delta \times^{\mathsf{RLD}} \uparrow \mathbb{R}_+$.

$\bigsqcup_{\mathcal{B}\in T}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \bigsqcup_{x\in\mathbb{R}_+}(\Delta \times^{\mathsf{RLD}} \uparrow \{x\})$.

We'll prove that $\bigsqcup_{x\in\mathbb{R}_+}(\Delta \times^{\mathsf{RLD}} \uparrow \{x\}) \neq \Delta \times^{\mathsf{RLD}} \uparrow \mathbb{R}_+$.

Consider $K = \bigcup_{x\in\mathbb{R}_+}(\{x\} \times]-1/x; 1/x[)$.

$K \in \operatorname{up}(\Delta \times^{\mathsf{RLD}} \uparrow \{x\})$ and thus $K \in \operatorname{up} \bigsqcup_{x\in\mathbb{R}_+}(\Delta \times^{\mathsf{RLD}} \uparrow \{x\})$. But $K \notin \operatorname{up}(\Delta \times^{\mathsf{RLD}} \uparrow \mathbb{R}_+)$. □

Theorem 1341. For a filter a we have $a \times^{\mathsf{RLD}} a \sqsubseteq 1^{\mathsf{RLD}}_{\operatorname{Base}(a)}$ only in the case if $a = \bot^{\mathscr{F}(\operatorname{Base}(a))}$ or a is a trivial ultrafilter.

Proof. If $a \times^{\mathsf{RLD}} a \sqsubseteq 1^{\mathsf{RLD}}_{\operatorname{Base}(a)}$ then there exists $m \in \operatorname{up}(a \times^{\mathsf{RLD}} a)$ such that $m \sqsubseteq 1^{\mathbf{Rel}}_{\operatorname{Base}(a)}$. Consequently there exist $A, B \in \operatorname{up} a$ such that $A \times B \sqsubseteq 1^{\mathbf{Rel}}_{\operatorname{Base}(a)}$ what is possible only in the case when $\uparrow A = \uparrow B = a$ is trivial a ultrafilter or the least filter. □

Corollary 1342. Reloidal product of a non-trivial atomic filter with itself is non-atomic.

Proof. Obviously $(a \times^{\mathsf{RLD}} a) \sqcap 1^{\mathsf{RLD}}_{\operatorname{Base}(a)} \neq \bot^{\mathsf{RLD}}$ and $(a \times^{\mathsf{RLD}} a) \sqcap 1^{\mathsf{RLD}}_{\operatorname{Base}(a)} \sqsubset a \times^{\mathsf{RLD}} a$. □

Example 1343. There exist two atomic reloids whose composition is non-atomic and non-empty.

Proof. Let a be a non-trivial ultrafilter on $\mathbb{N}$ and $x \in \mathbb{N}$. Then

$$(a \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}} \{x\}) \circ (\uparrow^{\mathbb{N}} \{x\} \times^{\mathsf{RLD}} a) = \prod_{A\in a}^{\mathsf{RLD}(\mathbb{N},\mathbb{N})} ((A \times \{x\}) \circ (\{x\} \times A) = \prod_{A\in a}^{\mathsf{RLD}(\mathbb{N},\mathbb{N})} (A \times A) = a \times^{\mathsf{RLD}} a$$

is non-atomic despite of $a \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}} \{x\}$ and $\uparrow^{\mathbb{N}} \{x\} \times^{\mathsf{RLD}} a$ are atomic. □

Example 1344. There exists non-monovalued atomic reloid.

Proof. From the previous example it follows that the atomic reloid $\uparrow^{\mathbb{N}} \{x\} \times^{\mathsf{RLD}} a$ is not monovalued. □

Example 1345. Non-convex reloids exist.

Proof. Let a be a non-trivial ultrafilter. Then $\operatorname{id}^{\mathsf{RLD}}_a$ is non-convex. This follows from the fact that only reloidal products which are below $1^{\mathsf{RLD}}_{\operatorname{Base}(a)}$ are reloidal products of ultrafilters and $\operatorname{id}^{\mathsf{RLD}}_a$ is not their join. □

Example 1346. There exists (atomic) composable funcoids f and g such that

$$H \in \operatorname{up}(g \circ f) \nRightarrow \exists F \in \operatorname{up} f, G \in \operatorname{up} g : H \sqsupseteq G \circ F.$$

Proof. Let a be a nontrivial ultrafilter and p be an arbitrary point, $f = a \times^{\mathsf{FCD}} \{p\}$, $g = \{p\} \times^{\mathsf{FCD}} a$. Then $g \circ f = a \times^{\mathsf{FCD}} a$. Take $H = 1$. Let $F \in \operatorname{up} f$ and $G \in \operatorname{up} g$. We have $F \in \operatorname{up}(A_0 \times^{\mathsf{FCD}} \{p\})$, $G \in \operatorname{up}(\{p\} \times^{\mathsf{FCD}} A_1)$ where $A_0, A_1 \in \operatorname{up} a$ (take $A_0 = \langle F \rangle^* @\{p\}$ and similarly for A_1). Thus $G \circ F \sqsupseteq A_0 \times A_1$ and so $H \notin \operatorname{up}(G \circ F)$. □

Example 1347. $(\mathsf{RLD})_{\mathrm{in}} f \neq (\mathsf{RLD})_{\mathrm{out}} f$ for a funcoid f.

PROOF. Let $f = 1^{\mathsf{FCD}}_{\mathbb{N}}$. Then $(\mathsf{RLD})_{\mathrm{in}} f = \bigsqcup_{a \in \operatorname{atoms}^{\mathscr{F}(\mathbb{N})}} (a \times^{\mathsf{RLD}} a)$ and $(\mathsf{RLD})_{\mathrm{out}} f = 1^{\mathsf{RLD}}_{\mathbb{N}}$. But we have shown above $a \times^{\mathsf{RLD}} a \not\sqsubseteq 1^{\mathsf{RLD}}_{\mathbb{N}}$ for non-trivial ultrafilter a, and so $(\mathsf{RLD})_{\mathrm{in}} f \not\sqsubseteq (\mathsf{RLD})_{\mathrm{out}} f$. $\square$

PROPOSITION 1348. $1^{\mathsf{FCD}}_{\mathfrak{U}} \sqcap \uparrow^{\mathsf{FCD}(\mathfrak{U},\mathfrak{U})} ((\mathfrak{U} \times \mathfrak{U}) \setminus \mathrm{id}_{\mathfrak{U}}) = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathfrak{U})} \neq \bot^{\mathsf{FCD}(\mathfrak{U},\mathfrak{U})}$ for every infinite set $\mathfrak{U}$.

PROOF. Note that $\left\langle \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathfrak{U})} \right\rangle \mathcal{X} = \mathcal{X} \sqcap \Omega(\mathfrak{U})$ for every filter $\mathcal{X}$ on $\mathfrak{U}$.

Let $f = 1^{\mathsf{FCD}}_{\mathfrak{U}}$, $g = \uparrow^{\mathsf{FCD}(\mathfrak{U},\mathfrak{U})} ((\mathfrak{U} \times \mathfrak{U}) \setminus \mathrm{id}_{\mathfrak{U}})$.

Let x be a non-trivial ultrafilter on $\mathfrak{U}$. If $X \in \operatorname{up} x$ then $\operatorname{card} X \geq 2$ (In fact, X is infinite but we don't need this.) and consequently $\langle g \rangle^* X = \top^{\mathscr{F}(\mathfrak{U})}$. Thus $\langle g \rangle x = \top^{\mathscr{F}(\mathfrak{U})}$. Consequently

$$\langle f \sqcap g \rangle x = \langle f \rangle x \sqcap \langle g \rangle x = x \sqcap \top^{\mathscr{F}(\mathfrak{U})} = x.$$

Also $\left\langle \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathfrak{U})} \right\rangle x = x \sqcap \Omega(\mathfrak{U}) = x$.

Let now x be a trivial ultrafilter. Then $\langle f \rangle x = x$ and $\langle g \rangle x = \top^{\mathscr{F}(\mathfrak{U})} \setminus x$. So

$$\langle f \sqcap g \rangle x = \langle f \rangle x \sqcap \langle g \rangle x = x \sqcap (\top^{\mathscr{F}(\mathfrak{U})} \setminus x) = \bot^{\mathscr{F}(\mathfrak{U})}.$$

Also $\left\langle \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathfrak{U})} \right\rangle x = x \sqcap \Omega(\mathfrak{U}) = \bot^{\mathscr{F}(\mathfrak{U})}$.

So $\langle f \sqcap g \rangle x = \left\langle \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathfrak{U})} \right\rangle x$ for every ultrafilter x on $\mathfrak{U}$. Thus $f \sqcap g = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathfrak{U})}$. $\square$

EXAMPLE 1349. There exist binary relations f and g such that $\uparrow^{\mathsf{FCD}(A,B)} f \sqcap \uparrow^{\mathsf{FCD}(A,B)} g \neq \uparrow^{\mathsf{FCD}(A,B)} (f \cap g)$ for some sets A, B such that $f, g \subseteq A \times B$.

PROOF. From the proposition above. $\square$

EXAMPLE 1350. There exists a principal funcoid which is not a complemented element of the lattice of funcoids.

PROOF. I will prove that quasi-complement of the funcoid $1^{\mathsf{FCD}}_{\mathbb{N}}$ is not its complement (it is enough by proposition 145). We have:

$$\begin{aligned}
(1^{\mathsf{FCD}}_{\mathbb{N}})^* &= \\
\bigsqcup \left\{ \frac{c \in \mathsf{FCD}(\mathbb{N},\mathbb{N})}{c \asymp 1^{\mathsf{FCD}}_{\mathbb{N}}} \right\} &\sqsupseteq \\
\bigsqcup \left\{ \frac{\uparrow^{\mathbb{N}} \{\alpha\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\beta\}}{\alpha, \beta \in \mathbb{N}, \uparrow^{\mathbb{N}} \{\alpha\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\beta\} \asymp 1^{\mathsf{FCD}}_{\mathbb{N}}} \right\} &= \\
\bigsqcup \left\{ \frac{\uparrow^{\mathbb{N}} \{\alpha\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\beta\}}{\alpha, \beta \in \mathbb{N}, \alpha \neq \beta} \right\} &= \\
\uparrow^{\mathsf{FCD}(\mathbb{N},\mathbb{N})} \bigcup \left\{ \frac{\{\alpha\} \times \{\beta\}}{\alpha, \beta \in \mathbb{N}, \alpha \neq \beta} \right\} &= \\
\uparrow^{\mathsf{FCD}(\mathbb{N},\mathbb{N})} (\mathbb{N} \times \mathbb{N} \setminus \mathrm{id}_{\mathbb{N}})
\end{aligned}$$

(used corollary 923). But by proved above $(1^{\mathsf{FCD}}_{\mathbb{N}})^* \sqcap 1^{\mathsf{FCD}}_{\mathbb{N}} \neq \bot^{\mathscr{F}(\mathbb{N})}$. $\square$

EXAMPLE 1351. There exists a funcoid h such that $\operatorname{up} h$ is not a filter.

PROOF. Consider the funcoid $h = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. We have (from the proof of proposition 1348) that $f \in \operatorname{up} h$ and $g \in \operatorname{up} h$, but $f \sqcap g \notin \operatorname{up} h$. $\square$

EXAMPLE 1352. There exists a funcoid $h \neq \bot^{\mathsf{FCD}(A,B)}$ such that $(\mathsf{RLD})_{\mathrm{out}} h = \bot^{\mathsf{RLD}(A,B)}$.

Proof. Consider $h = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. By proved above $h = f \sqcap g$ where $f = 1^{\mathsf{FCD}}_{\mathbb{N}} = \uparrow^{\mathsf{FCD}(\mathbb{N},\mathbb{N})} \mathrm{id}_{\mathbb{N}}$, $g = \uparrow^{\mathsf{FCD}(\mathbb{N},\mathbb{N})} (\mathbb{N} \times \mathbb{N} \setminus \mathrm{id}_{\mathbb{N}})$.

We have $\mathrm{id}_{\mathbb{N}}, \mathbb{N} \times \mathbb{N} \setminus \mathrm{id}_{\mathbb{N}} \in \mathrm{GR}\, h$.

So

$$(\mathsf{RLD})_{\mathrm{out}} h = \prod^{\mathsf{RLD}} \mathrm{up}\, h = \prod^{\mathsf{RLD}(\mathbb{N},\mathbb{N})} \mathrm{GR}\, h \sqsubseteq \uparrow^{\mathsf{RLD}(\mathbb{N},\mathbb{N})} (\mathrm{id}_{\mathbb{N}} \cap (\mathbb{N} \times \mathbb{N} \setminus \mathrm{id}_{\mathbb{N}})) = \bot^{\mathsf{RLD}(\mathbb{N},\mathbb{N})};$$

and thus $(\mathsf{RLD})_{\mathrm{out}} h = \bot^{\mathsf{RLD}(\mathbb{N},\mathbb{N})}$. □

Example 1353. There exists a funcoid h such that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}} h \neq h$.

Proof. It follows from the previous example. □

Example 1354. $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \neq f$ for some convex reloid f.

Proof. Let $f = 1^{\mathsf{RLD}}_{\mathbb{N}}$. Then $(\mathsf{FCD}) f = 1^{\mathsf{FCD}}_{\mathbb{N}}$. Let a be some non-trivial ultrafilter on $\mathbb{N}$. Then $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \sqsupseteq a \times^{\mathsf{RLD}} a \not\sqsubseteq 1^{\mathsf{RLD}}_{\mathbb{N}}$ and thus $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \not\sqsubseteq f$. □

Example 1355. There exist composable funcoids f and g such that

$$(\mathsf{RLD})_{\mathrm{out}}(g \circ f) \sqsupset (\mathsf{RLD})_{\mathrm{out}} g \circ (\mathsf{RLD})_{\mathrm{out}} f.$$

Proof. $f = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$ and $g = \top^{\mathscr{F}(\mathbb{N})} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}$ for some $\alpha \in \mathbb{N}$. Then $(\mathsf{RLD})_{\mathrm{out}} f = \bot^{\mathsf{RLD}(\mathbb{N},\mathbb{N})}$ and thus $(\mathsf{RLD})_{\mathrm{out}} g \circ (\mathsf{RLD})_{\mathrm{out}} f = \bot^{\mathsf{RLD}(\mathbb{N},\mathbb{N})}$.

We have $g \circ f = \Omega(\mathbb{N}) \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}$.

$(\mathsf{RLD})_{\mathrm{out}}(\Omega(\mathbb{N}) \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}} \{\alpha\}) = \Omega(\mathbb{N}) \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}} \{\alpha\}$ by properties of funcoidal reloids.

Thus $(\mathsf{RLD})_{\mathrm{out}}(g \circ f) = \Omega(\mathbb{N}) \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}} \{\alpha\} \neq \bot^{\mathsf{RLD}(\mathbb{N},\mathbb{N})}$. □

Conjecture 1356. For every composable funcoids f and g

$$(\mathsf{RLD})_{\mathrm{out}}(g \circ f) \sqsupseteq (\mathsf{RLD})_{\mathrm{out}} g \circ (\mathsf{RLD})_{\mathrm{out}} f.$$

Example 1357. (FCD) does not preserve binary meets.

Proof. $(\mathsf{FCD})(1^{\mathsf{RLD}}_{\mathbb{N}} \sqcap (\top^{\mathsf{RLD}(\mathbb{N},\mathbb{N})} \setminus 1^{\mathsf{RLD}}_{\mathbb{N}})) = (\mathsf{FCD})\bot^{\mathsf{RLD}(\mathbb{N},\mathbb{N})} = \bot^{\mathsf{FCD}(\mathbb{N},\mathbb{N})}$.

On the other hand,

$$(\mathsf{FCD}) 1^{\mathsf{RLD}}_{\mathbb{N}} \sqcap (\mathsf{FCD})(\top^{\mathsf{RLD}(\mathbb{N},\mathbb{N})} \setminus 1^{\mathsf{RLD}}_{\mathbb{N}}) = 1^{\mathsf{FCD}}_{\mathbb{N}} \sqcap \uparrow^{\mathsf{FCD}(\mathbb{N},\mathbb{N})} (\mathbb{N} \times \mathbb{N} \setminus \mathrm{id}_{\mathbb{N}}) = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \neq \bot^{\mathsf{FCD}(\mathbb{N},\mathbb{N})}$$

(used proposition 1064). □

Corollary 1358. (FCD) is not an upper adjoint (in general).

Considering restricting polynomials (considered as reloids) to ultrafilters, it is simple to prove that each that restriction is injective if not restricting a constant polynomial. Does this hold in general? No, see the following example:

Example 1359. There exists a monovalued reloid with atomic domain which is neither injective nor constant (that is not a restriction of a constant function).

Proof. (based on [31]) Consider the function $F \in \mathbb{N}^{\mathbb{N} \times \mathbb{N}}$ defined by the formula $(x, y) \mapsto x$.

Let ω_x be a non-trivial ultrafilter on the vertical line $\{x\} \times \mathbb{N}$ for every $x \in \mathbb{N}$.

Let T be the collection of such sets Y that $Y \cap (\{x\} \times \mathbb{N}) \in \omega_x$ for all but finitely many vertical lines. Obviously T is a filter.

Let $\omega \in \mathrm{atoms}\, T$.

For every $x \in \mathbb{N}$ we have some $Y \in T$ for which $(\{x\} \times \mathbb{N}) \cap Y = \emptyset$ and thus $\uparrow^{\mathbb{N} \times \mathbb{N}} (\{x\} \times \mathbb{N}) \sqcap \omega = \bot^{\mathscr{F}(\mathbb{N} \times \mathbb{N})}$.

Let $g = (\uparrow^{\mathsf{RLD}(\mathbb{N},\mathbb{N})} F)|_\omega$. If g is constant, then there exist a constant function $G \in \operatorname{up} g$ and $F \cap G$ is also constant. Obviously $\operatorname{dom} \uparrow^{\mathsf{RLD}(\mathbb{N}\times\mathbb{N},\mathbb{N})} (F \cap G) \sqsupseteq \omega$. The function $F \cap G$ cannot be constant because otherwise $\omega \sqsubseteq \operatorname{dom} \uparrow^{\mathsf{RLD}(\mathbb{N}\times\mathbb{N},\mathbb{N})} (F \cap G) \sqsubseteq \uparrow^{\mathbb{N}\times\mathbb{N}} (\{x\} \times \mathbb{N})$ for some $x \in \mathbb{N}$ what is impossible by proved above. So g is not constant.

Suppose that g is injective. Then there exists an injection $G \in \operatorname{up} g$. $F \sqcap G \in \operatorname{up} g$ is an injection which depends only on the first argument. So $\operatorname{dom}(F \sqcap G)$ intersects each vertical line by atmost one element that is $\overline{\operatorname{dom}(F \sqcap G)}$ intersects every vertical line by the whole line or the line without one element. Thus $\overline{\operatorname{dom}(F \sqcap G)} \in T \sqsupseteq \omega$ and consequently $\operatorname{dom}(F \sqcap G) \notin \omega$ what is impossible.

Thus g is neither injective nor constant. □

15.1. Second product. Oblique product

Definition 1360. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} = (\mathsf{RLD})_{\mathrm{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$ for every filters $\mathcal{A}$ and $\mathcal{B}$. I will call it *second product* of filters A and $\mathcal{B}$.

Remark 1361. The letter F is the above definition is from the word "funcoid". It signifies that it seems to be impossible to define $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ directly without referring to funcoidal product.

Definition 1362. *Oblique products* of filters $\mathcal{A}$ and $\mathcal{B}$ are defined as

$$\mathcal{A} \ltimes \mathcal{B} = \bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}} f}{f \in \mathbf{Rel}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{B})), \exists B \in \operatorname{up}\mathcal{B} : \uparrow^{\mathsf{FCD}} f \sqsupseteq \mathcal{A} \times^{\mathsf{FCD}} \uparrow B}\right\};$$

$$\mathcal{A} \rtimes \mathcal{B} = \bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}} f}{f \in \mathbf{Rel}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{B})), \exists A \in \operatorname{up}\mathcal{A} : \uparrow^{\mathsf{FCD}} f \sqsupseteq \uparrow A \times^{\mathsf{FCD}} \mathcal{B}}\right\}.$$

Proposition 1363.

1°. $\mathcal{A} \ltimes B = \mathcal{A} \times^{\mathsf{RLD}}_F B$ if $\mathcal{A}$ and B are filters and B is principal.

2°. $A \rtimes \mathcal{B} = A \times^{\mathsf{RLD}}_F \mathcal{B}$ if A and $\mathcal{B}$ are filters and A is principal.

Proof. $A \rtimes \mathcal{B} = \bigsqcap^{\mathsf{RLD}}\left\{\frac{f}{f \in \mathbf{Rel}, f \sqsupseteq A \times^{\mathsf{FCD}} \mathcal{B}}\right\} = A \times^{\mathsf{RLD}}_F \mathcal{B}$. The other is analogous. □

Proposition 1364. $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \sqsubseteq \mathcal{A} \ltimes \mathcal{B} \sqsubseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for every filters $\mathcal{A}$, $\mathcal{B}$.

Proof.

$$\mathcal{A} \ltimes \mathcal{B} \sqsubseteq \bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}} f}{f \in \mathbf{Rel}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{B})), \exists A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B} : \uparrow^{\mathsf{FCD}} f \sqsupseteq \uparrow A \times^{\mathsf{FCD}} \uparrow B}\right\} \sqsubseteq \bigsqcap\left\{\frac{\uparrow A \times^{\mathsf{RLD}} \uparrow B}{A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B}}\right\} = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}.$$

$$\mathcal{A} \ltimes \mathcal{B} \sqsupseteq \bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}} f}{f \in \mathbf{Rel}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{B})), \uparrow^{\mathsf{FCD}} f \sqsupseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}}\right\} = \bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}} f}{f \in \operatorname{up}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})}\right\} = (\mathsf{RLD})_{\mathrm{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}.$$

□

CONJECTURE 1365. $\mathcal{A} \times^{\mathsf{RLD}}_{F} \mathcal{B} \sqsubset \mathcal{A} \ltimes \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$.

A stronger conjecture:

CONJECTURE 1366. $\mathcal{A} \times^{\mathsf{RLD}}_{F} \mathcal{B} \sqsubset \mathcal{A} \ltimes \mathcal{B} \sqsubset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$. Particularly, is this formula true for $\mathcal{A} = \mathcal{B} = \Delta \sqcap \uparrow^{\mathbb{R}}]0; +\infty[$?

The above conjecture is similar to Fermat Last Theorem as having no value by itself but being somehow challenging to prove it (not expected to be as hard as FLT however).

EXAMPLE 1367. $\mathcal{A} \ltimes \mathcal{B} \sqsubset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$.

PROOF. It's enough to prove $\mathcal{A} \ltimes \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.
Let $\Delta_{+} = \Delta \sqcap \uparrow^{\mathbb{R}}]0; +\infty[$. Let $\mathcal{A} = \mathcal{B} = \Delta_{+}$.
Let $K = (\leq)|_{\mathbb{R} \times \mathbb{R}}$.
Obviously $K \notin \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$.
$\mathcal{A} \ltimes \mathcal{B} \sqsubseteq \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{B}))} K$ and thus $K \in \operatorname{up}(\mathcal{A} \ltimes \mathcal{B})$ because

$$\uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A}), \operatorname{Base}(\mathcal{B}))} K \sqsupseteq \Delta_{+} \times^{\mathsf{FCD}} \uparrow B = \mathcal{A} \times^{\mathsf{FCD}} \uparrow B$$

for $B =]0; +\infty[$ because for every $X \in \partial \Delta_{+}$ there is $x \in X$ such that $x \in]0; \epsilon[$ (for every positive ϵ) and thus $]\epsilon; +\infty[\subseteq \langle K \rangle^{*} \{x\}$ so having

$$\langle K \rangle^{*} X =]0; +\infty[\in \operatorname{GR} \left\langle \Delta_{+} \times^{\mathsf{FCD}} \uparrow B \right\rangle^{*} X.$$

Thus $\mathcal{A} \ltimes \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

EXAMPLE 1368. $\mathcal{A} \times^{\mathsf{RLD}}_{F} \mathcal{B} \sqsubset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$.

PROOF. This follows from the above example. □

CONJECTURE 1369. $(\mathcal{A} \ltimes \mathcal{B}) \sqcap (\mathcal{A} \rtimes \mathcal{B}) \neq \mathcal{A} \times^{\mathsf{RLD}}_{F} \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$.

(Earlier I presented a proof of the negation of this conjecture, but it was in error.)

EXAMPLE 1370. $(\mathcal{A} \ltimes \mathcal{B}) \sqcup (\mathcal{A} \rtimes \mathcal{B}) \sqsubset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some filters $\mathcal{A}$, $\mathcal{B}$.

PROOF. (based on [8]) Let $\mathcal{A} = \mathcal{B} = \Omega(\mathbb{N})$. It's enough to prove $(\mathcal{A} \ltimes \mathcal{B}) \sqcup (\mathcal{A} \rtimes \mathcal{B}) \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.
Let $X \in \operatorname{up} \mathcal{A}$, $Y \in \operatorname{up} \mathcal{B}$ that is $X \in \Omega(\mathbb{N})$, $Y \in \Omega(\mathbb{N})$.
Removing one element x from X produces a set P. Removing one element y from Y produces a set Q. Obviously $P \in \Omega(\mathbb{N})$, $Q \in \Omega(\mathbb{N})$.
Obviously $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \in \operatorname{up}((\mathcal{A} \ltimes \mathcal{B}) \sqcup (\mathcal{A} \rtimes \mathcal{B}))$.
$(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \not\supseteq X \times Y$ because $(x, y) \in X \times Y$ but $(x, y) \notin (P \times \mathbb{N}) \cup (\mathbb{N} \times Q)$ for every $X \in \operatorname{up} \mathcal{A}$, $Y \in \operatorname{up} \mathcal{B}$.
Thus some $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \notin \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$ by properties of filter bases. □

EXAMPLE 1371. $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD}) f \neq f$ for some convex reloid f.

PROOF. Let $f = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ where $\mathcal{A}$ and B are from example 1368.
$(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ by proposition 1074.
So $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = (\mathsf{RLD})_{\mathrm{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_{F} \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

CHAPTER 16

Funcoids are filters

The motto of this chapter is: "Funcoids are filters on a (boolean) lattice."

16.1. Rearrangement of collections of sets

Let Q be a set of sets.

Let $\equiv$ be the relation on $\bigcup Q$ defined by the formula

$$a \equiv b \Leftrightarrow \forall X \in Q : (a \in X \Leftrightarrow b \in X).$$

PROPOSITION 1372. $\equiv$ is an equivalence relation on $\bigcup Q$.

PROOF.

Reflexivity. Obvious.

Symmetry. Obvious.

Transitivity. Let $a \equiv b \wedge b \equiv c$. Then $a \in X \Leftrightarrow b \in X \Leftrightarrow c \in X$ for every $X \in Q$. Thus $a \equiv c$.

□

DEFINITION 1373. *Rearrangement* $\mathfrak{R}(Q)$ of Q is the set of equivalence classes of $\bigcup Q$ for $\equiv$.

OBVIOUS 1374. $\bigcup \mathfrak{R}(Q) = \bigcup Q$.

OBVIOUS 1375. $\emptyset \notin \mathfrak{R}(Q)$.

LEMMA 1376. $\operatorname{card} \mathfrak{R}(Q) \leq 2^{\operatorname{card} Q}$.

PROOF. Having an equivalence class C, we can find the set $f \in \mathscr{P}Q$ of all $X \in Q$ such that $a \in X$, for every $a \in C$.

$$b \equiv a \Leftrightarrow \forall X \in Q : (a \in X \Leftrightarrow b \in X) \Leftrightarrow \forall X \in Q : (X \in f \Leftrightarrow b \in X).$$

So $C = \left\{ \frac{b \in \bigcup Q}{b \equiv a} \right\}$ can be restored knowing f. Consequently there are no more than $\operatorname{card} \mathscr{P}Q = 2^{\operatorname{card} Q}$ classes. □

COROLLARY 1377. If Q is finite, then $\mathfrak{R}(Q)$ is finite.

PROPOSITION 1378. If $X \in Q$, $Y \in \mathfrak{R}(Q)$ then $X \cap Y \neq \emptyset \Leftrightarrow Y \subseteq X$.

PROOF. Let $X \cap Y \neq \emptyset$ and $x \in X \cap Y$. Then

$$y \in Y \Leftrightarrow x \equiv y \Leftrightarrow \forall X' \in Q : (x \in X' \Leftrightarrow y \in X') \Rightarrow (x \in X \Leftrightarrow y \in X) \Leftrightarrow y \in X$$

for every y. Thus $Y \subseteq X$.

$Y \subseteq X \Rightarrow X \cap Y \neq \emptyset$ because $Y \neq \emptyset$. □

PROPOSITION 1379. If $\emptyset \neq X \in Q$ then there exists $Y \in \mathfrak{R}(Q)$ such that $Y \subseteq X \wedge X \cap Y \neq \emptyset$.

Proof. Let $a \in X$. Then

$$[a] = \left\{ \frac{b \in \bigcup Q}{\forall X' \in Q : (a \in X' \Leftrightarrow b \in X')} \right\} \subseteq \left\{ \frac{b \in \bigcup Q}{a \in X \Leftrightarrow b \in X} \right\} = \left\{ \frac{b \in \bigcup Q}{b \in X} \right\} = X.$$

But $[a] \in \mathfrak{R}(Q)$.

$X \cap Y \neq \emptyset$ follows from $Y \subseteq X$ by the previous proposition. □

Proposition 1380. If $X \in Q$ then $X = \bigcup(\mathfrak{R}(Q) \cap \mathscr{P}X)$.

Proof. $\bigcup(\mathfrak{R}(Q) \cap \mathscr{P}X) \subseteq X$ is obvious.

Let $x \in X$. Then there is $Y \in \mathfrak{R}(Q)$ such that $x \in Y$. We have $Y \subseteq X$ that is $Y \in \mathscr{P}X$ by a proposition above. So $x \in Y$ where $Y \in \mathfrak{R}(Q) \cap \mathscr{P}X$ and thus $x \in \bigcup(\mathfrak{R}(Q) \cap \mathscr{P}X)$. We have $X \subseteq \bigcup(\mathfrak{R}(Q) \cap \mathscr{P}X)$. □

16.2. Finite unions of Cartesian products

Let A, B be sets.

I will denote $\overline{X} = A \setminus X$.

Let denote $\Gamma(A, B)$ the set of all finite unions $X_0 \times Y_0 \cup \ldots \cup X_{n-1} \times Y_{n-1}$ of Cartesian products, where $n \in \mathbb{N}$ and $X_i \in \mathscr{P}A$, $Y_i \in \mathscr{P}B$ for every $i = 0, \ldots, n-1$.

Proposition 1381. The following sets are pairwise equal:

1°. $\Gamma(A, B)$;
2°. the set of all sets of the form $\bigcup_{X \in S}(X \times Y_X)$ where S are finite collections on A and $Y_X \in \mathscr{P}B$ for every $X \in S$;
3°. the set of all sets of the form $\bigcup_{X \in S}(X \times Y_X)$ where S are finite partitions of A and $Y_X \in \mathscr{P}B$ for every $X \in S$;
4°. the set of all finite unions $\bigcup_{(X,Y) \in \sigma}(X \times Y)$ where σ is a relation between a partition of A and a partition of B (that is $\operatorname{dom}\sigma$ is a partition of A and $\operatorname{im}\sigma$ is a partition of B).
5°. the set of all finite intersections $\bigcap_{i=0,\ldots,n-1}\left(X_i \times Y_i \cup \overline{X_i} \times B\right)$ where $n \in \mathbb{N}$ and $X_i \in \mathscr{P}A$, $Y_i \in \mathscr{P}B$ for every $i = 0, \ldots, n-1$.

Proof.

1°⊇2°, 2°⊇3°. Obvious.

1°⊆2°. Let $Q \in \Gamma(A, B)$. Then $Q = X_0 \times Y_0 \cup \ldots \cup X_{n-1} \times Y_{n-1}$. Denote $S = \{X_0, \ldots, X_{n-1}\}$. We have $Q = \bigcup_{X' \in S}\left(X' \times \bigcup_{i=0,\ldots,n-1}\left\{ \frac{Y_i}{X_i = X'} \right\}\right) \in$ 2°.

2°⊆3°. Let $Q = \bigcup_{X \in S}(X \times Y_X)$ where S is a finite collection on A and $Y_X \in \mathscr{P}B$ for every $X \in S$. Let

$$P = \bigcup_{X' \in \mathfrak{R}(S)}\left(X' \times \bigcup_{X \in S}\left\{ \frac{Y_X}{\exists X \in S : X' \subseteq X} \right\}\right).$$

To finish the proof let's show $P = Q$.

$\langle P\rangle^*\{x\} = \bigcup_{X \in S}\left\{ \frac{Y_X}{\exists X \in S : X' \subseteq X} \right\}$ where $x \in X'$.

Thus $\langle P\rangle^*\{x\} = \bigcup\left\{ \frac{Y_X}{\exists X \in S : x \in X} \right\} = \langle Q\rangle^*\{x\}$. So $P = Q$.

4°⊆3°. $\bigcup_{(X,Y) \in \sigma}(X \times Y) = \bigcup_{X \in \operatorname{dom}\sigma}\left(X \times \bigcup\left\{ \frac{Y \in \mathscr{P}B}{(X,Y) \in \sigma} \right\}\right) \in$ 3°.

3°⊆4°.

$$\bigcup_{X\in S}(X\times Y_X)=\bigcup_{X\in S}\left(X\times\bigcup\left(\mathfrak{R}\left(\left\{\frac{Y_X}{X\in S}\right\}\right)\cap\mathscr{P}Y_X\right)\right)=$$

$$\bigcup_{X\in S}\left(X\times\bigcup\left\{\frac{Y'\in\mathfrak{R}(\{\frac{Y_X}{X\in S}\})}{Y'\subseteq Y_X}\right\}\right)=$$

$$\bigcup_{X\in S}\left(X\times\bigcup\left\{\frac{Y'\in\mathfrak{R}(\{\frac{Y_X}{X\in S}\})}{(X,Y')\in\sigma}\right\}\right)=\bigcup_{(X,Y)\in\sigma}(X\times Y)$$

where σ is a relation between S and $\mathfrak{R}(\{\frac{Y_X}{X\in S}\})$, and $(X,Y')\in\sigma\Leftrightarrow Y'\subseteq Y_X$.

5°⊆1°. Obvious.

3°⊆5°. Let $Q=\bigcup_{X\in S}(X\times Y_X)=\bigcup_{i=0,\dots,n-1}(X_i\times Y_i)$ for a partition $S=\{X_0,\dots,X_{n-1}\}$ of A. Then $Q=\bigcap_{i=0,\dots,n-1}\left(X_i\times Y_i\cup\overline{X_i}\times B\right)$.

□

Exercise 1382. Formulate the duals of these sets.

Proposition 1383. $\Gamma(A,B)$ is a boolean lattice, a sublattice of the lattice $\mathscr{P}(A\times B)$.

Proof. That it's a sublattice is obvious. That it has complement, is also obvious. Distributivity follows from distributivity of $\mathscr{P}(A\times B)$. □

16.3. Before the diagram

Next we will prove the below theorem 1399 (the theorem with a diagram). First we will present parts of this theorem as several lemmas, and then then state a statement about the diagram which concisely summarizes the lemmas (and their easy consequences).

Below for simplicity we will equate reloids with their graphs (that is with filters on binary cartesian products).

Obvious 1384. $\operatorname{up}^{\Gamma(\operatorname{Src}f,\operatorname{Dst}f)}f=(\operatorname{up}f)\cap\Gamma$ for every reloid f.

Conjecture 1385. $\upuparrows^{\mathfrak{F}(\mathfrak{B})}\operatorname{up}^{\mathfrak{A}}\mathcal{X}$ is not a filter for some filter $\mathcal{X}\in\mathfrak{F}\Gamma(A,B)$ for some sets A, B.

Remark 1386. About this conjecture see also:

- http://goo.gl/DHyuuU
- http://goo.gl/4a6wY6

Lemma 1387. Let A, B be sets. The following are mutually inverse order isomorphisms between $\mathfrak{F}\Gamma(A,B)$ and $\mathsf{FCD}(A,B)$:

1°. $\mathcal{A}\mapsto\bigsqcap^{\mathsf{FCD}}\operatorname{up}\mathcal{A}$;
2°. $f\mapsto\operatorname{up}^{\Gamma(A,B)}f$.

Proof. Let's prove that $\operatorname{up}^{\Gamma(A,B)}f$ is a filter for every funcoid f. We need to prove that $P\cap Q\in\operatorname{up}f$ whenever

$$P=\bigcap_{i=0,\dots,n-1}\left(X_i\times Y_i\cup\overline{X_i}\times B\right)\quad\text{and}\quad Q=\bigcap_{j=0,\dots,m-1}\left(X'_j\times Y'_j\cup\overline{X'_j}\times B\right).$$

This follows from $P\in\operatorname{up}f\Leftrightarrow\forall i\in 0,\dots,n-1:\langle f\rangle X_i\subseteq Y_i$ and likewise for Q, so having $\langle f\rangle(X_i\cap X'_j)\subseteq Y_i\cap Y'_j$ for every $i=0,\dots,n-1$ and $j=0,\dots,m-1$. From this it follows

$$((X_i\cap X'_j)\times(Y_i\cap Y'_j))\cup\left(\overline{X_i\cap X'_j}\times B\right)\supseteq f$$

and thus $P \cap Q \in \operatorname{up} f$.

Let $\mathcal{A}$, $\mathcal{B}$ be filters on Γ. Let $\bigsqcap^{\mathsf{FCD}} \operatorname{up} \mathcal{A} = \bigsqcap^{\mathsf{FCD}} \operatorname{up} \mathcal{B}$. We need to prove $\mathcal{A} = \mathcal{B}$. (The rest follows from proof of the lemma 924). We have:

$$\mathcal{A} = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{X \times Y \cup \overline{X} \times B \in \operatorname{up} \mathcal{A}}{X \in \mathscr{P}A, Y \in \mathscr{P}B} \right\} =$$

$$\bigsqcap^{\mathsf{FCD}} \left\{ \frac{X \times Y \cup \overline{X} \times B}{X \in \mathscr{P}A, Y \in \mathscr{P}B, \exists P \in \operatorname{up} \mathcal{A} : P \subseteq X \times Y \cup \overline{X} \times B} \right\} =$$

$$\bigsqcap^{\mathsf{FCD}} \left\{ \frac{X \times Y \cup \overline{X} \times B}{X \in \mathscr{P}A, Y \in \mathscr{P}B, \exists P \in \operatorname{up} \mathcal{A} : \langle P \rangle^* X \subseteq Y} \right\} = (*)$$

$$\bigsqcap^{\mathsf{FCD}} \left\{ \frac{X \times Y \cup \overline{X} \times B}{X \in \mathscr{P}A, Y \in \mathscr{P}B, \bigsqcap \left\{ \frac{\langle P \rangle^* X}{X \in \operatorname{up} \mathcal{A}} \right\} \sqsubseteq Y} \right\} =$$

$$\bigsqcap^{\mathsf{FCD}} \left\{ \frac{X \times Y \cup \overline{X} \times B}{X \in \mathscr{P}A, Y \in \mathscr{P}B, \bigsqcap \left\{ \frac{\langle P \rangle^* X}{X \in \operatorname{up} \bigsqcap^{\mathsf{RLD}} \operatorname{up} \mathcal{A}} \right\} \sqsubseteq Y} \right\} =$$

$$\bigsqcap^{\mathsf{FCD}} \left\{ \frac{X \times Y \cup \overline{X} \times B}{X \in \mathscr{P}A, Y \in \mathscr{P}B, \left\langle (\mathsf{FCD}) \bigsqcap^{\mathsf{RLD}} \operatorname{up} \mathcal{A} \right\rangle X \sqsubseteq Y} \right\} = (**)$$

$$\bigsqcap^{\mathsf{FCD}} \left\{ \frac{X \times Y \cup \overline{X} \times B}{X \in \mathscr{P}A, Y \in \mathscr{P}B, \left\langle \bigsqcap^{\mathsf{FCD}} \operatorname{up} \bigsqcap^{\mathsf{RLD}} \operatorname{up} \mathcal{A} \right\rangle X \sqsubseteq Y} \right\} =$$

$$\bigsqcap^{\mathsf{FCD}} \left\{ \frac{X \times Y \cup \overline{X} \times B}{X \in \mathscr{P}A, Y \in \mathscr{P}B, \left\langle \bigsqcap^{\mathsf{FCD}} \operatorname{up} \mathcal{A} \right\rangle X \sqsubseteq Y} \right\}.$$

(*) by properties of generalized filter bases, because $\left\{ \frac{\langle P \rangle^* X}{P \in \operatorname{up} \mathcal{A}} \right\}$ is a filter base.
(**) by theorem 1066.
Similarly

$$\mathcal{B} = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{X \times Y \cup \overline{X} \times B}{X \in \mathscr{P}A, Y \in \mathscr{P}B, \left\langle \bigsqcap^{\mathsf{FCD}} \operatorname{up} \mathcal{B} \right\rangle X \sqsubseteq Y} \right\}.$$

Thus $\mathcal{A} = \mathcal{B}$. □

PROPOSITION 1388. $g \circ f \in \Gamma(A, C)$ if $f \in \Gamma(A, B)$ and $g \in \Gamma(B, C)$ for some sets A, B, C.

PROOF. Because composition of Cartesian products is a Cartesian product. □

DEFINITION 1389. $g \circ f = \bigsqcap^{\mathfrak{F}\Gamma(A,C)} \left\{ \frac{G \circ F}{F \in \operatorname{up} f, G \in \operatorname{up} g} \right\}$ for $f \in \mathfrak{F}\Gamma(A, B)$ and $g \in \mathfrak{F}\Gamma(B, C)$ (for every sets A, B, C).

We define f^{-1} for $f \in \mathfrak{F}\Gamma(A, B)$ similarly to f^{-1} for reloids and similarly derive the formulas:

1°. $(f^{-1})^{-1} = f$;
2°. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

16.4. Associativity over composition

LEMMA 1390. $\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,C)}(g \circ f) = \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g\right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)}\right)$ for every $f \in \mathfrak{F}(\Gamma(A,B))$, $g \in \mathfrak{F}(\Gamma(B,C))$ (for every sets A, B, C).

PROOF. If $K \in \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,C)}(g \circ f)$ then $K \supseteq G \circ F$ for some $F \in f$, $G \in g$. But $F \in \operatorname{up}^{\Gamma(A,B)} f$, thus

$$F \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,B)} f$$

and similarly

$$G \in \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g.$$

So we have

$$K \supseteq G \circ F \in \operatorname{up}\left(\left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g\right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,B)} f\right)\right).$$

Let now

$$K \in \operatorname{up}\left(\left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g\right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,B)} f\right)\right).$$

Then there exist $F \in \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,B)} f$ and $G \in \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(B,C)} g$ such that $K \supseteq G \circ F$. By properties of generalized filter bases we can take $F \in \operatorname{up}^{\Gamma(A,B)} f$ and $G \in \operatorname{up}^{\Gamma(B,C)} g$. Thus $K \in \operatorname{up}^{\Gamma(A,C)}(g \circ f)$ and so $K \in \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up}^{\Gamma(A,C)}(g \circ f)$. □

LEMMA 1391. $(\mathsf{RLD})_{\mathrm{in}} X = X$ for $X \in \Gamma(A,B)$.

PROOF. $X = X_0 \times Y_0 \cup \ldots \cup X_n \times Y_n = (X_0 \times^{\mathsf{FCD}} Y_0) \sqcup^{\mathsf{FCD}} \ldots \sqcup^{\mathsf{FCD}} (X_n \times^{\mathsf{FCD}} Y_n)$.

$$\begin{aligned}(\mathsf{RLD})_{\mathrm{in}} X = {} & \\ & (\mathsf{RLD})_{\mathrm{in}}(X_0 \times^{\mathsf{FCD}} Y_0) \sqcup^{\mathsf{RLD}} \ldots \sqcup^{\mathsf{RLD}} (\mathsf{RLD})_{\mathrm{in}}(X_n \times^{\mathsf{FCD}} Y) = \\ & (X_0 \times^{\mathsf{RLD}} Y_0) \sqcup^{\mathsf{RLD}} \ldots \sqcup^{\mathsf{RLD}} (X_n \times^{\mathsf{RLD}} Y_n) = \\ & X_0 \times Y_0 \cup \ldots \cup X_n \times Y_n = X.\end{aligned}$$

□

LEMMA 1392. $\prod^{\mathsf{RLD}} f = (\mathsf{RLD})_{\mathrm{in}} \prod^{\mathsf{FCD}} f$ for every filter $f \in \mathfrak{F}\Gamma(A,B)$.

PROOF.

$$(\mathsf{RLD})_{\mathrm{in}} \prod^{\mathsf{FCD}} f = \prod^{\mathsf{RLD}} \langle (\mathsf{RLD})_{\mathrm{in}} \rangle^* f = (\text{by the previous lemma}) = \prod^{\mathsf{RLD}} f.$$

□

LEMMA 1393.

1°. $f \mapsto \prod^{\mathsf{RLD}} \operatorname{up} f$ and $\mathcal{A} \mapsto \Gamma(A,B) \cap \operatorname{up} \mathcal{A}$ are mutually inverse bijections between $\mathfrak{F}\Gamma(A,B)$ and a subset of reloids.

2°. These bijections preserve composition.

PROOF.

1°. That they are mutually inverse bijections is obvious.

2°.

$$\left(\prod^{\mathsf{RLD}} \operatorname{up} g\right) \circ \left(\prod^{\mathsf{RLD}} \operatorname{up} f\right) = \prod^{\mathsf{RLD}} \left\{ \frac{G \circ F}{F \in \prod^{\mathsf{RLD}} f, G \in \prod^{\mathsf{RLD}} g} \right\} =$$
$$\prod^{\mathsf{RLD}} \left\{ \frac{G \circ F}{F \in f, G \in g} \right\} = \prod^{\mathsf{RLD}} \prod^{\mathfrak{F}\Gamma(\operatorname{Src} f, \operatorname{Dst} g)} \left\{ \frac{G \circ F}{F \in f, G \in g} \right\} = \prod^{\mathsf{RLD}} (g \circ f).$$

So $\prod^{\mathsf{RLD}}$ preserves composition. That $\mathcal{A} \mapsto \Gamma(A, B) \cap \operatorname{up} \mathcal{A}$ preserves composition follows from properties of bijections.

□

Lemma 1394. Let A, B, C be sets.

1°. $\left(\prod^{\mathsf{FCD}} \operatorname{up} g\right) \circ \left(\prod^{\mathsf{FCD}} \operatorname{up} f\right) = \prod^{\mathsf{FCD}} \operatorname{up}(g \circ f)$ for every $f \in \mathfrak{F}\Gamma(A, B)$, $g \in \mathfrak{F}\Gamma(B, C)$;

2°. $(\operatorname{up}^{\Gamma(B,C)} g) \circ (\operatorname{up}^{\Gamma(A,B)} f) = \operatorname{up}^{\Gamma(A,B)}(g \circ f)$ for every funcoids $f \in \mathsf{FCD}(A, B)$ and $g \in \mathsf{FCD}(B : C)$.

Proof. It's enough to prove only the first formula, because of the bijection from lemma 1387.

Really:

$$\prod^{\mathsf{FCD}} \operatorname{up}(g \circ f) = \prod^{\mathsf{FCD}} \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up}(g \circ f) =$$
$$\prod^{\mathsf{FCD}} \operatorname{up}\left(\prod^{\mathsf{RLD}} \operatorname{up} g \circ \prod^{\mathsf{RLD}} \operatorname{up} f\right) = (\mathsf{FCD})\left(\prod^{\mathsf{RLD}} \operatorname{up} g \circ \prod^{\mathsf{RLD}} \operatorname{up} f\right) =$$
$$\left((\mathsf{FCD}) \prod^{\mathsf{RLD}} \operatorname{up} g\right) \circ \left((\mathsf{FCD}) \prod^{\mathsf{RLD}} \operatorname{up} f\right) =$$
$$\left(\prod^{\mathsf{FCD}} \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up} g\right) \circ \left(\prod^{\mathsf{FCD}} \operatorname{up} \prod^{\mathsf{RLD}} \operatorname{up} f\right) =$$
$$\left(\prod^{\mathsf{FCD}} \operatorname{up} g\right) \circ \left(\prod^{\mathsf{FCD}} \operatorname{up} f\right).$$

□

Corollary 1395. $(h \circ g) \circ f = h \circ (g \circ f)$ for every $f \in \mathfrak{F}(\Gamma(A, B))$, $g \in \mathfrak{F}\Gamma(B, C)$, $h \in \mathfrak{F}\Gamma(C, D)$ for every sets A, B, C, D.

Lemma 1396. $\Gamma(A, B) \cap \operatorname{GR} f$ is a filter on the lattice $\Gamma(A, B)$ for every reloid $f \in \mathsf{RLD}(A, B)$.

Proof. That it is an upper set, is obvious. If $A, B \in \Gamma(A, B) \cap \operatorname{GR} f$ then $A, B \in \Gamma(A, B)$ and $A, B \in \operatorname{GR} f$. Thus $A \cap B \in \Gamma(A, B) \cap \operatorname{GR} f$. □

Proposition 1397. If $Y \in \operatorname{up}\langle f\rangle\mathcal{X}$ for a funcoid f then there exists $A \in \operatorname{up}\mathcal{X}$ such that $Y \in \operatorname{up}\langle f\rangle A$.

Proof. $Y \in \operatorname{up} \prod^{\mathscr{F}}_{A \in \operatorname{up} a} \langle f\rangle A$. So by properties of generalized filter bases, there exists $A \in \operatorname{up} a$ such that $Y \in \operatorname{up}\langle f\rangle A$. □

Lemma 1398. $(\mathsf{FCD})f = \prod^{\mathsf{FCD}}(\Gamma(A, B) \cap \operatorname{GR} f)$ for every reloid $f \in \mathsf{RLD}(A, B)$.

Proof. Let a be an an atomic filter object. We need to prove

$$\langle(\mathsf{FCD})f\rangle a = \left\langle \bigsqcap^{\mathsf{FCD}}(\Gamma(A,B)\cap \operatorname{GR} f)\right\rangle a$$

that is

$$\left\langle \bigsqcap^{\mathsf{FCD}} \operatorname{up} f\right\rangle a = \left\langle \bigsqcap^{\mathsf{FCD}}(\Gamma(A,B)\cap \operatorname{GR} f)\right\rangle a$$

that is

$$\bigsqcap^{\mathscr{F}}_{F\in \operatorname{up} f}\langle F\rangle a = \bigsqcap^{\mathscr{F}}_{F\in\Gamma(A,B)\cap \operatorname{up} f}\langle F\rangle a.$$

For this it's enough to prove that $Y \in \operatorname{up}\langle F\rangle a$ for some $F \in \operatorname{up} f$ implies $Y \in \operatorname{up}\langle F'\rangle a$ for some $F' \in \Gamma(A,B)\cap \operatorname{GR} f$.

Let $Y \in \operatorname{up}\langle F\rangle a$. Then (proposition above) there exists $A \in \operatorname{up} a$ such that $Y \in \operatorname{up}\langle F\rangle A$.

$Y \in \operatorname{up}\left\langle A\times^{\mathsf{FCD}} Y \sqcup \overline{A}\times^{\mathsf{FCD}}\top\right\rangle a$; $\left\langle A\times^{\mathsf{FCD}} Y \sqcup \overline{A}\times^{\mathsf{FCD}}\top\right\rangle \mathcal{X} = Y \in \operatorname{up}\langle F\rangle\mathcal{X}$ if $\bot\neq\mathcal{X}\sqsubseteq A$ and $\left\langle A\times^{\mathsf{FCD}} Y \sqcup \overline{A}\times^{\mathsf{FCD}}\top\right\rangle \mathcal{X} = \top \in \operatorname{up}\langle F\rangle\mathcal{X}$ if $\mathcal{X}\not\sqsubseteq A$.

Thus $A\times^{\mathsf{FCD}} Y \sqcup \overline{A}\times^{\mathsf{FCD}}\top \sqsupseteq F$. So $A\times^{\mathsf{FCD}} Y \sqcup \overline{A}\times^{\mathsf{FCD}}\top$ is the sought for F'. □

16.5. The diagram

Theorem 1399. The diagram at the figure 1 is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity (therefore "parallel" arrows are mutually inverse). The arrows preserve order, composition, and reversal ($f\mapsto f^{-1}$).

$$\begin{tikzcd}
& \text{funcoids} \arrow[dl, shift left, "(\mathsf{RLD})_{\mathrm{in}}"] \arrow[dr, shift left, "\mathrm{up}^{\Gamma}"] & \\
\text{funcoidal reloids} \arrow[ur, shift left, "(\mathsf{FCD})"] \arrow[rr, shift left, "f\mapsto f\cap\Gamma"] & & \text{filters on }\Gamma \arrow[ul, shift left, "\bigsqcap^{\mathsf{FCD}}"] \arrow[ll, shift left, "\bigsqcap^{\mathsf{RLD}}"]
\end{tikzcd}$$

Figure 10

Proof. First we need to show that $\bigsqcap^{\mathsf{RLD}} f$ is a funcoidal reloid. But it follows from lemma 1392.

Next, we need to show that all morphisms depicted on the diagram are bijections and the depicted "opposite" morphisms are mutually inverse.

That (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ are mutually inverse was proved above in the book.

That $\bigsqcap^{\mathsf{RLD}}$ and $f\mapsto f\cap\Gamma$ are mutually inverse was proved above.

That $\bigsqcap^{\mathsf{FCD}}$ and up^{Γ} are mutually inverse was proved above.

That the morphisms preserve order and composition was proved above. That they preserve reversal is obvious.

So it remains to apply lemma 196 (taking into account lemma 1392). □

Another proof that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$ for every funcoid f:

Proof. For every filter $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$ we have $\langle(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}(\mathsf{RLD})_{\mathrm{in}} f}\langle F\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X}$.

Obviously $\bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X} \sqsupseteq \langle f\rangle\mathcal{X}$. So $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f \sqsupseteq f$.

Let $Y \in \mathrm{up}\langle f\rangle\mathcal{X}$. Then (proposition above) there exists $A \in \mathrm{up}\,\mathcal{X}$ such that $Y \in \mathrm{up}\langle f\rangle A$.

Thus $A\times Y \sqcup \overline{A}\times\top \in \mathrm{up}\, f$. So $\langle(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X} \sqsubseteq \langle A\times Y \sqcup \overline{A}\times\top\rangle\mathcal{X} = Y$. So $Y \in \mathrm{up}\langle(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f\rangle\mathcal{X}$ that is $\langle f\rangle\mathcal{X} \sqsupseteq \langle(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f\rangle\mathcal{X}$ that is $f \sqsupseteq (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f$. □

16.6. Some additional properties

Proposition 1400. For every funcoid $f \in \mathsf{FCD}(A,B)$ (for sets A, B):

1°. $\operatorname{dom} f = \bigsqcap^{\mathscr{F}(A)}\langle\operatorname{dom}\rangle^* \mathrm{up}^{\Gamma(A,B)} f$;
2°. $\operatorname{im} f = \bigsqcap^{\mathscr{F}(B)}\langle\operatorname{im}\rangle^* \mathrm{up}^{\Gamma(A,B)} f$.

Proof. Take $\left\{\frac{X\times Y}{X\in\mathscr{P}A, Y\in\mathscr{P}B, X\times Y\sqsupseteq f}\right\} \subseteq \mathrm{up}^{\Gamma(A,B)} f$. I leave the rest reasoning as an exercise. □

Theorem 1401. For every reloid f and $\mathcal{X} \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathscr{F}(\operatorname{Dst} f)$:

1°. $\mathcal{X}\ [(\mathsf{FCD})f]\ \mathcal{Y} \Leftrightarrow \forall F \in \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f : \mathcal{X}\ [F]\ \mathcal{Y}$;
2°. $\langle(\mathsf{FCD})f\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X}$.

Proof.

1°.

$$\begin{aligned}&\forall F \in \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f : \mathcal{X}\ [F]\ \mathcal{Y} \Leftrightarrow\\ &\qquad\forall F \in \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f : (\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \sqcap F \neq \bot \Leftrightarrow (*)\\ &\qquad(\mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) \sqcap \bigsqcap^{\mathsf{FCD}} \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f \neq \bot \Leftrightarrow\\ &\qquad\mathcal{X}\left[\bigsqcap^{\mathsf{FCD}} \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f\right]\mathcal{Y} \Leftrightarrow \mathcal{X}\ [(\mathsf{FCD})f]\ \mathcal{Y}.\end{aligned}$$

(*) by properties of generalized filter bases, taking into account that funcoids are isomorphic to filters.

2°. $\bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle a = \left\langle\bigsqcap^{\mathsf{FCD}} \mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f\right\rangle a = \langle(\mathsf{FCD})f\rangle a$ for every ultrafilter a.

It remains to prove that the function

$$\varphi = \lambda\mathcal{X} \in \mathscr{F}(\operatorname{Src} f) : \bigsqcap^{\mathscr{F}}_{F\in\mathrm{up}^{\Gamma(\operatorname{Src} f,\operatorname{Dst} f)} f}\langle F\rangle\mathcal{X}$$

is a component of a funcoid (from what follows that $\varphi = \langle(\mathsf{FCD})f\rangle$). To prove this, it's enough to show that it preserves finite joins and filtered meets.

$\varphi\bot = \bot$ is obvious. $\varphi(\mathcal{I}\sqcup\mathcal{J}) = \bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}^{\Gamma(\operatorname{Src}f,\operatorname{Dst}f)}f}(\langle F\rangle\mathcal{I}\sqcup\langle F\rangle\mathcal{J}) = \bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}^{\Gamma(\operatorname{Src}f,\operatorname{Dst}f)}f}\langle F\rangle\mathcal{I}\sqcup\bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}^{\Gamma(\operatorname{Src}f,\operatorname{Dst}f)}f}\langle F\rangle\mathcal{J} = \varphi\mathcal{I}\sqcup\varphi\mathcal{J}$. If S is a generalized filter base of $\operatorname{Src}f$, then

$$\varphi\bigsqcap^{\mathscr{F}}S = \bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}^{\Gamma(\operatorname{Src}f,\operatorname{Dst}f)}f}\langle F\rangle\bigsqcap^{\mathscr{F}}S = \bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}^{\Gamma(\operatorname{Src}f,\operatorname{Dst}f)}f}\bigsqcap^{\mathscr{F}}\langle\langle F\rangle\rangle^*S =$$
$$\bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}^{\Gamma(\operatorname{Src}f,\operatorname{Dst}f)}f}\bigsqcap^{\mathscr{F}}_{\mathcal{X}\in S}\langle F\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{\mathcal{X}\in S}\bigsqcap^{\mathscr{F}}_{F\in\operatorname{up}^{\Gamma(\operatorname{Src}f,\operatorname{Dst}f)}f}\langle F\rangle\mathcal{X} = \bigsqcap^{\mathscr{F}}_{\mathcal{X}\in S}\varphi\mathcal{X} = \bigsqcap^{\mathscr{F}}\langle\varphi\rangle^*S.$$

So φ is a component of a funcoid.

□

Definition 1402. $\boxdot f = \bigsqcap^{\mathsf{RLD}}\operatorname{up}^{\Gamma(\operatorname{Src}f,\operatorname{Dst}f)}f$ for reloid f.

Conjecture 1403. $\boxdot f = (\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})f$ for every reloid f.

Obvious 1404. $\boxdot f \sqsupseteq f$ for every reloid f.

Example 1405. $(\mathsf{RLD})_{\mathrm{in}}f \neq \boxdot(\mathsf{RLD})_{\mathrm{out}}f$ for some funcoid f.

Proof. Take $f = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. Then, as it was shown above, $(\mathsf{RLD})_{\mathrm{out}}f = \bot$ and thus $\boxdot(\mathsf{RLD})_{\mathrm{out}}f = \bot$. But $(\mathsf{RLD})_{\mathrm{in}}f \sqsupseteq (\mathsf{RLD})_{\mathrm{in}}f \neq \bot$. So $(\mathsf{RLD})_{\mathrm{in}}f \neq \boxdot(\mathsf{RLD})_{\mathrm{out}}f$. □

Another proof of the theorem "$\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f = \operatorname{dom}f$ and $\operatorname{im}(\mathsf{RLD})_{\mathrm{in}}f = \operatorname{im}f$ for every funcoid f.":

Proof. We have for every filter $\mathcal{X}\in\mathscr{F}(\operatorname{Src}f)$:

$$\begin{aligned}&\mathcal{X}\sqsupseteq\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f \Leftrightarrow \mathcal{X}\times^{\mathsf{RLD}}\top\sqsupseteq(\mathsf{RLD})_{\mathrm{in}}f\Leftrightarrow\\ &\quad\forall a\in\mathscr{F}(\operatorname{Src}f),b\in\mathscr{F}(\operatorname{Dst}f):(a\times^{\mathsf{FCD}}b\sqsubseteq f\Rightarrow a\times^{\mathsf{RLD}}b\sqsubseteq\mathcal{X}\times^{\mathsf{RLD}}\top)\Leftrightarrow\\ &\qquad\forall a\in\mathscr{F}(\operatorname{Src}f),b\in\mathscr{F}(\operatorname{Dst}f):(a\times^{\mathsf{FCD}}b\sqsubseteq f\Rightarrow a\sqsubseteq\mathcal{X})\end{aligned}$$

and

$$\begin{aligned}&\mathcal{X}\sqsupseteq\operatorname{dom}f \Leftrightarrow \mathcal{X}\times^{\mathsf{FCD}}\top\sqsupseteq f\Leftrightarrow\\ &\quad\forall a\in\mathscr{F}(\operatorname{Src}f),b\in\mathscr{F}(\operatorname{Dst}f):(a\times^{\mathsf{FCD}}b\sqsubseteq f\Rightarrow a\times^{\mathsf{FCD}}b\sqsubseteq\mathcal{X}\times^{\mathsf{FCD}}\top)\Leftrightarrow\\ &\qquad\forall a\in\mathscr{F}(\operatorname{Src}f),b\in\mathscr{F}(\operatorname{Dst}f):(a\times^{\mathsf{FCD}}b\sqsubseteq f\Rightarrow a\sqsubseteq\mathcal{X}).\end{aligned}$$

Thus $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f = \operatorname{dom}f$. The rest follows from symmetry. □

Another proof that $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f = \operatorname{dom}f$ and $\operatorname{im}(\mathsf{RLD})_{\mathrm{in}}f = \operatorname{im}f$ for every funcoid f:

Proof. $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f \sqsupseteq \operatorname{dom}f$ and $\operatorname{im}(\mathsf{RLD})_{\mathrm{in}}f \sqsupseteq \operatorname{im}f$ because $(\mathsf{RLD})_{\mathrm{in}}f \sqsupseteq (\mathsf{RLD})_{\mathrm{in}}$ and $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f = \operatorname{dom}f$ and $\operatorname{im}(\mathsf{RLD})_{\mathrm{in}}f = \operatorname{im}f$.

It remains to prove (as the rest follows from symmetry) that $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f \sqsubseteq \operatorname{dom}f$.

Really,

$$\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}}f \sqsubseteq \bigsqcap^{\mathscr{F}}\left\{\frac{X\in\operatorname{up}\operatorname{dom}f}{X\times\top\in\operatorname{up}f}\right\} =$$
$$\bigsqcap^{\mathscr{F}}\left\{\frac{X\in\operatorname{up}\operatorname{dom}f}{X\in\operatorname{up}\operatorname{dom}f}\right\} = \bigsqcap^{\mathscr{F}}\operatorname{up}\operatorname{dom}f = \operatorname{dom}f.$$

□

16.7. More on properties of funcoids

PROPOSITION 1406. $\Gamma(A, B)$ is the center of lattice $\mathsf{FCD}(A, B)$.

PROOF. Theorem 613. □

PROPOSITION 1407. $\mathrm{up}^{\Gamma(A,B)}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$ is defined by the filter base $\left\{ \frac{A \times B}{A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B}} \right\}$ on the lattice $\Gamma(A, B)$.

PROOF. It follows from the fact that $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{A \times B}{A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B}} \right\}$. □

PROPOSITION 1408. $\mathrm{up}^{\Gamma(A,B)}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathfrak{F}(\Gamma(A, B)) \cap \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$.

PROOF. It follows from the fact that $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} = \bigsqcap^{\mathsf{FCD}} \left\{ \frac{A \times B}{A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B}} \right\}$. □

PROPOSITION 1409. For every $f \in \mathfrak{F}(\Gamma(A, B))$:

1°. $f \circ f$ is defined by the filter base $\left\{ \frac{F \circ F}{F \in \mathrm{up}\, f} \right\}$ (if $A = B$);
2°. $f^{-1} \circ f$ is defined by the filter base $\left\{ \frac{F^{-1} \circ F}{F \in \mathrm{up}\, f} \right\}$;
3°. $f \circ f^{-1}$ is defined by the filter base $\left\{ \frac{F \circ F^{-1}}{F \in \mathrm{up}\, f} \right\}$.

PROOF. I will prove only 1° and 2° because 3° is analogous to 2°.

1°. It's enough to show that $\forall F, G \in \mathrm{up}\, f \exists H \in \mathrm{up}\, f : H \circ H \sqsubseteq G \circ F$. To prove it take $H = F \sqcap G$.

2°. It's enough to show that $\forall F, G \in \mathrm{up}\, f \exists H \in \mathrm{up}\, f : H^{-1} \circ H \sqsubseteq G^{-1} \circ F$. To prove it take $H = F \sqcap G$. Then $H^{-1} \circ H = (F \sqcap G)^{-1} \circ (F \sqcap G) \sqsubseteq G^{-1} \circ F$. □

THEOREM 1410. For every sets A, B, C if $g, h \in \mathfrak{F}\Gamma(A, B)$ then

1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$;
2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$.

PROOF. It follows from the order isomorphism above, which preserves composition. □

THEOREM 1411. $f \cap g = f \sqcap^{\mathsf{FCD}} g$ if $f, g \in \Gamma(A, B)$.

PROOF. Let $f = X_0 \times Y_0 \cup \ldots \cup X_n \times Y_n$ and $g = X'_0 \times Y'_0 \cup \ldots \cup X'_m \times Y'_m$. Then

$$f \cap g = \bigcup_{i=0,\ldots,n,j=0,\ldots,m} ((X_i \times Y_i) \cap (X'_j \times Y'_j)) = \bigcup_{i=0,\ldots,n,j=0,\ldots,m} ((X_i \cap X'_j) \times (Y_i \cap Y'_j)).$$

But $f = X_0 \times Y_0 \sqcup^{\mathsf{FCD}} \ldots \sqcup^{\mathsf{FCD}} X_n \times Y_n$ and $g = X'_0 \times Y'_0 \sqcup^{\mathsf{FCD}} \ldots \sqcup^{\mathsf{FCD}} X'_m \times Y'_m$;

$$f \sqcap^{\mathsf{FCD}} g = \bigsqcup_{i=0,\ldots,n,j=0,\ldots,m} ((X_i \times Y_i) \sqcap^{\mathsf{FCD}} (X'_j \times Y'_j)) = \bigsqcup_{i=0,\ldots,n,j=0,\ldots,m} ((X_i \sqcap X'_j) \times^{\mathsf{FCD}} (Y_i \sqcap Y'_j)).$$

COROLLARY 1412. If X and Y are finite binary relations, then

1°. $X \sqcap^{\mathsf{FCD}} Y = X \sqcap Y$;
2°. $(\top \setminus X) \sqcap^{\mathsf{FCD}} (\top \setminus Y) = (\top \setminus X) \sqcap (\top \setminus Y)$;
3°. $X \sqcap^{\mathsf{FCD}} (\top \setminus Y) = X \sqcap (\top \setminus Y)$.

Now it's obvious that $f \cap g = f \sqcap^{\mathsf{FCD}} g$. □

THEOREM 1413. The set of funcoids (from a given set A to a given set B) is with separable core.

PROOF. Let $f, g \in \mathsf{FCD}(A, B)$ (for some sets A,B).

Because filters on distributive lattices are with separable core, there exist $F, G \in \Gamma(A, B)$ such that $F \cap G = \emptyset$. Then by the previous theorem $F \sqcap^{\mathsf{FCD}} G = \bot$. □

THEOREM 1414. The coatoms of funcoids from a set A to a set B are exactly $(A \times B) \setminus (\{x\} \times \{y\})$ for $x \in A$, $y \in B$.

PROOF. That coatoms of $\Gamma(A, B)$ are exactly $(A \times B) \setminus (\{x\} \times \{y\})$ for $x \in A$, $y \in B$, is obvious. To show that coatoms of funcoids are the same, it remains to apply proposition 560. □

THEOREM 1415. The set of funcoids (for given A and B) is coatomic.

PROOF. Proposition 562. □

EXERCISE 1416. Prove that in general funcoids are not coatomistic.

16.8. Funcoid bases

This section will present mainly a counter-example against a statement you have not thought about anyway.

LEMMA 1417. If S is an upper set of principal funcoids, then $\bigsqcap^{\mathsf{FCD}}(S \cap \Gamma) = \bigsqcap^{\mathsf{FCD}} S$.

PROOF. $\bigsqcap^{\mathsf{FCD}}(S \cap \Gamma) \sqsupseteq \bigsqcap^{\mathsf{FCD}} S$ is obvious.

$\bigsqcap^{\mathsf{FCD}} S = \bigsqcap^{\mathsf{FCD}} \bigsqcap^{\mathsf{FCD}}_{K \in S} T_K \sqsupseteq \bigsqcap^{\mathsf{FCD}}(S \cap \Gamma)$. where $T_K \in \mathscr{P}(S \cap \Gamma)$. So $\bigsqcap^{\mathsf{FCD}}(S \cap \Gamma) = \bigsqcap^{\mathsf{FCD}} S$. □

THEOREM 1418. If S is a filter base on the set of binary relations then S is a base of $\bigsqcap^{\mathsf{FCD}} S$.

First prove a special case of our theorem to get the idea:

EXAMPLE 1419. Take the filter base $S = \left\{ \frac{\left\{ \frac{(x,y)}{|x-y|<\varepsilon} \right\}}{\varepsilon>0} \right\}$ and $K = \left\{ \frac{(x,y)}{|x-y|<\exp x} \right\}$ where x and y range real numbers. Then $K \notin \operatorname{up} \bigsqcap^{\mathsf{FCD}} S$.

PROOF. Take a nontrivial ultrafilter x on $\mathbb{R}$. We can for simplicity assume $x \sqsubseteq \mathbb{Z}$.

$$\left\langle \bigsqcap^{\mathsf{FCD}} S \right\rangle x = \bigsqcap^{\mathscr{F}}_{L \in S} \langle L \rangle x = \bigsqcap^{\mathscr{F}}_{L \in S, X \in \operatorname{up} x} \langle L \rangle^* X = \bigsqcap^{\mathscr{F}}_{\varepsilon>0, X \in \operatorname{up} x} \bigsqcup_{\alpha \in X}]\alpha - \varepsilon; \alpha + \varepsilon[.$$

$\langle K \rangle x = \bigsqcap^{\mathscr{F}}_{X \in \operatorname{up} x} \langle K \rangle^* X = \bigsqcap^{\mathscr{F}}_{X \in \operatorname{up} x} \bigsqcup_{\alpha \in X}]\alpha - \exp \alpha; \alpha + \exp \alpha[$.

Suppose for the contrary that $\langle K \rangle x \sqsupseteq \left\langle \bigsqcap^{\mathsf{FCD}} S \right\rangle x$.

Then

$\bigsqcup_{\alpha \in X}]\alpha - \exp \alpha; \alpha + \exp \alpha[\sqsupseteq \bigsqcap^{\mathscr{F}}_{\varepsilon>0, X \in \operatorname{up} x} \bigsqcup_{\alpha \in X}]\alpha - \varepsilon; \alpha + \varepsilon[$ for every $X \in \operatorname{up} x$; thus by properties of generalized filter bases ($\left\{ \frac{\bigsqcup_{\alpha \in X}]\alpha - \varepsilon; \alpha + \varepsilon[}{\varepsilon>0} \right\}$ is a filter base and even a chain)

$\bigsqcup_{\alpha\in X}]\alpha - \exp\alpha; \alpha + \exp\alpha[\sqsupseteq \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} \bigsqcup_{\alpha\in X}]\alpha-\varepsilon;\alpha+\varepsilon[$ for some $\varepsilon > 0$ and thus by properties of generalized filter bases ($\left\{ \frac{\bigsqcup_{\alpha\in X}]\alpha-\varepsilon;\alpha+\varepsilon[}{X\in\operatorname{up} x} \right\}$ is a filter base) for some $X' \in \operatorname{up} x$

$$\bigsqcup_{\alpha\in X}]\alpha - \exp\alpha; \alpha + \exp\alpha[\sqsupseteq \bigsqcup_{\alpha\in X'}]\alpha-\varepsilon;\alpha+\varepsilon[$$

what is impossible by the fact that $\exp\alpha$ goes infinitely small as $\alpha \to -\infty$ and the fact that we can take $X = \mathbb{Z}$ for some x. □

Now prove the general case:

Proof. Suppose that $K \in \operatorname{up} \bigsqcap^{\mathsf{FCD}} S$ and thus $\langle K\rangle x \sqsupseteq \left\langle \bigsqcap^{\mathsf{FCD}} S \right\rangle x$. We need to prove that there is some $L \in S$ such that $K \sqsupseteq L$.

Take an ultrafilter x.

$\left\langle \bigsqcap^{\mathsf{FCD}} S \right\rangle x = \bigsqcap^{\mathscr{F}}_{L\in S} \langle L\rangle x = \bigsqcap^{\mathscr{F}}_{L\in S, X\in\operatorname{up} x} \langle L\rangle^* X$.

$\langle K\rangle x = \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} \langle K\rangle^* X$.

Then $\langle K\rangle^* X \sqsupseteq \bigsqcap^{\mathscr{F}}_{L\in S, X\in\operatorname{up} x} \langle L\rangle^* X$ for every $X \in \operatorname{up} x$; thus by properties of generalized filter bases ($\left\{ \frac{\langle L\rangle^* X}{L\in S} \right\}$ is a filter base);

$\langle K\rangle^* X \sqsupseteq \bigsqcap^{\mathscr{F}}_{X\in\operatorname{up} x} \langle L\rangle^* X$ for some $L \in S$ and thus by properties of generalized filter bases ($\left\{ \frac{\langle L\rangle^* X}{X\in\operatorname{up} x} \right\}$ is a filter base) for some $X' \in \operatorname{up} x$

$\langle K\rangle^* X \sqsupseteq \langle L\rangle^* X' \sqsupseteq \langle L\rangle x$.

So $\langle K\rangle x \sqsupseteq \langle L\rangle x$ because this equality holds for every $X \in \operatorname{up} x$. Therefore $K \sqsupseteq L$. □

Example 1420. A base of a funcoid which is not a filter base.

Proof. Consider $f = \mathrm{id}^{\mathsf{FCD}}_\Omega$. We know that $\operatorname{up} f$ is not a filter base. But it is a base of a funcoid. □

Exercise 1421. Prove that a set S is a filter (on some set) iff

$$\forall X_0, \ldots, X_n \in S : \operatorname{up}(X_0 \sqcap \cdots \sqcap X_n) \subseteq S$$

for every natural n.

A similar statement does *not* hold for funcoids:

Example 1422. For a set S of binary relations

$$\forall X_0, \ldots, X_n \in S : \operatorname{up}(X_0 \sqcap^{\mathsf{FCD}} \cdots \sqcap^{\mathsf{FCD}} X_n) \subseteq S$$

does not imply that there exists funcoid f such that $S = \operatorname{up} f$.

Proof. Take $S_0 = \operatorname{up} 1^{\mathsf{FCD}}$ (where 1^{FCD} is the identity funcoid on any infinite set) and $S_1 = \bigcup_{F\in S_0} \left\{ \frac{\operatorname{up} G}{G\in\operatorname{up}^\Gamma F} \right\}$ (that is $S_1 = \bigcup_{F\in\operatorname{up}^\Gamma 1^{\mathsf{FCD}}} \operatorname{up} F$).

Both S_0 and S_1 are upper sets. $S_0 \neq S_1$ because $1^{\mathsf{FCD}} \in S_0$ and $1^{\mathsf{FCD}} \notin S_1$.

The formula in the example works for $S = S_0$ because $X_0, \ldots, X_n \in \operatorname{up} 1^{\mathsf{FCD}}$. It also holds for $S = S_1$ by the following reason:

Suppose $X_0, \ldots, X_n \in S_1$. Then $X_i \sqsupseteq F_i$ where $F_i \in S_0$. Consequently (take into account that Γ is a sublattice of FCD) $X_0, \ldots, X_n \sqsupseteq F_0 \sqcap^{\mathsf{FCD}} \cdots \sqcap^{\mathsf{FCD}} F_n$ and so $X_0 \sqcap^{\mathsf{FCD}} \cdots \sqcap^{\mathsf{FCD}} X_n = X_0 \sqcap \cdots \sqcap X_n \sqsupseteq F_0 \sqcap^{\mathsf{FCD}} \cdots \sqcap^{\mathsf{FCD}} F_n \sqsupseteq 1^{\mathsf{FCD}}$. Thus $X_0 \sqcap \cdots \sqcap X_n \in \operatorname{up}^\Gamma 1^{\mathsf{FCD}} \subseteq S_1$; $\operatorname{up}(X_0 \sqcap \cdots \sqcap X_n) \subseteq S_1$ as S_1 is an upper set.

To finish the proof suppose for the contrary that $\operatorname{up} f_0 = S_0$ and $\operatorname{up} f_1 = S_1$ for some funcoids f_0 and f_1. In this case $f_0 = \bigsqcap^{\mathsf{FCD}} S_0 = 1^{\mathsf{FCD}} = \bigsqcap^{\mathsf{FCD}} \operatorname{up}^\Gamma 1^{\mathsf{FCD}} = \bigsqcap^{\mathsf{FCD}} S_1 = f_1$ and thus $S_0 = S_1$, contradiction. □

PROPOSITION 1423. For a set S of binary relations

$$\forall X_0,\dots,X_n\in S:\operatorname{up}(X_0\sqcap^{\mathsf{FCD}}\dots\sqcap^{\mathsf{FCD}}X_n)\subseteq S$$

does not imply that S is a funcoid base.

PROOF. Suppose for the contrary that it does imply. Then, because S is an upper set (as follows from the condition, taking $n=0$), it implies that $S=\operatorname{up}f$ for a funcoid f, what contradicts to the above example. □

CONJECTURE 1424. Let $\forall X,Y\in S:\operatorname{up}(X\sqcap^{\mathsf{FCD}}Y)\subseteq S$.
Then

$$\forall X_0,\dots,X_n\in S:\operatorname{up}(X_0\sqcap^{\mathsf{FCD}}\dots\sqcap^{\mathsf{FCD}}X_n)\subseteq S.$$

EXERCISE 1425. $\operatorname{up}(f_0\sqcap^{\mathsf{FCD}}\ldots\sqcap^{\mathsf{FCD}}f_n)\subseteq\left\{\frac{F_0\sqcap\ldots\sqcap F_n}{F_0\in\operatorname{up}f_0\wedge\ldots\wedge F_n\in\operatorname{up}f_n}\right\}$ for every funcoids f_0, …, f_n ($n\in\mathbb{N}$).

16.9. Some (example) values

I will do some calculations of particular funcoids and reloids.

First note that $\sqcap^{\mathsf{FCD}}$ can be decomposed (see below for a short easy proof):

$$f\sqcap^{\mathsf{FCD}}g=(\mathsf{FCD})(((\mathsf{RLD})_{\mathrm{in}}f\sqcap(\mathsf{RLD})_{\mathrm{in}}g).$$

The above is a more understandable decomposition of the operation $\sqcap^{\mathsf{FCD}}$ which behaves in strange way, mapping meet of two binary relations into a funcoid which is not a binary relation ($1^{\mathsf{FCD}}\sqcap^{\mathsf{FCD}}(\top\setminus 1^{\mathsf{FCD}})=1^{\mathsf{FCD}}_{\Omega}$).

The last formula is easy to prove (and proved above in the book) but the result is counter-intuitive.

More generally:

$$\bigsqcap^{\mathsf{FCD}}S=(\mathsf{FCD})\bigsqcap^{\mathsf{RLD}}\langle(\mathsf{RLD})_{\mathrm{in}}\rangle^{*}S.$$

The above formulas follow from the fact that (FCD) is an upper adjoint and that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}f=f$ for every funcoid f.

Let FCD denote funcoids on a set U.

Consider a special case of the above formulas:

$$1^{\mathsf{FCD}}\sqcap^{\mathsf{FCD}}(\top\setminus 1^{\mathsf{FCD}})=(\mathsf{FCD})((\mathsf{RLD})_{\mathrm{in}}1^{\mathsf{FCD}}\sqcap(\mathsf{RLD})_{\mathrm{in}}(\top\setminus 1^{\mathsf{FCD}})).\tag{17}$$

We want to calculate terms of the formula (17) and more generally do some (probably useless) calculations for particular funcoids and reloids related to the above formula.

The left side is already calculated. The term $(\mathsf{RLD})_{\mathrm{in}}1^{\mathsf{FCD}}$ which I call "thick equality" above is well understood. Let's compute $(\mathsf{RLD})_{\mathrm{in}}(\top\setminus 1^{\mathsf{FCD}})$.

PROPOSITION 1426. $(\mathsf{RLD})_{\mathrm{in}}(\top\setminus 1^{\mathsf{FCD}})=\top\setminus 1^{\mathsf{FCD}}$.

PROOF. Consider funcoids on a set U. For any filters x and y (or without loss of generality ultrafilters x and y) we have:

$x\times^{\mathsf{FCD}}y\sqsubseteq\top\setminus 1^{\mathsf{FCD}}\Leftrightarrow$ (theorem 574 and the fact that funcoids are filters) $\Leftrightarrow x\times^{\mathsf{FCD}}y\asymp 1^{\mathsf{FCD}}\Leftrightarrow\neg\big(x\;[1^{\mathsf{FCD}}]\;y\big)\Leftrightarrow x\asymp y\Rightarrow\exists X\in\operatorname{up}x,Y\in\operatorname{up}y:X\asymp Y$.

Thus $(\mathsf{RLD})_{\mathrm{in}}(\top\setminus 1^{\mathsf{FCD}})=\bigsqcup\left\{\frac{X\times Y}{X,Y\in\mathscr{T}U,X\asymp Y}\right\}=\top\setminus 1^{\mathsf{FCD}}$. □

So, we have:

$$1^{\mathsf{FCD}}_{\Omega}=1^{\mathsf{FCD}}\sqcap^{\mathsf{FCD}}(\top\setminus 1^{\mathsf{FCD}})=(\mathsf{RLD})_{\mathrm{in}}1^{\mathsf{FCD}}\sqcap^{\mathsf{FCD}}(\top\setminus 1^{\mathsf{FCD}}).$$

PROPOSITION 1427. If $X_0\sqcup\ldots\sqcup X_n=\top$ then $(X_0\times X_0)\sqcup\ldots\sqcup(X_n\times X_n)\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}1^{\mathsf{FCD}}$.

PROOF. It's enough to prove $(X_0 \times X_0) \sqcup \ldots \sqcup (X_n \times X_n) \in \operatorname{up}(x \times x)$ for every ultrafilter x, what follows from the fact that $x \sqsubseteq X_i$ for some i and thus $x \times x \sqsubseteq X_i \times X_i$. □

PROPOSITION 1428. For finite tuples X, Y of typed sets

$$(X_0 \times Y_0) \sqcup \ldots \sqcup (X_n \times Y_n) \sqsupseteq 1 \Leftrightarrow (X_0 \sqcap Y_0) \sqcup \ldots \sqcup (X_n \sqcap Y_n) = \top.$$

PROOF. $(X_0 \times Y_0) \sqcup \ldots \sqcup (X_n \times Y_n) \sqsupseteq 1 \Leftrightarrow ((X_0 \times Y_0) \sqcup \ldots \sqcup (X_n \times Y_n)) \sqcap 1 = 1 \Leftrightarrow ((X_0 \times Y_0) \sqcap 1) \sqcup \ldots \sqcup ((X_n \times Y_n) \sqcap 1) = 1 \Leftrightarrow \mathrm{id}_{X_0 \sqcap Y_0} \sqcup \ldots \sqcup \mathrm{id}_{X_n \sqcap Y_n} = 1 \Leftrightarrow \mathrm{id}_{(X_0 \sqcap Y_0) \sqcup \ldots \sqcup (X_n \sqcap Y_n)} = 1 \Leftrightarrow (X_0 \sqcap Y_0) \sqcup \ldots \sqcup (X_n \sqcap Y_n) = \top$. □

COROLLARY 1429.

$$\operatorname{up}^\Gamma 1 = \left\{ \frac{(X_0 \times Y_0) \sqcup \ldots \sqcup (X_n \times Y_n)}{n \in \mathbb{N}, \forall i \in n : X_i, Y_i \in \mathscr{T} U, (X_0 \sqcap Y_0) \sqcup \ldots \sqcup (X_n \sqcap Y_n) = \top} \right\}.$$

COROLLARY 1430. The predicate $(X_0 \sqcap Y_0) \sqcup \ldots \sqcup (X_n \sqcap Y_n) = \top$ for an element $(X_0 \times Y_0) \sqcup \ldots \sqcup (X_n \times Y_n)$ of Γ does not depend on its representation $(X_0 \times Y_0) \sqcup \ldots \sqcup (X_n \times Y_n)$.

PROPOSITION 1431.

$$\operatorname{up}^\Gamma 1 = \bigcup \left\{ \frac{\operatorname{up}^\Gamma((X_0 \times X_0) \sqcup \ldots \sqcup (X_n \times X_n))}{n \in \mathbb{N}, \forall i \in n : X_i \in \mathscr{T} U, X_0 \sqcup \ldots \sqcup X_n = \top} \right\}.$$

PROOF. If $(X_0 \times Y_0) \sqcup \ldots \sqcup (X_n \times Y_n) \in \operatorname{up}^\Gamma 1$ then we have

$$(X_0 \times Y_0) \sqcup \ldots \sqcup (X_n \times Y_n) \sqsupseteq ((X_0 \sqcap Y_0) \times (X_0 \sqcap Y_0)) \sqcup \ldots \sqcup ((X_n \sqcap Y_n) \times (X_n \sqcap Y_n)) \in \operatorname{up}^\Gamma 1.$$

Thus

$$\operatorname{up}^\Gamma 1 \subseteq \bigcup \left\{ \frac{\operatorname{up}^\Gamma((X_0 \times X_0) \sqcup \ldots \sqcup (X_n \times X_n))}{n \in \mathbb{N}, \forall i \in n : X_i \in \mathscr{T} U, X_0 \sqcup \ldots \sqcup X_n = \top} \right\}.$$

The reverse inclusion is obvious. □

PROPOSITION 1432.

$$(\mathsf{RLD})_{\mathrm{in}} 1^{\mathsf{FCD}} = \bigsqcap^{\mathsf{RLD}} \left\{ \frac{(X_0 \times X_0) \sqcup \ldots \sqcup (X_n \times X_n)}{n \in \mathbb{N}, \forall i \in n : X_i \in \mathscr{T} U, X_0 \sqcup \ldots \sqcup X_n = \top} \right\}.$$

PROOF. By the diagram we have $(\mathsf{RLD})_{\mathrm{in}} 1^{\mathsf{FCD}} = \bigsqcap^{\mathsf{RLD}} \operatorname{up}^\Gamma 1$. So it follows from the previous proposition. □

PROPOSITION 1433. $\operatorname{up}^\Gamma (\mathsf{RLD})_{\mathrm{in}} 1^{\mathsf{FCD}} = \operatorname{up}^\Gamma 1$.

PROOF. If $K \in \operatorname{up}^\Gamma 1$ then $K \in \operatorname{up}^\Gamma((X_0 \times X_0) \sqcup \ldots \sqcup (X_n \times X_n))$ and thus $K \in \operatorname{up}^\Gamma (\mathsf{RLD})_{\mathrm{in}} 1^{\mathsf{FCD}}$ (see proposition 1427). Thus $\operatorname{up}^\Gamma 1 \subseteq \operatorname{up}^\Gamma (\mathsf{RLD})_{\mathrm{in}} 1^{\mathsf{FCD}}$. But $\operatorname{up}^\Gamma (\mathsf{RLD})_{\mathrm{in}} 1^{\mathsf{FCD}} \subseteq \operatorname{up}^\Gamma 1$ is obvious. □

CHAPTER 17

Generalized cofinite filters

The following is a straightforward generalization of cofinite filter.

DEFINITION 1434. $\Omega_{1a} = \bigsqcap^{\mathfrak{A}}_{X \in \operatorname{coatoms}^{\mathfrak{Z}}} X$; $\Omega_{1b} = \bigsqcap^{\mathfrak{A}}_{X \in \operatorname{coatoms}^{\mathfrak{A}}} X$.

PROPOSITION 1435. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $\Omega_{1a} = \Omega_{1b}$ for this filtrator.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Proposition 560.

□

PROPOSITION 1436. Let $(\mathfrak{A}, \mathfrak{Z})$ be a primary filtrator. Let $\mathfrak{Z}$ be a subset of $\mathscr{P}U$. Let it be a meet-semilattice with greatest element. Let also every non-coempty cofinite set lies in $\mathfrak{Z}$. Then

$$\partial\Omega = \left\{ \frac{Y \in \mathfrak{Z}}{\operatorname{card} \operatorname{atoms}^{\mathfrak{Z}} Y \geq \omega} \right\}. \tag{18}$$

PROOF. Ω exists by corollary 518.

$Y \in \partial\Omega \Leftrightarrow Y \not\asymp^{\mathfrak{A}} \bigsqcap^{\mathfrak{A}}_{X \in \operatorname{coatoms}^{\mathfrak{Z}}} X \Leftrightarrow$ (by properties of filter bases) $\Leftrightarrow \forall S \in \mathscr{P}_{\mathrm{fin}} \operatorname{coatoms}^{\mathfrak{Z}} : Y \not\asymp^{\mathfrak{A}} \bigsqcap^{\mathfrak{A}} S \Leftrightarrow$ (corollary 536) $\Leftrightarrow \forall S \in \mathscr{P}_{\mathrm{fin}} \operatorname{coatoms}^{\mathfrak{Z}} : Y \not\asymp \bigsqcap S \Leftrightarrow \forall K \in \mathscr{P}_{\mathrm{fin}} U : Y \setminus K \neq \emptyset \Leftrightarrow \operatorname{card} Y \geq \omega \Leftrightarrow \operatorname{card} \operatorname{atoms}^{\mathfrak{Z}} Y \geq \omega$. (Here $\mathscr{P}_{\mathrm{fin}}$ denotes the set of finite subsets.) □

COROLLARY 1437. Formula (18) holds for both reloids and funcoids.

PROOF. For reloids it's straightforward, for funcoids take that they are isomorphic to filters on lattice Γ. □

COROLLARY 1438. $\Omega^{\mathsf{FCD}} \neq \bot^{\mathsf{FCD}}$ (for $\mathsf{FCD}(A, B)$ where $A \times B$ is an infinite set).

PROPOSITION 1439. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over an atomic ideal base and $\forall \alpha \in \operatorname{atoms}^{\mathfrak{Z}} \exists X \in \operatorname{coatoms}^{\mathfrak{Z}} : a \not\sqsubseteq X$.
3°. Ω_{1a} and $\operatorname{Cor} \Omega_{1a}$ are defined, $\forall \alpha \in \operatorname{atoms}^{\mathfrak{Z}} \exists X \in \operatorname{coatoms}^{\mathfrak{Z}} : a \not\sqsubseteq X$ and $\mathfrak{Z}$ is an atomic poset.
4°. $\operatorname{Cor} \Omega_{1a} = \bot^{\mathfrak{Z}}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Obvious.

3°⇒4°. Suppose $\alpha \in \operatorname{atoms}^{\mathfrak{Z}} \operatorname{Cor} \Omega$. Then $\exists X \in \operatorname{up} \Omega : \alpha \not\sqsubseteq X$. Therefore $\alpha \notin \operatorname{atoms}^{\mathfrak{Z}} \operatorname{Cor} \Omega$. So $\operatorname{atoms}^{\mathfrak{Z}} \operatorname{Cor} \Omega_{1a} = \emptyset$ and thus by atomicity $\operatorname{Cor} \Omega_{1a} = \bot^{\mathfrak{Z}}$.

□

COROLLARY 1440. $\operatorname{Cor} \Omega^{\mathsf{FCD}} = \bot$.

PROPOSITION 1441. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over an atomic meet-semilattice with greatest element such that $\forall \alpha \in \operatorname{atoms}^{\mathfrak{Z}} \exists X \in \operatorname{coatoms}^{\mathfrak{Z}} : a \not\sqsubseteq X$.
3°. $\mathfrak{A}$ is a complete lattice, $\forall \alpha \in \operatorname{atoms}^{\mathfrak{Z}} \exists X \in \operatorname{coatoms}^{\mathfrak{Z}} : a \not\sqsubseteq X$ and $(\mathfrak{A}; \mathfrak{Z})$ is a filtered filtrator over an atomic poset.
4°. $\Omega_{1a} = \max\left\{ \frac{\mathcal{X} \in \mathfrak{A}}{\operatorname{Cor} \mathcal{X} = \bot^{\mathfrak{Z}}} \right\}$

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Obvious.
3°⇒4°. Due the last proposition, it is enough to show that $\operatorname{Cor} \mathcal{X} = \bot^{\mathfrak{Z}} \Rightarrow \mathcal{X} \sqsubseteq \Omega_{1a}$ for every $\mathcal{X} \in \mathfrak{A}$.

Let $\operatorname{Cor} \mathcal{X} = \bot^{\mathfrak{Z}}$ for some $\mathcal{X} \in \mathfrak{A}$. Because of our filtrator being filtered, it's enough to show $X \in \operatorname{up} \mathcal{X}$ for every $X \in \operatorname{up} \Omega_{1a}$. $X = a_0 \sqcap \ldots \sqcap a_n$ for a_i being coatoms of $\mathfrak{Z}$. $a_i \sqsupseteq \mathcal{X}$ because otherwise $a_i \not\sqsupseteq \operatorname{Cor} \mathcal{X}$. So $X \in \operatorname{up} \mathcal{X}$.

□

PROPOSITION 1442. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice.
3°. $\operatorname{up} \Omega_{1a} = \left\{ \frac{\bigsqcap S}{S \in \mathscr{P}_{\text{fin}} \operatorname{coatoms}^{\mathfrak{Z}}} \right\}$

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Because $\left\{ \frac{\bigsqcap S}{S \in \mathscr{P}_{\text{fin}} \operatorname{coatoms}^{\mathfrak{Z}}} \right\}$ is a filter.

□

COROLLARY 1443. $\operatorname{up} \Omega^{\mathsf{FCD}} = \operatorname{up} \Omega^{\mathsf{RLD}}$.

DEFINITION 1444. $\Omega_{1c} = \bigsqcup (\operatorname{atoms}^{\mathfrak{A}} \setminus \mathfrak{Z})$.

PROPOSITION 1445. The following is an implications tuple:

1°. $(\mathfrak{A}; \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}; \mathfrak{Z})$ is a down-aligned filtered complete lattice filtrator over an atomistic poset and $\forall \alpha \in \operatorname{atoms}^{\mathfrak{Z}} \exists X \in \operatorname{coatoms}^{\mathfrak{Z}} : a \not\sqsubseteq X$.
3°. $\Omega_{1c} = \Omega_{1a}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. For $x \in \operatorname{atoms}^{\mathfrak{A}} \setminus \mathfrak{Z}$ we have $\operatorname{Cor} x = \bot$ because otherwise $\bot \neq \operatorname{Cor} x \sqsubset x$. Thus by previous $x \sqsubseteq \Omega_{1a}$ and so $\Omega_{1c} = \bigsqcup (\operatorname{atoms}^{\mathfrak{A}} \setminus \mathfrak{Z}) \sqsubseteq \Omega_{1a}$.

If $x \in \operatorname{atoms} \Omega_{1a}$ then $x \notin \mathfrak{Z}$ because otherwise $\operatorname{Cor} x \neq \bot$. So

$$\Omega_{1a} = \bigsqcup \operatorname{atoms} \Omega_{1a} = \bigsqcup (\operatorname{atoms} \Omega_{1a} \setminus \mathfrak{Z}) \sqsubseteq \bigsqcup (\operatorname{atoms}^{\mathfrak{A}} \setminus \mathfrak{Z}) = \Omega_{1c}.$$

□

THEOREM 1446. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete atomic boolean lattice.
3°. All of the following:
  (a) $\mathfrak{A}$ is atomistic complete starrish lattice.
  (b) $\mathfrak{Z}$ is a complete atomistic lattice.
  (c) $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned filtrator with binarily meet-closed core.
4°. Cor' is the lower adjoint of $\Omega_{1c} \sqcup^{\mathfrak{A}} -$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Obvious.
3°⇒4°. It with join-closed core by theorem 534.
We will prove $\operatorname{Cor}' \mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \Omega_{1c} \sqcup \mathcal{Y}$.

By atomisticity it is equivalent to: $\operatorname{atoms}^{\mathfrak{A}} \operatorname{Cor}' \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y} \Leftrightarrow \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{A}}(\Omega_{1c} \sqcup \mathcal{Y})$; (theorem 603) $\operatorname{atoms}^{\mathfrak{A}} \operatorname{Cor}' \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y} \Leftrightarrow \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{A}} \Omega_{1c} \cup \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y}$; what by below is equivalent to: $\operatorname{atoms}^{\mathfrak{Z}} \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{Z}} \mathcal{Y} \Leftrightarrow \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{A}} \Omega_{1c} \cup \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y}$.

$\operatorname{Cor}' \mathcal{X} \sqsubseteq \mathcal{Y} \Leftrightarrow \operatorname{atoms}^{\mathfrak{A}} \operatorname{Cor}' \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y} \Rightarrow \operatorname{atoms}^{\mathfrak{Z}} \operatorname{Cor}' \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{Z}} \mathcal{Y} \Leftrightarrow \operatorname{atoms}^{\mathfrak{Z}} \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{Z}} \mathcal{Y}$;

$\operatorname{atoms}^{\mathfrak{Z}} \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{Z}} \mathcal{Y} \Rightarrow$ (theorem 599) $\Rightarrow \operatorname{Cor}' \mathcal{X} \sqsubseteq \operatorname{Cor}' \mathcal{Y} \Rightarrow$ (theorem 543) $\Rightarrow \operatorname{Cor}' \mathcal{X} \sqsubseteq \mathcal{Y}$.

Finishing the proof $\operatorname{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{A}} \Omega_{1c} \cup \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y} \Leftrightarrow \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} \subseteq (\operatorname{atoms}^{\mathfrak{A}} \setminus \mathfrak{Z}) \cup \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y} \Leftrightarrow \operatorname{atoms}^{\mathfrak{Z}} \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{A}} \mathcal{Y} \Leftrightarrow \operatorname{atoms}^{\mathfrak{Z}} \mathcal{X} \subseteq \operatorname{atoms}^{\mathfrak{Z}} \mathcal{Y}$.

□

Next there is an alternative proof of the above theorem. This alternative proof requires additional condition $\forall \alpha \in \operatorname{atoms}^{\mathfrak{Z}} \exists X \in \operatorname{coatoms}^{\mathfrak{Z}} : a \not\sqsubseteq X$ however.

PROOF. Define $\Omega = \Omega_{1a} = \Omega_{1c}$.
It with join-closed core by theorem 534.
It's enough to prove that

$$\mathcal{X} \sqsubseteq \Omega \sqcup^{\mathfrak{A}} \operatorname{Cor}' \mathcal{X} \quad \text{and} \quad \operatorname{Cor}'(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) \sqsubseteq \mathcal{Y}.$$

$\operatorname{Cor}'(\Omega \sqcup^{\mathfrak{A}} \mathcal{Y}) =$ (theorem 603) $= \operatorname{Cor}' \Omega \sqcup^{\mathfrak{Z}} \operatorname{Cor}' \mathcal{Y} =$ (proposition 1439) $= \bot^{\mathfrak{Z}} \sqcup^{\mathfrak{Z}} \operatorname{Cor}' \mathcal{Y} \sqsubseteq$ (theorem 543) $\sqsubseteq \mathcal{Y}$.

$\Omega \sqcup^{\mathfrak{A}} \operatorname{Cor}' \mathcal{X} = \bigsqcup \operatorname{atoms}(\Omega \sqcup^{\mathfrak{A}} \operatorname{Cor}' \mathcal{X}) = \bigsqcup(\operatorname{atoms} \Omega \cup \operatorname{Cor}' \mathcal{X}) = \bigsqcup \operatorname{atoms} \Omega \sqcup \bigsqcup \operatorname{atoms} \mathcal{X}) \sqsupseteq \bigsqcup(\operatorname{atoms} \mathcal{X} \setminus \mathfrak{Z}) \sqcup \bigsqcup(\operatorname{atoms} \mathcal{X} \cap \mathfrak{Z}) = \bigsqcup((\operatorname{atoms} \mathcal{X} \setminus \mathfrak{Z}) \cup (\operatorname{atoms} \mathcal{X} \cap \mathfrak{Z}) = \bigsqcup \operatorname{atoms} \mathcal{X} = \mathcal{X}$. □

COROLLARY 1447. Under conditions of the last theorem $\operatorname{Cor}' \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{A}} \langle \operatorname{Cor}' \rangle^* S$.

PROPOSITION 1448. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete atomic boolean lattice.
3°. All of the following:
  (a) $\mathfrak{A}$ is atomistic complete co-brouwerian lattice.
  (b) $\mathfrak{Z}$ is a complete atomistic lattice.
  (c) $(\mathfrak{A}, \mathfrak{Z})$ is a filtered down-aligned filtrator with binarily meet-closed core.
4°. $\operatorname{Cor}' \mathcal{X} = \mathcal{X} \setminus^* \Omega_{1c}$

PROOF.

1°⇒2° Obvious.
2°⇒3° Because complete atomic boolean lattice is isomorphic to a powerset.
3°⇒4° Theorems 1446 and 154.

□

PROPOSITION 1449.
1°. $\langle\Omega^{\mathsf{FCD}}\rangle\{x\} = \Omega^U$;
2°. $\langle\Omega^{\mathsf{FCD}}\rangle p = \top$ for every nontrivial atomic filter p.

PROOF. $\langle\Omega^{\mathsf{FCD}}\rangle\{x\} = \bigsqcap^{\mathfrak{A}}_{y\in U}(U\setminus\{y\}) = \Omega^U$; $\langle\Omega^{\mathsf{FCD}}\rangle p = \bigsqcap^{\mathfrak{A}}_{y\in U}\top = \top$. □

PROPOSITION 1450. $(\mathsf{FCD})\Omega^{\mathsf{RLD}} = \Omega^{\mathsf{FCD}}$.

PROOF. $(\mathsf{FCD})\Omega^{\mathsf{RLD}} = \bigsqcap^{\mathsf{FCD}}\operatorname{up}\Omega^{\mathsf{RLD}} = \Omega^{\mathsf{FCD}}$. □

PROPOSITION 1451. $(\mathsf{RLD})_{\mathrm{out}}\Omega^{\mathsf{FCD}} = \Omega^{\mathsf{RLD}}$.

PROOF. $(\mathsf{RLD})_{\mathrm{out}}\Omega^{\mathsf{FCD}} = \bigsqcap^{\mathsf{RLD}}\operatorname{up}\Omega^{\mathsf{FCD}} = \bigsqcap^{\mathsf{RLD}}\operatorname{up}\Omega^{\mathsf{RLD}} = \Omega^{\mathsf{RLD}}$. □

PROPOSITION 1452. $(\mathsf{RLD})_{\mathrm{in}}\Omega^{\mathsf{FCD}} = \Omega^{\mathsf{RLD}}$.

PROOF.
$$(\mathsf{RLD})_{\mathrm{in}}\Omega^{\mathsf{FCD}} = \bigsqcup\left\{\frac{a\times^{\mathsf{RLD}} b}{a\in\operatorname{atoms}^{\mathscr{F}}, b\in\operatorname{atoms}^{\mathscr{F}}, a\times^{\mathsf{FCD}} b\sqsubseteq\Omega^{\mathsf{FCD}}}\right\} =$$
$$\bigsqcup\left\{\frac{a\times^{\mathsf{RLD}} b}{a\in\operatorname{atoms}^{\mathscr{F}}, b\in\operatorname{atoms}^{\mathscr{F}}, \text{not } a \text{ and } b \text{ both trivial}}\right\} =$$
$$\bigsqcup\left\{\frac{\bigsqcup\operatorname{atoms}(a\times^{\mathsf{RLD}} b)}{a\in\operatorname{atoms}^{\mathscr{F}}, b\in\operatorname{atoms}^{\mathscr{F}}, \text{not } a \text{ and } b \text{ both trivial}}\right\} =$$
$$\bigsqcup\bigcup\left\{\frac{\operatorname{atoms}(a\times^{\mathsf{RLD}} b)}{a\in\operatorname{atoms}^{\mathscr{F}}, b\in\operatorname{atoms}^{\mathscr{F}}, \text{not } a \text{ and } b \text{ both trivial}}\right\} =$$
$$\bigsqcup(\text{nontrivial atomic reloids under } A\times B) = \Omega^{\mathsf{RLD}}.$$

□

CHAPTER 18

Convergence of funcoids

18.1. Convergence

The following generalizes the well-known notion of a filter convergent to a point or to a set:

DEFINITION 1453. A filter $\mathcal{F} \in \mathscr{F}(\operatorname{Dst}\mu)$ *converges* to a filter $\mathcal{A} \in \mathscr{F}(\operatorname{Src}\mu)$ regarding a funcoid μ ($\mathcal{F} \xrightarrow{\mu} \mathcal{A}$) iff $\mathcal{F} \sqsubseteq \langle\mu\rangle\mathcal{A}$.

DEFINITION 1454. A funcoid f *converges* to a filter $\mathcal{A} \in \mathscr{F}(\operatorname{Src}\mu)$ regarding a funcoid μ where $\operatorname{Dst} f = \operatorname{Dst}\mu$ (denoted $f \xrightarrow{\mu} \mathcal{A}$) iff $\operatorname{im} f \sqsubseteq \langle\mu\rangle\mathcal{A}$ that is iff $\operatorname{im} f \xrightarrow{\mu} \mathcal{A}$.

DEFINITION 1455. A funcoid f *converges* to a filter $\mathcal{A} \in \mathscr{F}(\operatorname{Src}\mu)$ on a filter $\mathcal{B} \in \mathscr{F}(\operatorname{Src} f)$ regarding a funcoid μ where $\operatorname{Dst} f = \operatorname{Dst}\mu$ iff $f|_{\mathcal{B}} \xrightarrow{\mu} \mathcal{A}$.

OBVIOUS 1456. A funcoid f converges to a filter $\mathcal{A} \in \mathscr{F}(\operatorname{Src}\mu)$ on a filter $\mathcal{B} \in \mathscr{F}(\operatorname{Src} f)$ regarding a funcoid μ iff $\langle f\rangle\mathcal{B} \sqsubseteq \langle\mu\rangle\mathcal{A}$.

REMARK 1457. We can define also convergence for a reloid f: $f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \operatorname{im} f \sqsubseteq \langle\mu\rangle\mathcal{A}$ or what is the same $f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow (\mathsf{FCD})f \xrightarrow{\mu} \mathcal{A}$.

THEOREM 1458. Let f, g be funcoids, μ, ν be endofuncoids, $\operatorname{Dst} f = \operatorname{Src} g = \operatorname{Ob}\mu$, $\operatorname{Dst} g = \operatorname{Ob}\nu$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$. If $f \xrightarrow{\mu} \mathcal{A}$,

$$g|_{\langle\mu\rangle\mathcal{A}} \in \mathrm{C}(\mu \sqcap (\langle\mu\rangle\mathcal{A} \times^{\mathsf{FCD}} \langle\mu\rangle\mathcal{A}), \nu),$$

and $\langle\mu\rangle\mathcal{A} \sqsupseteq \mathcal{A}$, then $g \circ f \xrightarrow{\nu} \langle g\rangle\mathcal{A}$.

PROOF.

$$\operatorname{im} f \sqsubseteq \langle\mu\rangle\mathcal{A};$$
$$\langle g\rangle \operatorname{im} f \sqsubseteq \langle g\rangle\langle\mu\rangle\mathcal{A};$$
$$\operatorname{im}(g \circ f) \sqsubseteq \left\langle g|_{\langle\mu\rangle\mathcal{A}}\right\rangle\langle\mu\rangle\mathcal{A};$$
$$\operatorname{im}(g \circ f) \sqsubseteq \left\langle g|_{\langle\mu\rangle\mathcal{A}}\right\rangle\left\langle\mu \sqcap (\langle\mu\rangle\mathcal{A} \times^{\mathsf{FCD}} \langle\mu\rangle\mathcal{A})\right\rangle\mathcal{A};$$
$$\operatorname{im}(g \circ f) \sqsubseteq \left\langle g|_{\langle\mu\rangle\mathcal{A}} \circ (\mu \sqcap (\langle\mu\rangle\mathcal{A} \times^{\mathsf{FCD}} \langle\mu\rangle\mathcal{A}))\right\rangle\mathcal{A};$$
$$\operatorname{im}(g \circ f) \sqsubseteq \left\langle \nu \circ g|_{\langle\mu\rangle\mathcal{A}}\right\rangle\mathcal{A};$$
$$\operatorname{im}(g \circ f) \sqsubseteq \langle \nu \circ g\rangle\mathcal{A};$$
$$g \circ f \xrightarrow{\nu} \langle g\rangle\mathcal{A}.$$

□

COROLLARY 1459. Let f, g be funcoids, μ, ν be endofuncoids, $\operatorname{Dst} f = \operatorname{Src} g = \operatorname{Ob}\mu$, $\operatorname{Dst} g = \operatorname{Ob}\nu$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$. If $f \xrightarrow{\mu} \mathcal{A}$, $g \in \mathrm{C}(\mu, \nu)$, and $\langle\mu\rangle\mathcal{A} \sqsupseteq \mathcal{A}$ then $g \circ f \xrightarrow{\nu} \langle g\rangle\mathcal{A}$.

PROOF. From the last theorem and theorem 1187. □

18.2. Relationships between convergence and continuity

LEMMA 1460. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$, $\operatorname{Src} f = \operatorname{Ob}\mu$, $\operatorname{Dst} f = \operatorname{Ob}\nu$. If $f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu)$ then

$$f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}}\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A}.$$

PROOF.

$$f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A} \Leftrightarrow \operatorname{im} f|_{\langle\mu\rangle\mathcal{A}} \sqsubseteq \langle\nu\rangle\langle f\rangle\mathcal{A} \Leftrightarrow$$
$$\langle f\rangle\langle\mu\rangle\mathcal{A} \sqsubseteq \langle\nu\rangle\langle f\rangle\mathcal{A} \Leftrightarrow \langle f \circ \mu\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}}\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A}.$$

□

THEOREM 1461. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$, $\operatorname{Src} f = \operatorname{Ob}\mu$, $\operatorname{Dst} f = \operatorname{Ob}\nu$. If $f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu)$ then $f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A}$.

PROOF.

$$f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A} \Leftrightarrow \text{(by the lemma)} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}}\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A} \Leftarrow$$
$$f \circ \mu|_{\mathcal{A}} \sqsubseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu).$$

□

COROLLARY 1462. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$, $\operatorname{Src} f = \operatorname{Ob}\mu$, $\operatorname{Dst} f = \operatorname{Ob}\nu$. If $f \in \mathrm{C}(\mu, \nu)$ then $f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A}$.

THEOREM 1463. Let μ, ν be endofuncoids, $f \in \mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\nu)$, $\mathcal{A} \in \mathscr{F}(\operatorname{Ob}\mu)$ be an ultrafilter, $\operatorname{Src} f = \operatorname{Ob}\mu$, $\operatorname{Dst} f = \operatorname{Ob}\nu$. $f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu)$ iff $f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A}$.

PROOF.

$$f|_{\langle\mu\rangle\mathcal{A}} \xrightarrow{\nu} \langle f\rangle\mathcal{A} \Leftrightarrow \text{(by the lemma)} \Leftrightarrow \langle f \circ \mu|_{\mathcal{A}}\rangle\mathcal{A} \sqsubseteq \langle \nu \circ f\rangle\mathcal{A} \Leftrightarrow$$
$$\text{(used the fact that } \mathcal{A} \text{ is an ultrafilter)}$$
$$f \circ \mu|_{\mathcal{A}} \sqsubseteq \nu \circ f|_{\mathcal{A}} \Leftrightarrow f \circ \mu|_{\mathcal{A}} \sqsubseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu|_{\mathcal{A}}, \nu).$$

□

18.3. Convergence of join

PROPOSITION 1464. $\bigsqcup S \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall \mathcal{F} \in S : \mathcal{F} \xrightarrow{\mu} \mathcal{A}$ for every collection S of filters on $\operatorname{Dst}\mu$ and filter $\mathcal{A}$ on $\operatorname{Src}\mu$, for every funcoid μ.

PROOF.
$$\bigsqcup S \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \bigsqcup S \sqsubseteq \langle\mu\rangle\mathcal{A} \Leftrightarrow \forall \mathcal{F} \in S : \mathcal{F} \sqsubseteq \langle\mu\rangle\mathcal{A} \Leftrightarrow \forall \mathcal{F} \in S : \mathcal{F} \xrightarrow{\mu} \mathcal{A}.$$

□

COROLLARY 1465. $\bigsqcup F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in F : f \xrightarrow{\mu} \mathcal{A}$ for every collection F of funcoids f such that $\operatorname{Dst} f = \operatorname{Dst}\mu$ and filter $\mathcal{A}$ on $\operatorname{Src}\mu$, for every funcoid μ.

PROOF. By corollary 896 we have

$$\bigsqcup F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \operatorname{im} \bigsqcup F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \bigsqcup \langle \operatorname{im}\rangle^* F \xrightarrow{\mu} \mathcal{A} \Leftrightarrow$$
$$\forall f \in \langle \operatorname{im}\rangle^* F : \mathcal{F} \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in F : \operatorname{im} f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \forall f \in F : f \xrightarrow{\mu} \mathcal{A}.$$

□

THEOREM 1466. $f|_{\mathcal{B}_0 \sqcup \mathcal{B}_1} \xrightarrow{\mu} \mathcal{A} \Leftrightarrow f|_{\mathcal{B}_0} \xrightarrow{\mu} \mathcal{A} \wedge f|_{\mathcal{B}_1} \xrightarrow{\mu} \mathcal{A}$. for all filters $\mathcal{A}$, $\mathcal{B}_0$, $\mathcal{B}_1$ and funcoids μ, f and g on suitable sets.

PROOF. As easily follows from distributivity of the lattices of funcoids we have $f|_{\mathcal{B}_0 \sqcup \mathcal{B}_1} = f|_{\mathcal{B}_0} \sqcup f|_{\mathcal{B}_1}$. Thus our theorem follows from the previous corollary. □

18.4. Limit

DEFINITION 1467. $\lim^{\mu} f = a$ iff $f \xrightarrow{\mu} \uparrow^{\operatorname{Src}\mu} \{a\}$ for a T_2-separable funcoid μ and a non-empty funcoid f such that $\operatorname{Dst} f = \operatorname{Dst}\mu$.

It is defined correctly, that is f has no more than one limit.

PROOF. Let $\lim^{\mu} f = a$ and $\lim^{\mu} f = b$. Then $\operatorname{im} f \sqsubseteq \langle\mu\rangle^{*} @\{a\}$ and $\operatorname{im} f \sqsubseteq \langle\mu\rangle^{*} @\{b\}$.

Because $f \neq \bot^{\mathsf{FCD}(\operatorname{Src} f, \operatorname{Dst} f)}$ we have $\operatorname{im} f \neq \bot^{\mathscr{F}(\operatorname{Dst} f)}$; $\langle\mu\rangle^{*} @\{a\} \sqcap \langle\mu\rangle^{*} @\{b\} \neq \bot^{\mathscr{F}(\operatorname{Dst} f)}$; $\uparrow^{\operatorname{Src}\mu} \{b\} \sqcap \langle\mu^{-1}\rangle\langle\mu\rangle^{*} @\{a\} \neq \bot^{\mathscr{F}(\operatorname{Src}\mu)}$; $\uparrow^{\operatorname{Src}\mu} \{b\} \sqcap \langle\mu^{-1} \circ \mu\rangle @\{a\} \neq \bot^{\mathscr{F}(\operatorname{Src}\mu)}$; $@\{a\} \left[\mu^{-1} \circ \mu\right] @\{b\}$. Because μ is T_2-separable we have $a = b$. □

DEFINITION 1468. $\lim^{\mu}_{\mathcal{B}} f = \lim^{\mu}(f|_{\mathcal{B}})$.

REMARK 1469. We can also in an obvious way define limit of a reloid.

18.5. Generalized limit

18.5.1. Definition. Let μ and ν be endofuncoids. Let G be a transitive permutation group on $\operatorname{Ob}\mu$.

For an element $r \in G$ we will denote $\uparrow r = \uparrow^{\mathsf{FCD}(\operatorname{Ob}\mu, \operatorname{Ob}\mu)} r$.

We require that μ and every $r \in G$ commute, that is

$$\mu \circ \uparrow r = \uparrow r \circ \mu.$$

We require for every $y \in \operatorname{Ob}\nu$

$$\nu \sqsupseteq \langle\nu\rangle^{*} @\{y\} \times^{\mathsf{FCD}} \langle\nu\rangle^{*} @\{y\}. \tag{19}$$

PROPOSITION 1470. Formula (19) follows from $\nu \sqsupseteq \nu \circ \nu^{-1}$.

PROOF. Let $\nu \sqsupseteq \nu \circ \nu^{-1}$. Then

$$\begin{aligned} \langle\nu\rangle^{*} @\{y\} \times^{\mathsf{FCD}} \langle\nu\rangle^{*} @\{y\} &= \\ \langle\nu\rangle @\{y\} \times^{\mathsf{FCD}} \langle\nu\rangle @\{y\} &= \\ \nu \circ (\uparrow^{\operatorname{Ob}\nu} \{y\} \times^{\mathsf{FCD}} \uparrow^{\operatorname{Ob}\nu} \{y\}) \circ \nu^{-1} &= \\ \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Ob}\nu, \operatorname{Ob}\nu)} (\{y\} \times \{y\}) \circ \nu^{-1} &\sqsubseteq \\ \nu \circ 1^{\mathsf{FCD}}_{\operatorname{Ob}\nu} \circ \nu^{-1} &= \\ \nu \circ \nu^{-1} &\sqsubseteq \nu. \end{aligned}$$

□

REMARK 1471. The formula (19) usually works if ν is a proximity. It does not work if μ is a pretopology or preclosure.

We are going to consider (generalized) limits of arbitrary functions acting from $\operatorname{Ob}\mu$ to $\operatorname{Ob}\nu$. (The functions in consideration are not required to be continuous.)

REMARK 1472. Most typically G is the group of translations of some topological vector space.

Generalized limit is defined by the following formula:

DEFINITION 1473. $\operatorname{xlim} f \stackrel{\text{def}}{=} \left\{ \frac{\nu \circ f \circ \uparrow r}{r \in G} \right\}$ for any funcoid f.

REMARK 1474. Generalized limit technically is a set of funcoids.

We will assume that $\operatorname{dom} f \sqsupseteq \langle\mu\rangle^{*} @\{x\}$.

DEFINITION 1475. $\operatorname{xlim}_x f = \operatorname{xlim} f|_{\langle\mu\rangle^{*} @\{x\}}$.

Obvious 1476. $\operatorname{xlim}_x f = \left\{ \frac{\nu \circ f|_{\langle \mu \rangle^* @\{x\}} \circ \uparrow r}{r \in G} \right\}$.

Remark 1477. $\operatorname{xlim}_x f$ is the same for funcoids μ and $\operatorname{Compl} \mu$.

The function τ will define an injection from the set of points of the space ν ("numbers", "points", or "vectors") to the set of all (generalized) limits (i.e. values which $\operatorname{xlim}_x f$ may take).

Definition 1478. $\tau(y) \stackrel{\text{def}}{=} \left\{ \frac{\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\}}{x \in D} \right\}$.

Proposition 1479. $\tau(y) = \left\{ \frac{(\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\}) \circ \uparrow r}{r \in G} \right\}$ for every (fixed) $x \in D$.

Proof.

$$\begin{aligned}
&(\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\}) \circ \uparrow r = \\
&\left\langle \uparrow r^{-1} \right\rangle \langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\} = \\
&\langle \mu \rangle \left\langle \uparrow r^{-1} \right\rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\} = \\
&\quad \langle \mu \rangle^* @\{r^{-1} x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\} \in \left\{ \frac{\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\}}{x \in D} \right\}.
\end{aligned}$$

Reversely $\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\} = (\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\}) \circ \uparrow e$ where e is the identify element of G. $\square$

Proposition 1480. $\tau(y) = \operatorname{xlim}(\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\operatorname{Ob} \nu)} \{y\})$ (for every x). Informally: Every $\tau(y)$ is a generalized limit of a constant funcoid.

Proof.

$$\begin{gathered}
\operatorname{xlim}(\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\operatorname{Ob} \nu)} \{y\}) = \\
\left\{ \frac{\nu \circ (\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\operatorname{Ob} \nu)} \{y\}) \circ \uparrow r}{r \in G} \right\} = \\
\left\{ \frac{(\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle^* @\{y\}) \circ \uparrow r}{r \in G} \right\} = \tau(y).
\end{gathered}$$

$\square$

Theorem 1481. If f is a function and $f|_{\langle \mu \rangle^* @\{x\}} \in \mathrm{C}(\mu, \nu)$ and $\langle \mu \rangle^* @\{x\} \sqsupseteq \uparrow^{\operatorname{Ob} \mu} \{x\}$ then $\operatorname{xlim}_x f = \tau(f x)$.

Proof. $f|_{\langle \mu \rangle^* @\{x\}} \circ \mu \sqsubseteq \nu \circ f|_{\langle \mu \rangle^* @\{x\}} \sqsubseteq \nu \circ f$; thus $\langle f \rangle \langle \mu \rangle^* @\{x\} \sqsubseteq \langle \nu \rangle \langle f \rangle^* @\{x\}$; consequently we have

$$\nu \sqsupseteq \langle \nu \rangle \langle f \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle \langle f \rangle^* @\{x\} \sqsupseteq \langle f \rangle \langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle \langle f \rangle^* @\{x\}.$$

$$\begin{aligned}
&\nu \circ f|_{\langle \mu \rangle^* @\{x\}} \sqsupseteq \\
&(\langle f \rangle \langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle \langle f \rangle^* @\{x\}) \circ f|_{\langle \mu \rangle^* @\{x\}} = \\
&(f|_{\langle \mu \rangle^* \{x\}})^{-1} \langle f \rangle \langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle \langle f \rangle^* @\{x\} \sqsupseteq \\
&\left\langle \mathrm{id}^{\mathsf{FCD}}_{\operatorname{dom} f|_{\langle \mu \rangle^* \{x\}}} \right\rangle \langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle \langle f \rangle^* @\{x\} \sqsupseteq \\
&\operatorname{dom} f|_{\langle \mu \rangle^* @\{x\}} \times^{\mathsf{FCD}} \langle \nu \rangle \langle f \rangle^* @\{x\} = \\
&\langle \mu \rangle^* @\{x\} \times^{\mathsf{FCD}} \langle \nu \rangle \langle f \rangle^* @\{x\}.
\end{aligned}$$

$\operatorname{im}(\nu \circ f|_{\langle\mu\rangle^*@\{x\}}) = \langle\nu\rangle\langle f\rangle^*@\{x\}$;

$$\nu \circ f|_{\langle\mu\rangle^*@\{x\}} \sqsubseteq \langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \operatorname{im}(\nu \circ f|_{\langle\mu\rangle^*@\{x\}}) = \langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\langle f\rangle^*@\{x\}.$$

So $\nu \circ f|_{\langle\mu\rangle^*@\{x\}} = \langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\langle f\rangle^*@\{x\}$.

Thus $\operatorname{xlim}_x f = \left\{ \frac{(\langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\langle f\rangle^*@\{x\}) \circ \uparrow r}{r \in G} \right\} = \tau(fx)$. □

REMARK 1482. Without the requirement of $\langle\mu\rangle^*@\{x\} \sqsupseteq \uparrow^{\operatorname{Ob}\mu} \{x\}$ the last theorem would not work in the case of removable singularity.

THEOREM 1483. Let $\nu \sqsubseteq \nu \circ \nu$. If $f|_{\langle\mu\rangle^*@\{x\}} \xrightarrow{\nu} \uparrow^{\operatorname{Ob}\mu} \{y\}$ then $\operatorname{xlim}_x f = \tau(y)$.

PROOF. $\operatorname{im} f|_{\langle\mu\rangle^*@\{x\}} \sqsubseteq \langle\nu\rangle^*@\{y\}$; $\langle f\rangle\langle\mu\rangle^*@\{x\} \sqsubseteq \langle\nu\rangle^*@\{y\}$;

$$\begin{aligned}\nu \circ f|_{\langle\mu\rangle^*@\{x\}} &\sqsupseteq \\ (\langle\nu\rangle^*@\{y\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\}) \circ f|_{\langle\mu\rangle^*@\{x\}} &= \\ \left\langle (f|_{\langle\mu\rangle^*@\{x\}})^{-1} \right\rangle \langle\nu\rangle^*@\{y\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\} &= \\ \left\langle \operatorname{id}^{\mathsf{FCD}}_{\langle\mu\rangle^*\{x\}} \circ f^{-1} \right\rangle \langle\nu\rangle^*@\{y\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\} &\sqsupseteq \\ \left\langle \operatorname{id}^{\mathsf{FCD}}_{\langle\mu\rangle^*\{x\}} \circ f^{-1} \right\rangle \langle f\rangle\langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\} &= \\ \left\langle \operatorname{id}^{\mathsf{FCD}}_{\langle\mu\rangle^*\{x\}} \right\rangle \langle f^{-1} \circ f\rangle\langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\} &\sqsupseteq \\ \left\langle \operatorname{id}^{\mathsf{FCD}}_{\langle\mu\rangle^*\{x\}} \right\rangle \left\langle \operatorname{id}^{\mathsf{FCD}}_{\langle\mu\rangle^*\{x\}} \right\rangle \langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\} &= \\ \langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\}&.\end{aligned}$$

On the other hand, $f|_{\langle\mu\rangle^*@\{x\}} \sqsubseteq \langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\}$;
$\nu \circ f|_{\langle\mu\rangle^*@\{x\}} \sqsubseteq \langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle\langle\nu\rangle^*@\{y\} \sqsubseteq \langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\}$.
So $\nu \circ f|_{\langle\mu\rangle^*@\{x\}} = \langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\}$.
$\operatorname{xlim}_x f = \left\{ \frac{\nu \circ f|_{\langle\mu\rangle^*@\{x\}} \circ \uparrow r}{r \in G} \right\} = \left\{ \frac{(\langle\mu\rangle^*@\{x\} \times^{\mathsf{FCD}} \langle\nu\rangle^*@\{y\}) \circ \uparrow r}{r \in G} \right\} = \tau(y)$. □

COROLLARY 1484. If $\lim^{\nu}_{\langle\mu\rangle^*@\{x\}} f = y$ then $\operatorname{xlim}_x f = \tau(y)$ (provided that $\nu \sqsubseteq \nu \circ \nu$).

We have injective τ if $\langle\nu\rangle^*@\{y_1\} \sqcap \langle\nu\rangle^*@\{y_2\} = \bot^{\mathscr{F}(\operatorname{Ob}\mu)}$ for every distinct $y_1, y_2 \in \operatorname{Ob}\nu$ that is if ν is T_2-separable.

18.6. Expressing limits as implications

When you studied limits in the school, you was told that $\lim_{x\to\alpha} f(x) = \beta$ when $x \to \alpha$ implies $f(x) \to \beta$. Now let us formalize this.

PROPOSITION 1485. The following are pairwise equivalent for funcoids μ, ν, f of suitable ("compatible") sources and destinations:

1°. $f|_{\langle\mu\rangle^*\{\alpha\}} \xrightarrow{\nu} \beta$;

2°. $\forall x \in \mathscr{F}(\operatorname{Ob}\mu) : \left(x \xrightarrow{\mu} \alpha \Rightarrow \langle f\rangle x \xrightarrow{\nu} \beta \right)$;

3°. $\forall x \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Ob}\mu)} : \left(x \xrightarrow{\mu} \alpha \Rightarrow \langle f\rangle x \xrightarrow{\nu} \beta \right)$.

PROOF.

1°⇔2°. $\forall x \in \mathscr{F}(\operatorname{Ob}\mu) : \left(x \xrightarrow{\mu} \alpha \Rightarrow \langle f\rangle x \xrightarrow{\nu} \beta \right) \Leftrightarrow \forall x \in \mathscr{F}(\operatorname{Ob}\mu) : (x \sqsubseteq \langle\mu\rangle\alpha \Rightarrow \langle f\rangle x \sqsubseteq \langle\nu\rangle\beta) \Leftrightarrow \langle f\rangle\langle\mu\rangle\alpha \sqsubseteq \langle\nu\rangle\beta \Leftrightarrow f|_{\langle\mu\rangle^*\{\alpha\}} \xrightarrow{\nu} \beta$.

2°⇒3°. Obvious.

3°⇒2°. Let 3° hold. Then for $x \in \mathscr{F}(\operatorname{Ob}\mu)$ we have $x \xrightarrow{\mu} \alpha \Leftrightarrow x \sqsubseteq \langle\mu\rangle\alpha \Leftrightarrow \forall x' \in \operatorname{atoms} x : x' \sqsubseteq \langle\mu\rangle\alpha \Leftrightarrow \forall x' \in \operatorname{atoms} x : x' \xrightarrow{\mu} \alpha \Rightarrow \forall x' \in \operatorname{atoms} x : \langle f\rangle x' \xrightarrow{\nu} \beta \Leftrightarrow \forall x' \in \operatorname{atoms} x : \langle f\rangle x' \sqsubseteq \langle\nu\rangle\beta \Leftrightarrow \bigsqcup_{x' \in \operatorname{atoms} x}\langle f\rangle x' \sqsubseteq \langle\nu\rangle\beta \Leftrightarrow \langle f\rangle x \sqsubseteq \langle\nu\rangle\beta \Leftrightarrow \langle f\rangle x \xrightarrow{\nu} \beta$.

□

LEMMA 1486. If f is an enterely defined monovalued funcoid and x is an ultrafilter, y is a filter, then $\langle f\rangle x \sqsubseteq y \Leftrightarrow x \sqsubseteq \langle f^{-1}\rangle y$.

PROOF. $\langle f\rangle x$ is an ultrafilter. $\langle f\rangle x \sqsubseteq y \Leftrightarrow \langle f\rangle x \not\asymp y \Leftrightarrow x \not\asymp \langle f^{-1}\rangle y \Leftrightarrow x \sqsubseteq \langle f^{-1}\rangle y$. □

PROPOSITION 1487. The following are pairwise equivalent for funcoids μ, ν, f, g of suitable ("compatible") sources and destinations provided that g is entirely defined and monovalued:

1°. $(f \circ g^{-1})|_{\langle\mu\rangle^*\{\alpha\}} \xrightarrow{\nu} \beta$;

2°. $\forall x \in \mathscr{F}(\operatorname{Ob}\mu) : \left(\langle g\rangle x \xrightarrow{\mu} \alpha \Rightarrow \langle f\rangle x \xrightarrow{\nu} \beta\right)$;

3°. $\forall x \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Ob}\mu)} : \left(\langle g\rangle x \xrightarrow{\mu} \alpha \Rightarrow \langle f\rangle x \xrightarrow{\nu} \beta\right)$.

PROOF.

1°⇔3°. Equivalently transforming: $(f \circ g^{-1})|_{\langle\mu\rangle^*\{\alpha\}} \xrightarrow{\nu} \beta$; $\langle f\rangle\langle g^{-1}\rangle^*\langle\mu\rangle^*\{\alpha\} \sqsubseteq \langle\nu\rangle^*\{\beta\}$; for every $x \in \operatorname{atoms}^{\mathscr{F}(\operatorname{Ob}\mu)}$ we have $x \sqsubseteq \langle g^{-1}\rangle\langle\mu\rangle^*\{\alpha\} \Rightarrow \langle f\rangle x \sqsubseteq \langle\nu\rangle^*\{\beta\}$; what by the lemma is equivalent to $\langle g\rangle x \sqsubseteq \langle\mu\rangle^*\{\alpha\} \Rightarrow \langle f\rangle x \sqsubseteq \langle\nu\rangle^*\{\beta\}$ that is $\langle g\rangle x \xrightarrow{\mu} \alpha \Rightarrow \langle f\rangle x \xrightarrow{\nu} \beta$.

3°⇔2°. Let $x \in \mathscr{F}(\operatorname{Ob}\mu)$ and 3° holds. Let $\langle g\rangle x \xrightarrow{\mu} \alpha$. Then $\forall x' \in \operatorname{atoms} x : \langle g\rangle x' \xrightarrow{\mu} \alpha$ and thus $\langle f\rangle x' \xrightarrow{\nu} \beta$ that is $\langle f\rangle x' \sqsubseteq \langle\nu\rangle\beta$. $\langle f\rangle x = \bigsqcup_{x' \in \operatorname{atoms} x}\langle f\rangle x' \sqsubseteq \langle\nu\rangle\beta$ that is $\langle f\rangle x \xrightarrow{\nu} \beta$.

□

PROBLEM 1488. Can the theorem be strenhtened for: a. non-monovalued; b. not entirely defined g? (The problem seems easy but I have not checked it.)

Part 3

Pointfree funcoids and reloids

CHAPTER 19

Pointfree funcoids

This chapter is based on [29].

This is a routine chapter. There is almost nothing creative here. I just generalize theorems about funcoids to the maximum extent for *pointfree funcoids* (defined below) preserving the proof idea. The main idea behind this chapter is to find weakest theorem conditions enough for the same theorem statement as for above theorems for funcoids.

For these who know pointfree topology: Pointfree topology notions of frames and locales is a non-trivial generalization of topological spaces. Pointfree funcoids are different: I just replace the set of filters on a set with an arbitrary poset, this readily gives the definition of *pointfree funcoid*, almost no need of creativity here.

Pointfree funcoids are used in the below definitions of products of funcoids.

19.1. Definition

Definition 1489. *Pointfree funcoid* is a quadruple $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$ where $\mathfrak{A}$ and $\mathfrak{B}$ are posets, $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ and $\beta \in \mathfrak{A}^{\mathfrak{B}}$ such that

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (y \not\asymp \alpha x \Leftrightarrow x \not\asymp \beta y).$$

Definition 1490. The *source* $\operatorname{Src}(\mathfrak{A}, \mathfrak{B}, \alpha, \beta) = \mathfrak{A}$ and *destination* $\operatorname{Dst}(\mathfrak{A}, \mathfrak{B}, \alpha, \beta) = \mathfrak{B}$ for every pointfree funcoid $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$.

To every funcoid (A, B, α, β) corresponds pointfree funcoid $(\mathscr{P}A, \mathscr{P}B, \alpha, \beta)$. Thus pointfree funcoids are a generalization of funcoids.

Definition 1491. I will denote $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ the set of pointfree funcoids from $\mathfrak{A}$ to $\mathfrak{B}$ (that is with source $\mathfrak{A}$ and destination $\mathfrak{B}$), for every posets $\mathfrak{A}$ and $\mathfrak{B}$.

$\langle(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)\rangle \overset{\text{def}}{=} \alpha$ for every pointfree funcoid $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$.

Definition 1492. $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)^{-1} = (\mathfrak{B}, \mathfrak{A}, \beta, \alpha)$ for every pointfree funcoid $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$.

Proposition 1493. If f is a pointfree funcoid then f^{-1} is also a pointfree funcoid.

Proof. It follows from symmetry in the definition of pointfree funcoid. □

Obvious 1494. $(f^{-1})^{-1} = f$ for every pointfree funcoid f.

Definition 1495. The relation $[f] \in \mathscr{P}(\operatorname{Src} f \times \operatorname{Dst} f)$ is defined by the formula (for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$)

$$x \mathrel{[f]} y \overset{\text{def}}{=} y \not\asymp \langle f\rangle x.$$

Obvious 1496. $x \mathrel{[f]} y \Leftrightarrow y \not\asymp \langle f\rangle x \Leftrightarrow x \not\asymp \langle f^{-1}\rangle y$ for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$.

Obvious 1497. $[f^{-1}] = [f]^{-1}$ for every pointfree funcoid f.

Theorem 1498. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. Then:

1°. If $\mathfrak{A}$ is separable, for given value of $\langle f \rangle$ there exists no more than one $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

2°. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable, for given value of $[f]$ there exists no more than one $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

Proof. Let $f, g \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

1°. Let $\langle f \rangle = \langle g \rangle$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have

$$x \not\asymp \langle f^{-1} \rangle y \Leftrightarrow y \not\asymp \langle f \rangle x \Leftrightarrow y \not\asymp \langle g \rangle x \Leftrightarrow x \not\asymp \langle g^{-1} \rangle y$$

and thus by separability of $\mathfrak{A}$ we have $\langle f^{-1} \rangle y = \langle g^{-1} \rangle y$ that is $\langle f^{-1} \rangle = \langle g^{-1} \rangle$ and so $f = g$.

2°. Let $[f]$=$[g]$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have

$$x \not\asymp \langle f^{-1} \rangle y \Leftrightarrow x \, [f] \, y \Leftrightarrow x \, [g] \, y \Leftrightarrow x \not\asymp \langle g^{-1} \rangle y$$

and thus by separability of $\mathfrak{A}$ we have $\langle f^{-1} \rangle y = \langle g^{-1} \rangle y$ that is $\langle f^{-1} \rangle = \langle g^{-1} \rangle$. Similarly we have $\langle f \rangle = \langle g \rangle$. Thus $f = g$.

□

Proposition 1499. If $\operatorname{Src} f$ and $\operatorname{Dst} f$ have least elements, then $\langle f \rangle \bot^{\operatorname{Src} f} = \bot^{\operatorname{Dst} f}$ for every pointfree funcoid f.

Proof. $y \not\asymp \langle f \rangle \bot^{\operatorname{Src} f} \Leftrightarrow \bot^{\operatorname{Src} f} \not\asymp \langle f^{-1} \rangle y \Leftrightarrow 0$ for every $y \in \operatorname{Dst} f$. Thus $\langle f \rangle \bot^{\operatorname{Src} f} \asymp \langle f \rangle \bot^{\operatorname{Src} f}$. So $\langle f \rangle \bot^{\operatorname{Src} f} = \bot^{\operatorname{Dst} f}$. □

Proposition 1500. If $\operatorname{Dst} f$ is strongly separable then $\langle f \rangle$ is a monotone function (for a pointfree funcoid f).

Proof.

$$\begin{aligned}
a \sqsubseteq b &\Rightarrow \\
\forall x \in \operatorname{Dst} f : (a \not\asymp \langle f^{-1} \rangle x \Rightarrow b \not\asymp \langle f^{-1} \rangle x) &\Rightarrow \\
\forall x \in \operatorname{Dst} f : (x \not\asymp \langle f \rangle a \Rightarrow x \not\asymp \langle f \rangle b) &\Leftrightarrow \\
\star \langle f \rangle a \subseteq \star \langle f \rangle b &\Rightarrow \\
\langle f \rangle a \sqsubseteq \langle f \rangle b.&
\end{aligned}$$

□

Theorem 1501. Let f be a pointfree funcoid from a starrish join-semilattice $\operatorname{Src} f$ to a separable starrish join-semilattice $\operatorname{Dst} f$. Then $\langle f \rangle (i \sqcup j) = \langle f \rangle i \sqcup \langle f \rangle j$ for every $i, j \in \operatorname{Src} f$.

Proof.

$$\begin{aligned}
\star \langle f \rangle (i \sqcup j) &= \\
\left\{ \frac{y \in \operatorname{Dst} f}{y \not\asymp \langle f \rangle (i \sqcup j)} \right\} &= \\
\left\{ \frac{y \in \operatorname{Dst} f}{i \sqcup j \not\asymp \langle f^{-1} \rangle y} \right\} &= \\
\left\{ \frac{y \in \operatorname{Dst} f}{i \not\asymp \langle f^{-1} \rangle y \vee j \not\asymp \langle f^{-1} \rangle y} \right\} &= \\
\left\{ \frac{y \in \operatorname{Dst} f}{y \not\asymp \langle f \rangle i \vee y \not\asymp \langle f \rangle j} \right\} &= \\
\left\{ \frac{y \in \operatorname{Dst} f}{y \not\asymp \langle f \rangle i \sqcup \langle f \rangle j} \right\} &= \\
\star (\langle f \rangle i \sqcup \langle f \rangle j).&
\end{aligned}$$

Thus $\langle f\rangle(i \sqcup j) = \langle f\rangle i \sqcup \langle f\rangle j$ by separability. □

PROPOSITION 1502. Let f be a pointfree funcoid. Then:

1°. $k\ [f]\ i \sqcup j \Leftrightarrow k\ [f]\ i \vee k\ [f]\ j$ for every $i, j \in \operatorname{Dst} f$, $k \in \operatorname{Src} f$ if $\operatorname{Dst} f$ is a starrish join-semilattice.
2°. $i \sqcup j\ [f]\ k \Leftrightarrow i\ [f]\ k \vee j\ [f]\ k$ for every $i, j \in \operatorname{Src} f$, $k \in \operatorname{Dst} f$ if $\operatorname{Src} f$ is a starrish join-semilattice.

PROOF.
1°. $k\ [f]\ i \sqcup j \Leftrightarrow i \sqcup j \not\asymp \langle f\rangle k \Leftrightarrow i \not\asymp \langle f\rangle k \vee j \not\asymp \langle f\rangle k \Leftrightarrow k\ [f]\ i \vee k\ [f]\ j$.
2°. Similar.

□

19.2. Composition of pointfree funcoids

DEFINITION 1503. *Composition* of pointfree funcoids is defined by the formula

$$(\mathfrak{B}, \mathfrak{C}, \alpha_2, \beta_2) \circ (\mathfrak{A}, \mathfrak{B}, \alpha_1, \beta_1) = (\mathfrak{A}, \mathfrak{C}, \alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2).$$

DEFINITION 1504. I will call funcoids f and g *composable* when $\operatorname{Dst} f = \operatorname{Src} g$.

PROPOSITION 1505. If f, g are composable pointfree funcoids then $g \circ f$ is pointfree funcoid.

PROOF. Let $f = (\mathfrak{A}, \mathfrak{B}, \alpha_1, \beta_1)$, $g = (\mathfrak{B}, \mathfrak{C}, \alpha_2, \beta_2)$. For every $x, y \in \mathfrak{A}$ we have

$$y \not\asymp (\alpha_2 \circ \alpha_1)x \Leftrightarrow y \not\asymp \alpha_2\alpha_1 x \Leftrightarrow \alpha_1 x \not\asymp \beta_2 y \Leftrightarrow x \not\asymp \beta_1\beta_2 y \Leftrightarrow x \not\asymp (\beta_1 \circ \beta_2)y.$$

So $(\mathfrak{A}, \mathfrak{C}, \alpha_2 \circ \alpha_1, \beta_1 \circ \beta_2)$ is a pointfree funcoid. □

OBVIOUS 1506. $\langle g \circ f\rangle = \langle g\rangle \circ \langle f\rangle$ for every composable pointfree funcoids f and g.

THEOREM 1507. $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every composable pointfree funcoids f and g.

PROOF.

$$\left\langle (g \circ f)^{-1}\right\rangle = \left\langle f^{-1}\right\rangle \circ \left\langle g^{-1}\right\rangle = \left\langle f^{-1} \circ g^{-1}\right\rangle;$$
$$\left\langle ((g \circ f)^{-1})^{-1}\right\rangle = \langle g \circ f\rangle = \left\langle (f^{-1} \circ g^{-1})^{-1}\right\rangle.$$

□

PROPOSITION 1508. $(h \circ g) \circ f = h \circ (g \circ f)$ for every composable pointfree funcoids f, g, h.

PROOF. $\langle (h \circ g) \circ f\rangle = \langle h \circ g\rangle \circ \langle f\rangle = \langle h\rangle \circ \langle g\rangle \circ \langle f\rangle = \langle h\rangle \circ \langle g \circ f\rangle = \langle h \circ (g \circ f)\rangle$;

$$\left\langle ((h \circ g) \circ f)^{-1}\right\rangle = \left\langle f^{-1} \circ (h \circ g)^{-1}\right\rangle = \left\langle f^{-1} \circ g^{-1} \circ h^{-1}\right\rangle = \left\langle (g \circ f)^{-1} \circ h^{-1}\right\rangle = \left\langle (h \circ (g \circ f))^{-1}\right\rangle.$$

□

EXERCISE 1509. Generalize section 7.4 for pointfree funcoids.

19.3. Pointfree funcoid as continuation

PROPOSITION 1510. Let f be a pointfree funcoid. Then for every $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$ we have

1°. If $(\operatorname{Src} f, \mathfrak{Z})$ is a filtrator with separable core then $x \mathrel{[f]} y \Leftrightarrow \forall X \in \operatorname{up}^{\mathfrak{Z}} x : X \mathrel{[f]} y$.

2°. If $(\operatorname{Dst} f, \mathfrak{Z})$ is a filtrator with separable core then $x \mathrel{[f]} y \Leftrightarrow \forall Y \in \operatorname{up}^{\mathfrak{Z}} y : x \mathrel{[f]} Y$.

PROOF. We will prove only the second because the first is similar.

$$x \mathrel{[f]} y \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow \forall Y \in \operatorname{up}^{\mathfrak{Z}} y : Y \not\asymp \langle f \rangle x \Leftrightarrow \forall Y \in \operatorname{up}^{\mathfrak{Z}} y : x \mathrel{[f]} Y.$$

□

COROLLARY 1511. Let f be a pointfree funcoid and $(\operatorname{Src} f, \mathfrak{Z}_0)$, $(\operatorname{Dst} f, \mathfrak{Z}_1)$ be filtrators with separable core. Then

$$x \mathrel{[f]} y \Leftrightarrow \forall X \in \operatorname{up}^{\mathfrak{Z}_0} x, Y \in \operatorname{up}^{\mathfrak{Z}_1} y : X \mathrel{[f]} Y.$$

PROOF. Apply the proposition twice. □

THEOREM 1512. Let f be a pointfree funcoid. Let $(\operatorname{Src} f, \mathfrak{Z}_0)$ be a binarily meet-closed filtrator with separable core which is a meet-semilattice and $\forall x \in \operatorname{Src} f : \operatorname{up}^{\mathfrak{Z}_0} x \neq \emptyset$ and $(\operatorname{Dst} f, \mathfrak{Z}_1)$ be a primary filtrator over a boolean lattice.

$$\langle f \rangle x = \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{\mathfrak{Z}_0} x.$$

PROOF. By the previous proposition for every $y \in \operatorname{Dst} f$:

$$y \not\asymp^{\operatorname{Dst} f} \langle f \rangle x \Leftrightarrow x \mathrel{[f]} y \Leftrightarrow \forall X \in \operatorname{up}^{\mathfrak{Z}_0} x : X \mathrel{[f]} y \Leftrightarrow \forall X \in \operatorname{up}^{\mathfrak{Z}_0} x : y \not\asymp^{\operatorname{Dst} f} \langle f \rangle X.$$

Let's denote $W = \left\{ \frac{y \sqcap^{\operatorname{Dst} f} \langle f \rangle X}{X \in \operatorname{up}^{\mathfrak{Z}_0} x} \right\}$. We will prove that W is a generalized filter base over $\mathfrak{Z}_1$. To prove this enough to show that $V = \left\{ \frac{\langle f \rangle X}{X \in \operatorname{up}^{\mathfrak{Z}_0} x} \right\}$ is a generalized filter base.

Let $\mathcal{P}, \mathcal{Q} \in V$. Then $\mathcal{P} = \langle f \rangle A$, $\mathcal{Q} = \langle f \rangle B$ where $A, B \in \operatorname{up}^{\mathfrak{Z}_0} x$; $A \sqcap^{\mathfrak{Z}_0} B \in \operatorname{up}^{\mathfrak{Z}_0} x$ (used the fact that it is a binarily meet-closed and theorem 535) and $\mathcal{R} \sqsubseteq \mathcal{P} \sqcap^{\operatorname{Dst} f} \mathcal{Q}$ for $\mathcal{R} = \langle f \rangle (A \sqcap^{\mathfrak{Z}_0} B) \in V$ because $\operatorname{Dst} f$ is strongly separable by proposition 579. So V is a generalized filter base and thus W is a generalized filter base.

$\bot^{\operatorname{Dst} f} \notin W \Leftrightarrow \bot^{\operatorname{Dst} f} \notin \bigsqcap^{\operatorname{Dst} f} W$ by theorem 572. That is

$$\forall X \in \operatorname{up}^{\mathfrak{Z}_0} x : y \sqcap^{\operatorname{Dst} f} \langle f \rangle X \neq \bot^{\operatorname{Dst} f} \Leftrightarrow y \sqcap^{\operatorname{Dst} f} \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{\mathfrak{Z}_0} x \neq \bot^{\operatorname{Dst} f}.$$

Comparing with the above,

$$y \sqcap^{\operatorname{Dst} f} \langle f \rangle x \neq \bot^{\operatorname{Dst} f} \Leftrightarrow y \sqcap^{\operatorname{Dst} f} \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{\mathfrak{Z}_0} x \neq \bot^{\operatorname{Dst} f}.$$

So $\langle f \rangle x = \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* \operatorname{up}^{\mathfrak{Z}_0} x$ because $\operatorname{Dst} f$ is separable (proposition 579 and the fact that $\mathfrak{Z}_1$ is a boolean lattice). □

THEOREM 1513. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices.

1°. A function $\alpha \in \mathfrak{B}^{\mathfrak{Z}_0}$ conforming to the formulas (for every $I, J \in \mathfrak{Z}_0$)

$$\alpha \bot^{\mathfrak{Z}_0} = \bot^{\mathfrak{B}}, \quad \alpha (I \sqcup J) = \alpha I \sqcup \alpha J$$

can be continued to the function $\langle f\rangle$ for a unique $f\in\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$;

$$\langle f\rangle\mathcal{X}=\prod^{\mathfrak{B}}\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}\tag{20}$$

for every $\mathcal{X}\in\mathfrak{A}$.

2°. A relation $\delta\in\mathscr{P}(\mathfrak{Z}_0\times\mathfrak{Z}_1)$ conforming to the formulas (for every $I,J,K\in\mathfrak{Z}_0$ and $I',J',K'\in\mathfrak{Z}_1$)

$$\begin{aligned}\neg(\bot^{\mathfrak{Z}_0}\ \delta\ I'),\quad & I\sqcup J\ \delta\ K'\Leftrightarrow I\ \delta\ K'\vee J\ \delta\ K',\\ \neg(I\ \delta\ \bot^{\mathfrak{Z}_1}),\quad & K\ \delta\ I'\sqcup J'\Leftrightarrow K\ \delta\ I'\vee K\ \delta\ J'\end{aligned}\tag{21}$$

can be continued to the relation $[f]$ for a unique $f\in\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$;

$$\mathcal{X}\ [f]\ \mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X},Y\in\operatorname{up}^{\mathfrak{Z}_1}\mathcal{Y}:X\ \delta\ Y\tag{22}$$

for every $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{B}$.

Proof. Existence of no more than one such pointfree funcoids and formulas (20) and (22) follow from two previous theorems.

2°. $\left\{\frac{Y\in\mathfrak{Z}_1}{X\delta Y}\right\}$ is obviously a free star for every $X\in\mathfrak{Z}_0$. By properties of filters on boolean lattices, there exist a unique filter αX such that $\partial(\alpha X)=\left\{\frac{Y\in\mathfrak{Z}_1}{X\delta Y}\right\}$ for every $X\in\mathfrak{Z}_0$. Thus $\alpha\in\mathfrak{B}^{\mathfrak{Z}_0}$. Similarly it can be defined $\beta\in\mathfrak{A}^{\mathfrak{Z}_1}$ by the formula $\partial(\beta Y)=\left\{\frac{X\in\mathfrak{Z}_0}{X\delta Y}\right\}$. Let's continue the functions α and β to $\alpha'\in\mathfrak{B}^{\mathfrak{A}}$ and $\beta'\in\mathfrak{A}^{\mathfrak{B}}$ by the formulas

$$\alpha'\mathcal{X}=\prod^{\mathfrak{B}}\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}\quad\text{and}\quad\beta'\mathcal{Y}=\prod^{\mathfrak{A}}\langle\beta\rangle^{*}\operatorname{up}^{\mathfrak{Z}_1}\mathcal{Y}$$

and δ to $\delta'\in\mathscr{P}(\mathfrak{A}\times\mathfrak{B})$ by the formula

$$\mathcal{X}\ \delta'\ \mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X},Y\in\operatorname{up}^{\mathfrak{Z}_1}\mathcal{Y}:\mathcal{X}\ \delta\ \mathcal{Y}.$$

$\mathcal{Y}\sqcap\alpha'\mathcal{X}\neq\bot^{\mathfrak{B}}\Leftrightarrow\mathcal{Y}\sqcap\prod\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}\neq\bot^{\mathfrak{B}}\Leftrightarrow\prod\langle\mathcal{Y}\sqcap\rangle^{*}\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}\neq\bot^{\mathfrak{B}}$. Let's prove that

$$W=\langle\mathcal{Y}\sqcap\rangle^{*}\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}$ is a generalized filter base.

If $\mathcal{A},\mathcal{B}\in\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}$ then exist $X_1,X_2\in\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}$ such that $\mathcal{A}=\alpha X_1$ and $\mathcal{B}=\alpha X_2$. Then $\alpha(X_1\sqcap^{\mathfrak{Z}_0}X_2)\in\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}$. So $\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

By properties of generalized filter bases, $\prod\langle\mathcal{Y}\sqcap\rangle^{*}\langle\alpha\rangle^{*}\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}\neq\bot^{\mathfrak{B}}$ is equivalent to

$$\forall X\in\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X}:\mathcal{Y}\sqcap\alpha X\neq\bot^{\mathfrak{B}},$$

what is equivalent to

$$\begin{gathered}\forall X\in\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X},Y\in\operatorname{up}^{\mathfrak{Z}_1}\mathcal{Y}:Y\sqcap^{\mathfrak{B}}\alpha X\neq\bot^{\mathfrak{B}}\Leftrightarrow\\ \forall X\in\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X},Y\in\operatorname{up}^{\mathfrak{Z}_1}\mathcal{Y}:Y\in\partial(\alpha X)\Leftrightarrow\\ \forall X\in\operatorname{up}^{\mathfrak{Z}_0}\mathcal{X},Y\in\operatorname{up}^{\mathfrak{Z}_1}\mathcal{Y}:X\ \delta\ Y.\end{gathered}$$

Combining the equivalencies we get $\mathcal{Y}\sqcap\alpha'\mathcal{X}\neq\bot^{\mathfrak{B}}\Leftrightarrow\mathcal{X}\ \delta'\ \mathcal{Y}$. Analogously $\mathcal{X}\sqcap\beta'\mathcal{Y}\neq\bot^{\mathfrak{A}}\Leftrightarrow\mathcal{X}\ \delta'\ \mathcal{Y}$. So $\mathcal{Y}\sqcap\alpha'\mathcal{X}\neq\bot^{\mathfrak{B}}\Leftrightarrow\mathcal{X}\sqcap\beta'\mathcal{Y}\neq\bot^{\mathfrak{A}}$, that is $(\mathfrak{A},\mathfrak{B},\alpha',\beta')$ is a pointfree funcoid. From the formula $\mathcal{Y}\sqcap\alpha'\mathcal{X}\neq\bot^{\mathfrak{B}}\Leftrightarrow\mathcal{X}\ \delta'\ \mathcal{Y}$ it follows that $[(\mathfrak{A},\mathfrak{B},\alpha',\beta')]$ is a continuation of δ.

1°. Let define the relation $\delta\in\mathscr{P}(\mathfrak{Z}_0\times\mathfrak{Z}_1)$ by the formula $X\ \delta\ Y\Leftrightarrow Y\sqcap^{\mathfrak{B}}\alpha X\neq\bot^{\mathfrak{B}}$.

That $\neg(\bot^{\mathfrak{Z}_0}\ \delta\ I')$ and $\neg(I\ \delta\ \bot^{\mathfrak{Z}_1})$ is obvious. We have

$$\begin{aligned} & K \mathrel{\delta} I' \sqcup^{\mathfrak{Z}_1} J' \Leftrightarrow \\ & (I' \sqcup^{\mathfrak{Z}_1} J') \sqcap^{\mathfrak{B}} \alpha K \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ & (I' \sqcup^{\mathfrak{B}} J') \sqcap \alpha K \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ & (I' \sqcap^{\mathfrak{B}} \alpha K) \sqcup (J' \sqcap^{\mathfrak{B}} \alpha K) \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ & I' \sqcap^{\mathfrak{B}} \alpha K \neq \bot^{\mathfrak{B}} \vee J' \sqcap^{\mathfrak{B}} \alpha K \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ & K \mathrel{\delta} I' \vee K \mathrel{\delta} J' \end{aligned}$$

and

$$\begin{aligned} & I \sqcup^{\mathfrak{Z}_0} J \mathrel{\delta} K' \Leftrightarrow \\ & K' \sqcap^{\mathfrak{B}} \alpha(I \sqcup^{\mathfrak{Z}_0} J) \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ & K' \sqcap^{\mathfrak{B}} (\alpha I \sqcup \alpha J) \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ & (K' \sqcap^{\mathfrak{B}} \alpha I) \sqcup (K' \sqcap^{\mathfrak{B}} \alpha J) \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ & K' \sqcap^{\mathfrak{B}} \alpha I \neq \bot^{\mathfrak{B}} \vee K' \sqcap^{\mathfrak{B}} \alpha J \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ & I \mathrel{\delta} K' \vee J \mathrel{\delta} K'. \end{aligned}$$

That is the formulas (21) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

$\forall X \in \mathfrak{Z}_0, Y \in \mathfrak{Z}_1 : (Y \sqcap^{\mathfrak{B}} \langle f \rangle X \neq \bot^{\mathfrak{B}} \Leftrightarrow X \mathrel{[f]} Y \Leftrightarrow Y \sqcap^{\mathfrak{B}} \alpha X \neq \bot^{\mathfrak{B}})$, consequently $\forall X \in \mathfrak{Z}_0 : \alpha X = \langle f \rangle X$ because our filtrator is with separable core. So $\langle f \rangle$ is a continuation of α.

□

THEOREM 1514. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. If $\alpha \in \mathfrak{B}^{\mathfrak{Z}_0}$, $\beta \in \mathfrak{A}^{\mathfrak{Z}_1}$ are functions such that $Y \not\asymp \alpha X \Leftrightarrow X \not\asymp \beta Y$ for every $X \in \mathfrak{Z}_0$, $Y \in \mathfrak{Z}_1$, then there exists exactly one pointfree funcoid $f : \mathfrak{A} \to \mathfrak{B}$ such that $\langle f \rangle|_{\mathfrak{Z}_0} = \alpha$, $\langle f^{-1} \rangle|_{\mathfrak{Z}_1} = \beta$.

PROOF. Prove $\alpha(I \sqcup J) = \alpha I \sqcup \alpha J$. Really, $Y \not\asymp \alpha(I \sqcup J) \Leftrightarrow I \sqcup J \not\asymp \beta Y \Leftrightarrow I \not\asymp \beta Y \vee J \not\asymp \beta Y \Leftrightarrow Y \not\asymp \alpha I \vee Y \not\asymp \alpha J \Leftrightarrow Y \not\asymp \alpha I \sqcup \alpha J$. So $\alpha(I \sqcup J) = \alpha I \sqcup \alpha J$ by star-separability. Similarly $\beta(I \sqcup J) = \beta I \sqcup \beta J$.

Thus by the theorem above there exists a pointfree funcoid f such that $\langle f \rangle|_{\mathfrak{Z}_0} = \alpha$, $\langle f^{-1} \rangle|_{\mathfrak{Z}_1} = \beta$.

That this pointfree funcoid is unique, follows from the above. □

PROPOSITION 1515. Let $(\operatorname{Src} f, \mathfrak{Z}_0)$ be a primary filtrator over a bounded distributive lattice and $(\operatorname{Dst} f, \mathfrak{Z}_1)$ be a primary filtrator over boolean lattice. If S is a generalized filter base on $\operatorname{Src} f$ then $\langle f \rangle \bigsqcap^{\operatorname{Src} f} S = \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* S$ for every pointfree funcoid f.

PROOF. First the meets $\bigsqcap^{\operatorname{Src} f} S$ and $\bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* S$ exist by corollary 518.

$(\operatorname{Src} f, \mathfrak{Z}_0)$ is a binarily meet-closed filtrator by corollary 536 and with separable core by theorem 537; thus we can apply theorem 1512 (up $x \neq \emptyset$ is obvious).

$\langle f \rangle \bigsqcap^{\operatorname{Src} f} S \sqsubseteq \langle f \rangle X$ for every $X \in S$ because $\operatorname{Dst} f$ is strongly separable by proposition 579 and thus $\langle f \rangle \bigsqcap^{\operatorname{Src} f} S \sqsubseteq \bigsqcap^{\operatorname{Dst} f} \langle \langle f \rangle \rangle^* S$.

Taking into account properties of generalized filter bases:

$$\begin{aligned}
&\langle f\rangle \bigsqcap^{\mathrm{Src}\, f} S = \\
&\bigsqcap^{\mathrm{Dst}\, f} \langle\langle f\rangle\rangle^* \operatorname{up} \bigsqcap S = \\
&\bigsqcap^{\mathrm{Dst}\, f} \langle\langle f\rangle\rangle^* \left\{ \frac{X}{\exists \mathcal{P} \in S : X \in \operatorname{up} \mathcal{P}} \right\} = \\
&\bigsqcap^{\mathrm{Dst}\, f} \left\{ \frac{\langle f\rangle^* X}{\exists \mathcal{P} \in S : X \in \operatorname{up} \mathcal{P}} \right\} \sqsupseteq \text{ (because } \mathrm{Dst}\, f \text{ is a strongly separable poset)} \\
&\bigsqcap^{\mathrm{Dst}\, f} \left\{ \frac{\langle f\rangle \mathcal{P}}{\mathcal{P} \in S} \right\} = \\
&\bigsqcap^{\mathrm{Dst}\, f} \langle\langle f\rangle\rangle^* S.
\end{aligned}$$

□

PROPOSITION 1516. $\mathcal{X}\ [f] \bigsqcap S \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X}\ [f]\ \mathcal{Y}$ if f is a pointfree funcoid, $\mathrm{Dst}\, f$ is a meet-semilattice with least element and S is a generalized filter base on $\mathrm{Dst}\, f$.

PROOF.

$$\begin{aligned}
&\mathcal{X}\ [f] \bigsqcap S \Leftrightarrow \bigsqcap S \sqcap \langle f\rangle \mathcal{X} \neq \bot \Leftrightarrow \bigsqcap \langle\langle f\rangle \mathcal{X} \sqcap\rangle^* S \neq \bot \Leftrightarrow \\
&\qquad \text{(by properties of generalized filter bases)} \Leftrightarrow \\
&\exists \mathcal{Y} \in \langle\langle f\rangle \mathcal{X} \sqcap\rangle^* S : \mathcal{Y} \neq \bot \Leftrightarrow \exists \mathcal{Y} \in S : \langle f\rangle \mathcal{X} \sqcap \mathcal{Y} \neq \bot \Leftrightarrow \exists \mathcal{Y} \in S : \mathcal{X}\ [f]\ \mathcal{Y}.
\end{aligned}$$

□

THEOREM 1517. A function $\varphi : \mathfrak{A} \to \mathfrak{B}$, where $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ are primary filtrators over boolean lattices, preserves finite joins (including nullary joins) and filtered meets iff there exists a pointfree funcoid f such that $\langle f\rangle = \varphi$.

PROOF. Backward implication follows from above.

Let $\psi = \varphi|_{\mathfrak{Z}_0}$. Then ψ preserves bottom element and binary joins. Thus there exists a funcoid f such that $\langle f\rangle^* = \psi$.

It remains to prove that $\langle f\rangle = \varphi$.

Really, $\langle f\rangle \mathcal{X} = \bigsqcap \langle\langle f\rangle\rangle^* \operatorname{up} \mathcal{X} = \bigsqcap \langle \psi\rangle^* \operatorname{up} \mathcal{X} = \bigsqcap \langle \varphi\rangle^* \operatorname{up} \mathcal{X} = \varphi \bigsqcap \operatorname{up} \mathcal{X} = \varphi \mathcal{X}$ for every $\mathcal{X} \in \mathscr{F}(\mathrm{Src}\, f)$. □

COROLLARY 1518. Pointfree funcoids f from a lattice $\mathfrak{A}$ of fitlters on a boolean lattice to a lattice $\mathfrak{B}$ of fitlters on a boolean lattice bijectively correspond by the formula $\langle f\rangle = \varphi$ to functions $\varphi : \mathfrak{A} \to \mathfrak{B}$ preserving finite joins and filtered meets.

THEOREM 1519. The set of pointfree funcoids between sets of filters on boolean lattices is a co-frame.

PROOF. Theorems 1513 and 533. □

19.4. The order of pointfree funcoids

DEFINITION 1520. The order of pointfree funcoids $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is defined by the formula:

$$f \sqsubseteq g \Leftrightarrow \forall x \in \mathfrak{A} : \langle f\rangle x \sqsubseteq \langle g\rangle x \wedge \forall y \in \mathfrak{B} : \langle f^{-1}\rangle y \sqsubseteq \langle g^{-1}\rangle y.$$

PROPOSITION 1521. It is really a partial order on the set $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

PROOF.

Reflexivity. Obvious.
Transitivity. It follows from transitivity of the order relations on $\mathfrak{A}$ and $\mathfrak{B}$.
Antisymmetry. It follows from antisymmetry of the order relations on $\mathfrak{A}$ and $\mathfrak{B}$.

□

REMARK 1522. It is enough to define order of pointfree funcoids on every set $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ where $\mathfrak{A}$ and $\mathfrak{B}$ are posets. We do not need to compare pointfree funcoids with different sources or destinations.

OBVIOUS 1523. $f \sqsubseteq g \Rightarrow [f] \subseteq [g]$ for every $f, g \in \mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ for every posets $\mathfrak{A}$ and $\mathfrak{B}$.

THEOREM 1524. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable posets then $f \sqsubseteq g \Leftrightarrow [f] \subseteq [g]$.

PROOF. From the theorem 1498. □

PROPOSITION 1525. If $\mathfrak{A}$ and $\mathfrak{B}$ have least elements, then $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ has least element.

PROOF. It is $(\mathfrak{A}, \mathfrak{B}, \mathfrak{A} \times \{\bot^{\mathfrak{B}}\}, \mathfrak{B} \times \{\bot^{\mathfrak{A}}\})$. □

THEOREM 1526. If $\mathfrak{A}$ and $\mathfrak{B}$ are bounded posets, then $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ is bounded.

PROOF. That $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ has least element was proved above. I will demonstrate that $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$ is the greatest element of $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ for

$$\alpha X = \begin{cases} \bot^{\mathfrak{B}} & \text{if } X = \bot^{\mathfrak{A}} \\ \top^{\mathfrak{B}} & \text{if } X \neq \bot^{\mathfrak{A}} \end{cases}; \quad \beta Y = \begin{cases} \bot^{\mathfrak{A}} & \text{if } Y = \bot^{\mathfrak{B}} \\ \top^{\mathfrak{A}} & \text{if } Y \neq \bot^{\mathfrak{B}} \end{cases}.$$

First prove $Y \not\asymp \alpha X \Leftrightarrow X \not\asymp \beta Y$.

If $\top^{\mathfrak{B}} = \bot^{\mathfrak{B}}$ then $Y \not\asymp \alpha X \Leftrightarrow Y \not\asymp \bot^{\mathfrak{B}} \Leftrightarrow 0 \Leftrightarrow X \not\asymp \bot^{\mathfrak{A}} \Leftrightarrow X \not\asymp \beta\bot^{\mathfrak{A}}$ (proposition 1499). The case $\top^{\mathfrak{A}} = \bot^{\mathfrak{A}}$ is similar. So we can assume $\top^{\mathfrak{A}} \neq \bot^{\mathfrak{A}}$ and $\top^{\mathfrak{B}} \neq \bot^{\mathfrak{B}}$.

Consider all variants:

$X = \bot^{\mathfrak{A}}$ and $Y = \bot^{\mathfrak{B}}$. $Y \not\asymp \alpha X \Leftrightarrow 0 \Leftrightarrow X \not\asymp \beta Y$.
$X \neq \bot^{\mathfrak{A}}$ and $Y \neq \bot^{\mathfrak{B}}$. $\alpha X = \top^{\mathfrak{B}}$ and $\beta Y = \top^{\mathfrak{A}}$; $Y \not\asymp \alpha X \Leftrightarrow Y \not\asymp \top^{\mathfrak{B}} \Leftrightarrow 1 \Leftrightarrow X \not\asymp \top^{\mathfrak{A}} \Leftrightarrow X \not\asymp \beta Y$ (used that $\top^{\mathfrak{A}} \neq \bot^{\mathfrak{A}}$ and $\top^{\mathfrak{B}} \neq \bot^{\mathfrak{B}}$).
$X = \bot^{\mathfrak{A}}$ and $Y \neq \bot^{\mathfrak{B}}$. $\alpha X = \bot^{\mathfrak{B}}$ (proposition 1499) and $\beta Y = \top^{\mathfrak{A}}$; $Y \not\asymp \alpha X \Leftrightarrow Y \not\asymp \bot^{\mathfrak{B}} \Leftrightarrow 0 \Leftrightarrow \bot^{\mathfrak{A}} \not\asymp \beta Y \Leftrightarrow X \not\asymp \beta Y$.
$X = \bot^{\mathfrak{A}}$ and $Y \neq \bot^{\mathfrak{B}}$. Similar.

It's easy to show that both α and β are the greatest possible components of a pointfree funcoid taking into account proposition 1499. □

THEOREM 1527. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then for $R \in \mathscr{P}\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ and $X \in \mathfrak{Z}_0$, $Y \in \mathfrak{Z}_1$ we have:

1°. $X \left[\bigsqcup R\right] Y \Leftrightarrow \exists f \in R : X\ [f]\ Y$;
2°. $\langle \bigsqcup R \rangle X = \bigsqcup_{f \in R} \langle f \rangle X$.

PROOF.

2°. $\alpha X \stackrel{\text{def}}{=} \bigsqcup_{f\in R}\langle f\rangle X$ (by corollary 518 all joins on $\mathfrak{B}$ exist). We have $\alpha\bot^{\mathfrak{A}} = \bot^{\mathfrak{B}}$;

$$\begin{aligned}&\alpha(I \sqcup^{\mathfrak{Z}_0} J) = \\ &\bigsqcup\left\{\frac{\langle f\rangle(I \sqcup^{\mathfrak{Z}_0} J)}{f\in R}\right\} = \\ &\bigsqcup\left\{\frac{\langle f\rangle(I \sqcup^{\mathfrak{A}} J)}{f\in R}\right\} = \\ &\bigsqcup\left\{\frac{\langle f\rangle I \sqcup^{\mathfrak{B}} \langle f\rangle J}{f\in R}\right\} = \\ &\bigsqcup\left\{\frac{\langle f\rangle I}{f\in R}\right\} \sqcup^{\mathfrak{B}} \bigsqcup\left\{\frac{\langle f\rangle J}{f\in R}\right\} = \\ &\alpha I \sqcup^{\mathfrak{B}} \alpha J\end{aligned}$$

(used theorem 1501). By theorem 1513 the function α can be continued to $\langle h\rangle$ for an $h \in \mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$. Obviously

$$\forall f \in R : h \sqsupseteq f. \tag{23}$$

And h is the least element of $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ for which the condition (23) holds. So $h = \bigsqcup R$.

1°.

$$\begin{aligned}&X \left[\bigsqcup R\right] Y \Leftrightarrow \\ &Y \sqcap^{\mathfrak{B}} \left\langle\bigsqcup R\right\rangle X \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ &Y \sqcap^{\mathfrak{B}} \bigsqcup\left\{\frac{\langle f\rangle X}{f\in R}\right\} \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ &\exists f\in R : Y \sqcap^{\mathfrak{B}} \langle f\rangle X \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ &\exists f \in R : X\,[f]\,Y\end{aligned}$$

(used theorem 610).

□

Corollary 1528. If $(\mathfrak{A},\mathfrak{Z}_0)$ and $(\mathfrak{B},\mathfrak{Z}_1)$ are primary filtrators over boolean lattices then $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ is a complete lattice.

Proof. Apply [27]. □

Theorem 1529. Let $\mathfrak{A}$ and $\mathfrak{B}$ be starrish join-semilattices. Then for $f, g \in \mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$:

1°. $\langle f \sqcup g\rangle x = \langle f\rangle x \sqcup \langle g\rangle x$ for every $x \in \mathfrak{A}$;
2°. $[f \sqcup g]=[f] \cup [g]$.

Proof. □

1°. Let $\alpha\mathcal{X} \stackrel{\text{def}}{=} \langle f\rangle x \sqcup \langle g\rangle x$; $\beta\mathcal{Y} \stackrel{\text{def}}{=} \langle f^{-1}\rangle y \sqcup \langle g^{-1}\rangle y$ for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$. Then

$$\begin{aligned}&y \not\asymp^{\mathfrak{B}} \alpha x \Leftrightarrow \\ &y \not\asymp \langle f\rangle x \vee y \not\asymp \langle g\rangle x \Leftrightarrow \\ &x \not\asymp \langle f^{-1}\rangle y \vee x \not\asymp \langle g^{-1}\rangle y \Leftrightarrow \\ &x \not\asymp \langle f^{-1}\rangle y \sqcup \langle g^{-1}\rangle y \Leftrightarrow \\ &x \not\asymp \beta y.\end{aligned}$$

So $h = (\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$ is a pointfree funcoid. Obviously $h \sqsupseteq f$ and $h \sqsupseteq g$. If $p \sqsupseteq f$ and $p \sqsupseteq g$ for some $p \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $\langle p \rangle x \sqsupseteq \langle f \rangle x \sqcup \langle g \rangle x = \langle h \rangle x$ and $\langle p^{-1} \rangle y \sqsupseteq \langle f^{-1} \rangle y \sqcup \langle g^{-1} \rangle y = \langle h^{-1} \rangle y$ that is $p \sqsupseteq h$. So $f \sqcup g = h$.

2°.

$$\begin{aligned} x \,[f \sqcup g]\, y &\Leftrightarrow \\ y \not\asymp \langle f \sqcup g \rangle x &\Leftrightarrow \\ y \not\asymp \langle f \rangle x \sqcup \langle g \rangle x &\Leftrightarrow \\ y \not\asymp \langle f \rangle x \vee y \not\asymp \langle g \rangle x &\Leftrightarrow \\ x \,[f]\, y \vee x \,[g]\, y & \end{aligned}$$

for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$.

19.5. Domain and range of a pointfree funcoid

DEFINITION 1530. Let $\mathfrak{A}$ be a poset. The *identity pointfree funcoid* $1^{\mathsf{pFCD}}_{\mathfrak{A}} = (\mathfrak{A}, \mathfrak{A}, \mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}})$.

It is trivial that identity funcoid is really a pointfree funcoid.

Let now $\mathfrak{A}$ be a meet-semilattice.

DEFINITION 1531. Let $a \in \mathfrak{A}$. The *restricted identity pointfree funcoid* $\mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_a = (\mathfrak{A}, \mathfrak{A}, a \sqcap^{\mathfrak{A}}, a \sqcap^{\mathfrak{A}})$.

PROPOSITION 1532. The restricted pointfree funcoid is a pointfree funcoid.

PROOF. We need to prove that $(a \sqcap^{\mathfrak{A}} x) \not\asymp^{\mathfrak{A}} y \Leftrightarrow (a \sqcap^{\mathfrak{A}} y) \not\asymp^{\mathfrak{A}} x$ what is obvious. □

OBVIOUS 1533. $(\mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_a)^{-1} = \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_a$.

OBVIOUS 1534. $x \left[\mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_a\right] y \Leftrightarrow a \not\asymp^{\mathfrak{A}} x \sqcap^{\mathfrak{A}} y$ for every $x, y \in \mathfrak{A}$.

DEFINITION 1535. I will define *restricting* of a pointfree funcoid f to an element $a \in \operatorname{Src} f$ by the formula $f|_a \overset{\text{def}}{=} f \circ \mathrm{id}^{\mathsf{pFCD}(\operatorname{Src} f)}_a$.

DEFINITION 1536. Let f be a pointfree funcoid whose source is a set with greatest element. *Image* of f will be defined by the formula $\operatorname{im} f = \langle f \rangle \top$.

PROPOSITION 1537. $\operatorname{im} f \sqsupseteq \langle f \rangle x$ for every $x \in \operatorname{Src} f$ whenever $\operatorname{Dst} f$ is a strongly separable poset with greatest element.

PROOF. $\langle f \rangle \top$ is greater than every $\langle f \rangle x$ (where $x \in \operatorname{Src} f$) by proposition 1500. □

DEFINITION 1538. *Domain* of a pointfree funcoid f is defined by the formula $\operatorname{dom} f = \operatorname{im} f^{-1}$.

PROPOSITION 1539. $\langle f \rangle \operatorname{dom} f = \operatorname{im} f$ if f is a pointfree funcoid and $\operatorname{Src} f$ is a strongly separable poset with greatest element and $\operatorname{Dst} f$ is a separable poset with greatest element.

PROOF. For every $y \in \operatorname{Dst} f$

$$y \not\asymp \langle f \rangle \operatorname{dom} f \Leftrightarrow \operatorname{dom} f \not\asymp \langle f^{-1} \rangle y \Leftrightarrow \langle f^{-1} \rangle \top \not\asymp \langle f^{-1} \rangle y \Leftrightarrow$$

$$\text{(by strong separability of } \operatorname{Src} f)$$

$$\langle f^{-1} \rangle y \text{ is not least} \Leftrightarrow \top \not\asymp \langle f^{-1} \rangle y \Leftrightarrow y \not\asymp \langle f \rangle \top \Leftrightarrow y \not\asymp \operatorname{im} f.$$

So $\langle f \rangle \operatorname{dom} f = \operatorname{im} f$ by separability of $\operatorname{Dst} f$. □

PROPOSITION 1540. $\langle f\rangle x = \langle f\rangle(x\sqcap\operatorname{dom} f)$ for every $x\in\operatorname{Src} f$ for a pointfree funcoid f whose source is a bounded separable meet-semilattice and destination is a bounded separable poset.

PROOF. $\operatorname{Src} f$ is strongly separable by theorem 225. For every $y\in\operatorname{Dst} f$ we have

$$y\not\asymp\langle f\rangle(x\sqcap\operatorname{dom} f)\Leftrightarrow x\sqcap\operatorname{dom} f\sqcap\left\langle f^{-1}\right\rangle y\neq\bot^{\operatorname{Src} f}\Leftrightarrow$$
$$x\sqcap\operatorname{im} f^{-1}\sqcap\left\langle f^{-1}\right\rangle y\neq\bot^{\operatorname{Src} f}\Leftrightarrow$$
$$\text{(by strong separability of } \operatorname{Src} f)$$
$$x\sqcap\left\langle f^{-1}\right\rangle y\neq\bot^{\operatorname{Src} f}\Leftrightarrow y\not\asymp\langle f\rangle x.$$

Thus $\langle f\rangle x = \langle f\rangle(x\sqcap\operatorname{dom} f)$ by separability of $\operatorname{Dst} f$. □

PROPOSITION 1541. $x\not\asymp\operatorname{dom} f\Leftrightarrow(\langle f\rangle x$ is not least) for every pointfree funcoid f and $x\in\operatorname{Src} f$ if $\operatorname{Dst} f$ has greatest element $\top$.

PROOF. $x\not\asymp\operatorname{dom} f\Leftrightarrow x\not\asymp\left\langle f^{-1}\right\rangle\top^{\operatorname{Dst} f}\Leftrightarrow\top^{\operatorname{Dst} f}\not\asymp\langle f\rangle x\Leftrightarrow$ $(\langle f\rangle x$ is not least). □

PROPOSITION 1542. $\operatorname{dom} f=\bigsqcup\left\{\frac{a\in\operatorname{atoms}^{\operatorname{Src} f}}{\langle f\rangle a\neq\bot^{\operatorname{Dst} f}}\right\}$ for every pointfree funcoid f whose destination is a bounded strongly separable poset and source is an atomistic poset.

PROOF. For every $a\in\operatorname{atoms}^{\operatorname{Src} f}$ we have

$$a\not\asymp\operatorname{dom} f\Leftrightarrow a\not\asymp\left\langle f^{-1}\right\rangle\top^{\operatorname{Dst} f}\Leftrightarrow\top^{\operatorname{Dst} f}\not\asymp\langle f\rangle a\Leftrightarrow\langle f\rangle a\neq\bot^{\operatorname{Dst} f}.$$

So $\operatorname{dom} f=\bigsqcup\left\{\frac{a\in\operatorname{atoms}^{\operatorname{Src} f}}{a\not\asymp\operatorname{dom} f}\right\}=\bigsqcup\left\{\frac{a\in\operatorname{atoms}^{\operatorname{Src} f}}{\langle f\rangle a\neq\bot^{\operatorname{Dst} f}}\right\}$. □

PROPOSITION 1543. $\operatorname{dom}(f|_a)=a\sqcap\operatorname{dom} f$ for every pointfree funcoid f and $a\in\operatorname{Src} f$ where $\operatorname{Src} f$ is a meet-semilattice and $\operatorname{Dst} f$ has greatest element.

PROOF.

$$\operatorname{dom}(f|_a)=\operatorname{im}(\operatorname{id}_a^{\mathsf{pFCD}(\operatorname{Src} f)}\circ f^{-1})=$$
$$\left\langle\operatorname{id}_a^{\mathsf{pFCD}(\operatorname{Src} f)}\right\rangle\left\langle f^{-1}\right\rangle\top^{\operatorname{Dst} f}=a\sqcap\left\langle f^{-1}\right\rangle\top^{\operatorname{Dst} f}=a\sqcap\operatorname{dom} f.$$

□

PROPOSITION 1544. For every composable pointfree funcoids f and g

1°. If $\operatorname{im} f\sqsupseteq\operatorname{dom} g$ then $\operatorname{im}(g\circ f)=\operatorname{im} g$, provided that the posets $\operatorname{Src} f$, $\operatorname{Dst} f=\operatorname{Src} g$ and $\operatorname{Dst} g$ have greatest elements and $\operatorname{Src} g$ and $\operatorname{Dst} g$ are strongly separable.

2°. If $\operatorname{im} f\sqsubseteq\operatorname{dom} g$ then $\operatorname{dom}(g\circ f)=\operatorname{dom} g$, provided that the posets $\operatorname{Dst} g$, $\operatorname{Dst} f=\operatorname{Src} g$ and $\operatorname{Src} f$ have greatest elements and $\operatorname{Dst} f$ and $\operatorname{Src} f$ are strongly separable.

PROOF.

1°. $\operatorname{im}(g\circ f)=\langle g\circ f\rangle\top^{\operatorname{Src} f}=\langle g\rangle\langle f\rangle\top^{\operatorname{Src} f}\sqsubseteq\operatorname{im} g$ by strong separability of $\operatorname{Dst} g$; $\operatorname{im}(g\circ f)=\langle g\circ f\rangle\top^{\operatorname{Src} f}=\langle g\rangle\operatorname{im} f\sqsupseteq\langle g\rangle\operatorname{dom} g=\operatorname{im} g$ by strong separability of $\operatorname{Dst} g$ and proposition 1539.

2°. $\operatorname{dom}(g\circ f)=\operatorname{im}(f^{-1}\circ g^{-1})$ what by the proved is equal to $\operatorname{im} f^{-1}$ that is $\operatorname{dom} f$.

□

19.6. Specifying funcoids by functions or relations on atomic filters

THEOREM 1545. Let $\mathfrak{A}$ be an atomic poset and $(\mathfrak{B}, \mathfrak{Z}_1)$ is a primary filtrator over a boolean lattice. Then for every $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{X} \in \mathfrak{A}$ we have

$$\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}.$$

PROOF. For every $Y \in \mathfrak{Z}_1$ we have

$$Y \not\asymp^{\mathfrak{B}} \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \Leftrightarrow \\ \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : Y \not\asymp^{\mathfrak{B}} \langle f \rangle x.$$

Thus $\partial \langle f \rangle \mathcal{X} = \bigcup \langle \partial \rangle^* \langle \langle f \rangle \rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} = \partial \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ (used corollary 569). Consequently $\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ by the corollary 568. □

PROPOSITION 1546. Let f be a pointfree funcoid. Then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

1°. $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X} : x \, [f] \, \mathcal{Y}$ if $\operatorname{Src} f$ is an atomic poset.
2°. $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \exists y \in \operatorname{atoms} \mathcal{Y} : \mathcal{X} \, [f] \, y$ if $\operatorname{Dst} f$ is an atomic poset.

PROOF. I will prove only the second as the first is similar.

If $\mathcal{X} \, [f] \, \mathcal{Y}$, then $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$, consequently exists $y \in \operatorname{atoms} \mathcal{Y}$ such that $y \not\asymp \langle f \rangle \mathcal{X}$, $\mathcal{X} \, [f] \, y$. The reverse is obvious. □

COROLLARY 1547. If f is a pointfree funcoid with both source and destination being atomic posets, then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Y} \in \operatorname{Dst} f$

$$\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y} : x \, [f] \, y.$$

PROOF. Apply the theorem twice. □

COROLLARY 1548. If $\mathfrak{A}$ is a separable atomic poset and $\mathfrak{B}$ is a separable poset then $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is determined by the values of $\langle f \rangle X$ for $X \in \operatorname{atoms}^{\mathfrak{A}}$.

PROOF.

$$y \not\asymp \langle f \rangle x \Leftrightarrow x \not\asymp \langle f^{-1} \rangle y \Leftrightarrow \exists X \in \operatorname{atoms} x : X \not\asymp \langle f^{-1} \rangle y \Leftrightarrow \exists X \in \operatorname{atoms} x : y \not\asymp \langle f \rangle X.$$

Thus by separability of $\mathfrak{B}$ we have $\langle f \rangle$ is determined by $\langle f \rangle X$ for $X \in \operatorname{atoms} x$.

By separability of $\mathfrak{A}$ we infer that f can be restored from $\langle f \rangle$ (theorem 1498). □

THEOREM 1549. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices.

1°. A function $\alpha \in \mathfrak{B}^{\operatorname{atoms}^{\mathfrak{A}}}$ such that (for every $a \in \operatorname{atoms}^{\mathfrak{A}}$)

$$\alpha a \sqsubseteq \bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \circ \operatorname{atoms}^{\mathfrak{A}} \right\rangle^* \operatorname{up}^{\mathfrak{Z}_0} a \tag{24}$$

can be continued to the function $\langle f \rangle$ for a unique $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$;

$$\langle f \rangle \mathcal{X} = \bigsqcup \langle \alpha \rangle^* \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} \tag{25}$$

for every $\mathcal{X} \in \mathfrak{A}$.

2°. A relation $\delta \in \mathscr{P}(\operatorname{atoms}^{\mathfrak{A}} \times \operatorname{atoms}^{\mathfrak{B}})$ such that (for every $a \in \operatorname{atoms}^{\mathfrak{A}}$, $b \in \operatorname{atoms}^{\mathfrak{B}}$)

$$\forall X \in \operatorname{up}^{\mathfrak{Z}_0} a, Y \in \operatorname{up}^{\mathfrak{Z}_1} b \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y \Rightarrow a \, \delta \, b \tag{26}$$

can be continued to the relation $[f]$ for a unique $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$;

$$\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y} : x \, \delta \, y \tag{27}$$

for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$.

Proof. Existence of no more than one such funcoids and formulas (25) and (27) follow from theorem 1545 and corollary 1547 and the fact that our filtrators are separable.

1°. Consider the function $\alpha' \in \mathfrak{B}^{\mathfrak{Z}_0}$ defined by the formula (for every $X \in \mathfrak{Z}_0$)

$$\alpha' X = \bigsqcup \langle \alpha \rangle^* \operatorname{atoms}^{\mathfrak{A}} X.$$

Obviously $\alpha' \bot^{\mathfrak{Z}_0} = \bot^{\mathfrak{B}}$. For every $I, J \in \mathfrak{Z}_0$

$$\begin{aligned}
&\alpha'(I \sqcup J) = \\
&\bigsqcup \langle \alpha \rangle^* \operatorname{atoms}^{\mathfrak{A}} (I \sqcup J) = \\
&\bigsqcup \langle \alpha \rangle^* (\operatorname{atoms}^{\mathfrak{A}} I \cup \operatorname{atoms}^{\mathfrak{A}} J) = \\
&\bigsqcup (\langle \alpha \rangle^* \operatorname{atoms}^{\mathfrak{A}} I \cup \langle \alpha \rangle^* \operatorname{atoms}^{\mathfrak{A}} J) = \\
&\bigsqcup \langle \alpha \rangle^* \operatorname{atoms}^{\mathfrak{A}} I \sqcup \bigsqcup \langle \alpha \rangle^* \operatorname{atoms}^{\mathfrak{A}} J = \\
&\alpha' I \sqcup \alpha' J.
\end{aligned}$$

Let continue α' till a pointfree funcoid f (by the theorem 1513): $\langle f \rangle \mathcal{X} = \bigsqcap \langle \alpha' \rangle^* \operatorname{up}^{\mathfrak{Z}_0} \mathcal{X}$.

Let's prove the reverse of (24):

$$\begin{aligned}
&\bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \circ \operatorname{atoms}^{\mathfrak{A}} \right\rangle^* \operatorname{up}^{\mathfrak{Z}_0} a = \\
&\bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \right\rangle^* \left\langle \operatorname{atoms}^{\mathfrak{A}} \right\rangle^* \operatorname{up}^{\mathfrak{Z}_0} a \sqsubseteq \\
&\bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \right\rangle^* \{\{a\}\} = \\
&\bigsqcap \left\{ \left(\bigsqcup \circ \langle \alpha \rangle^* \right) \{a\} \right\} = \\
&\bigsqcap \left\{ \bigsqcup \langle \alpha \rangle^* \{a\} \right\} = \\
&\bigsqcap \left\{ \bigsqcup \{\alpha a\} \right\} = \\
&\bigsqcap \{\alpha a\} = \alpha a.
\end{aligned}$$

Finally,

$$\alpha a = \bigsqcap \left\langle \bigsqcup \circ \langle \alpha \rangle^* \circ \operatorname{atoms}^{\mathfrak{A}} \right\rangle^* \operatorname{up}^{\mathfrak{Z}_0} a = \bigsqcap \langle \alpha' \rangle^* \operatorname{up}^{\mathfrak{Z}_0} a = \langle f \rangle a,$$

so $\langle f \rangle$ is a continuation of α.

2°. Consider the relation $\delta' \in \mathscr{P}(\mathfrak{Z}_0 \times \mathfrak{Z}_1)$ defined by the formula (for every $X \in \mathfrak{Z}_0$, $Y \in \mathfrak{Z}_1$)

$$X \mathrel{\delta'} Y \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \mathrel{\delta} y.$$

Obviously $\neg(X \mathrel{\delta'} \bot^{\mathfrak{Z}_1})$ and $\neg(\bot^{\mathfrak{Z}_0} \mathrel{\delta'} Y)$.

$$\begin{aligned}
&I \sqcup J \mathrel{\delta'} Y \Leftrightarrow \\
&\exists x \in \operatorname{atoms}^{\mathfrak{A}} (I \sqcup J), y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \mathrel{\delta} y \Leftrightarrow \\
&\exists x \in \operatorname{atoms}^{\mathfrak{A}} I \cup \operatorname{atoms}^{\mathfrak{A}} J, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \mathrel{\delta} y \Leftrightarrow \\
&\exists x \in \operatorname{atoms}^{\mathfrak{A}} I, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \mathrel{\delta} y \vee \exists x \in \operatorname{atoms}^{\mathfrak{A}} J, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \mathrel{\delta} y \Leftrightarrow \\
&I \mathrel{\delta'} Y \vee J \mathrel{\delta'} Y;
\end{aligned}$$

similarly $X\ \delta'\ I \sqcup J \Leftrightarrow X\ \delta'\ I \vee X\ \delta'\ J$. Let's continue δ' till a funcoid f (by the theorem 1513):

$$\mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}^{\mathfrak{Z}_0} \mathcal{X}, Y \in \operatorname{up}^{\mathfrak{Z}_1} \mathcal{Y} : X\ \delta'\ Y.$$

The reverse of (26) implication is trivial, so

$$\forall X \in \operatorname{up}^{\mathfrak{Z}_0} a, Y \in \operatorname{up}^{\mathfrak{Z}_1} b \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x\ \delta\ y \Leftrightarrow a\ \delta\ b;$$

$$\begin{aligned} &\forall X \in \operatorname{up}^{\mathfrak{Z}_0} a, Y \in \operatorname{up}^{\mathfrak{Z}_1} b \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x\ \delta\ y \Leftrightarrow \\ &\forall X \in \operatorname{up}^{\mathfrak{Z}_0} a, Y \in \operatorname{up}^{\mathfrak{Z}_1} b : X\ \delta'\ Y \Leftrightarrow \\ &a\ [f]\ b. \end{aligned}$$

So $a\ \delta\ b \Leftrightarrow a\ [f]\ b$, that is $[f]$ is a continuation of δ.

□

THEOREM 1550. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. If $R \in \mathscr{P}\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $x \in \operatorname{atoms}^{\mathfrak{A}}$, $y \in \operatorname{atoms}^{\mathfrak{B}}$, then

1°. $\langle \bigsqcap R \rangle x = \bigsqcap_{f \in R} \langle f \rangle x$;

2°. $x\ [\bigsqcap R]\ y \Leftrightarrow \forall f \in R : x\ [f]\ y$.

PROOF.

2°. Let denote $x\ \delta\ y \Leftrightarrow \forall f \in R : x\ [f]\ y$.

$$\begin{aligned} &\forall X \in \operatorname{up}^{\mathfrak{Z}_0} a, Y \in \operatorname{up}^{\mathfrak{Z}_1} b \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x\ \delta\ y \Rightarrow \\ &\forall f \in R, X \in \operatorname{up}^{\mathfrak{Z}_0} a, Y \in \operatorname{up}^{\mathfrak{Z}_1} b \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x\ [f]\ y \Rightarrow \\ &\forall f \in R, X \in \operatorname{up}^{\mathfrak{Z}_0} a, Y \in \operatorname{up}^{\mathfrak{Z}_1} b : X\ [f]\ Y \Rightarrow \\ &\forall f \in R : a\ [f]\ b \Leftrightarrow \\ &a\ \delta\ b. \end{aligned}$$

So by theorem 1549, δ can be continued till $[p]$ for some $p \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

For every $q \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ such that $\forall f \in R : q \sqsubseteq f$ we have $x\ [q]\ y \Rightarrow \forall f \in R : x\ [f]\ y \Leftrightarrow x\ \delta\ y \Leftrightarrow x\ [p]\ y$, so $q \sqsubseteq p$. Consequently $p = \bigsqcap R$.

From this $x\ [\bigsqcap R]\ y \Leftrightarrow \forall f \in R : x\ [f]\ y$.

1°. From the former

$$\begin{aligned} y \in \operatorname{atoms}^{\mathfrak{B}} \left\langle \bigsqcap R \right\rangle x \Leftrightarrow y \sqcap \left\langle \bigsqcap R \right\rangle x \neq \bot^{\mathfrak{B}} \Leftrightarrow \forall f \in R : y \sqcap \langle f \rangle x \neq \bot^{\mathfrak{B}} \Leftrightarrow \\ y \in \bigcap \langle \operatorname{atoms}^{\mathfrak{B}} \rangle^* \left\{ \frac{\langle f \rangle x}{f \in R} \right\} \Leftrightarrow y \in \operatorname{atoms} \bigsqcap \left\{ \frac{\langle f \rangle x}{f \in R} \right\} \end{aligned}$$

for every $y \in \operatorname{atoms}^{\mathfrak{B}}$.

$\mathfrak{B}$ is atomically separable by the corollary 582. Thus $\langle \bigsqcap R \rangle x = \bigsqcap_{f \in R} \langle f \rangle x$.

□

19.7. More on composition of pointfree funcoids

PROPOSITION 1551. $[g \circ f] = [g] \circ \langle f \rangle = \left\langle g^{-1} \right\rangle^{-1} \circ [f]$ for every composable pointfree funcoids f and g.

PROOF. For every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$

$$x\ [g \circ f]\ y \Leftrightarrow y \not\asymp \langle g \circ f \rangle x \Leftrightarrow y \not\asymp \langle g \rangle \langle f \rangle x \Leftrightarrow \langle f \rangle x\ [g]\ y \Leftrightarrow x\ ([g] \circ \langle f \rangle)\ y.$$

Thus $[g \circ f] = [g] \circ \langle f \rangle$.

$$[g \circ f] = \left[(f^{-1} \circ g^{-1})^{-1} \right] = \left[f^{-1} \circ g^{-1} \right]^{-1} = (\left[f^{-1} \right] \circ \left\langle g^{-1} \right\rangle)^{-1} = \left\langle g^{-1} \right\rangle^{-1} \circ [f].$$

□

THEOREM 1552. Let f and g be pointfree funcoids and $\mathfrak{A} = \operatorname{Dst} f = \operatorname{Src} g$ be an atomic poset. Then for every $\mathcal{X} \in \operatorname{Src} f$ and $\mathcal{Z} \in \operatorname{Dst} g$

$$\mathcal{X} \; [g \circ f] \; \mathcal{Z} \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{X} \; [f] \; y \wedge y \; [g] \; \mathcal{Z}).$$

PROOF.

$$\begin{aligned}
\exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{X} \; [f] \; y \wedge y \; [g] \; \mathcal{Z}) \Leftrightarrow \\
\exists y \in \operatorname{atoms}^{\mathfrak{A}} : (\mathcal{Z} \not\asymp \langle g \rangle y \wedge y \not\asymp \langle f \rangle \mathcal{X}) \Leftrightarrow \\
\exists y \in \operatorname{atoms}^{\mathfrak{A}} : (y \not\asymp \langle g^{-1} \rangle \mathcal{Z} \wedge y \not\asymp \langle f \rangle \mathcal{X}) \Leftrightarrow \\
\langle g^{-1} \rangle \mathcal{Z} \not\asymp \langle f \rangle \mathcal{X} \Leftrightarrow \\
\mathcal{X} \; [g \circ f] \; \mathcal{Z}.
\end{aligned}$$

□

THEOREM 1553. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be separable starrish join-semilattices and $\mathfrak{B}$ is atomic. Then:

1°. $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ for $g, h \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $f \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$.
2°. $(g \sqcup h) \circ f = g \circ f \sqcup h \circ f$ for $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $g, h \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$.

PROOF. I will prove only the first equality because the other is analogous.
We can apply theorem 1529.
For every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{C}$

$$\begin{aligned}
\mathcal{X} \; [f \circ (g \sqcup h)] \; \mathcal{Z} \Leftrightarrow \\
\exists y \in \operatorname{atoms}^{\mathfrak{B}} : (\mathcal{X} \; [g \sqcup h] \; y \wedge y \; [f] \; \mathcal{Z}) \Leftrightarrow \\
\exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((\mathcal{X} \; [g] \; y \vee \mathcal{X} \; [h] \; y) \wedge y \; [f] \; \mathcal{Z}) \Leftrightarrow \\
\exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((\mathcal{X} \; [g] \; y \wedge y \; [f] \; \mathcal{Z}) \vee (\mathcal{X} \; [h] \; y \wedge y \; [f] \; \mathcal{Z})) \Leftrightarrow \\
\exists y \in \operatorname{atoms}^{\mathfrak{B}} : (\mathcal{X} \; [g] \; y \wedge y \; [f] \; \mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (\mathcal{X} \; [h] \; y \wedge y \; [f] \; \mathcal{Z}) \Leftrightarrow \\
\mathcal{X} \; [f \circ g] \; \mathcal{Z} \vee \mathcal{X} \; [f \circ h] \; \mathcal{Z} \Leftrightarrow \\
\mathcal{X} \; [f \circ g \sqcup f \circ h] \; \mathcal{Z}.
\end{aligned}$$

Thus $f \circ (g \sqcup h) = f \circ g \sqcup f \circ h$ by theorem 1498. □

THEOREM 1554. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be posets of filters over some boolean lattices, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$, $g \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$, $h \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{C})$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

PROOF.

$$\begin{aligned}
g \circ f \not\asymp h \Leftrightarrow \\
\exists a \in \operatorname{atoms}^{\mathfrak{A}}, c \in \operatorname{atoms}^{\mathfrak{C}} : a \; [(g \circ f) \sqcap h] \; c \Leftrightarrow \\
\exists a \in \operatorname{atoms}^{\mathfrak{A}}, c \in \operatorname{atoms}^{\mathfrak{C}} : (a \; [g \circ f] \; c \wedge a \; [h] \; c) \Leftrightarrow \\
\exists a \in \operatorname{atoms}^{\mathfrak{A}}, b \in \operatorname{atoms}^{\mathfrak{B}}, c \in \operatorname{atoms}^{\mathfrak{C}} : (a \; [f] \; b \wedge b \; [g] \; c \wedge a \; [h] \; c) \Leftrightarrow \\
\exists b \in \operatorname{atoms}^{\mathfrak{B}}, c \in \operatorname{atoms}^{\mathfrak{C}} : (b \; [g] \; c \wedge b \; \left[h \circ f^{-1}\right] \; c) \Leftrightarrow \\
\exists b \in \operatorname{atoms}^{\mathfrak{B}}, c \in \operatorname{atoms}^{\mathfrak{C}} : b \; \left[g \sqcap (h \circ f^{-1})\right] \; c \Leftrightarrow \\
g \not\asymp h \circ f^{-1}.
\end{aligned}$$

□

19.8. Funcoidal product of elements

DEFINITION 1555. *Funcoidal product* $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ where $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$ and $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements is a pointfree funcoid such that for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$

$$\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \not\asymp \mathcal{A}; \\ \bot^{\mathfrak{B}} & \text{if } \mathcal{X} \asymp \mathcal{A}; \end{cases} \quad \text{and} \quad \langle (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})^{-1} \rangle \mathcal{Y} = \begin{cases} \mathcal{A} & \text{if } \mathcal{Y} \not\asymp \mathcal{B}; \\ \bot^{\mathfrak{A}} & \text{if } \mathcal{Y} \asymp \mathcal{B}. \end{cases}$$

PROPOSITION 1556. $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ is really a pointfree funcoid and

$$\mathcal{X} \left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right] \mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp \mathcal{A} \wedge \mathcal{Y} \not\asymp \mathcal{B}.$$

PROOF. Obvious. □

PROPOSITION 1557. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets with least elements, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$, $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$. Then

$$f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Rightarrow \operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}.$$

PROOF. If $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ then $\operatorname{dom} f \sqsubseteq \operatorname{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \sqsubseteq \mathcal{A}$, $\operatorname{im} f \sqsubseteq \operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \sqsubseteq \mathcal{B}$. □

THEOREM 1558. Let $\mathfrak{A}$ and $\mathfrak{B}$ be strongly separable bounded posets, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$, $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$. Then

$$f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}.$$

PROOF. One direction is the proposition above. The other:

If $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$ then $\mathcal{X}\ [f]\ \mathcal{Y} \Rightarrow \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X} \Rightarrow \mathcal{Y} \not\asymp \operatorname{im} f \Rightarrow \mathcal{Y} \not\asymp \mathcal{B}$ (strong separability used) and similarly $\mathcal{X}\ [f]\ \mathcal{Y} \Rightarrow \mathcal{X} \not\asymp \mathcal{A}$.

So $[f] \subseteq \left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right]$ and thus using separability $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

THEOREM 1559. Let $\mathfrak{A}$, $\mathfrak{B}$ be bounded separable meet-semilattices. For every $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$

$$f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathrm{id}_{\mathcal{B}}^{\mathsf{pFCD}(\mathfrak{B})} \circ f \circ \mathrm{id}_{\mathcal{A}}^{\mathsf{pFCD}(\mathfrak{A})}.$$

PROOF. $h \stackrel{\text{def}}{=} \mathrm{id}_{\mathcal{B}}^{\mathsf{pFCD}(\mathfrak{B})} \circ f \circ \mathrm{id}_{\mathcal{A}}^{\mathsf{pFCD}(\mathfrak{A})}$. For every $\mathcal{X} \in \mathfrak{A}$

$$\langle h \rangle \mathcal{X} = \left\langle \mathrm{id}_{\mathcal{B}}^{\mathsf{pFCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle \mathrm{id}_{\mathcal{A}}^{\mathsf{pFCD}(\mathfrak{A})} \right\rangle \mathcal{X} = \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \mathcal{X})$$

and

$$\langle h^{-1} \rangle \mathcal{X} = \left\langle \mathrm{id}_{\mathcal{A}}^{\mathsf{pFCD}(\mathfrak{A})} \right\rangle \langle f^{-1} \rangle \left\langle \mathrm{id}_{\mathcal{B}}^{\mathsf{pFCD}(\mathfrak{B})} \right\rangle \mathcal{X} = \mathcal{A} \sqcap \langle f^{-1} \rangle (\mathcal{B} \sqcap \mathcal{X}).$$

From this, as easy to show, $h \sqsubseteq f$ and $h \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. If $g \sqsubseteq f \wedge g \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a $g \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $\operatorname{dom} g \sqsubseteq \mathcal{A}$. $\mathfrak{A}$ and $\mathfrak{B}$ are are strongly separable by theorem 225. Thus by propositions 1540 we have:

$$\langle g \rangle \mathcal{X} = \langle g \rangle (\mathcal{X} \sqcap \operatorname{dom} g) = \langle g \rangle (\mathcal{X} \sqcap \mathcal{A}) = \mathcal{B} \sqcap \langle g \rangle (\mathcal{A} \sqcap \mathcal{X}) \sqsubseteq \mathcal{B} \sqcap \langle f \rangle (\mathcal{A} \sqcap \mathcal{X}) = \left\langle \mathrm{id}_{\mathcal{B}}^{\mathsf{pFCD}(\mathfrak{B})} \right\rangle \langle f \rangle \left\langle \mathrm{id}_{\mathcal{A}}^{\mathsf{pFCD}(\mathfrak{A})} \right\rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

and similarly $\langle g^{-1} \rangle \mathcal{Y} \sqsubseteq \langle h^{-1} \rangle \mathcal{Y}$. Thus $g \sqsubseteq h$.

So $h = f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

COROLLARY 1560. Let $\mathfrak{A}$, $\mathfrak{B}$ be bounded separable meet-semilattices. For every $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$ we have $f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \top^{\mathfrak{B}})$.

PROOF. $f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \top^{\mathfrak{B}}) = \mathrm{id}_{\top^{\mathfrak{B}}}^{\mathsf{pFCD}(\mathfrak{B})} \circ f \circ \mathrm{id}_{\mathcal{A}}^{\mathsf{pFCD}(\mathfrak{A})} = f \circ \mathrm{id}_{\mathcal{A}}^{\mathsf{pFCD}(\mathfrak{A})} = f|_{\mathcal{A}}$. □

COROLLARY 1561. Let $\mathfrak{A}$, $\mathfrak{B}$ be bounded separable meet-semilattices. For every $f \in \mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$ we have

$$f \not\asymp \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \mathcal{A}\, [f]\, \mathcal{B}.$$

PROOF. Existence of $f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$ follows from the above theorem.

$$\begin{aligned} f \not\asymp \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} &\Leftrightarrow \\ f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \neq \bot^{\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})} &\Leftrightarrow \\ \langle f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})\rangle \top^{\mathfrak{A}} \neq \bot^{\mathfrak{B}} &\Leftrightarrow \\ \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\mathcal{B}} \circ f \circ \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}} \right\rangle \top^{\mathfrak{A}} \neq \bot^{\mathfrak{B}} &\Leftrightarrow \\ \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\mathcal{B}} \right\rangle \langle f\rangle \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\mathcal{A}} \right\rangle \top^{\mathfrak{A}} \neq \bot^{\mathfrak{B}} &\Leftrightarrow \\ \mathcal{B} \sqcap \langle f\rangle(\mathcal{A} \sqcap \top^{\mathfrak{A}}) \neq \bot^{\mathfrak{B}} &\Leftrightarrow \\ \mathcal{B} \sqcap \langle f\rangle\mathcal{A} \neq \bot^{\mathfrak{B}} &\Leftrightarrow \\ \mathcal{A}\, [f]\, \mathcal{B}. & \end{aligned}$$

□

THEOREM 1562. Let $\mathfrak{A}$, $\mathfrak{B}$ be bounded separable meet-semilattices. Then the poset $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ is separable.

PROOF. Let $f, g \in \mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ and $f \neq g$. By the theorem 1498 $[f] \neq [g]$. That is there exist $x, y \in \mathfrak{A}$ such that $x\, [f]\, y \not\Leftrightarrow x\, [g]\, y$ that is $f \sqcap (x \times^{\mathsf{FCD}} y) \neq \bot^{\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})} \not\Leftrightarrow g \sqcap (x \times^{\mathsf{FCD}} y) \neq \bot^{\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})}$. Thus $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ is separable. □

COROLLARY 1563. Let $\mathfrak{A}$, $\mathfrak{B}$ be atomic bounded separable meet-semilattices. The poset $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ is:

1°. separable;
2°. strongly separable;
3°. atomically separable;
4°. conforming to Wallman's disjunction property.

PROOF. By the theorem 233. □

REMARK 1564. For more ways to characterize (atomic) separability of the lattice of pointfree funcoids see subsections "Separation subsets and full stars" and "Atomically Separable Lattices".

COROLLARY 1565. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. The poset $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ is an atomistic lattice.

PROOF. By the corollary 1528 $\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ is a complete lattice. We can use theorem 231. □

THEOREM 1566. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets of filters over boolean lattices. If $S \in \mathscr{P}(\mathfrak{A} \times \mathfrak{B})$ then

$$\bigsqcap_{(\mathcal{A},\mathcal{B})\in S} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \bigsqcap \operatorname{dom} S \times^{\mathsf{FCD}} \bigsqcap \operatorname{im} S.$$

PROOF. If $x \in \operatorname{atoms}^{\mathfrak{A}}$ then by the theorem 1550

$$\left\langle \bigsqcap_{(\mathcal{A},\mathcal{B})\in S} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \right\rangle x = \bigsqcap \left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\rangle x}{(\mathcal{A},\mathcal{B}) \in S} \right\}.$$

If $x \sqcap \bigsqcap \operatorname{dom} S \neq \bot^{\mathfrak{A}}$ then

$$\forall (\mathcal{A}, \mathcal{B}) \in S : (x \sqcap \mathcal{A} \neq \bot^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \mathcal{B});$$

$$\left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}, \mathcal{B}) \in S} \right\} = \operatorname{im} S;$$

if $x \sqcap \bigsqcap \operatorname{dom} S = \bot^{\mathfrak{A}}$ then

$$\exists (\mathcal{A}, \mathcal{B}) \in S : (x \sqcap \mathcal{A} = \bot^{\mathfrak{A}} \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \bot^{\mathcal{B}});$$

$$\left\{ \frac{\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x}{(\mathcal{A}, \mathcal{B}) \in S} \right\} \ni \bot^{\mathfrak{B}}.$$

So

$$\left\langle \bigsqcap_{(\mathcal{A}, \mathcal{B}) \in S} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \right\rangle x = \begin{cases} \bigsqcap \operatorname{im} S & \text{if } x \sqcap \bigsqcap \operatorname{dom} S \neq \bot^{\mathfrak{A}}; \\ \bot^{\mathfrak{B}} & \text{if } x \sqcap \bigsqcap \operatorname{dom} S = \bot^{\mathfrak{A}}. \end{cases}$$

From this by theorem 1549 the statement of our theorem follows. □

COROLLARY 1567. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets of filters over boolean lattices. For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{A}$ and $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{B}$

$$(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1).$$

PROOF. $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \sqcap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = \bigsqcap \{ \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \}$ what is by the last theorem equal to $(\mathcal{A}_0 \sqcap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \sqcap \mathcal{B}_1)$. □

THEOREM 1568. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. If $\mathcal{A} \in \mathfrak{A}$ then $\mathcal{A} \times^{\mathsf{FCD}}$ is a complete homomorphism from the lattice $\mathfrak{A}$ to the lattice $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$, if also $\mathcal{A} \neq \bot^{\mathfrak{A}}$ then it is an order embedding.

PROOF. Let $S \in \mathscr{P}\mathfrak{A}$, $X \in \mathfrak{Z}_0$, $x \in \operatorname{atoms}^{\mathfrak{A}}$.

$$\begin{gathered} \left\langle \bigsqcup \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle^* S \right\rangle X = \\ \bigsqcup_{\mathcal{B} \in S} \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle X = \\ \begin{cases} \bigsqcup S & \text{if } X \sqcap^{\mathfrak{A}} \mathcal{A} \neq \bot^{\mathfrak{A}} \\ \bot^{\mathfrak{B}} & \text{if } X \sqcap^{\mathfrak{A}} \mathcal{A} = \bot^{\mathfrak{A}} \end{cases} = \\ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup S \right\rangle X. \end{gathered}$$

Thus $\bigsqcup \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle^* S = \mathcal{A} \times^{\mathsf{FCD}} \bigsqcup S$ by theorem 1512.

$$\begin{gathered} \left\langle \bigsqcap \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle^* S \right\rangle x = \\ \bigsqcap_{\mathcal{B} \in S} \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \\ \begin{cases} \bigsqcap S & \text{if } X \sqcap^{\mathfrak{A}} \mathcal{A} \neq \bot^{\mathfrak{A}} \\ \bot^{\mathfrak{B}} & \text{if } X \sqcap^{\mathfrak{A}} \mathcal{A} = \bot^{\mathfrak{A}} \end{cases} = \\ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigsqcap S \right\rangle x. \end{gathered}$$

Thus $\bigsqcap \langle \mathcal{A} \times^{\mathsf{FCD}} \rangle^* S = \mathcal{A} \times^{\mathsf{FCD}} \bigsqcap S$ by theorem 1545.

If $\mathcal{A} \neq \bot^{\mathfrak{A}}$ then obviously $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{X} \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{Y} \Leftrightarrow \mathcal{X} \sqsubseteq \mathcal{Y}$, because $\operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{X}) = \mathcal{X}$ and $\operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{Y}) = \mathcal{Y}$. □

PROPOSITION 1569. Let $\mathfrak{A}$ be a meet-semilattice with least element and $\mathfrak{B}$ be a poset with least element. If a is an atom of $\mathfrak{A}$, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $f|_a = a \times^{\mathsf{FCD}} \langle f \rangle a$.

PROOF. Let $\mathcal{X} \in \mathfrak{A}$.

$$\mathcal{X} \sqcap a \neq \bot^{\mathfrak{A}} \Rightarrow \langle f|_a \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \sqcap a = \bot^{\mathfrak{A}} \Rightarrow \langle f|_a \rangle \mathcal{X} = \bot^{\mathfrak{B}}.$$

□

PROPOSITION 1570. $f \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \langle f \rangle \mathcal{B}$ for elements $\mathcal{A} \in \mathfrak{A}$ and $B \in \mathfrak{B}$ of some posets $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ with least elements and $f \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$.

PROOF. Let $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$.

$$\langle f \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \rangle \mathcal{X} = \left(\begin{cases} \langle f \rangle \mathcal{B} & \text{if } \mathcal{X} \not\asymp \mathcal{A} \\ \bot & \text{if } \mathcal{X} \asymp \mathcal{A} \end{cases} \right) = \langle \mathcal{A} \times^{\mathsf{FCD}} \langle f \rangle \mathcal{B} \rangle \mathcal{X};$$

$$\begin{gathered} \left\langle (f \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}))^{-1} \right\rangle \mathcal{Y} = \\ \left\langle (\mathcal{B} \times^{\mathsf{FCD}} \mathcal{A}) \circ f^{-1} \right\rangle \mathcal{Y} = \\ \left(\begin{cases} \mathcal{A} & \text{if } \langle f^{-1} \rangle \mathcal{Y} \not\asymp \mathcal{B} \\ \bot & \text{if } \langle f^{-1} \rangle \mathcal{Y} \asymp \mathcal{B} \end{cases} \right) = \\ \left(\begin{cases} \mathcal{A} & \text{if } \mathcal{Y} \not\asymp \langle f \rangle \mathcal{B} \\ \bot & \text{if } \mathcal{Y} \asymp \langle f \rangle \mathcal{B} \end{cases} \right) = \\ \langle \langle f \rangle \mathcal{B} \times^{\mathsf{FCD}} \mathcal{A} \rangle \mathcal{Y} = \\ \left\langle (\mathcal{A} \times^{\mathsf{FCD}} \langle f \rangle \mathcal{B})^{-1} \right\rangle \mathcal{Y}. \end{gathered}$$

□

19.9. Category of pointfree funcoids

I will define the category pFCD of pointfree funcoids:

- The class of objects are small posets.
- The set of morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.
- The composition is the composition of pointfree funcoids.
- Identity morphism for an object $\mathfrak{A}$ is $(\mathfrak{A}, \mathfrak{A}, \mathrm{id}_{\mathfrak{A}}, \mathrm{id}_{\mathfrak{A}})$.

To prove that it is really a category is trivial.

The *category of pointfree funcoid quintuples* is defined as follows:

- Objects are pairs $(\mathfrak{A}, \mathcal{A})$ where $\mathfrak{A}$ is a small meet-semilattice and $\mathcal{A} \in \mathfrak{A}$.
- The morphisms from an object $(\mathfrak{A}, \mathcal{A})$ to an object $(\mathfrak{B}, \mathcal{B})$ are tuples $(\mathfrak{A}, \mathfrak{B}, \mathcal{A}, \mathcal{B}, f)$ where $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and
$$\forall x \in \mathfrak{A} : \langle f \rangle x \sqsubseteq \mathcal{A}, \quad \forall y \in \mathfrak{B} : \left\langle f^{-1} \right\rangle y \sqsubseteq \mathcal{B}. \tag{28}$$
- The composition is defined by the formula
$$(\mathfrak{B}, \mathfrak{C}, \mathcal{B}, \mathcal{C}, g) \circ (\mathfrak{A}, \mathfrak{B}, \mathcal{A}, \mathcal{B}, f) = (\mathfrak{A}, \mathfrak{C}, \mathcal{A}, \mathcal{C}, g \circ f).$$
- Identity morphism for an object $(\mathfrak{A}, \mathcal{A})$ is $\mathrm{id}_{\mathcal{A}}^{\mathsf{pFCD}(\mathfrak{A})}$. (Note: this is defined only for meet-semilattices.)

To prove that it is really a category is trivial.

PROPOSITION 1571. For strongly separated and bounded $\mathfrak{A}$ and $\mathfrak{B}$ formula (28) is equivalent to each of the following:

1°. $\operatorname{dom} f \sqsubseteq \mathcal{A} \wedge \operatorname{im} f \sqsubseteq \mathcal{B}$;
2°. $f \sqsubseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$.

PROOF. Because $\langle f \rangle x \sqsubseteq \operatorname{im} f$, $\left\langle f^{-1} \right\rangle y \sqsubseteq \operatorname{dom} f$, and theorem 1558. □

19.10. Atomic pointfree funcoids

THEOREM 1572. Let $\mathfrak{A}$, $\mathfrak{B}$ be atomic bounded separable meet-semilattices. An $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is an atom of the poset $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ iff there exist $a \in \operatorname{atoms}^{\mathfrak{A}}$ and $b \in \operatorname{atoms}^{\mathfrak{B}}$ such that $f = a \times^{\mathsf{FCD}} b$.

PROOF.

$\Rightarrow$. Let f be an atom of the poset $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$. Let's get elements $a \in \operatorname{atoms} \operatorname{dom} f$ and $b \in \operatorname{atoms}\langle f \rangle a$. Then for every $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \asymp a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = \bot^{\mathfrak{B}} \sqsubseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = b \sqsubseteq \langle f \rangle \mathcal{X}.$$

So $\langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} \sqsubseteq \langle f \rangle \mathcal{X}$ and similarly $\langle b \times^{\mathsf{FCD}} a \rangle \mathcal{Y} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y}$ for every $\mathcal{Y} \in \mathfrak{B}$ thus $a \times^{\mathsf{FCD}} b \sqsubseteq f$; because f is atomic we have $f = a \times^{\mathsf{FCD}} b$.

$\Leftarrow$. Let $a \in \operatorname{atoms}^{\mathfrak{A}}$, $b \in \operatorname{atoms}^{\mathfrak{B}}$, $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$. If $b \asymp^{\mathfrak{B}} \langle f \rangle a$ then $\neg(a \; [f] \; b)$, $f \sqcap (a \times^{\mathsf{FCD}} b) = \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})}$ (by corollary 1561 because $\mathfrak{A}$ and $\mathfrak{B}$ are bounded meet-semilattices); if $b \sqsubseteq \langle f \rangle a$, then for every $\mathcal{X} \in \mathfrak{A}$

$$\mathcal{X} \asymp a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = \bot^{\mathfrak{B}} \sqsubseteq \langle f \rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} = b \sqsubseteq \langle f \rangle \mathcal{X}$$

that is $\langle a \times^{\mathsf{FCD}} b \rangle \mathcal{X} \sqsubseteq \langle f \rangle \mathcal{X}$ and likewise $\langle b \times^{\mathsf{FCD}} a \rangle \mathcal{Y} \sqsubseteq \langle f^{-1} \rangle \mathcal{Y}$ for every $\mathcal{Y} \in \mathfrak{B}$, so $f \sqsupseteq a \times^{\mathsf{FCD}} b$. Consequently $f \sqcap (a \times^{\mathsf{FCD}} b) = \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})} \vee f \sqsupseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atomic pointfree funcoid.

□

THEOREM 1573. Let $\mathfrak{A}$, $\mathfrak{B}$ be atomic bounded separable meet-semilattices. Then $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is atomic.

PROOF. Let $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ and $f \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})}$. Then $\operatorname{dom} f \neq \bot^{\mathfrak{A}}$, thus exists $a \in \operatorname{atoms} \operatorname{dom} f$. So $\langle f \rangle a \neq \bot^{\mathfrak{B}}$ thus exists $b \in \operatorname{atoms}\langle f \rangle a$. Finally the atomic pointfree funcoid $a \times^{\mathsf{FCD}} b \sqsubseteq f$. □

PROPOSITION 1574. Let $\mathfrak{A}$, $\mathfrak{B}$ be starrish bounded separable lattices. $\operatorname{atoms}(f \sqcup g) = \operatorname{atoms} f \cup \operatorname{atoms} g$ for every $f, g \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

PROOF.

$$\begin{aligned}(a \times^{\mathsf{FCD}} b) \sqcap (f \sqcup g) \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})} \Leftrightarrow \text{(corollary 1561)} \Leftrightarrow& \\ a \; [f \sqcup g] \; b \Leftrightarrow \text{(theorem 1529)} \Leftrightarrow& \\ a \; [f] \; b \vee a \; [g] \; b \Leftrightarrow \text{(corollary 1561)} \Leftrightarrow& \\ (a \times^{\mathsf{FCD}} b) \sqcap f \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})} \vee (a \times^{\mathsf{FCD}} b) \sqcap g \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})}&\end{aligned}$$

for every $a \in \operatorname{atoms}^{\mathfrak{A}}$ and $b \in \operatorname{atoms}^{\mathfrak{B}}$. □

THEOREM 1575. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is a co-frame.

PROOF. Theorems 1513 and 533. □

COROLLARY 1576. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is a co-brouwerian lattice.

PROPOSITION 1577. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ be atomic bounded separable meet-semilattices, and $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$, $g \in \mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})$. Then

$$\operatorname{atoms}(g \circ f) = \left\{ \frac{x \times^{\mathsf{FCD}} z}{\begin{array}{l} x \in \operatorname{atoms}^{\mathfrak{A}}, z \in \operatorname{atoms}^{\mathfrak{C}}, \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms} g) \end{array}} \right\}.$$

PROOF.

$$\begin{aligned}(x \times^{\mathsf{FCD}} z) \sqcap (g \circ f) \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{C})} &\Leftrightarrow \\ x \, [g \circ f] \, z &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : (x \, [f] \, y \wedge y \, [g] \, z) &\Leftrightarrow \\ \exists y \in \operatorname{atoms}^{\mathfrak{B}} : ((x \times^{\mathsf{FCD}} y) \sqcap f \neq \bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})} \wedge (y \times^{\mathsf{FCD}} z) \sqcap g \neq \bot^{\mathsf{pFCD}(\mathfrak{B}, \mathfrak{C})})&\end{aligned}$$

(were used corollary 1561 and theorem 1552). □

THEOREM 1578. Let f be a pointfree funcoid between atomic bounded separable meet-semilattices $\mathfrak{A}$ and $\mathfrak{B}$.

1°. $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{A}$, $\mathcal{Y} \in \mathfrak{B}$;

2°. $\langle f \rangle \mathcal{X} = \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{A}$ provided that $\mathfrak{B}$ is a complete lattice.

PROOF.

1°.

$$\begin{aligned}\exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y} &\Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathfrak{A}}, b \in \operatorname{atoms}^{\mathfrak{B}} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge \mathcal{X} \left[a \times^{\mathsf{FCD}} b\right] \mathcal{Y}) &\Leftrightarrow \\ \exists a \in \operatorname{atoms}^{\mathfrak{A}}, b \in \operatorname{atoms}^{\mathfrak{B}} : (a \times^{\mathsf{FCD}} b \not\asymp f \wedge a \times^{\mathsf{FCD}} b \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) &\Leftrightarrow \\ \exists F \in \operatorname{atoms} f : (F \not\asymp f \wedge F \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y}) &\Leftrightarrow \\ (\text{by theorem } 1573)& \\ f \not\asymp \mathcal{X} \times^{\mathsf{FCD}} \mathcal{Y} &\Leftrightarrow \\ \mathcal{X} \, [f] \, \mathcal{Y}.&\end{aligned}$$

2°. Let $\mathcal{Y} \in \mathfrak{B}$. Suppose $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$. Then $\mathcal{X} \, [f] \, \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{X} \, [F] \, \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{Y} \not\asymp \langle F \rangle \mathcal{X}$; and (taking into account that $\mathfrak{B}$ is strongly separable by theorem 225) $\mathcal{Y} \not\asymp \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. So $\langle f \rangle \mathcal{X} \sqsubseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ by strong separability. The contrary $\langle f \rangle \mathcal{X} \sqsupseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ is obvious.

□

19.11. Complete pointfree funcoids

DEFINITION 1579. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. A pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is *complete*, when for every $S \in \mathscr{P}\mathfrak{A}$ whenever both $\bigsqcup S$ and $\bigsqcup \langle \langle f \rangle \rangle^* S$ are defined we have

$$\langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle^* S.$$

DEFINITION 1580. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be filtrators. I will call a *co-complete pointfree funcoid* a pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ such that $\langle f \rangle X \in \mathfrak{Z}_1$ for every $X \in \mathfrak{Z}_0$.

PROPOSITION 1581. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Co-complete pointfree funcoids $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ bijectively correspond to functions $\mathfrak{Z}_1^{\mathfrak{Z}_0}$ preserving finite joins, where the bijection is $f \mapsto \langle f \rangle|_{\mathfrak{Z}_0}$.

PROOF. It follows from the theorem 1513. □

THEOREM 1582. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ be a down-aligned, with join-closed, binarily meet-closed and separable core which is a complete boolean lattice.

Let $(\mathfrak{B}, \mathfrak{Z}_1)$ be a star-separable filtrator.

The following conditions are equivalent for every pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$:

1°. f^{-1} is co-complete;

2°. $\forall S \in \mathscr{P}\mathfrak{A}, J \in \mathfrak{Z}_1 : (\bigsqcup^{\mathfrak{A}} S \mathrel{[f]} J \Rightarrow \exists \mathcal{I} \in S : \mathcal{I} \mathrel{[f]} J)$;

3°. $\forall S \in \mathscr{P}\mathfrak{Z}_0, J \in \mathfrak{Z}_1 : (\bigsqcup^{\mathfrak{Z}_0} S \mathrel{[f]} J \Rightarrow \exists I \in S : I \mathrel{[f]} J)$;

4°. f is complete;

5°. $\forall S \in \mathscr{P}\mathfrak{Z}_0 : \langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$.

PROOF. First note that the theorem 583 applies to the filtrator $(\mathfrak{A}, \mathfrak{Z}_0)$.

3°⇒1°. For every $S \in \mathscr{P}\mathfrak{Z}_0$, $J \in \mathfrak{Z}_1$

$$\bigsqcup^{\mathfrak{Z}_0} S \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle J \neq \bot^{\mathfrak{A}} \Rightarrow \exists I \in S : I \sqcap^{\mathfrak{A}} \langle f^{-1} \rangle J \neq \bot^{\mathfrak{A}}, \tag{29}$$

consequently by the theorem 583 we have $\langle f^{-1} \rangle J \in \mathfrak{Z}_0$.

1°⇒2°. For every $S \in \mathscr{P}\mathfrak{A}$, $J \in \mathfrak{Z}_1$ we have $\langle f^{-1} \rangle J \in \mathfrak{Z}_0$, consequently

$$\forall S \in \mathscr{P}\mathfrak{A}, J \in \mathfrak{Z}_1 : \left(\bigsqcup^{\mathfrak{A}} S \not\asymp \langle f^{-1} \rangle J \Rightarrow \exists \mathcal{I} \in S : \mathcal{I} \not\asymp \langle f^{-1} \rangle J \right).$$

From this follows 2°.

2°⇒4°. Let $\langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S$ and $\bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$ be defined. We have $\langle f \rangle \bigsqcup^{\mathfrak{A}} S = \langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S$.

$$\begin{aligned} J \sqcap^{\mathfrak{B}} \langle f \rangle \bigsqcup^{\mathfrak{A}} S \neq \bot^{\mathfrak{B}} &\Leftrightarrow \\ \bigsqcup^{\mathfrak{A}} S \mathrel{[f]} J &\Leftrightarrow \\ \exists \mathcal{I} \in S : \mathcal{I} \mathrel{[f]} J &\Leftrightarrow \\ \exists \mathcal{I} \in S : J \sqcap^{\mathfrak{B}} \langle f \rangle \mathcal{I} \neq \bot^{\mathfrak{B}} &\Leftrightarrow \\ J \sqcap^{\mathfrak{B}} \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S \neq \bot^{\mathfrak{B}} & \end{aligned}$$

(used theorem 583). Thus $\langle f \rangle \bigsqcup^{\mathfrak{A}} S = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$ by star-separability of $(\mathfrak{B}, \mathfrak{Z}_1)$.

5°⇒3°. Let $\langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S$ be defined. Then $\bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$ is also defined because $\langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S$. Then

$$\bigsqcup^{\mathfrak{Z}_0} S \mathrel{[f]} J \Leftrightarrow J \sqcap^{\mathfrak{B}} \langle f \rangle \bigsqcup^{\mathfrak{Z}_0} S \neq \bot^{\mathfrak{B}} \Leftrightarrow J \sqcap^{\mathfrak{B}} \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle^* S \neq \bot^{\mathfrak{B}}$$

what by theorem 583 is equivalent to $\exists I \in S : J \sqcap^{\mathfrak{B}} \langle f \rangle I \neq \bot^{\mathfrak{B}}$ that is $\exists I \in S : I \mathrel{[f]} J$.

2°⇒3°, 4°⇒5°. By join-closedness of the core of $(\mathfrak{A}, \mathfrak{Z}_0)$.

□

THEOREM 1583. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. If R is a set of co-complete pointfree funcoids in $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ then $\bigsqcup R$ is a co-complete pointfree funcoid.

PROOF. Let R be a set of co-complete pointfree funcoids. Then for every $X \in \mathfrak{Z}_0$

$$\left\langle \bigsqcup R \right\rangle X = \bigsqcup_{f \in R}^{\mathfrak{B}} \langle f \rangle X = \bigsqcup_{f \in R}^{\mathfrak{Z}_1} \langle f \rangle X \in \mathfrak{Z}_1$$

(used the theorems 1527 and 534). $\square$

Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets with least elements. I will denote $\mathsf{ComplpFCD}(\mathfrak{A}, \mathfrak{B})$ and $\mathsf{CoComplpFCD}(\mathfrak{A}, \mathfrak{B})$ the sets of complete and co-complete funcoids correspondingly from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

PROPOSITION 1584.

1°. Let $f \in \mathsf{ComplpFCD}(\mathfrak{A}, \mathfrak{B})$ and $g \in \mathsf{ComplpFCD}(\mathfrak{B}, \mathfrak{C})$ where $\mathfrak{A}$ and $\mathfrak{C}$ are posets with least elements and $\mathfrak{B}$ is a complete lattice. Then $g \circ f \in \mathsf{ComplpFCD}(\mathfrak{A}, \mathfrak{C})$.

2°. Let $f \in \mathsf{CoComplpFCD}(\mathfrak{A}, \mathfrak{B})$ and $g \in \mathsf{CoComplpFCD}(\mathfrak{B}, \mathfrak{C})$ where $(\mathfrak{A}, \mathfrak{Z}_0)$, $(\mathfrak{B}, \mathfrak{Z}_1)$, $(\mathfrak{C}, \mathfrak{Z}_2)$ are filtrators. Then $g \circ f \in \mathsf{CoComplpFCD}(\mathfrak{A}, \mathfrak{C})$.

PROOF.

1°. Let $\bigsqcup S$ and $\bigsqcup \langle \langle g \circ f \rangle \rangle^* S$ be defined. Then

$$\langle g \circ f \rangle \bigsqcup S = \langle g \rangle \langle f \rangle \bigsqcup S = \langle g \rangle \bigsqcup \langle \langle f \rangle \rangle^* S = \bigsqcup \langle \langle g \rangle \rangle^* \langle \langle f \rangle \rangle^* S = \bigsqcup \langle \langle g \circ f \rangle \rangle^* S.$$

2°. $\langle g \circ f \rangle \mathfrak{Z}_0 = \langle g \rangle \langle f \rangle \mathfrak{Z}_0 \in \mathfrak{Z}_2$ because $\langle f \rangle \mathfrak{Z}_0 \in \mathfrak{Z}_1$.

$\square$

PROPOSITION 1585. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. Then $\mathsf{CoComplpFCD}(\mathfrak{A}, \mathfrak{B})$ (with induced order) is a complete lattice.

PROOF. Follows from the theorem 1583. $\square$

THEOREM 1586. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators where $\mathfrak{Z}_0$ and $\mathfrak{Z}_1$ are boolean lattices. Let R be a set of pointfree funcoids from $\mathfrak{A}$ to $\mathfrak{B}$.

$g \circ (\bigsqcup R) = \bigsqcup_{g \in R} (g \circ f) = \bigsqcup \langle g \circ \rangle^* R$ if g is a complete pointfree funcoid from $\mathfrak{B}$.

PROOF. For every $X \in \mathfrak{A}$

$$\begin{aligned}
&\left\langle g \circ \left(\bigsqcup R \right) \right\rangle X = \\
&\langle g \rangle \left\langle \bigsqcup R \right\rangle X = \\
&\langle g \rangle \bigsqcup_{f \in R} \langle f \rangle X = \\
&\bigsqcup_{f \in R} \langle g \rangle \langle f \rangle X = \\
&\bigsqcup_{f \in R} \langle g \circ f \rangle X = \\
&\left\langle \bigsqcup_{f \in R} (g \circ f) \right\rangle X = \\
&\left\langle \bigsqcup \langle g \circ \rangle^* R \right\rangle X.
\end{aligned}$$

So $g \circ (\bigsqcup R) = \bigsqcup \langle g \circ \rangle^* R$. $\square$

19.12. Completion and co-completion

DEFINITION 1587. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices and $\mathfrak{Z}_1$ is a complete atomistic lattice.

Co-completion of a pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is pointfree funcoid $\operatorname{CoCompl} f$ defined by the formula (for every $X \in \mathfrak{Z}_0$)

$$\langle \operatorname{CoCompl} f \rangle X = \operatorname{Cor}\langle f \rangle X.$$

PROPOSITION 1588. Above defined co-completion always exists.

PROOF. Existence of $\operatorname{Cor}\langle f \rangle X$ follows from completeness of $\mathfrak{Z}_1$.

We may apply the theorem 1513 because

$$\operatorname{Cor}\langle f \rangle (X \sqcup^{\mathfrak{Z}_0} Y) = \operatorname{Cor}(\langle f \rangle X \sqcup^{\mathfrak{B}} \langle f \rangle Y) = \operatorname{Cor}\langle f \rangle X \sqcup^{\mathfrak{Z}_1} \operatorname{Cor}\langle f \rangle Y$$

by theorem 603. □

OBVIOUS 1589. Co-completion is always co-complete.

PROPOSITION 1590. For above defined always $\operatorname{CoCompl} f \sqsubseteq f$.

PROOF. By proposition 542. □

19.13. Monovalued and injective pointfree funcoids

DEFINITION 1591. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. Let $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

The pointfree funcoid f is:

- *monovalued* when $f \circ f^{-1} \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{B}}$.
- *injective* when $f^{-1} \circ f \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{A}}$.

Monovaluedness is dual of injectivity.

PROPOSITION 1592. Let $\mathfrak{A}$ and $\mathfrak{B}$ be posets. Let $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

The pointfree funcoid f is:

- monovalued iff $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f}$, if $\mathfrak{A}$ has greatest element and $\mathfrak{B}$ is a strongly separable meet-semilattice;
- injective iff $f^{-1} \circ f \sqsubseteq \mathrm{id}^{\mathsf{pFCD}(\mathfrak{A})}_{\operatorname{dom} f}$, if $\mathfrak{B}$ has greatest element and $\mathfrak{A}$ is a strongly separable meet-semilattice.

PROOF. It's enough to prove $f \circ f^{-1} \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{B}} \Leftrightarrow f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f}$. $\operatorname{im} f$ is defined because $\mathfrak{A}$ has greatest element. $\mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f}$ is defined because $\mathfrak{B}$ is a meet-semilattice.

$\Leftarrow$. Obvious.

$\Rightarrow$. Let $f \circ f^{-1} \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{B}}$. Then $\langle f \circ f^{-1} \rangle x \sqsubseteq x$; $\langle f \circ f^{-1} \rangle x \sqsubseteq \operatorname{im} f$ (proposition 1500). Thus $\langle f \circ f^{-1} \rangle x \sqsubseteq x \sqcap \operatorname{im} f = \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f} \right\rangle x$.

$\langle (f \circ f^{-1})^{-1} \rangle x \sqsubseteq x$ and $\langle (f \circ f^{-1})^{-1} \rangle x = \langle f \circ f^{-1} \rangle x \sqsubseteq \operatorname{im} f$. Thus $\langle (f \circ f^{-1})^{-1} \rangle x \sqsubseteq x \sqcap \operatorname{im} f = \left\langle \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f} \right\rangle x$.

Thus $f \circ f^{-1} \sqsubseteq \mathrm{id}^{\mathsf{pFCD}(\mathfrak{B})}_{\operatorname{im} f}$.

□

THEOREM 1593. Let $\mathfrak{A}$ be an atomistic meet-semilattice with least element, $\mathfrak{B}$ be an atomistic bounded meet-semilattice. The following statements are equivalent for every $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$:

1°. f is monovalued.
2°. $\forall a \in \operatorname{atoms}^{\mathfrak{A}} : \langle f \rangle a \in \operatorname{atoms}^{\mathfrak{B}} \cup \{\bot^{\mathfrak{B}}\}$.
3°. $\forall i, j \in \mathfrak{B} : \langle f^{-1} \rangle (i \sqcap j) = \langle f^{-1} \rangle i \sqcap \langle f^{-1} \rangle j$.

PROOF.

2°⇒3°. Let $a \in \operatorname{atoms}^{\mathfrak{A}}$, $\langle f \rangle a = b$. Then because $b \in \operatorname{atoms}^{\mathfrak{B}} \cup \{\bot^{\mathfrak{B}}\}$

$$(i \sqcap j) \sqcap b \neq \bot^{\mathfrak{B}} \Leftrightarrow i \sqcap b \neq \bot^{\mathfrak{B}} \wedge j \sqcap b \neq \bot^{\mathfrak{B}};$$
$$a\ [f]\ i \sqcap j \Leftrightarrow a\ [f]\ i \wedge a\ [f]\ j;$$
$$i \sqcap j\ \left[f^{-1}\right]\ a \Leftrightarrow i\ \left[f^{-1}\right]\ a \wedge j\ \left[f^{-1}\right]\ a;$$
$$a \sqcap^{\mathfrak{A}} \left\langle f^{-1} \right\rangle (i \sqcap j) \neq \bot^{\mathfrak{A}} \Leftrightarrow a \sqcap \left\langle f^{-1} \right\rangle i \neq \bot^{\mathfrak{A}} \wedge a \sqcap \left\langle f^{-1} \right\rangle j \neq \bot^{\mathfrak{A}};$$
$$a \sqcap^{\mathfrak{A}} \left\langle f^{-1} \right\rangle (i \sqcap j) \neq \bot^{\mathfrak{A}} \Leftrightarrow a \sqcap \left\langle f^{-1} \right\rangle i \sqcap \left\langle f^{-1} \right\rangle j \neq \bot^{\mathfrak{A}};$$
$$\left\langle f^{-1} \right\rangle (i \sqcap j) = \left\langle f^{-1} \right\rangle i \sqcap \left\langle f^{-1} \right\rangle j.$$

3°⇒1°. $\left\langle f^{-1} \right\rangle a \sqcap \left\langle f^{-1} \right\rangle b = \left\langle f^{-1} \right\rangle (a \sqcap b) = \left\langle f^{-1} \right\rangle \bot^{\mathfrak{B}} = \bot^{\mathfrak{A}}$ (by proposition 1499) for every two distinct $a, b \in \operatorname{atoms}^{\mathfrak{B}}$. This is equivalent to $\neg(\left\langle f^{-1} \right\rangle a\ [f]\ b)$; $b \sqcap \langle f \rangle \left\langle f^{-1} \right\rangle a = \bot^{\mathfrak{B}}$; $b \sqcap \left\langle f \circ f^{-1} \right\rangle a = \bot^{\mathfrak{B}}$; $\neg(a\ \left[f \circ f^{-1}\right]\ b)$. So $a\ \left[f \circ f^{-1}\right]\ b \Rightarrow a = b$ for every $a, b \in \operatorname{atoms}^{\mathfrak{B}}$. This is possible only (corollary 1547 and the fact that $\mathfrak{B}$ is atomic) when $f \circ f^{-1} \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{B}}$.

¬2°⇒ ¬1°. Suppose $\langle f \rangle a \notin \operatorname{atoms}^{\mathfrak{B}} \cup \{\bot^{\mathfrak{B}}\}$ for some $a \in \operatorname{atoms}^{\mathfrak{A}}$. Then there exist two atoms $p \neq q$ such that $\langle f \rangle a \sqsupseteq p \wedge \langle f \rangle a \sqsupseteq q$. Consequently $p \sqcap \langle f \rangle a \neq \bot^{\mathfrak{B}}$; $a \sqcap \left\langle f^{-1} \right\rangle p \neq \bot^{\mathfrak{A}}$; $a \sqsubseteq \left\langle f^{-1} \right\rangle p$; $\left\langle f \circ f^{-1} \right\rangle p = \langle f \rangle \left\langle f^{-1} \right\rangle p \sqsupseteq \langle f \rangle a \sqsupseteq q$ (by proposition 1500 because $\mathfrak{B}$ is separable by proposition 234 and thus strongly separable by theorem 225); $\left\langle f \circ f^{-1} \right\rangle p \not\sqsubseteq p$ and $\left\langle f \circ f^{-1} \right\rangle p \neq \bot^{\mathfrak{B}}$. So it cannot be $f \circ f^{-1} \sqsubseteq 1^{\mathsf{pFCD}}_{\mathfrak{B}}$.

□

THEOREM 1594. The following is equivalent for primary filtrators $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ over boolean lattices and pointfree funcoids $f : \mathfrak{A} \to \mathfrak{B}$:

1°. f is monovalued.
2°. f is metamonovalued.
3°. f is weakly metamonovalued.

PROOF.

2°⇒3°. Obvious.

1°⇒2°.

$$\left\langle \left(\bigsqcap G\right) \circ f \right\rangle x = \left\langle \left(\bigsqcap G\right) \right\rangle \langle f \rangle x = \bigsqcap_{g \in G} \langle g \rangle \langle f \rangle x = \bigsqcap_{g \in G} \langle g \circ f \rangle x = \left\langle \bigsqcap_{g \in G} (g \circ f) \right\rangle x$$

for every atomic filter object $x \in \operatorname{atoms}^{\mathfrak{A}}$. Thus $(\bigsqcap G) \circ f = \bigsqcap_{g \in G} (g \circ f)$.

3°⇒1°. Take $g = a \times^{\mathsf{FCD}} y$ and $h = b \times^{\mathsf{FCD}} y$ for arbitrary atomic filter objects $a \neq b$ and y. We have $g \sqcap h = \bot$; thus $(g \circ f) \sqcap (h \circ f) = (g \sqcap h) \circ f = \bot$ and thus impossible $x\ [f]\ a \wedge x\ [f]\ b$ as otherwise $x\ [g \circ f]\ y$ and $x\ [h \circ f]\ y$ so $x\ [(g \circ f) \sqcap (h \circ f)]\ y$. Thus f is monovalued.

□

THEOREM 1595. Let $(\mathfrak{A}, \mathfrak{Z}_0)$ and $(\mathfrak{B}, \mathfrak{Z}_1)$ be primary filtrators over boolean lattices. A pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ is monovalued iff

$$\forall I, J \in \mathfrak{Z}_1 : \left\langle f^{-1} \right\rangle (I \sqcap^{\mathfrak{Z}_1} J) = \left\langle f^{-1} \right\rangle I \sqcap \left\langle f^{-1} \right\rangle J.$$

PROOF. $\mathfrak{A}$ and $\mathfrak{B}$ are complete lattices (corollary 518).
$(\mathfrak{B}, \mathfrak{Z}_1)$ is a filtrator with separable core by theorem 537.
$(\mathfrak{B}, \mathfrak{Z}_1)$ is binarily meet-closed by corollary 536.
$\mathfrak{A}$ and $\mathfrak{B}$ are starrish by corollary 531.
$(\mathfrak{A}, \mathfrak{Z}_0)$ is with separable core by theorem 537.

We are under conditions of theorem 1512 for the pointfree funcoid f^{-1}.

$\Rightarrow$. Obvious (taking into account that $(\mathfrak{B}, \mathfrak{Z}_1)$ is binarily meet-closed).

$\Leftarrow$.

$$\begin{aligned}
\langle f^{-1}\rangle(\mathcal{I}\sqcap\mathcal{J}) &= \\
\bigsqcap\langle\langle f^{-1}\rangle\rangle^* \operatorname{up}^{\mathfrak{Z}_1}(\mathcal{I}\sqcap\mathcal{J}) &= \\
\bigsqcap\langle\langle f^{-1}\rangle\rangle^* \left\{\frac{I\sqcap^{\mathfrak{Z}_1} J}{I\in\operatorname{up}\mathcal{I}, J\in\operatorname{up}\mathcal{J}}\right\} &= \\
\bigsqcap\left\{\frac{\langle f^{-1}\rangle(I\sqcap^{\mathfrak{Z}_1} J)}{I\in\operatorname{up}\mathcal{I}, J\in\operatorname{up}\mathcal{J}}\right\} &= \\
\bigsqcap\left\{\frac{\langle f^{-1}\rangle I\sqcap\langle f^{-1}\rangle J}{I\in\operatorname{up}\mathcal{I}, J\in\operatorname{up}\mathcal{J}}\right\} &= \\
\bigsqcap\left\{\frac{\langle f^{-1}\rangle I}{I\in\operatorname{up}\mathcal{I}}\right\}\sqcap\bigsqcap\left\{\frac{\langle f^{-1}\rangle J}{J\in\operatorname{up}\mathcal{J}}\right\} &= \\
\langle f^{-1}\rangle\mathcal{I}\sqcap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{J}
\end{aligned}$$

(used theorem 1512, corollary 521, theorem 1501).

□

PROPOSITION 1596. Let $\mathfrak{A}$ be an atomistic meet-semilattice with least element, $\mathfrak{B}$ be an atomistic bounded meet-semilattice. Then if f, g are pointfree funcoids from $\mathfrak{A}$ to $\mathfrak{B}$, $f\sqsubseteq g$ and g is monovalued then $g|_{\operatorname{dom} f} = f$.

PROOF. Obviously $g|_{\operatorname{dom} f}\sqsupseteq f$. Suppose for contrary that $g|_{\operatorname{dom} f}\sqsubset f$. Then there exists an atom $a\in\operatorname{atoms}\operatorname{dom} f$ such that $\langle g|_{\operatorname{dom} f}\rangle a\neq\langle f\rangle a$ that is $\langle g\rangle a\sqsubset\langle f\rangle a$ what is impossible. □

19.14. Elements closed regarding a pointfree funcoid

Let $\mathfrak{A}$ be a poset. Let $f\in\mathsf{pFCD}(\mathfrak{A},\mathfrak{A})$.

DEFINITION 1597. Let's call *closed* regarding a pointfree funcoid f such element $a\in\mathfrak{A}$ that $\langle f\rangle a\sqsubseteq a$.

PROPOSITION 1598. If i and j are closed (regarding a pointfree funcoid $f\in\mathsf{pFCD}(\mathfrak{A},\mathfrak{A})$), S is a set of closed elements (regarding f), then

1°. $i\sqcup j$ is a closed element, if $\mathfrak{A}$ is a separable starrish join-semilattice;
2°. $\bigsqcap S$ is a closed element if $\mathfrak{A}$ is a strongly separable complete lattice.

PROOF. $\langle f\rangle(i\sqcup j)=\langle f\rangle i\sqcup\langle f\rangle j\sqsubseteq i\sqcup j$ (theorem 1501), $\langle f\rangle\bigsqcap S\sqsubseteq\bigsqcap\langle\langle f\rangle\rangle^* S\sqsubseteq\bigsqcap S$ (used strong separability of $\mathfrak{A}$ twice). Consequently the elements $i\sqcup j$ and $\bigsqcap S$ are closed. □

PROPOSITION 1599. If S is a set of elements closed regarding a complete pointfree funcoid f with strongly separable destination which is a complete lattice, then the element $\bigsqcup S$ is also closed regarding our funcoid.

PROOF. $\langle f\rangle\bigsqcup S=\bigsqcup\langle\langle f\rangle\rangle^* S\sqsubseteq\bigsqcup S$. □

19.15. Connectedness regarding a pointfree funcoid

Let $\mathfrak{A}$ be a poset with least element. Let $\mu\in\mathsf{pFCD}(\mathfrak{A},\mathfrak{A})$.

Definition 1600. An element $a \in \mathfrak{A}$ is called *connected* regarding a pointfree funcoid μ over $\mathfrak{A}$ when

$$\forall x, y \in \mathfrak{A} \setminus \{\bot^{\mathfrak{A}}\} : (x \sqcup y = a \Rightarrow x \; [\mu] \; y).$$

Proposition 1601. Let $(\mathfrak{A}, \mathfrak{Z})$ be a co-separable filtrator with finitely join-closed core. An $A \in \mathfrak{Z}$ is connected regarding a funcoid μ iff

$$\forall X, Y \in \mathfrak{Z} \setminus \{\bot^{\mathfrak{Z}}\} : (X \sqcup^{\mathfrak{Z}} Y = A \Rightarrow X \; [\mu] \; Y).$$

Proof.

$\Rightarrow$. Obvious.

$\Leftarrow$. Follows from co-separability.

□

Obvious 1602. For $\mathfrak{A}$ being a set of filters over a boolean lattice, an element $a \in \mathfrak{A}$ is connected regarding a pointfree funcoid μ iff it is connected regarding the funcoid $\mu \sqcap (a \times^{\mathsf{FCD}} a)$.

Exercise 1603. Consider above without requirement of existence of least element.

19.16. Boolean funcoids

I call *boolean funcoids* pointfree funcoids between boolean lattices.

Proposition 1604. Every pointfree funcoid, whose source is a complete and completely starrish and whose destination is complete and completely starrish and separable, is complete.

Proof. It's enough to prove $\langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle^* S$ for our pointfree funcoid f for every $S \in \mathscr{P} \operatorname{Src} f$.

Really, $Y \not\asymp \langle f \rangle \bigsqcup S \Leftrightarrow \bigsqcup S \not\asymp \langle f^{-1} \rangle Y \Leftrightarrow \exists X \in S : X \not\asymp \langle f^{-1} \rangle Y \Leftrightarrow \exists X \in S : Y \not\asymp \langle f \rangle X \Leftrightarrow Y \not\asymp \bigsqcup \langle \langle f \rangle \rangle^* S$ for every $Y \in \operatorname{Dst} f$ and thus we have $\langle f \rangle \bigsqcup S = \bigsqcup \langle \langle f \rangle \rangle^* S$ because $\operatorname{Dst} f$ is separable. □

Remark 1605. It seems that this theorem can be generalized for non-complete lattices.

Corollary 1606. Every boolean funcoid is complete and co-complete.

Proof. Using proposition 226 and corollary 89. □

Theorem 1607. Let $\mathfrak{A}$, $\mathfrak{B}$ be complete boolean lattices.

A function $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ is equal to the component $\langle f \rangle$ of a pointree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ iff α is preserving all joins (= lower adjoint).

Proof. Let $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ and preserves all joins. Then $\alpha \in \mathscr{F}(\mathfrak{B})^{\mathfrak{A}}$ (We equate principal filters of the set $\mathscr{F}\mathfrak{A}$ of filters on $\mathfrak{A}$ with elements of $\mathfrak{A}$). Thus (theorem 1513) $\alpha = \langle g \rangle^*$ for some $g \in \mathsf{pFCD}(\mathscr{F}\mathfrak{A}, \mathscr{F}\mathfrak{B})$.

$\langle g^{-1} \rangle \in \mathscr{F}(\mathfrak{A})^{\mathscr{F}(\mathfrak{B})}$.

Let $y \in \mathfrak{B}$. We need to prove $\langle g^{-1} \rangle y \in \mathfrak{A}$ that is $\bigsqcup S \not\asymp \langle g^{-1} \rangle y \Leftrightarrow \exists x \in S : \langle g^{-1} \rangle y \not\asymp x$ for every $S \in \mathscr{P}\mathfrak{A}$.

Really, $\bigsqcup S \not\asymp \langle g^{-1} \rangle y \Leftrightarrow y \not\asymp \langle g \rangle \bigsqcup S \Leftrightarrow y \not\asymp \bigsqcup \langle \langle g \rangle \rangle^* S \Leftrightarrow \exists x \in S : y \not\asymp \langle g \rangle x \Leftrightarrow \exists x \in S : \langle g^{-1} \rangle y \not\asymp x$.

Take $\beta = \langle g^{-1} \rangle^*$. We have $\beta \in \mathfrak{A}^{\mathfrak{B}}$.

$x \not\asymp \beta y \Leftrightarrow x \not\asymp \langle g^{-1} \rangle y \Leftrightarrow y \not\asymp \langle g \rangle x \Leftrightarrow y \not\asymp \alpha x$.

So $(\mathfrak{A}, \mathfrak{B}, \alpha, \beta)$ is a pointfree funcoid.

The other direction: Let now $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$. We need to prove that it preserves all joins. But it was proved above. □

CONJECTURE 1608. Let $\mathfrak{A}$, $\mathfrak{B}$ be boolean lattices.

A function $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ is equal to the component $\langle f \rangle$ of a pointfree funcoid $f \in \mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$ iff α is a lower adjoint.

It is tempting to conclude that $\langle f \rangle$ is a lower adjoint to $\langle f^{-1} \rangle$. But that's false: We should disprove that $\langle f \rangle X \sqsubseteq Y \Leftrightarrow X \sqsubseteq \langle f^{-1} \rangle Y$.

For a counter-example, take $f = \{0\} \times \mathbb{N}$. Then our condition takes form $Y = \mathbb{N} \Leftrightarrow X \sqsubseteq \{0\}$ for $X \ni 0$, $Y \ni 0$ what obviously does not hold.

19.17. Binary relations are pointfree funcoids

Below for simplicity we will equate $\mathscr{T} A$ with $\mathscr{P} A$.

THEOREM 1609. Pointfree funcoids f between powerset posets $\mathscr{T} A$ and $\mathscr{T} B$ bijectively (moreover this bijection is an order-isomorphism) correspond to morphisms $p \in \mathbf{Rel}(A, B)$ by the formulas:

$$\langle f \rangle = \langle p \rangle^*, \quad \langle f^{-1} \rangle = \langle p^{-1} \rangle^*; \tag{30}$$

$$(x, y) \in \operatorname{GR} p \Leftrightarrow y \in \langle f \rangle \{x\} \Leftrightarrow x \in \langle f^{-1} \rangle \{y\}. \tag{31}$$

PROOF. Suppose $p \in \mathbf{Rel}(A, B)$ and prove that there is a pointfree funcoid f conforming to (30). Really, for every $X \in \mathscr{T} A$, $Y \in \mathscr{T} B$

$$Y \not\asymp \langle f \rangle X \Leftrightarrow Y \not\asymp \langle p \rangle^* X \Leftrightarrow Y \not\asymp \langle p \rangle X \Leftrightarrow \\ X \not\asymp \langle p^{-1} \rangle Y \Leftrightarrow X \not\asymp \langle p^{-1} \rangle^* Y \Leftrightarrow X \not\asymp \langle f^{-1} \rangle Y.$$

Now suppose $f \in \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$ and prove that the relation defined by the formula (31) exists. To prove it, it's enough to show that $y \in \langle f \rangle \{x\} \Leftrightarrow x \in \langle f^{-1} \rangle \{y\}$. Really,

$$y \in \langle f \rangle \{x\} \Leftrightarrow \{y\} \not\asymp \langle f \rangle \{x\} \Leftrightarrow \{x\} \not\asymp \langle f^{-1} \rangle \{y\} \Leftrightarrow x \in \langle f^{-1} \rangle \{y\}.$$

It remains to prove that functions defined by (30) and (31) are mutually inverse. (That these functions are monotone is obvious.)

Let $p_0 \in \mathbf{Rel}(A, B)$ and $f \in \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$ corresponds to p_0 by the formula (30); let $p_1 \in \mathbf{Rel}(A, B)$ corresponds to f by the formula (31). Then $p_0 = p_1$ because

$$(x, y) \in \operatorname{GR} p_0 \Leftrightarrow y \in \langle p_0 \rangle^* \{x\} \Leftrightarrow y \in \langle f \rangle \{x\} \Leftrightarrow (x, y) \in \operatorname{GR} p_1.$$

Let now $f_0 \in \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$ and $p \in \mathbf{Rel}(A, B)$ corresponds to f_0 by the formula (31); let $f_1 \in \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$ corresponds to p by the formula (30). Then $(x, y) \in \operatorname{GR} p \Leftrightarrow y \in \langle f_0 \rangle \{x\}$ and $\langle f_1 \rangle = \langle p \rangle^*$; thus

$$y \in \langle f_1 \rangle \{x\} \Leftrightarrow y \in \langle p \rangle^* \{x\} \Leftrightarrow (x, y) \in \operatorname{GR} p \Leftrightarrow y \in \langle f_0 \rangle \{x\}.$$

So $\langle f_0 \rangle = \langle f_1 \rangle$. Similarly $\langle f_0^{-1} \rangle = \langle f_1^{-1} \rangle$. □

PROPOSITION 1610. The bijection defined by the theorem 1609 preserves composition and identities, that is is a functor between categories $\mathbf{Rel}$ and $(A, B) \mapsto \mathsf{pFCD}(\mathscr{T} A, \mathscr{T} B)$.

PROOF. Let $\langle f \rangle = \langle p \rangle^*$ and $\langle g \rangle = \langle q \rangle^*$. Then $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle = \langle q \rangle^* \circ \langle p \rangle^* = \langle q \circ p \rangle^*$. Likewise $\langle (g \circ f)^{-1} \rangle = \langle (q \circ p)^{-1} \rangle^*$. So it preserves composition.

Let $p = 1^A_{\mathbf{Rel}}$ for some set A. Then $\langle f \rangle = \langle p \rangle^* = \langle 1^A_{\mathbf{Rel}} \rangle^* = \mathrm{id}_{\mathscr{P} A}$ and likewise $\langle f^{-1} \rangle = \mathrm{id}_{\mathscr{P} A}$, that is f is an identity pointfree funcoid. So it preserves identities. □

PROPOSITION 1611. The bijection defined by theorem 1609 preserves reversal.

PROOF. $\langle f^{-1} \rangle = \langle p^{-1} \rangle^*$. □

Proposition 1612. The bijection defined by theorem 1609 preserves monovaluedness and injectivity.

Proof. Because it is a functor which preserves reversal. $\square$

Proposition 1613. The bijection defined by theorem 1609 preserves domain an image.

Proof. $\operatorname{im} f = \langle f \rangle \top = \langle p \rangle^* \top = \operatorname{im} p$, likewise for domain. $\square$

Proposition 1614. The bijection defined by theorem 1609 maps cartesian products to corresponding funcoidal products.

Proof. $\left\langle A \times^{\mathsf{FCD}} B \right\rangle X = \begin{cases} B & \text{if } X \not\asymp A \\ \bot & \text{if } X \asymp A \end{cases} = \langle A \times B \rangle^* X$. Likewise $\left\langle (A \times^{\mathsf{FCD}} B)^{-1} \right\rangle Y = \left\langle (A \times B)^{-1} \right\rangle^* Y$. $\square$

»»»> master

CHAPTER 20

Alternative representations of binary relations

Theorem 1615. Let A and B be fixed sets. The diagram at the figure 1 is a commutative diagram (in category **Set**), every arrow in this diagram is an isomorphism. Every cycle in this diagram is an identity. All "parallel" arrows are mutually inverse.

For a Galois connection f I denote f_0 the lower adjoint and f_1 the upper adjoint. For simplicity, in the diagram I equate $\mathscr{P}A$ and $\mathscr{T}A$.

Proof. First, note that despite we use the notation Ψ_i^{-1}, it is not yet proved that Ψ_i^{-1} is the inverse of Ψ_i. We will prove it below.

Now prove a list of claims. First concentrate on the upper "triangle" of the diagram (the lower one will be considered later).

Claim: $\left\{\frac{(x,y)}{y\in f_0\{x\}}\right\} = \left\{\frac{(x,y)}{x\in f_1\{y\}}\right\}$ when f is an antitone Galois connection between $\mathscr{P}A$ and $\mathscr{P}B$.

Proof: $y\in f_0\{x\} \Leftrightarrow \{y\}\sqsubseteq f_0\{x\} \Leftrightarrow \{x\}\sqsubseteq f_1\{y\} \Leftrightarrow x\in f_1\{y\}$. ∎

Claim: $\left(X\mapsto\bigsqcap_{x\in\mathscr{T}X\setminus\{\bot\}}\langle f\rangle x, Y\mapsto\bigsqcap_{y\in\mathscr{T}Y\setminus\{\bot\}}\langle f^{-1}\rangle y\right)=\left(X\mapsto\bigsqcap_{x\in X}\langle f\rangle\{x\}, Y\mapsto\bigsqcap_{y\in Y}\langle f^{-1}\rangle\{y\}\right)$ when f is a pointfree funcoid between $\mathscr{P}A$ and $\mathscr{P}B$.

Proof: It is enough to prove $\bigsqcap_{x\in\mathscr{T}X\setminus\{\bot\}}\langle f\rangle x = \bigsqcap_{x\in X}\langle f\rangle\{x\}$ (the rest follows from symmetry). $\bigsqcap_{x\in\mathscr{T}X\setminus\{\bot\}}\langle f\rangle x \sqsubseteq \bigsqcap_{x\in X}\langle f\rangle\{x\}$ because $\mathscr{T}X\setminus\{\bot\}\supseteq\left\{\frac{\{x\}}{x\in X}\right\}$. $\bigsqcap_{x\in\mathscr{T}X\setminus\{\bot\}}\langle f\rangle x \sqsupseteq \bigsqcap_{x\in X}\langle f\rangle\{x\}$ because if $x\in\mathscr{T}X\setminus\{\bot\}$ then we can take $x'\in x$ that is $\{x'\}\subseteq x$ and thus $\langle f\rangle x\sqsupseteq\langle f\rangle\{x'\}$, so $\bigsqcap_{x\in\mathscr{T}X\setminus\{\bot\}}\langle f\rangle x \sqsupseteq \bigsqcap_{x\in\mathscr{T}X\setminus\{\bot\}}\langle f\rangle\{x'\}\sqsupseteq\bigsqcap_{x\in X}\langle f\rangle\{x\}$. ∎

Claim: $\left(\mathscr{P}A,\mathscr{P}B, X\mapsto\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} f_0x, Y\mapsto\bigsqcup_{y\in\mathscr{T}Y\setminus\{\bot\}} f_1y\right) = \left(\mathscr{P}A,\mathscr{P}B, X\mapsto\bigsqcup_{x\in X} f_0\{x\}, Y\mapsto\bigsqcup_{y\in Y} f_1\{y\}\right)$ when f is an antitone Galois connection between $\mathscr{P}A$ and $\mathscr{P}B$.

Proof: It is enough to prove $\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} f_0x = \bigsqcup_{x\in X} f_0\{x\}$ (the rest follows from symmetry). We have $\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} f_0x \sqsupseteq \bigsqcup_{x\in X} f_0\{x\}$ because $\{x\}\in\mathscr{T}X\setminus\{\bot\}$. Let $x\in\mathscr{T}X\setminus\{\bot\}$. Take $x'\in X$. We have $f_0x\sqsubseteq f_0\{x'\}$ and thus $f_0x\sqsubseteq\bigsqcup_{x\in X} f_0\{x\}$. So $\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} f_0x\sqsubseteq\bigsqcup_{x\in X} f_0\{x\}$. ∎

Claim: $\Psi_3^{-1} = \Psi_2\circ\Psi_1$.

Proof: $\Psi_2\Psi_1 f = \left(\mathscr{P}A,\mathscr{P}B, X\mapsto\left\{\frac{y}{\exists x\in X:(x,y)\in\Psi_1 f}\right\}, Y\mapsto\left\{\frac{x}{\exists y\in Y:(x,y)\in\Psi_1 f}\right\}\right) = \left(\mathscr{P}A,\mathscr{P}B, X\mapsto\left\{\frac{y}{\exists x\in X: y\in f_0\{x\}}\right\}, Y\mapsto\left\{\frac{x}{\exists y\in Y: x\in f_1\{y\}}\right\}\right) = \left(\mathscr{P}A,\mathscr{P}B, X\mapsto\bigsqcup_{x\in X} f_0\{x\}, Y\mapsto\bigsqcup_{y\in Y} f_1\{y\}\right) = \Psi_3^{-1}f$. ∎

Claim: $\Psi_3 = \Psi_1^{-1}\circ\Psi_2^{-1}$.

Proof: $\Psi_1^{-1}\Psi_2^{-1}f = \left(X\mapsto\left\{\frac{y\in B}{\forall x\in X:\{x\}[f]\{y\}}\right\}, Y\mapsto\left\{\frac{x\in A}{\forall y\in Y:\{x\}[f]\{y\}}\right\}\right) = \left(X\mapsto\left\{\frac{y\in B}{\forall x\in X: y\in\langle f\rangle\{x\}}\right\}, Y\mapsto\left\{\frac{x\in A}{\forall y\in Y: x\in\langle f^{-1}\rangle\{y\}}\right\}\right) = \left(X\mapsto\bigsqcap_{x\in X}\langle f\rangle\{x\}, Y\mapsto\bigsqcap_{y\in Y}\langle f^{-1}\rangle\{y\}\right) = \Psi_3 f$. ∎

$$\begin{tikzcd}
 & \text{binary relations between } A \text{ and } B \arrow[dl, shift left, "\Psi_2"] \arrow[dr, shift left, "\Psi_1^{-1}"] & \\
\text{pointfree funcoids between } \mathscr{P}A \text{ and } \mathscr{P}B \arrow[ur, shift left, "\Psi_2^{-1}"] \arrow[rr, shift left, "\Psi_3"] \arrow[dr, shift left, "\Psi_4"] & & \text{antitone Galois connections between } \mathscr{P}A \text{ and } \mathscr{P}B \arrow[ul, shift left, "\Psi_1"] \arrow[ll, shift left, "\Psi_3^{-1}"] \arrow[dl, shift left, "\Psi_5=\Psi_5^{-1}"] \\
 & \text{Galois connections between } \mathscr{P}A \text{ and } \mathscr{P}B \arrow[ul, shift left, "\Psi_4^{-1}"] &
\end{tikzcd}$$

Ψ_1. $f \mapsto \left\{ \frac{(x,y)}{y \in f_0\{x\}} \right\} = \left\{ \frac{(x,y)}{x \in f_1\{y\}} \right\}$
Ψ_1^{-1}. $r \mapsto \left(X \mapsto \left\{ \frac{y \in B}{\forall x \in X : x r y} \right\}, Y \mapsto \left\{ \frac{x \in A}{\forall y \in Y : x r y} \right\} \right)$
Ψ_2. $r \mapsto (\mathscr{P}A, \mathscr{P}B, \langle r \rangle^*, \langle r^{-1} \rangle^*)$
Ψ_2^{-1}. $f \mapsto \left\{ \frac{(x,y)}{\{x\}[f]\{y\}} \right\}$
Ψ_3. $f \mapsto \left(X \mapsto \bigsqcap_{x \in \mathscr{T}X \setminus \{\bot\}} \langle f \rangle x, Y \mapsto \bigsqcap_{y \in \mathscr{T}Y \setminus \{\bot\}} \langle f^{-1} \rangle y \right) = \left(X \mapsto \bigsqcap_{x \in X} \langle f \rangle \{x\}, Y \mapsto \bigsqcap_{y \in Y} \langle f^{-1} \rangle \{y\} \right)$
Ψ_3^{-1}. $f \mapsto \left(\mathscr{P}A, \mathscr{P}B, X \mapsto \bigsqcup_{x \in \mathscr{T}X \setminus \{\bot\}} f_0 x, Y \mapsto \bigsqcup_{y \in \mathscr{T}Y \setminus \{\bot\}} f_1 y \right) = \left(\mathscr{P}A, \mathscr{P}B, X \mapsto \bigsqcup_{x \in X} f_0\{x\}, Y \mapsto \bigsqcup_{y \in Y} f_1\{y\} \right)$
Ψ_4. $f \mapsto \left(X \mapsto \neg \bigsqcap_{x \in \mathscr{T}X \setminus \{\bot\}} \langle f \rangle x, Y \mapsto \bigsqcap_{y \in \mathscr{T}Y \setminus \{\bot\}} \langle f^{-1} \rangle \neg y \right) = \left(X \mapsto \bigsqcup_{x \in \mathscr{T}X \setminus \{\bot\}} \neg \langle f \rangle x, Y \mapsto \bigsqcap_{y \in \mathscr{T}Y \setminus \{\bot\}} \langle f^{-1} \rangle \neg y \right) = \left(X \mapsto \neg \bigsqcap_{x \in X} \langle f \rangle \{x\}, Y \mapsto \bigsqcap_{y \in Y} \langle f^{-1} \rangle \neg \{y\} \right) = \left(X \mapsto \bigsqcup_{x \in X} \neg \langle f \rangle \{x\}, Y \mapsto \bigsqcap_{y \in Y} \langle f^{-1} \rangle \neg \{y\} \right)$
Ψ_4^{-1}. $f \mapsto \left(\mathscr{P}A, \mathscr{P}B, X \mapsto \bigsqcup_{x \in \mathscr{T}X \setminus \{\bot\}} \neg f_0 x, Y \mapsto \bigsqcup_{y \in \mathscr{T}Y \setminus \{\bot\}} f_1 \neg y \right) = \left(\mathscr{P}A, \mathscr{P}B, X \mapsto \neg \bigsqcap_{x \in \mathscr{T}X \setminus \{\bot\}} f_0 x, Y \mapsto \bigsqcup_{y \in \mathscr{T}Y \setminus \{\bot\}} f_1 \neg y \right) = \left(\mathscr{P}A, \mathscr{P}B, X \mapsto \bigsqcup_{x \in X} \neg f_0\{x\}, Y \mapsto \bigsqcup_{y \in Y} f_1 \neg \{y\} \right) = \left(\mathscr{P}A, \mathscr{P}B, X \mapsto \neg \bigsqcap_{x \in X} f_0\{x\}, Y \mapsto \bigsqcup_{y \in Y} f_1 \neg \{y\} \right)$
$\Psi_5 = \Psi_5^{-1}$. $f \mapsto (\neg \circ f_0, f_1 \circ \neg)$

FIGURE 11

Claim: Ψ_1 maps antitone Galois connections between $\mathscr{P}A$ and $\mathscr{P}B$ into binary relations between A and B.
Proof: Obvious. ■
Claim: Ψ_1^{-1} maps binary relations between A and B into antitone Galois connections between $\mathscr{P}A$ and $\mathscr{P}B$.
Proof: We need to prove $Y \subseteq \left\{ \frac{y \in B}{\forall x \in X : x r y} \right\} \Leftrightarrow X \subseteq \left\{ \frac{x \in A}{\forall y \in Y : x r y} \right\}$. After we equivalently rewrite it:

$$\forall y \in Y \forall x \in X : x \; r \; y \Leftrightarrow \forall x \in X \forall y \in Y : x \; r \; y$$

it becomes obvious. ■

Claim: Ψ_2 maps binary relations between A and B into pointfree funcoids between $\mathscr{P}A$ and $\mathscr{P}B$.

Proof: We need to prove that $f = (\mathscr{P}A, \mathscr{P}B, \langle f\rangle, \langle f^{-1}\rangle)$ is a pointfree funcoids that is $Y \not\asymp \langle f\rangle X \Leftrightarrow X \not\asymp \langle f^{-1}\rangle Y$. Really, for every $X \in \mathscr{T}A$, $Y \in \mathscr{T}B$

$$Y \not\asymp \langle f\rangle X \Leftrightarrow Y \not\asymp \langle r\rangle^* X \Leftrightarrow Y \not\asymp \langle r\rangle X \Leftrightarrow X \not\asymp \langle r^{-1}\rangle Y \Leftrightarrow X \not\asymp \langle r^{-1}\rangle^* Y \Leftrightarrow X \not\asymp \langle f^{-1}\rangle Y.$$

■

Claim: Ψ_2^{-1} maps pointfree funcoids between $\mathscr{P}A$ and $\mathscr{P}B$ into binary relations between A and B.

Proof: Suppose $f \in \mathsf{pFCD}(\mathscr{T}A, \mathscr{T}B)$ and prove that the relation defined by the formula Ψ_2^{-1} exists. To prove it, it's enough to show that $y \in \langle f\rangle\{x\} \Leftrightarrow x \in \langle f^{-1}\rangle\{y\}$. Really,

$$y \in \langle f\rangle\{x\} \Leftrightarrow \{y\} \not\asymp \langle f\rangle\{x\} \Leftrightarrow \{x\} \not\asymp \langle f^{-1}\rangle\{y\} \Leftrightarrow x \in \langle f^{-1}\rangle\{y\}.$$

■

Claim: Ψ_3 maps pointfree funcoids between $\mathscr{P}A$ and $\mathscr{P}B$ into antitone Galois connections between $\mathscr{P}A$ and $\mathscr{P}B$.

Proof: Because $\Psi_3 = \Psi_1^{-1} \circ \Psi_2^{-1}$. ■

Claim: Ψ_3^{-1} maps antitone Galois connections between $\mathscr{P}A$ and $\mathscr{P}B$ into pointfree funcoids between $\mathscr{P}A$ and $\mathscr{P}B$.

Proof: Because $\Psi_3^{-1} = \Psi_2 \circ \Psi_1$. ■

Claim: Ψ_2 and Ψ_2^{-1} are mutually inverse.

Proof: Let $r_0 \in \mathscr{P}(A \times B)$ and $f \in \mathsf{pFCD}(\mathscr{T}A, \mathscr{T}B)$ corresponds to r_0 by the formula Ψ_2; let $r_1 \in \mathscr{P}(A \times B)$ corresponds to f by the formula Ψ_2^{-1}. Then $r_0 = r_1$ because

$$(x, y) \in r_0 \Leftrightarrow y \in \langle r_0\rangle^*\{x\} \Leftrightarrow y \in \langle f\rangle\{x\} \Leftrightarrow (x, y) \in r_1.$$

Let now $f_0 \in \mathsf{pFCD}(\mathscr{T}A, \mathscr{T}B)$ and $r \in \mathscr{P}(A \times B)$ corresponds to f_0 by the formula Ψ_2^{-1}; let $f_1 \in \mathsf{pFCD}(\mathscr{T}A, \mathscr{T}B)$ corresponds to r by the formula Ψ_2. Then $(x, y) \in r \Leftrightarrow y \in \langle f_0\rangle\{x\}$ and $\langle f_1\rangle = \langle r\rangle^*$; thus

$$y \in \langle f_1\rangle\{x\} \Leftrightarrow y \in \langle r\rangle^*\{x\} \Leftrightarrow (x, y) \in r \Leftrightarrow y \in \langle f_0\rangle\{x\}.$$

So $\langle f_0\rangle = \langle f_1\rangle$. Similarly $\langle f_0^{-1}\rangle = \langle f_1^{-1}\rangle$. ■

Claim: Ψ_1 and Ψ_1^{-1} are mutually inverse.

Proof: Let $r_0 \in \mathscr{P}(A\times B)$ and $f \in \mathscr{T}A \otimes \mathscr{T}B$ corresponds to r_0 by the formula Ψ_1^{-1}; let $r_1 \in \mathscr{P}(A \times B)$ corresponds to f by the formula Ψ_1. Then $r_0 = r_1$ because

$$(x, y) \in r_1 \Leftrightarrow y \in f_0\{x\} \Leftrightarrow y \in \left\{\frac{y \in B}{x \mathrel{r_0} y}\right\} \Leftrightarrow x \mathrel{r_0} y.$$

Let now $f_0 \in \mathscr{T}A \otimes \mathscr{T}B$ and $r \in \mathscr{P}(A\times B)$ corresponds to f_0 by the formula Ψ_1; let $f_1 \in \mathscr{T}A \otimes \mathscr{T}B$ corresponds to r by the formula Ψ_1^{-1}. Then $f_0 = f_1$ because

$$f_{10}X = \left\{\frac{y \in B}{\forall x \in X : x \mathrel{r} y}\right\} = \left\{\frac{y \in B}{\forall x \in X : y \in f_{00}\{x\}}\right\} = \prod_{x\in X} f_{00}\{x\} = (\text{obvious } 142) = f_{00}X.$$

■

Claim: Ψ_3 and Ψ_3^{-1} are mutually inverse.

Proof: Because $\Psi_3^{-1} = \Psi_2 \circ \Psi_1$ and $\Psi_3 = \Psi_1^{-1} \circ \Psi_2^{-1}$ and that Ψ_2^{-1} is the inverse of Ψ_2 and Ψ_3^{-1} is the inverse of Ψ_3 were proved above. ■

Now switch to the lower "triangle":

Claim: $\left(X \mapsto \bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} \neg f_0 x, Y \mapsto \bigsqcup_{y\in\mathscr{T}Y\setminus\{\bot\}} f_1 \neg y\right) = \left(X \mapsto \bigsqcup_{x\in X} \neg f_0\{x\}, Y \mapsto \bigsqcup_{y\in Y} f_1 \neg\{y\}\right)$.

Proof: It is enough to prove $\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} \neg f_0 x = \bigsqcup_{x\in X} \neg f_0\{x\}$ for a Galois connection f (the rest follows from symmetry).

$\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} \neg f_0 x \sqsupseteq \bigsqcup_{x\in X} \neg f_0\{x\}$ because $\{x\} \in \mathscr{T}X\setminus\{\bot\}$. If $x \in \mathscr{T}X\setminus\{\bot\}$ then there exists $x' \in \{x\}$ and thus $\neg f_0\{x'\} \sqsupseteq \neg f_0 x$. Thus $\neg f_0 x \sqsubseteq \bigsqcup_{x\in X} \neg f_0\{x\}$ and so $\bigsqcup_{x\in\mathscr{T}X\setminus\{\bot\}} \neg f_0 x \sqsubseteq \bigsqcup_{x\in X} \neg f_0\{x\}$. ■

Claim: Ψ_5 is self-inverse.

Proof: Obvious. ■

Claim: $\Psi_4 = \Psi_5 \circ \Psi_3$.

Proof: Easily follows from symmetry. ■

Claim: $\Psi_4^{-1} = \Psi_3^{-1} \circ \Psi_5^{-1}$.

Proof: Easily follows from symmetry. ■

Claim: Ψ_4 and Ψ_4^{-1} are mutually inverse.

Proof: From two above claims and the fact that Ψ_3^{-1} is the inverse of Ψ_3 and Ψ_5^{-1} is the inverse of Ψ_5 proved above. ■

Note that now we have proved that Ψ_i and Ψ_i^{-1} are mutually inverse for all $i = 1, 2, 3, 4, 5$.

Claim: For every path of the diagram on figure 2 started with the circled node, the corresponding morphism is with which the node is labeled.

$$\begin{tikzcd}
 & 1 \arrow[dl, shift left, "\Psi_2"] \arrow[dr, shift left, "\Psi_1"] & \\
\Psi_2 \arrow[ur, shift left, "\Psi_2^{-1}"] \arrow[rr, shift left, "\Psi_3"] \arrow[dr, shift left, "\Psi_4"] & & \Psi_1^{-1} \arrow[ul, shift left, "\Psi_1^{-1}"] \arrow[ll, shift left, "\Psi_3^{-1}"] \arrow[dl, leftrightarrow, "\Psi_5=\Psi_5^{-1}"] \\
 & \Psi_5 \circ \Psi_1^{-1} \arrow[ul, shift left, "\Psi_4^{-1}"] &
\end{tikzcd}$$

FIGURE 12

Proof: Take into account that $\Psi_3^{-1} = \Psi_2 \circ \Psi_1$, $\Psi_4 = \Psi_5 \circ \Psi_3$ and thus also $\Psi_4 \circ \Psi_2 = \Psi_5 \circ \Psi_1^{-1}$. Now prove it by induction on path length. ■

Claim: Every cycle in the diagram at figure 1 is identity.

Proof: For cycles starting at the top node it follows from the previous claim. For arbitrary cycles it follows from theorem 195. ■

Claim: The diagram at figure 1 is commutative.

Proof: From the previous claim. ■

□

PROPOSITION 1616. We equate the set of binary relations between A and B with $\mathbf{Rld}(A, B)$. Ψ_2 and Ψ_2^{-1} from the diagram at figure 1 preserve composition and identities (that are functors between categories $\mathbf{Rel}$ and $(A, B) \mapsto \mathsf{pFCD}(\mathscr{T}A, \mathscr{T}B)$) and also reversal ($f \mapsto f^{-1}$).

PROOF. Let $\langle f\rangle = \langle p\rangle^*$ and $\langle g\rangle = \langle q\rangle^*$. Then $\langle g \circ f\rangle = \langle g\rangle \circ \langle f\rangle = \langle q\rangle^* \circ \langle p\rangle^* = \langle q \circ p\rangle^*$. Likewise $\left\langle (g \circ f)^{-1}\right\rangle = \left\langle (q \circ p)^{-1}\right\rangle^*$. So Φ_2 preserves composition.

Let $p = 1^A_{\mathbf{Rel}}$ for some set A. Then $\langle f\rangle = \langle p\rangle^* = \left\langle 1^A_{\mathbf{Rel}}\right\rangle^* = \mathrm{id}_{\mathscr{P}A}$ and likewise $\left\langle f^{-1}\right\rangle = \mathrm{id}_{\mathscr{P}A}$, that is f is an identity pointfree funcoid. So Φ_2 preserves identities.

That Φ_2^{-1} preserves composition and identities follows from the fact that it is an isomorphism.

That is preserves reversal follows from the formula $\langle f^{-1}\rangle = \langle p^{-1}\rangle^*$. ☐

PROPOSITION 1617. The bijections Ψ_2 and Ψ_2^{-1} from the diagram at figure 1 preserves monovaluedness and injectivity.

PROOF. Because it is a functor which preserves reversal. ☐

PROPOSITION 1618. The bijections Ψ_2 and Ψ_2^{-1} from the diagram at figure 1 preserves domain an image.

PROOF. $\operatorname{im} f = \langle f\rangle\top = \langle p\rangle^*\top = \operatorname{im} p$, likewise for domain. ☐

PROPOSITION 1619. The bijections Ψ_2 and Ψ_2^{-1} from the diagram at figure 1 maps cartesian products to corresponding funcoidal products.

PROOF. $\langle A \times^{\mathsf{FCD}} B\rangle X = \begin{cases} B & \text{if } X \not\asymp A \\ \bot & \text{if } X \asymp A \end{cases} = \langle A \times B\rangle^* X$. Likewise $\langle (A \times^{\mathsf{FCD}} B)^{-1}\rangle Y = \langle (A \times B)^{-1}\rangle^* Y$. ☐

Let Φ map a pointfree funcoid whose first component is c into the Galois connection whose lower adjoint is c. Then Φ is an isomorphism (theorem 1607) and Φ^{-1} maps a Galois connection whose lower adjoint is c into the pointfree funcoid whose first component is c.

Informally speaking, Φ replaces a relation r with its complement relations $\neg r$. Formally:

PROPOSITION 1620.

1°. For every path P in the diagram at figure 1 from binary relations between A and B to pointfree funcoids between $\mathscr{P}A$ and $\mathscr{P}B$ and every path Q in the diagram at figure 1 from Galois connections between $\mathscr{P}A$ and $\mathscr{P}B$ to binary relations between A and B, we have $Q\Phi Pr = \neg r$.

2°. For every path Q in the diagram at figure 1 from binary relations between A and B to pointfree funcoids between $\mathscr{P}A$ and $\mathscr{P}B$ and every path P in the diagram at figure 1 from Galois connections between $\mathscr{P}A$ and $\mathscr{P}B$ to binary relations between A and B, we have $P\Phi^{-1}Qr = \neg r$.

PROOF. We will prove only the second ($P \circ \Phi^{-1} \circ Q = \neg$), because the first ($Q \circ \Phi \circ P = \neg$) can be obtained from it by inverting the morphisms (and variable replacement).

Because the diagram is commutative, it is enough to prove it for some fixed P and Q. For example, we will prove $\Psi_2^{-1}\Phi^{-1}\Psi_4\Psi_2 r = \neg r$.

$\Psi_4\Psi_2 r = \left(X \mapsto \neg\bigsqcap_{x\in X}\langle r\rangle^*\{x\}, Y \mapsto \bigsqcap_{y\in Y}\langle r\rangle^*\neg\{y\}\right)$.

$\Phi^{-1}\Psi_4\Psi_2 r$ is pointfree funcoid f with $\langle f\rangle = X \mapsto \neg\bigsqcap_{x\in X}\langle r\rangle^*\{x\}$.

$\Psi_2^{-1}\Phi^{-1}\Psi_4\Psi_2 r$ is the relation consisting of (x, y) such that $\{x\}\ [f]\ \{y\}$ what is equivalent to: $\{y\} \not\asymp \langle f\rangle\{x\}$; $\{y\} \not\asymp \neg\langle r\rangle^*\{x\}$; $\{y\} \not\sqsubseteq \langle r\rangle^*\{x\}$; $y \notin \langle r\rangle^*\{x\}$.

So $\Psi_2^{-1}\Phi^{-1}\Psi_4\Psi_2 r = \neg r$. ☐

PROPOSITION 1621. Φ and Φ^{-1} preserve composition.

PROOF. By definitions of compositions and the fact that both pointfree funcoids and Galois connections are determined by the first component. ☐

Part 4

Staroids and multifuncoids

CHAPTER 21

Multifuncoids and staroids

21.1. Product of two funcoids

21.1.1. Definition.

DEFINITION 1622. I will call a *quasi-invertible category* a partially ordered dagger category such that it holds

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^\dagger \tag{32}$$

for every morphisms $f \in \operatorname{Hom}(A, B)$, $g \in \operatorname{Hom}(B, C)$, $h \in \operatorname{Hom}(A, C)$, where A, B, C are objects of this category.

Inverting this formula, we get $f^\dagger \circ g^\dagger \not\asymp h^\dagger \Leftrightarrow g^\dagger \not\asymp f \circ h^\dagger$. After replacement of variables, this gives: $f^\dagger \circ g \not\asymp h \Leftrightarrow g \not\asymp f \circ h$.

EXERCISE 1623. Prove that every ordered groupoid is quasi-invertible category if we define the dagger as the inverse morphism.

As it follows from above, the categories **Rel** of binary relations (proposition 283), FCD of funcoids (theorem 882) and RLD of reloids (theorem 1009) are quasi-invertible (taking $f^\dagger = f^{-1}$). Moreover the category of pointfree funcoids between lattices of filters on boolean lattices is quasi-invertible (theorem 1554).

DEFINITION 1624. The *cross-composition product* of morphisms f and g of a quasi-invertible category is the pointfree funcoid $\operatorname{Hom}(\operatorname{Src} f, \operatorname{Src} g) \to \operatorname{Hom}(\operatorname{Dst} f, \operatorname{Dst} g)$ defined by the formulas (for every $a \in \operatorname{Hom}(\operatorname{Src} f, \operatorname{Src} g)$ and $b \in \operatorname{Hom}(\operatorname{Dst} f, \operatorname{Dst} g)$):

$$\left\langle f \times^{(C)} g \right\rangle a = g \circ a \circ f^\dagger \quad \text{and} \quad \left\langle (f \times^{(C)} g)^{-1} \right\rangle b = g^\dagger \circ b \circ f.$$

We need to prove that it is really a pointfree funcoid that is that

$$b \not\asymp \left\langle f \times^{(C)} g \right\rangle a \Leftrightarrow a \not\asymp \left\langle (f \times^{(C)} g)^{-1} \right\rangle b.$$

This formula means $b \not\asymp g \circ a \circ f^\dagger \Leftrightarrow a \not\asymp g^\dagger \circ b \circ f$ and can be easily proved applying formula (32) twice.

PROPOSITION 1625. $a \left[f \times^{(C)} g \right] b \Leftrightarrow a \circ f^\dagger \not\asymp g^\dagger \circ b$.

PROOF. From the definition. □

PROPOSITION 1626. $a \left[f \times^{(C)} g \right] b \Leftrightarrow f \left[a \times^{(C)} b \right] g$.

PROOF. $f \left[a \times^{(C)} b \right] g \Leftrightarrow f \circ a^\dagger \not\asymp b^\dagger \circ g \Leftrightarrow a \circ f^\dagger \not\asymp g^\dagger \circ b \Leftrightarrow a \left[f \times^{(C)} g \right] b$. □

THEOREM 1627. $(f \times^{(C)} g)^{-1} = f^\dagger \times^{(C)} g^\dagger$.

PROOF. For every morphisms $a \in \operatorname{Hom}(\operatorname{Src} f, \operatorname{Src} g)$ and $b \in \operatorname{Hom}(\operatorname{Dst} f, \operatorname{Dst} g)$ we have:

$\langle (f \times^{(C)} g)^{-1} \rangle b = g^\dagger \circ b \circ f = \langle f^\dagger \times^{(C)} g^\dagger \rangle b.$

$\langle ((f \times^{(C)} g)^{-1})^{-1} \rangle a = \langle f \times^{(C)} g \rangle a = g \circ a \circ f^\dagger = \langle (f^\dagger \times^{(C)} g^\dagger)^{-1} \rangle a.$ □

THEOREM 1628. Let f, g be pointfree funcoids between filters on boolean lattices. Then for every filters $\mathcal{A}_0 \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{B}_0 \in \mathscr{F}(\operatorname{Src} g)$

$$\left\langle f \times^{(C)} g\right\rangle(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) = \langle f\rangle\mathcal{A}_0 \times^{\mathsf{FCD}} \langle g\rangle\mathcal{B}_0.$$

PROOF. For every atom $a_1 \times^{\mathsf{FCD}} b_1$ ($a_1 \in \operatorname{atoms}^{\operatorname{Dst} f}$, $b_1 \in \operatorname{atoms}^{\operatorname{Dst} g}$) (see theorem 1572) of the lattice of funcoids we have:

$$\begin{aligned}
a_1 \times^{\mathsf{FCD}} b_1 \not\asymp \left\langle f \times^{(C)} g\right\rangle(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) &\Leftrightarrow \\
\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g\right] a_1 \times^{\mathsf{FCD}} b_1 &\Leftrightarrow \\
(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \circ f^{-1} \not\asymp g^{-1} \circ (a_1 \times^{\mathsf{FCD}} b_1) &\Leftrightarrow \\
\langle f\rangle\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \not\asymp a_1 \times^{\mathsf{FCD}} \langle g^{-1}\rangle b_1 &\Leftrightarrow \\
\langle f\rangle\mathcal{A}_0 \not\asymp a_1 \wedge \langle g^{-1}\rangle b_1 \not\asymp \mathcal{B}_0 &\Leftrightarrow \\
\langle f\rangle\mathcal{A}_0 \not\asymp a_1 \wedge \langle g\rangle\mathcal{B}_0 \not\asymp b_1 &\Leftrightarrow \\
\langle f\rangle\mathcal{A}_0 \times^{\mathsf{FCD}} \langle g\rangle\mathcal{B}_0 \not\asymp a_1 \times^{\mathsf{FCD}} b_1&.
\end{aligned}$$

Thus $\left\langle f \times^{(C)} g\right\rangle(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) = \langle f\rangle\mathcal{A}_0 \times^{\mathsf{FCD}} \langle g\rangle\mathcal{B}_0$ because the lattice $\mathsf{pFCD}(\mathscr{F}(\operatorname{Dst} f), \mathscr{F}(\operatorname{Dst} g))$ is atomically separable (corollary 1563). □

COROLLARY 1629. $\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g\right] \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \Leftrightarrow \mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1$ for every $\mathcal{A}_0 \in \mathscr{F}(\operatorname{Src} f)$, $\mathcal{A}_1 \in \mathscr{F}(\operatorname{Dst} f)$, $\mathcal{B}_0 \in \mathscr{F}(\operatorname{Src} g)$, $\mathcal{B}_1 \in \mathscr{F}(\operatorname{Dst} g)$ where $\operatorname{Src} f$, $\operatorname{Dst} f$, $\operatorname{Src} g$, $\operatorname{Dst} g$ are boolean lattices.

PROOF.

$$\begin{aligned}
\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g\right] \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 &\Leftrightarrow \\
\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \not\asymp \left\langle f \times^{(C)} g\right\rangle\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 &\Leftrightarrow \\
\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \not\asymp \langle f\rangle\mathcal{A}_0 \times^{\mathsf{FCD}} \langle g\rangle\mathcal{B}_0 &\Leftrightarrow \\
\mathcal{A}_1 \not\asymp \langle f\rangle\mathcal{A}_0 \wedge \mathcal{B}_1 \not\asymp \langle g\rangle\mathcal{B}_0 &\Leftrightarrow \\
\mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1&.
\end{aligned}$$

□

21.2. Definition of staroids

It follows from the above theorem 831 that funcoids are essentially the same as relations δ between sets A and B, such that $\left\{\frac{Y \in \mathscr{P} B}{\exists X \in \mathscr{P} A : X \delta Y}\right\}$ and $\left\{\frac{X \in \mathscr{P} A}{\exists Y \in \mathscr{P} B : X \delta Y}\right\}$ are free stars. This inspires the below definition of staroids (switching from two sets X and Y to a (potentially infinite) family of posets).

Whilst I have (mostly) thoroughly studied basic properties of funcoids, *staroids* (defined below) are yet much a mystery. For example, we do not know whether the set of staroids on powersets is atomic.

Let n be a set. As an example, n may be an ordinal, n may be a natural number, considered as a set by the formula $n = \{0, \ldots, n-1\}$. Let $\mathfrak{A} = \mathfrak{A}_{i \in n}$ be a family of posets indexed by the set n.

DEFINITION 1630. I will call an *anchored relation* a pair $f = (\operatorname{form} f, \operatorname{GR} f)$ of a family $\operatorname{form}(f)$ of relational structures indexed by some index set and a relation $\operatorname{GR}(f) \in \mathscr{P} \prod \operatorname{form}(f)$. I call $\operatorname{GR}(f)$ the *graph* of the anchored relation f. I denote $\operatorname{Anch}(\mathfrak{A})$ the set of anchored relations of the form $\mathfrak{A}$.

DEFINITION 1631. *Infinitary anchored relation* is such an anchored relation whose arity is infinite; *finitary anchored relation* is such an anchored relation whose arity is finite.

DEFINITION 1632. An anchored relation *on powersets* is an anchored relation f such that every $(\operatorname{form} f)_i$ is a powerset.

I will denote $\operatorname{arity} f = \operatorname{dom}\operatorname{form} f$.

DEFINITION 1633. $[f]^*$ is the relation between typed elements $\mathfrak{T}(\operatorname{form} f)_i$ (for $i \in \operatorname{arity} f$) defined by the formula $L \in [f]^* \Leftrightarrow \mathfrak{T} \circ L \in \operatorname{GR} f$.

Every set of anchored relations of the same form constitutes a poset by the formula $f \sqsubseteq g \Leftrightarrow \operatorname{GR} f \subseteq \operatorname{GR} g$.

DEFINITION 1634. An anchored relation is an *anchored relation between posets* when every $(\operatorname{form} f)_i$ is a poset.

DEFINITION 1635. $(\operatorname{val} f)_i L = \left\{ \frac{X \in (\operatorname{form} f)_i}{L \cup \{(i,X)\} \in \operatorname{GR} f} \right\}$.

PROPOSITION 1636. f can be restored knowing $\operatorname{form}(f)$ and $(\operatorname{val} f)_i$ for some $i \in \operatorname{arity} f$.

PROOF.

$$\operatorname{GR} f = \left\{ \frac{K \in \prod \operatorname{form} f}{K \in \operatorname{GR} f} \right\} =$$

$$\left\{ \frac{L \cup \{(i,X)\}}{L \in \prod (\operatorname{form} f)|_{(\operatorname{arity} f)\setminus\{i\}}, X \in (\operatorname{form} f)_i, L \cup \{(i,X)\} \in \operatorname{GR} f} \right\} =$$

$$\left\{ \frac{L \cup \{(i,X)\}}{L \in \prod (\operatorname{form} f)|_{(\operatorname{arity} f)\setminus\{i\}}, X \in (\operatorname{val} f)_i L} \right\}.$$

□

DEFINITION 1637. A *prestaroid* is an anchored relation f between posets such that $(\operatorname{val} f)_i L$ is a free star for every $i \in \operatorname{arity} f$, $L \in \prod (\operatorname{form} f)|_{(\operatorname{arity} f)\setminus\{i\}}$.

DEFINITION 1638. A *staroid* is a prestaroid whose graph is an upper set (on the poset $\prod \operatorname{form}(f)$).

DEFINITION 1639. A *(pre)staroid on power sets* is such a (pre)staroid f that every $(\operatorname{form} f)_i$ is a lattice of all subsets of some set.

PROPOSITION 1640. If $L \in \prod \operatorname{form} f$ and $L_i = \perp^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin \operatorname{GR} f$ if f is a prestaroid.

PROOF. Let $K = L|_{(\operatorname{arity} f)\setminus\{i\}}$. We have $\perp \notin (\operatorname{val} f)_i K$; $K \cup \{(i,\perp)\} \notin \operatorname{GR} f$; $L \notin \operatorname{GR} f$. □

Next we will define *completary staroids*. First goes the general case, next simpler case for the special case of join-semilattices instead of arbitrary posets.

DEFINITION 1641. A *completary staroid* is an anchored relation between posets conforming to the formulas:

1°. $\forall K \in \prod \operatorname{form} f : (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \operatorname{GR} f)$ is equivalent to $\exists c \in \{0,1\}^n : (\lambda i \in n : L_{c(i)} i) \in \operatorname{GR} f$ for every $L_0, L_1 \in \prod \operatorname{form} f$.

2°. If $L \in \prod \operatorname{form} f$ and $L_i = \perp^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin \operatorname{GR} f$.

LEMMA 1642. Every graph of completary staroid is an upper set.

PROOF. Let f be a completary staroid. Let $L_0 \sqsubseteq L_1$ for some $L_0, L_1 \in \prod \operatorname{form} f$ and $L_0 \in \operatorname{GR} f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n : L_{c(i)} i = \lambda i \in n : L_0 i = L_0 \in \operatorname{GR} f$ and thus $L_1 \in \operatorname{GR} f$ because $L_1 \sqsupseteq L_0 \wedge L_1 \sqsupseteq L_1$. □

PROPOSITION 1643. An anchored relation f between posets whose form is a family of join-semilattices is a completary staroid iff both:

1°. $L_0 \sqcup L_1 \in \operatorname{GR} f \Leftrightarrow \exists c \in \{0,1\}^n : (\lambda i \in n : L_{c(i)} i) \in \operatorname{GR} f$ for every $L_0, L_1 \in \prod \operatorname{form} f$.
2°. If $L \in \prod \operatorname{form} f$ and $L_i = \bot^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin \operatorname{GR} f$.

PROOF. Let the formulas 1° and 2° hold. Then f is an upper set: Let $L_0 \sqsubseteq L_1$ for some $L_0, L_1 \in \prod \operatorname{form} f$ and $L_0 \in f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n : L_{c(i)} i = \lambda i \in n : L_0 i = L_0 \in \operatorname{GR} f$ and thus $L_1 = L_0 \sqcup L_1 \in \operatorname{GR} f$.

Thus to finish the proof it is enough to show that

$$L_0 \sqcup L_1 \in \operatorname{GR} f \Leftrightarrow \forall K \in \prod \operatorname{form} f : (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \operatorname{GR} f)$$

under condition that $\operatorname{GR} f$ is an upper set. But this equivalence is obvious in both directions. □

PROPOSITION 1644. Every completary staroid is a staroid.

PROOF. Let f be a completary staroid.

Let $i \in \operatorname{arity} f$, $K \in \prod_{i \in (\operatorname{arity} f) \setminus \{i\}} (\operatorname{form} f)_i$. Let $L_0 = K \cup \{(i, X_0)\}$, $L_1 = K \cup \{(i, X_1)\}$ for some $X_0, X_1 \in \mathfrak{A}_i$.

Let

$$\forall Z \in \mathfrak{A}_i : (Z \sqsupseteq X_0 \wedge Z \sqsupseteq X_1 \Rightarrow Z \in (\operatorname{val} f)_i K);$$

then

$$\forall Z \in \mathfrak{A}_i : (Z \sqsupseteq X_0 \wedge Z \sqsupseteq X_1 \Rightarrow K \cup \{(i, Z)\} \in \operatorname{GR} f).$$

If $z \sqsupseteq L_0 \wedge z \sqsupseteq L_1$ then $z \sqsupseteq K \cup \{(i, z_i)\}$, thus taking into account that $\operatorname{GR} f$ is an upper set,

$$\forall z \in \prod \mathfrak{A} : (z \sqsupseteq L_0 \wedge z \sqsupseteq L_1 \Rightarrow K \cup \{(i, z_i)\} \in \operatorname{GR} f).$$
$$\forall z \in \prod \mathfrak{A} : (z \sqsupseteq L_0 \wedge z \sqsupseteq L_1 \Rightarrow z \in \operatorname{GR} f).$$

Thus, by the definition of completary staroid, $L_0 \in \operatorname{GR} f \vee L_1 \in \operatorname{GR} f$ that is

$$X_0 \in (\operatorname{val} f)_i K \vee X_1 \in (\operatorname{val} f)_i K.$$

So $(\operatorname{val} f)_i K$ is a free star (taken into account that $z_i = \bot^{(\operatorname{form} f)_i} \Rightarrow z \notin \operatorname{GR} f$ and that $(\operatorname{val} f)_i K$ is an upper set). □

EXERCISE 1645. Write a simplified proof for the case if every $(\operatorname{form} f)_i$ is a join-semilattice.

LEMMA 1646. Every finitary prestaroid is completary.

PROOF.

$$\exists c\in\{0,1\}^n:(\lambda i\in n:L_{c(i)}i)\in\operatorname{GR}f\Leftrightarrow$$

$$\exists c\in\{0,1\}^{n-1}:\begin{pmatrix}(\{(n-1,L_0(n-1))\}\cup(\lambda i\in n-1:L_{c(i)}i))\in\operatorname{GR}f\vee\\(\{(n-1,L_1(n-1))\}\cup(\lambda i\in n-1:L_{c(i)}i))\in\operatorname{GR}f\end{pmatrix}\Leftrightarrow$$

$$\exists c\in\{0,1\}^{n-1}:\begin{pmatrix}L_0(n-1)\in(\operatorname{val}f)_{n-1}(\lambda i\in n-1:L_{c(i)}i)\vee\\L_1(n-1)\in(\operatorname{val}f)_{n-1}(\lambda i\in n-1:L_{c(i)}i)\end{pmatrix}\Leftrightarrow$$

$$\exists c\in\{0,1\}^{n-1}\forall K\in(\operatorname{form}f)_i:\begin{pmatrix}K\sqsupseteq L_0(n-1)\vee K\sqsupseteq L_1(n-1)\Rightarrow\\K\in(\operatorname{val}f)_{n-1}(\lambda i\in n-1:L_{c(i)}i)\end{pmatrix}\Leftrightarrow$$

$$\exists c\in\{0,1\}^{n-1}\forall K\in(\operatorname{form}f)_i:\begin{pmatrix}K\sqsupseteq L_0(n-1)\vee K\sqsupseteq L_1(n-1)\Rightarrow\\\{(n-1,K)\}\cup(\lambda i\in n-1:L_{c(i)}i)\in\operatorname{GR}f\end{pmatrix}\Leftrightarrow$$

$$\ldots$$

$$\forall K\in\prod\operatorname{form}f:(K\sqsupseteq L_0\wedge K\sqsupseteq L_1\Rightarrow K\in\operatorname{GR}f).$$

□

EXERCISE 1647. Prove the simpler special case of the above theorem when the form is a family of join-semilattices.

THEOREM 1648. For finite arity the following are the same:

1°. prestaroids;
2°. staroids;
3°. completary staroids.

PROOF. f is a finitary prestaroid $\Rightarrow$ f is a finitary completary staroid.
f is a finitary completary staroid $\Rightarrow$ f is a finitary staroid.
f is a finitary staroid $\Rightarrow$ f is a finitary prestaroid. □

DEFINITION 1649. We will denote the set of staroids of a form $\mathfrak{A}$ as $\mathsf{Strd}(\mathfrak{A})$.

21.3. Upgrading and downgrading a set regarding a filtrator

Let fix a filtrator $(\mathfrak{A},\mathfrak{Z})$.

DEFINITION 1650. $\downdownarrows f=f\cap\mathfrak{Z}$ for every $f\in\mathscr{P}\mathfrak{A}$ (*downgrading* f).

DEFINITION 1651. $\upuparrows f=\left\{\frac{L\in\mathfrak{A}}{\operatorname{up}L\subseteq f}\right\}$ for every $f\in\mathscr{P}\mathfrak{Z}$ (*upgrading* f).

OBVIOUS 1652. $a\in\upuparrows f\Leftrightarrow\operatorname{up}a\subseteq f$ for every $f\in\mathscr{P}\mathfrak{Z}$ and $a\in\mathfrak{A}$.

PROPOSITION 1653. $\downdownarrows\upuparrows f=f$ if f is an upper set for every $f\in\mathscr{P}\mathfrak{Z}$.

PROOF. $\downdownarrows\upuparrows f=\upuparrows f\cap\mathfrak{Z}=\left\{\frac{L\in\mathfrak{Z}}{\operatorname{up}L\subseteq f}\right\}=\left\{\frac{L\in\mathfrak{Z}}{L\in f}\right\}=f\cap\mathfrak{Z}=f$. □

21.3.1. Upgrading and downgrading staroids.

Let fix a family $(\mathfrak{A},\mathfrak{Z})$ of filtrators.

For a graph f of an anchored relation between posets define $\downdownarrows f$ and $\upuparrows f$ taking the filtrator of $(\prod\mathfrak{A},\prod\mathfrak{Z})$.

For a anchored relation between posets f define:

$$\operatorname{form}\downdownarrows f=\mathfrak{Z}\quad\text{and}\quad\operatorname{GR}\downdownarrows f=\downdownarrows\operatorname{GR}f;$$
$$\operatorname{form}\upuparrows f=\mathfrak{A}\quad\text{and}\quad\operatorname{GR}\upuparrows f=\upuparrows\operatorname{GR}f.$$

Below we will show that under certain conditions upgraded staroid is a staroid, see theorem 1678.

PROPOSITION 1654. $(\operatorname{val} \downdownarrows f)_i L = (\operatorname{val} f)_i L \cap \mathfrak{Z}_i$ for every $L \in \prod \mathfrak{Z}|_{(\operatorname{arity} f)\setminus\{i\}}$.

PROOF. $(\operatorname{val} \downdownarrows f)_i L = \left\{ \frac{X \in \mathfrak{Z}_i}{L \cup \{(i,X)\} \in \operatorname{GR} f \cap \prod \mathfrak{Z}} \right\} = \left\{ \frac{X \in \mathfrak{Z}_i}{L \cup \{(i,X)\} \in \operatorname{GR} f} \right\} = (\operatorname{val} f)_i L \cap \mathfrak{Z}_i$. □

PROPOSITION 1655. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be binarily join-closed filtrators with both the base and the core being join-semilattices. If f is a staroid of the form $\mathfrak{A}$, then $\downdownarrows f$ is a staroid of the form $\mathfrak{Z}$.

PROOF. Let f be a staroid.

We need to prove that $(\operatorname{val} \downdownarrows f)_i L$ is a free star. It follows from the last proposition and the fact that it is binarily join-closed. □

PROPOSITION 1656. Let each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ for $i \in n$ (where n is an index set) be a binarily join-closed filtrator, such that each $\mathfrak{A}_i$ and each $\mathfrak{Z}_i$ are join-semilattices. If f is a completary staroid of the form $\mathfrak{A}$ then $\downdownarrows f$ is a completary staroid of the form $\mathfrak{Z}$.

PROOF.

$$\begin{aligned} L_0 \sqcup^{\mathfrak{Z}} L_1 \in \operatorname{GR} \downdownarrows f \Leftrightarrow L_0 \sqcup^{\mathfrak{Z}} L_1 \in \operatorname{GR} f \Leftrightarrow L_0 \sqcup^{\mathfrak{A}} L_1 \in \operatorname{GR} f \Leftrightarrow& \\ \exists c \in \{0,1\}^n : (\lambda i \in n : L_{c(i)} i) \in \operatorname{GR} f \Leftrightarrow& \\ \exists c \in \{0,1\}^n : (\lambda i \in n : L_{c(i)} i) \in \operatorname{GR} \downdownarrows f& \end{aligned}$$

for every $L_0, L_1 \in \prod \mathfrak{Z}$. □

21.4. Principal staroids

DEFINITION 1657. The *staroid generated* by an anchored relation F is the staroid $f = \uparrow^{\mathsf{Strd}} F$ on powersets such that $\uparrow \circ L \in \operatorname{GR} f \Leftrightarrow \prod L \not\asymp F$ and $(\operatorname{form} f)_i = \mathscr{T}(\operatorname{form} F)_i$ for every $L \in \prod_{i \in \operatorname{arity} f} \mathscr{T}(\operatorname{form} F)_i$.

REMARK 1658. Below we will prove that staroid generated by an anchored relation is a staroid and moreover a completary staroid.

DEFINITION 1659. A *principal staroid* is a staroid generated by some anchored relation.

PROPOSITION 1660. Every principal staroid is a completary staroid.

PROOF. That $L \notin \operatorname{GR} f$ if $L_i = \bot^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ is obvious. It remains to prove

$$\prod (L_0 \sqcup L_1) \not\asymp F \Leftrightarrow \exists c \in \{0,1\}^{\operatorname{arity} f} : \prod_{i \in n} L_{c(i)} i \not\asymp F.$$

Really

$$\prod (L_0 \sqcup L_1) \not\asymp F \Leftrightarrow$$
$$\exists x \in \prod (L_0 \sqcup L_1) : x \in F \Leftrightarrow$$
$$\exists x \in \prod_{i \in \operatorname{arity} f} (\operatorname{form} f)_i \forall i \in \operatorname{arity} f : (x_i \in L_0 i \sqcup L_1 i \wedge x \in F) \Leftrightarrow$$
$$\exists x \in \prod_{i \in \operatorname{arity} f} (\operatorname{form} f)_i \forall i \in \operatorname{arity} f : ((x_i \in L_0 i \vee x_i \in L_1 i) \wedge x \in F) \Leftrightarrow$$
$$\exists x \in \prod_{i \in \operatorname{arity} f} (\operatorname{form} f)_i \left(\exists c \in \{0,1\}^{\operatorname{arity} f} : x \in \prod_{i \in \operatorname{arity} f} L_{c(i)} i \wedge x \in F \right) \Leftrightarrow$$
$$\exists c \in \{0,1\}^{\operatorname{arity} f} : \prod_{i \in n} L_{c(i)} i \not\asymp F.$$

□

Definition 1661. The *upgraded staroid generated* by an anchored relation F is the anchored relation $\upuparrows \uparrow^{\mathsf{Strd}} F$.

Proposition 1662. $\uparrow^{\mathsf{Strd}} F = \downdownarrows \upuparrows \uparrow^{\mathsf{Strd}} F$.

Proof. Because $\mathrm{GR} \uparrow^{\mathsf{Strd}} F$ is an upper set. □

Example 1663. There is such anchored relation f that $\upuparrows f$ is not a completary staroid. This also proves existence of non-completary staroids (but not on powersets).

Proof. (based on an Andreas Blass's proof) Take f the set of functions $x : \mathbb{N} \to \mathbb{N}$ where x_0 is an arbitrary natural number and $x_i = \begin{cases} 0 & \text{if } n \leqslant x_0 \\ 1 & \text{if } n > x_0 \end{cases}$ for $i = 1, 2, 3, \ldots$. Thus f is the graph of a staroid of the form $\lambda i \in \mathbb{N} : \mathscr{P}\mathbb{N}$ (on powersets).

Let $\mathcal{L}_0(0) = \mathcal{L}_1(0) = \Omega(\mathbb{N})$, $\mathcal{L}_0(i) = \uparrow \{0\}$ and $\mathcal{L}_1(i) = \uparrow \{1\}$ for $i > 0$.

Let $X \in \operatorname{up}(\mathcal{L}_0 \sqcup \mathcal{L}_1)$ that is $X \in \operatorname{up} \mathcal{L}_0 \cap \operatorname{up} \mathcal{L}_1$.

X_0 contains all but finitely many elements of $\mathbb{N}$.

For $i > 0$ we have $\{0, 1\} \subseteq X_i$.

Evidently, $\prod X$ contains an element of f, that is $\operatorname{up}(\mathcal{L}_0 \sqcup \mathcal{L}_1) \in f$ what means $\mathcal{L}_0 \sqcup \mathcal{L}_1 \in \upuparrows f$.

Now consider any fixed $c \in \{0,1\}^{\mathbb{N}}$. There is at most one $k \in \mathbb{N}$ such that the sequence $x = \llbracket k, c(1), c(2), \ldots \rrbracket$ (i.e. c with $c(0)$ replaced by k) is in f. Let $Q = \mathbb{N} \setminus \{k\}$ if there is such a k and $Q = \mathbb{N}$ otherwise.

Take $Y_i = \begin{cases} Q & \text{if } i = 0 \\ \{c(i)\} & \text{if } i > 0 \end{cases}$ for $i = 0, 1, 2, \ldots$. We have $Y \in \operatorname{up}(\lambda i \in \mathbb{N} : \mathcal{L}_{c(i)}(i))$ for every $c \in \{0,1\}^n$.

But evidently $\prod Y$ does not contain an element of f. Thus, $\prod Y \asymp f$ that is $Y \notin f$; $\operatorname{up} Y \nsubseteq f$; $Y \notin \mathrm{GR} \upuparrows f$ what is impossible if $\upuparrows f$ is completary. □

Example 1664. There exists such an (infinite) set N and N-ary relation f that $\mathcal{P} \in \mathrm{GR} \upuparrows f$ but there is no indexed family $a \in \prod_{i \in N} \operatorname{atoms} \mathcal{P}_i$ of atomic filters such that $a \in \mathrm{GR} \upuparrows f$ that is $\forall A \in \operatorname{up} a : f \not\asymp \prod A$.

Proof. Take $\mathcal{L}_0$, $\mathcal{L}_1$ and f from the proof of example 1663. Take $\mathcal{P} = \mathcal{L}_0 \sqcup \mathcal{L}_1$. If $a \in \prod_{i \in N} \operatorname{atoms} \mathcal{P}_i$ then there exists $c \in \{0,1\}^N$ such that $a_i \sqsubseteq \mathcal{L}_{c(i)}(i)$ (because $\mathcal{L}_{c(i)}(i) \neq \bot$). Then from that example it follows that $(\lambda i \in N : \mathcal{L}_{c(i)}(i)) \notin \mathrm{GR} \upuparrows f$ and thus $a \notin \mathrm{GR} \upuparrows f$. □

CONJECTURE 1665. Filtrators of staroids on powersets are join-closed.

21.5. Multifuncoids

DEFINITION 1666. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ (where $i \in n$ for an index set n) be an indexed family of filtrators.

I call a *mult* f of the form $(\mathfrak{A}_i, \mathfrak{Z}_i)$ the triple $f = (\operatorname{base} f, \operatorname{core} f, \langle f \rangle^*)$ of n-indexed families of posets $\operatorname{base} f$ and $\operatorname{core} f$ and $\langle f \rangle^*$ of functions where for every $i \in n$

$$\langle f \rangle_i^* : \prod (\operatorname{core} f)_i|_{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}} \to (\operatorname{base} f)_i.$$

I call $(\operatorname{base} f, \operatorname{core} f)$ the *form* of the mult f.

REMARK 1667. I call it *mult* because it comprises multiple functions $\langle f \rangle_i^*$.

DEFINITION 1668. A mult *on powersets* is a mult such that every $((\operatorname{base} f)_i, (\operatorname{core} f)_i)$ is a powerset filtrator.

DEFINITION 1669. I will call a *relational mult* a mult f such that every $(\operatorname{base} f)_i$ is a set and for every $i, j \in n$ and $L \in \prod \operatorname{core} f$

$$L_i \in \langle f \rangle_i^* L|_{(\operatorname{dom} L) \setminus \{i\}} \Leftrightarrow L_j \in \langle f \rangle_j^* L|_{(\operatorname{dom} L) \setminus \{j\}}.$$

I denote $\operatorname{arity} f = n$.

DEFINITION 1670. *Prestaroidal mult* is a relational mult of the form $(\mathfrak{A}, \lambda i \in \operatorname{dom} \mathfrak{A} : \mathfrak{S}(\mathfrak{A}_i))$ (where $\mathfrak{A}$ is a poset), that is such that $\langle f \rangle_i^* L$ is a free star for every $i \in n$ and $L \in \prod_{i \in (\operatorname{dom} L) \setminus \{i\}} \operatorname{core} f_i$.

DEFINITION 1671. I will call a *multifuncoid* a mult f such that $(\operatorname{core} f)_i \subseteq (\operatorname{base} f)_i$ (thus having a filtrator $((\operatorname{base} f)_i, (\operatorname{core} f)_i)$) for each $i \in n$ and for every $i, j \in n$ and $L \in \prod \operatorname{core} f$

$$L_i \not\asymp \langle f \rangle_i^* L|_{(\operatorname{dom} L) \setminus \{i\}} \Leftrightarrow L_j \not\asymp \langle f \rangle_j^* L|_{(\operatorname{dom} L) \setminus \{j\}}. \tag{33}$$

I denote the set of multifuncoids for a family $(\mathfrak{A}, \mathfrak{Z})$ of filtrators as $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{Z})$ or just $\mathsf{pFCD}(\mathfrak{A})$ when $\mathfrak{Z}$ is clear from context.

DEFINITION 1672. To every multifuncoid f corresponds an anchored relation g by the formula (with arbitrary $i \in \operatorname{arity} f$)

$$L \in \operatorname{GR} g \Leftrightarrow L_i \not\asymp \langle f \rangle_i^* L|_{(\operatorname{dom} L) \setminus \{i\}}.$$

PROPOSITION 1673. Prestaroidal mults $\Lambda g = f$ of the form $(\mathfrak{Z}, \lambda i \in \operatorname{dom} \mathfrak{Z} : \mathfrak{S}(\mathfrak{Z}_i))$ bijectively correspond to pre-staroids g of the form $\mathfrak{Z}$ by the formulas (for every $K \in \prod \mathfrak{Z}$, $i \in \operatorname{dom} \mathfrak{Z}$, $L \in \prod_{j \in (\operatorname{dom} \mathfrak{A}) \setminus \{i\}} \mathfrak{Z}_j$, $X \in \mathfrak{Z}_i$)

$$K \in \operatorname{GR} g \Leftrightarrow K_i \in \langle f \rangle_i^* K|_{(\operatorname{dom} L) \setminus \{i\}}; \tag{34}$$

$$X \in \langle f \rangle_i^* L \Leftrightarrow L \cup \{(i, X)\} \in \operatorname{GR} g. \tag{35}$$

PROOF. If f is a prestaroidal mult, then obviously formula (34) defines an anchored relation between posets. $(\operatorname{val} g)_i = \langle f \rangle_i^* L$ is a free star. Thus g is a prestaroid.

If g is a prestaroid, then obviously formula (35) defines a relational mult. This mult is obviously prestaroidal.

It remains to prove that these correspondences are inverse of each other.

Let f_0 be a prestaroidal mult, g be the pre-staroid corresponding to f by formula (34), and f_1 be the prestaroidal mult corresponding to g by formula (35). Let's prove $f_0 = f_1$. Really,

$$X \in \langle f_1 \rangle_i^* L \Leftrightarrow L \cup \{(i, X)\} \in \operatorname{GR} g \Leftrightarrow X \in \langle f_0 \rangle_i^* L.$$

Let now g_0 be a prestaroid, f be a prestaroidal mult corresponding to g_0 by formula (35), and g_1 be a prestaroid corresponding to f by formula (34). Let's prove $g_0 = g_1$. Really,

$$K \in \operatorname{GR} g_1 \Leftrightarrow K_i \in \langle f\rangle_i^* K|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow K|_{(\operatorname{dom} L)\setminus\{i\}} \cup \{(i, K_i)\} \in \operatorname{GR} g_0 \Leftrightarrow K \in \operatorname{GR} g_0.$$

□

Definition 1674. I will denote $[f]^* = \operatorname{GR} g$ for the prestaroidal mult f corresponding to anchored relation g.

Proposition 1675. For a form $(\mathfrak{Z}, \lambda i \in \operatorname{dom}\mathfrak{Z} : \mathfrak{S}(\mathfrak{Z}_i))$ where each $\mathfrak{Z}_i$ is a boolean lattice, relational mults are the same as multifuncoids (if we equate poset elements with principal free stars).

Proof.

$$\begin{aligned}(L_i \not\asymp \langle f\rangle_i^* L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \not\asymp \langle f\rangle_j^* L|_{(\operatorname{dom} L)\setminus\{j\}}) \Leftrightarrow \\ (L_i \in \partial\langle f\rangle_i^* L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \in \partial\langle f\rangle_j^* L|_{(\operatorname{dom} L)\setminus\{j\}}) \Leftrightarrow \\ (L_i \in \langle f\rangle_i^* L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \in \langle f\rangle_j^* L|_{(\operatorname{dom} L)\setminus\{j\}}).\end{aligned}$$

□

Theorem 1676. Fix some indexed family $\mathfrak{Z}$ of join semi-lattices.

$$(\operatorname{val} f)_j(L \cup \{(i, X \sqcup Y)\}) = (\operatorname{val} f)_j(L \cup \{(i, X)\}) \sqcup (\operatorname{val} f)_j(L \cup \{(i, Y)\})$$

for every prestaroid f of the form $\mathfrak{Z}$ and $i, j \in \operatorname{arity} f$, $i \neq j$, $L \in \prod_{k \in L\setminus\{i,j\}} \mathfrak{Z}_k$, $X, Y \in \mathfrak{Z}_i$.

Proof. Let $i, j \in \operatorname{arity} f$, $i \neq j$ and $L \in \prod_{k \in L\setminus\{i,j\}} \mathfrak{Z}_k$. Let $Z \in \mathfrak{Z}_i$.

$$\begin{aligned}Z \in (\operatorname{val} f)_j(L \cup \{(i, X \sqcup Y)\}) \Leftrightarrow \\ L \cup \{(i, X \sqcup Y), (j, Z)\} \in \operatorname{GR} f \Leftrightarrow \\ X \sqcup Y \in (\operatorname{val} f)_i(L \cup \{(j, Z)\}) \Leftrightarrow \\ X \in (\operatorname{val} f)_i(L \cup \{(j, Z)\}) \vee Y \in (\operatorname{val} f)_i(L \cup \{(j, Z)\}) \Leftrightarrow \\ L \cup \{(i, X), (j, Z)\} \in \operatorname{GR} f \vee L \cup \{(i, Y), (j, Z)\} \in \operatorname{GR} f \Leftrightarrow \\ Z \in (\operatorname{val} f)_j(L \cup \{(i, X)\}) \vee Z \in (\operatorname{val} f)_j(L \cup \{(i, Y)\}) \Leftrightarrow \\ Z \in (\operatorname{val} f)_j(L \cup \{(i, X)\}) \cup (\operatorname{val} f)_j(L \cup \{(i, Y)\}) \Leftrightarrow \\ Z \in (\operatorname{val} f)_j(L \cup \{(i, X)\}) \sqcup (\operatorname{val} f)_j(L \cup \{(i, Y)\})\end{aligned}$$

Thus $(\operatorname{val} f)_j(L \cup \{(i, X \sqcup Y)\}) = (\operatorname{val} f)_j(L \cup \{(i, X)\}) \sqcup (\operatorname{val} f)_j(L \cup \{(i, Y)\})$. □

Let us consider the filtrator $\left(\prod_{i \in \operatorname{arity} f} \mathfrak{S}((\operatorname{form} f)_i), \prod_{i \in \operatorname{arity} f} (\operatorname{form} f)_i\right)$.

Conjecture 1677. A finitary anchored relation between join-semilattices is a staroid iff $(\operatorname{val} f)_j(L \cup \{(i, X \sqcup Y)\}) = (\operatorname{val} f)_j(L \cup \{(i, X)\}) \sqcup (\operatorname{val} f)_j(L \cup \{(i, Y)\})$ for every $i, j \in \operatorname{arity} f$ $(i \neq j)$ and $X, Y \in (\operatorname{form} f)_i$.

Theorem 1678. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be a family of join-closed down-aligned filtrators whose both base and core are join-semilattices. Let f be a staroid of the form $\mathfrak{Z}$. Then $\upuparrows f$ is a staroid of the form $\mathfrak{A}$.

Proof. First prove that $\upuparrows f$ is a prestaroid. We need to prove that $\bot \notin (\operatorname{GR} \upuparrows f)_i$ (that is $\operatorname{up} \bot \nsubseteq (\operatorname{GR} f)_i$ that is $\bot \notin (\operatorname{GR} f)_i$ what is true by the theorem conditions) and that for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{A}_i$ and $\mathcal{L} \in \prod_{i \in (\operatorname{arity} f)\setminus\{i\}} \mathfrak{A}_i$ where $i \in \operatorname{arity} f$

$$\mathcal{L} \cup \{(i, \mathcal{X} \sqcup \mathcal{Y})\} \in \operatorname{GR} \upuparrows f \Leftrightarrow \mathcal{L} \cup \{(i, \mathcal{X})\} \in \operatorname{GR} \upuparrows f \vee \mathcal{L} \cup \{(i, \mathcal{Y})\} \in \operatorname{GR} \upuparrows f.$$

The reverse implication is obvious. Let $\mathcal{L} \cup \{(i, \mathcal{X} \sqcup \mathcal{Y})\} \in \mathrm{GR} \upuparrows f$. Then for every $L \in \mathrm{up}\,\mathcal{L}$ and $X \in \mathrm{up}\,\mathcal{X}$, $Y \in \mathrm{up}\,\mathcal{Y}$ we have $L \cup \{(i, X \sqcup^{\mathfrak{Z}_i} Y)\} \in \mathrm{GR}\, f$ and thus

$$L \cup \{(i, X)\} \in \mathrm{GR}\, f \vee L \cup \{(i, Y)\} \in \mathrm{GR}\, f$$

consequently $\mathcal{L} \cup \{(i, \mathcal{X})\} \in \mathrm{GR} \upuparrows f \vee \mathcal{L} \cup \{(i, \mathcal{Y})\} \in \mathrm{GR} \upuparrows f$.

It is left to prove that $\upuparrows f$ is an upper set, but this is obvious. □

There is a conjecture similar to the above theorems:

Conjecture 1679. $L \in \upuparrows [f]^* \Rightarrow \upuparrows [f]^* \cap \prod_{i \in \mathrm{dom}\,\mathfrak{A}} \mathrm{atoms}\, L_i \neq \emptyset$ for every multifuncoid f for the filtrator $(\mathscr{F}^n, \mathfrak{Z}^n)$.

Conjecture 1680. Let $(\mathfrak{A}, \mathfrak{Z})$ be a powerset filtrator, let n be an index set. Consider the filtrator $(\mathscr{F}^n, \mathfrak{Z}^n)$. Then if f is a completary staroid of the form $\mathfrak{Z}^n$, then $\upuparrows f$ is a completary staroid of the form $\mathfrak{A}^n$.

Example 1681. There is such an anchored relation f that for some $k \in \mathrm{dom}\, f$

$$\langle \upuparrows\uparrow f \rangle^*{}_k \mathcal{L} \neq \bigsqcup_{a \in \prod_{i \in (\mathrm{dom}\, f) \setminus \{k\}} \mathrm{atoms}\, \mathcal{L}_i} \langle \upuparrows\uparrow f \rangle^*_k a.$$

Proof. Take $\mathcal{P} \in \mathrm{GR}\, f$ from the counter-example 1664. We have

$$\forall a \in \prod_{i \in \mathrm{dom}\, f} \mathrm{atoms}\, \mathcal{P}_i : a \notin \mathrm{GR}\, \mathcal{P}.$$

Take $k = 1$.

Let $\mathcal{L} = \mathcal{P}|_{(\mathrm{dom}\, f) \setminus \{k\}}$. Then $a \notin \mathrm{GR} \upuparrows\uparrow f$ and thus $a_k \asymp \langle \upuparrows\uparrow f \rangle^*{}_k a|_{(\mathrm{dom}\, f) \setminus \{k\}}$.

Consequently $\mathcal{P}_k \asymp \langle \upuparrows\uparrow f \rangle^*{}_k a|_{(\mathrm{dom}\, f) \setminus \{k\}}$ and thus $\mathcal{P}_k \asymp \bigsqcup_{a \in \prod_{i \in (\mathrm{dom}\, f) \setminus \{k\}} \mathrm{atoms}\, \mathcal{L}_i} \langle \upuparrows\uparrow f \rangle^*{}_k a$ because $\mathcal{P}_k$ is principal.

But $\mathcal{P}_k \not\asymp \langle \upuparrows\uparrow f \rangle^*{}_k \mathcal{L}$. Thus follows $\langle \upuparrows\uparrow f \rangle^*_k \mathcal{L} \neq \bigsqcup_{a \in \prod_{i \in (\mathrm{dom}\, f) \setminus \{k\}} \mathrm{atoms}\, \mathcal{L}_i} \langle \upuparrows\uparrow f \rangle^*{}_k a$. □

21.6. Join of multifuncoids

Mults are ordered by the formula $f \sqsubseteq g \Leftrightarrow \langle f \rangle^* \sqsubseteq \langle g \rangle^*$ where $\sqsubseteq$ in the right part of this formula is the product order. I will denote $\sqcap$, $\sqcup$, $\prod$, $\bigsqcup$ (without an index) the order poset operations on the poset of mults.

Remark 1682. To describe this, the definition of product order is used twice. Let f and g be mults of the same form $(\mathfrak{A}, \mathfrak{Z})$

$$\langle f \rangle^* \sqsubseteq \langle g \rangle^* \Leftrightarrow \forall i \in \mathrm{dom}\, \mathfrak{Z} : \langle f \rangle^*_i \sqsubseteq \langle g \rangle^*_i;$$

$$\langle f \rangle^*_i \sqsubseteq \langle g \rangle^*_i \Leftrightarrow \forall L \in \prod \mathfrak{Z}|_{(\mathrm{dom}\, \mathfrak{Z}) \setminus \{i\}} : \langle f \rangle^*_i L \sqsubseteq \langle g \rangle^*_i L.$$

Obvious 1683. $(\bigsqcup F) K = \bigsqcup_{f \in F} f K$ for every set F of mults of the same form $\mathfrak{Z}$ and $K \in \prod \mathfrak{Z}$ whenever every $\bigsqcup_{f \in F} f K$ is defined.

Theorem 1684. $f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g = f \sqcup g$ for every multifuncoids f and g for the same indexed family of starrish join-semilattices filtrators.

Proof. $\alpha_i x \overset{\mathrm{def}}{=} \langle f_i \rangle^* x \sqcup \langle g_i \rangle^* x$. It is enough to prove that α is a multifuncoid. We need to prove:

$$L_i \not\asymp \alpha_i L|_{(\mathrm{dom}\, L) \setminus \{i\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\mathrm{dom}\, L) \setminus \{j\}}.$$

Really,

$$\begin{aligned}
L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \\
L_i \not\asymp \langle f_i\rangle^* L|_{(\operatorname{dom} L)\setminus\{i\}} \sqcup \langle g_i\rangle^* L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \\
L_i \not\asymp \langle f_i\rangle^* L|_{(\operatorname{dom} L)\setminus\{i\}} \vee L_i \not\asymp \langle g_i\rangle^* L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \\
L_j \not\asymp \langle f_j\rangle^* L|_{(\operatorname{dom} L)\setminus\{j\}} \vee L_j \not\asymp \langle g_j\rangle^* L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow \\
L_j \not\asymp \langle f_j\rangle^* L|_{(\operatorname{dom} L)\setminus\{j\}} \sqcup \langle g_j\rangle^* L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow \\
L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}.
\end{aligned}$$

□

THEOREM 1685. $\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F = \bigsqcup F$ for every set F of multifuncoids for the same indexed family of join infinite distributive complete lattices filtrators.

PROOF. $\alpha_i x \overset{\text{def}}{=} \bigsqcup_{f\in F}\langle f\rangle_i^* x$. It is enough to prove that α is a multifuncoid.

We need to prove:

$$L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}.$$

Really,

$$\begin{aligned}
L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \\
L_i \not\asymp \bigsqcup_{f\in F} \langle f_i\rangle^* L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \\
\exists f \in F : L_i \not\asymp \langle f_i\rangle^* L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \\
\exists f \in F : L_j \not\asymp \langle f_j\rangle^* L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow \\
L_j \not\asymp \bigsqcup_{f\in F} \langle f_j\rangle^* L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow \\
L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}.
\end{aligned}$$

□

THEOREM 1686. If f, g are multifuncoids for a primary filtrator $(\mathfrak{A}_i, \mathfrak{Z}_i)$ where $\mathfrak{Z}_i$ are separable starrish posets, then $f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g \in \mathsf{pFCD}(\mathfrak{A})$.

PROOF. Let $A \in \left[f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right]^*$ and $B \sqsupseteq A$. Then for every $k \in \operatorname{dom}\mathfrak{A}$

$A_k \not\asymp \left\langle f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right\rangle^* A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}$; $A_k \not\asymp \langle f \sqcup g\rangle^* A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}$; $A_k \not\asymp \langle f\rangle^*(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \sqcup \langle g\rangle^*(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$.

Thus $A_k \not\asymp \langle f\rangle^*(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \vee A_k \not\asymp \langle g\rangle^*(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $A \in [f]^* \vee A \in [g]^*$; $B \in [f]^* \vee B \in [g]^*$; $B_k \not\asymp \langle f\rangle^*(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \vee B_k \not\asymp \langle g\rangle^*(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$;

$B_k \not\asymp \langle f\rangle^*(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \sqcup \langle g\rangle^*(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $B_k \not\asymp \langle f \sqcup g\rangle^* B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \left\langle f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right\rangle^* B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}$.

Thus $B \in \left[f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right]^*$. □

THEOREM 1687. If F is a set of multifuncoids for the same indexed family of join infinite distributive complete lattices filtrators, then $\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F \in \mathsf{pFCD}(\mathfrak{A})$.

PROOF. Let $A \in \left[\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right]^*$ and $B \sqsupseteq A$. Then for every $k \in \operatorname{dom}\mathfrak{A}$

$A_k \not\asymp \left\langle\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right\rangle^* A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \langle\bigsqcup F\rangle^* A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \bigsqcup_{f\in F}\langle f\rangle^*(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$.

Thus $\exists f \in F : A_k \not\asymp \langle f\rangle^*(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $\exists f \in F : A \in [f]^*$; $B \in [f]^*$ for some $f \in F$; $\exists f \in F : B_k \not\asymp \langle f\rangle^*(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $B_k \not\asymp \bigsqcup_{f\in F}\langle f\rangle^*(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) = \left\langle\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right\rangle^* B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}$. Thus $B \in \left[\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right]^*$. □

21.7. Infinite product of poset elements

Let A_i be a family of elements of a family $\mathfrak{A}_i$ of posets. The *staroidal product* $\prod^{\mathsf{Strd}(\mathfrak{A})} A$ is defined by the formula (for every $L \in \prod \mathfrak{A}$)

$$\operatorname{form} \prod^{\mathsf{Strd}(\mathfrak{A})} A = \mathfrak{A} \quad \text{and} \quad L \in \mathrm{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A} : A_i \not\asymp L_i.$$

PROPOSITION 1688. If $\mathfrak{A}_i$ are powerset algebras, staroidal product of principal filters is essentially equivalent to Cartesian product. More precisely, $\prod^{\mathsf{Strd}}_{i \in \operatorname{dom} A} \uparrow^{\mathscr{F}} A_i = \upuparrows \uparrow^{\mathsf{Strd}} \prod A$ for an indexed family A of sets.

PROOF.

$$\begin{aligned} L \in \mathrm{GR} \upuparrows \uparrow^{\mathsf{Strd}} \prod A &\Leftrightarrow \\ \operatorname{up} L \subseteq \mathrm{GR} \uparrow^{\mathsf{Strd}} \prod A &\Leftrightarrow \\ \forall X \in \operatorname{up} L : \prod X \not\asymp \prod A &\Leftrightarrow \\ \forall X \in \operatorname{up} L, i \in \operatorname{dom} A : X_i \not\asymp A_i &\Leftrightarrow \\ \forall i \in \operatorname{dom} A : L_i \not\asymp \uparrow^{\mathscr{F}} A_i &\Leftrightarrow \\ L \in \mathrm{GR} \prod^{\mathsf{Strd}}_{i \in \operatorname{dom} A} \uparrow^{\mathscr{F}} A_i&. \end{aligned}$$

□

COROLLARY 1689. Staroidal product of principal filters is an upgraded principal staroid.

PROPOSITION 1690. $\prod^{\mathsf{Strd}} a = \upuparrows \downdownarrows \prod^{\mathsf{Strd}} a$ if each $a_i \in \mathfrak{A}_i$ (for $i \in n$ where n is some index set) where each $(\mathfrak{A}_{i \in n}, \mathfrak{Z}_{i \in n})$ is a filtrator with separable core.

PROOF.

$$\begin{aligned} \mathrm{GR} \upuparrows \downdownarrows \prod^{\mathsf{Strd}} a &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\operatorname{up} L \subseteq \mathfrak{Z} \cap \mathrm{GR} \prod^{\mathsf{Strd}} a} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\operatorname{up} L \subseteq \mathrm{GR} \prod^{\mathsf{Strd}} a} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\forall K \in \operatorname{up} L : K \in \mathrm{GR} \prod^{\mathsf{Strd}} a} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\forall K \in \operatorname{up} L, i \in n : K_i \not\asymp a_i} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in n, K \in \operatorname{up} L : K_i \not\asymp a_i} \right\} &= \\ \left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in n : L_i \not\asymp a_i} \right\} &= \\ \mathrm{GR} \prod^{\mathsf{Strd}} a& \end{aligned}$$

(taken into account that our filtrators are with a separable core). □

THEOREM 1691. Staroidal product is a completary staroid (if our posets are starrish join-semilattices).

PROOF. We need to prove

$$\forall i \in \operatorname{dom}\mathfrak{A} : A_i \not\asymp (L_0 i \sqcup L_1 i) \Leftrightarrow \exists c \in \{0,1\}^n \forall i \in \operatorname{dom}\mathfrak{A} : A_i \not\asymp L_{c(i)} i.$$

Really,

$$\forall i \in \operatorname{dom}\mathfrak{A} : A_i \not\asymp (L_0 i \sqcup L_1 i) \Leftrightarrow \forall i \in \operatorname{dom}\mathfrak{A} : (A_i \not\asymp L_0 i \vee A_i \not\asymp L_1 i) \Leftrightarrow \\ \exists c \in \{0,1\}^{\operatorname{dom}\mathfrak{A}} \forall i \in \operatorname{dom}\mathfrak{A} : A_i \not\asymp L_{c(i)} i.$$

□

DEFINITION 1692. Let $(\mathfrak{A}_i, \mathfrak{Z}_i)$ be an indexed family of filtrators and every $\mathfrak{A}_i$ has least element.

Then for every $A \in \prod \mathfrak{A}$ *funcoidal product* is multifuncoid $\prod^{\mathsf{FCD}(\mathfrak{A})} A$ defined by the formula (for every $L \in \prod \mathfrak{Z}$):

$$\left\langle \prod^{\mathsf{FCD}(\mathfrak{A})} A \right\rangle_k^* L = \begin{cases} A_k & \text{if } \forall i \in (\operatorname{dom}\mathfrak{A}) \setminus \{k\} : A_i \not\asymp L_i \\ \bot^{\mathfrak{A}} & \text{otherwise.} \end{cases}$$

PROPOSITION 1693. $\operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A = \left[\prod^{\mathsf{FCD}(\mathfrak{A})} A \right]^*$.

PROOF.

$$\begin{aligned} L \in \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A \Leftrightarrow \\ \forall i \in \operatorname{dom}\mathfrak{A} : A_i \not\asymp L_i \Leftrightarrow \\ \forall i \in (\operatorname{dom}\mathfrak{A}) \setminus \{k\} : A_i \not\asymp L_i \wedge L_k \not\asymp A_k \Leftrightarrow \\ L_k \not\asymp \left\langle \prod^{\mathsf{FCD}(\mathfrak{A})} A \right\rangle_k^* L|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} \Leftrightarrow \\ L \in \left[\prod^{\mathsf{FCD}(\mathfrak{A})} A \right]^*. \end{aligned}$$

□

COROLLARY 1694. Funcoidal product is a completary multifuncoid.

PROOF. It is enough to prove that funcoidal product is a multifuncoid. Really,

$$L_i \not\asymp \left\langle \prod^{\mathsf{FCD}(\mathfrak{A})} A \right\rangle_i^* L|_{(\operatorname{dom}\mathfrak{A})\setminus\{i\}} \Leftrightarrow \forall i \in \operatorname{dom}\mathfrak{A} : A_i \not\asymp L_i \Leftrightarrow L_j \not\asymp \left\langle \prod^{\mathsf{FCD}(\mathfrak{A})} A \right\rangle_j^* L|_{(\operatorname{dom}\mathfrak{A})\setminus\{j\}}.$$

□

THEOREM 1695. If our each filtrator $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is with separable core and $A \in \prod \mathfrak{Z}$, then $\uparrow\uparrow \prod^{\mathsf{Strd}(\mathfrak{Z})} A = \prod^{\mathsf{Strd}(\mathfrak{A})} A$.

PROOF.

$$\mathrm{GR} \upuparrows \prod^{\mathsf{Strd}(\mathfrak{Z})} A =$$

$$\left\{ \frac{L \in \prod \mathfrak{A}}{\mathrm{up}\, L \subseteq \prod^{\mathsf{Strd}(\mathfrak{Z})} A} \right\} =$$

$$\left\{ \frac{L \in \prod \mathfrak{A}}{\forall K \in \mathrm{up}\, L, i \in \mathrm{dom}\, \mathfrak{A} : A_i \not\asymp K_i} \right\} =$$

$$\left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in \mathrm{dom}\, \mathfrak{A}, K \in \mathrm{up}\, L_i : A_i \not\asymp K} \right\} =$$

$$\left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in \mathrm{dom}\, \mathfrak{A} : A_i \not\asymp L_i} \right\} =$$

$$\mathrm{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A.$$

□

PROPOSITION 1696. Let $(\prod \mathfrak{A}, \prod \mathfrak{Z})$ be a meet-closed filtrator, $A \in \prod \mathfrak{Z}$. Then $\downdownarrows \prod^{\mathsf{Strd}(\mathfrak{A})} A = \prod^{\mathsf{Strd}(\mathfrak{Z})} A$.

PROOF.

$$\mathrm{GR} \downdownarrows \prod^{\mathsf{Strd}(\mathfrak{A})} A =$$

$$\downdownarrows \mathrm{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A =$$

$$\downdownarrows \left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in \mathrm{dom}\, \mathfrak{A} : A_i \not\asymp L_i} \right\} =$$

$$\left\{ \frac{L \in \prod \mathfrak{A}}{\forall i \in \mathrm{dom}\, \mathfrak{A} : A_i \not\asymp L_i} \right\} \cap \prod \mathfrak{Z} =$$

$$\left\{ \frac{L \in \prod \mathfrak{Z}}{\forall i \in \mathrm{dom}\, \mathfrak{A} : A_i \not\asymp L_i} \right\} =$$

$$\mathrm{GR} \prod^{\mathsf{Strd}(\mathfrak{Z})} A.$$

□

COROLLARY 1697. If each $(\mathfrak{A}_i, \mathfrak{Z}_i)$ is a powerset filtrator and $A \in \prod \mathfrak{Z}$, then $\downdownarrows \prod^{\mathsf{Strd}(\mathfrak{A})} A$ is a principal staroid.

PROOF. Use the "obvious" fact above. □

THEOREM 1698. Let $\mathscr{F}$ be a family of sets of filters on meet-semilattices with least elements. Let $a \in \prod \mathscr{F}$, $S \in \mathscr{P} \prod \mathscr{F}$, and every $\Pr_i S$ be a generalized filter base, $\bigsqcap S = a$. Then

$$\prod^{\mathsf{Strd}(\mathscr{F})} a = \bigsqcap_{A \in S} \prod^{\mathsf{Strd}(\mathscr{F})} A.$$

PROOF. That $\prod^{\mathsf{Strd}(\mathscr{F})} a$ is a lower bound for $\left\{ \frac{\prod^{\mathsf{Strd}(\mathscr{F})} A}{A \in S} \right\}$ is obvious.

Let f be a lower bound for $\left\{ \frac{\prod^{\mathsf{Strd}(\mathscr{F})} A}{A \in S} \right\}$. Thus $\forall A \in S : \mathrm{GR}\, f \subseteq \mathrm{GR} \prod^{\mathsf{Strd}(\mathscr{F})} A$. Thus for every $A \in S$ we have $L \in \mathrm{GR}\, f$ implies $\forall i \in \mathrm{dom}\, \mathfrak{A}$:

$A_i \not\asymp L_i$. Then, by properties of generalized filter bases, $\forall i \in \operatorname{dom} \mathfrak{A} : a_i \not\asymp L_i$ that is $L \in \operatorname{GR} \prod^{\mathsf{Strd}(\mathscr{F})} a$. So $f \sqsubseteq \prod^{\mathsf{Strd}(\mathscr{F})} a$ and thus $\prod^{\mathsf{Strd}(\mathscr{F})} a$ is the greatest lower bound of $\left\{ \frac{\prod^{\mathsf{Strd}(\mathscr{F})} A}{A \in S} \right\}$. □

Conjecture 1699. Let $\mathscr{F}$ be a family of sets of filters on meet-semilattices with least elements. Let $a \in \prod \mathscr{F}$, $S \in \mathscr{P} \prod \mathscr{F}$ be a generalized filter base, $\prod S = a$, f is a staroid of the form $\prod \mathscr{F}$. Then

$$\prod^{\mathsf{Strd}(\mathscr{F})} a \not\asymp f \Leftrightarrow \forall A \in S : \prod^{\mathsf{Strd}(\mathfrak{Z})} A \not\asymp f.$$

21.8. On products of staroids

Definition 1700. $\prod^{(D)} F = \left\{ \frac{\operatorname{uncurry} z}{z \in \prod F} \right\}$ (*reindexation product*) for every indexed family F of relations.

Definition 1701. *Reindexation product* of an indexed family F of anchored relations is defined by the formulas:

$$\operatorname{form} \prod^{(D)} F = \operatorname{uncurry}(\operatorname{form} \circ F) \quad \text{and} \quad \operatorname{GR} \prod^{(D)} F = \prod^{(D)} (\operatorname{GR} \circ F).$$

Obvious 1702.

1°. $\operatorname{form} \prod^{(D)} F = \left\{ \frac{((i,j),(\operatorname{form} F_i)_j)}{i \in \operatorname{dom} F, j \in \operatorname{arity} F_i} \right\}$;

2°. $\operatorname{GR} \prod^{(D)} F = \left\{ \frac{\left\{ \frac{((i,j),(zi)j)}{i \in \operatorname{dom} F, j \in \operatorname{arity} F_i} \right\}}{z \in \prod (\operatorname{GR} \circ F)} \right\}$.

Proposition 1703. $\prod^{(D)} F$ is an anchored relation if every F_i is an anchored relation.

Proof. We need to prove $\operatorname{GR} \prod^{(D)} F \in \mathscr{P} \prod \operatorname{form} \left(\prod^{(D)} F \right)$ that is $\operatorname{GR} \prod^{(D)} F \subseteq \prod \operatorname{form} \left(\prod^{(D)} F \right)$; $\left\{ \frac{\left\{ \frac{((i,j),(zi)j)}{i \in \operatorname{dom} F, j \in \operatorname{arity} F_i} \right\}}{z \in \prod (\operatorname{GR} \circ F)} \right\} \subseteq \prod \left\{ \frac{((i,j),(\operatorname{form} F_i)_j)}{i \in \operatorname{dom} F, j \in \operatorname{arity} F_i} \right\}$;
$\forall z \in \prod (\operatorname{GR} \circ F), i \in \operatorname{dom} F, j \in \operatorname{arity} F_i : (zi)j \in (\operatorname{form} F_i)_j$.
Really, $zi \in \operatorname{GR} F_i \subseteq \prod (\operatorname{form} F_i)$ and thus $(zi)j \in (\operatorname{form} F_i)_j$. □

Obvious 1704. $\operatorname{arity} \prod^{(D)} F = \coprod_{i \in \operatorname{dom} F} \operatorname{arity} F_i = \left\{ \frac{(i,j)}{i \in \operatorname{dom} F, j \in \operatorname{arity} F_i} \right\}$.

Definition 1705. $f \times^{(D)} g = \prod^{(D)} [\![f, g]\!]$.

Lemma 1706. $\prod^{(D)} F$ is an upper set if every F_i is an upper set.

Proof. We need to prove that $\prod^{(D)} F$ is an upper set. Let $a \in \prod^{(D)} F$ and an anchored relation $b \sqsupseteq a$ of the same form as a. We have $a = \operatorname{uncurry} z$ for some $z \in \prod F$ that is $a(i,j) = (zi)j$ for all $i \in \operatorname{dom} F$ and $j \in \operatorname{dom} F_i$ where $zi \in F_i$. Also $b(i,j) \sqsupseteq a(i,j)$. Thus $(\operatorname{curry} b)i \sqsupseteq zi$; $\operatorname{curry} b \in \prod F$ because every F_i is an upper set and so $b \in \prod^{(D)} F$. □

Proposition 1707. Let F be an indexed family of anchored relations and every $(\operatorname{form} F)_i$ be a join-semilattice.

1°. $\prod^{(D)} F$ is a prestaroid if every F_i is a prestaroid.
2°. $\prod^{(D)} F$ is a staroid if every F_i is a staroid.
3°. $\prod^{(D)} F$ is a completary staroid if every F_i is a completary staroid.

PROOF.

1°. Let $q \in \operatorname{arity} \prod^{(D)} F$ that is $q = (i, j)$ where $i \in \operatorname{dom} F$, $j \in \operatorname{arity} F_i$; let

$$L \in \prod \left(\left(\operatorname{form} \prod^{(D)} F \right) |_{\left(\operatorname{arity} \prod^{(D)} F \right) \setminus \{q\}} \right)$$

that is $L_{(i',j')} \in \left(\operatorname{form} \prod^{(D)} F \right)_{(i',j')}$ for every $(i', j') \in \left(\operatorname{arity} \prod^{(D)} F \right) \setminus \{q\}$, that is $L_{(i',j')} \in (\operatorname{form} F_{i'})_{j'}$. We have $X \in \left(\operatorname{form} \prod^{(D)} F \right)_{(i,j)} \Leftrightarrow X \in (\operatorname{form} F_i)_j$. So

$$\left(\operatorname{val} \prod^{(D)} F \right)_{(i,j)} L = \left\{ \frac{X \in (\operatorname{form} F_i)_j}{L \cup \{((i,j), X)\} \in \operatorname{GR} \prod^{(D)} F} \right\} =$$

$$\left\{ \frac{X \in (\operatorname{form} F_i)_j}{\exists z \in \prod (\operatorname{GR} \circ F) : L \cup \{((i,j), X)\} = \operatorname{uncurry} z} \right\} =$$

$$\left\{ \frac{X \in (\operatorname{form} F_i)_j}{\exists z \in \prod \left((\operatorname{GR} \circ F)|_{\left(\operatorname{arity} \prod^{(D)} F \right) \setminus \{(i,j)\}} \right), v \in \operatorname{GR} F_i : (L = \operatorname{uncurry} z \wedge v_j = X)} \right\} =$$

$$\left\{ \frac{X \in (\operatorname{form} F_i)_j}{\exists z \in \prod \left((\operatorname{GR} \circ F)|_{\left(\operatorname{arity} \prod^{(D)} F \right) \setminus \{(i,j)\}} \right) : L = \operatorname{uncurry} z \wedge \exists v \in \operatorname{GR} F_i : v_j = X} \right\}.$$

If $\exists z \in \prod \left((\operatorname{GR} \circ F)|_{\left(\operatorname{arity} \prod^{(D)} F \right) \setminus \{(i,j)\}} \right) : L = \operatorname{uncurry} z$ is false then $\left(\operatorname{val} \prod^{(D)} F \right)_{(i,j)} L = \emptyset$ is a free star. We can assume it is true. So

$$\left(\operatorname{val} \prod^{(D)} F \right)_{(i,j)} L = \left\{ \frac{X \in (\operatorname{form} F_i)_j}{\exists v \in \operatorname{GR} F_i : v_j = X} \right\} =$$

$$\left\{ \frac{X \in (\operatorname{form} F_i)_j}{\exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} : K \cup \{(j, X)\} \in \operatorname{GR} F_i} \right\} =$$

$$\left\{ \frac{X \in (\operatorname{form} F_i)_j}{\exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} : X \in (\operatorname{val} F_i)_j K} \right\}.$$

Thus

$$A \sqcup B \in \left(\operatorname{val} \prod^{(D)} F \right)_{(i,j)} L \Leftrightarrow$$

$$\exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} : A \sqcup B \in (\operatorname{val} F_i)_j K \Leftrightarrow$$

$$\exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} : (A \in (\operatorname{val} F_i)_j K \vee B \in (\operatorname{val} F_i)_j K) \Leftrightarrow$$

$$\begin{array}{c} \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} : A \in (\operatorname{val} F_i)_j K \vee \\ \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} : B \in (\operatorname{val} F_i)_j K \end{array} \Leftrightarrow$$

$$A \in \left(\operatorname{val} \prod^{(D)} F \right)_{(i,j)} L \vee B \in \left(\operatorname{val} \prod^{(D)} F \right)_{(i,j)} L.$$

Least element $\bot$ is not in $\left(\operatorname{val} \prod^{(D)} F \right)_{(i,j)} L$ because $K \cup \{(j, \bot)\} \notin \operatorname{GR} F_i$.

2°. From the lemma.

3°. We need to prove

$$L_0 \sqcup L_1 \in \operatorname{GR} \prod^{(D)} F \Leftrightarrow$$
$$\exists c \in \{0,1\}^{\operatorname{arity} \prod^{(D)} F} : \left(\lambda i \in \operatorname{arity} \prod^{(D)} F : L_{c(i)} i\right) \in \operatorname{GR} \prod^{(D)} F$$

for every $L_0, L_1 \in \prod \operatorname{form} \prod^{(D)} F$ that is $L_0, L_1 \in \prod \operatorname{uncurry}(\operatorname{form} \circ F)$.

Really $L_0 \sqcup L_1 \in \operatorname{GR} \prod^{(D)} F \Leftrightarrow L_0 \sqcup L_1 \in \left\{ \frac{\operatorname{uncurry} z}{z \in \prod(\operatorname{GR} \circ F)} \right\}$.

$$\exists c \in \{0,1\}^{\operatorname{arity} \prod^{(D)} F} : \left(\lambda i \in \operatorname{arity} \prod^{(D)} F : L_{c(i)} i\right) \in \operatorname{GR} \prod^{(D)} F \Leftrightarrow$$
$$\exists c \in \{0,1\}^{\operatorname{arity} \prod^{(D)} F} : \left(\lambda i \in \operatorname{arity} \prod^{(D)} F : L_{c(i)} i\right) \in \left\{ \frac{\operatorname{uncurry} z}{z \in \prod(\operatorname{GR} \circ F)} \right\} \Leftrightarrow$$
$$\exists c \in \{0,1\}^{\operatorname{arity} \prod^{(D)} F} : \operatorname{curry}\left(\lambda i \in \operatorname{arity} \prod^{(D)} F : L_{c(i)} i\right) \in \prod(\operatorname{GR} \circ F) \Leftrightarrow$$
$$\exists c \in \{0,1\}^{\operatorname{arity} \prod^{(D)} F} : \operatorname{curry}\left(\lambda (i,j) \in \operatorname{arity} \prod^{(D)} F : L_{c(i,j)} (i,j)\right) \in \prod(\operatorname{GR} \circ F) \Leftrightarrow$$
$$\exists c \in \{0,1\}^{\operatorname{arity} \prod^{(D)} F} : (\lambda i \in \operatorname{dom} F : (\lambda j \in \operatorname{dom} F_i : L_{c(i,j)}(i,j))) \in \prod(\operatorname{GR} \circ F) \Leftrightarrow$$
$$\exists c \in \{0,1\}^{\operatorname{arity} \prod^{(D)} F} \forall i \in \operatorname{dom} F : (\lambda j \in \operatorname{dom} F_i : L_{c(i,j)}(i,j)) \in \operatorname{GR} F_i \Leftrightarrow$$
$$\forall i \in \operatorname{dom} F \exists c \in \{0,1\}^{\operatorname{dom} F_i} : (\lambda j \in \operatorname{dom} F_i : L_{c(j)}(i,j)) \in \operatorname{GR} F_i \Leftrightarrow$$
$$\forall i \in \operatorname{dom} F \exists c \in \{0,1\}^{\operatorname{dom} F_i} : (\lambda j \in \operatorname{dom} F_i : (\operatorname{curry}(L_{c(j)}) i) j) \in \operatorname{GR} F_i \Leftrightarrow$$
$$\forall i \in \operatorname{dom} F : \operatorname{curry}(L_0) i \sqcup \operatorname{curry}(L_1) i \in \operatorname{GR} F_i \Leftrightarrow$$
$$\forall i \in \operatorname{dom} F : (\operatorname{curry}(L_0) \sqcup \operatorname{curry}(L_1)) i \in \operatorname{GR} F_i \Leftrightarrow$$
$$\forall i \in \operatorname{dom} F : \operatorname{curry}(L_0 \sqcup L_1) i \in \operatorname{GR} F_i \Leftrightarrow$$
$$L_0 \sqcup L_1 \in \left\{ \frac{\operatorname{uncurry} z}{z \in \prod(\operatorname{GR} \circ F)} \right\} \Leftrightarrow$$
$$L_0 \sqcup L_1 \in \operatorname{GR} \prod^{(D)} F.$$

□

For staroids it is defined *ordinated product* $\prod^{(\mathrm{ord})}$ as defined in the section 3.7.4 above.

Obvious 1708. If f and g are anchored relations and there exists a bijection φ from $\operatorname{arity} g$ to $\operatorname{arity} f$ such that $\left\{ \frac{F \circ \varphi}{F \in \operatorname{GR} f} \right\} = \operatorname{GR} g$, then:

1°. f is a prestaroid iff g is a prestaroid.

2°. f is a staroid iff g is a staroid.

3°. f is a completary staroid iff g is a completary staroid.

Corollary 1709. Let F be an indexed family of anchored relations and every $(\operatorname{form} F)_i$ be a join-semilattice.

1°. $\prod^{(\mathrm{ord})} F$ is a prestaroid if every F_i is a prestaroid.
2°. $\prod^{(\mathrm{ord})} F$ is a staroid if every F_i is a staroid.
3°. $\prod^{(\mathrm{ord})} F$ is a completary staroid if every F_i is a completary staroid.

Proof. Use the fact that $\mathrm{GR} \prod^{(\mathrm{ord})} F = \left\{ \frac{F \circ \left(\bigoplus (\mathrm{dom} \circ F)\right)^{-1}}{F \in \mathrm{GR} \prod^{(D)} f} \right\}$. □

Definition 1710. $f \times^{(\mathrm{ord})} g = \prod^{(\mathrm{ord})} \llbracket f, g \rrbracket$.

Remark 1711. If f and g are binary funcoids, then $f \times^{(\mathrm{ord})} g$ is ternary.

21.9. Star categories

Definition 1712. A *precategory with star-morphisms* consists of

1°. a precategory C (*the base precategory*);
2°. a set M (*star-morphisms*);
3°. a function "arity" defined on M (how many objects are connected by this star-morphism);
4°. a function $\mathrm{Obj}_m : \mathrm{arity}\, m \to \mathrm{Obj}(C)$ defined for every $m \in M$;
5°. a function (*star composition*) $(m, f) \mapsto \mathrm{StarComp}(m, f)$ defined for $m \in M$ and f being an $(\mathrm{arity}\, m)$-indexed family of morphisms of C such that $\forall i \in \mathrm{arity}\, m : \mathrm{Src}\, f_i = \mathrm{Obj}_m\, i$ ($\mathrm{Src}\, f_i$ is the source object of the morphism f_i) such that

such that it holds:

1°. $\mathrm{StarComp}(m, f) \in M$;
2°. $\mathrm{arity}\, \mathrm{StarComp}(m, f) = \mathrm{arity}\, m$;
3°. $\mathrm{Obj}_{\mathrm{StarComp}(m,f)}\, i = \mathrm{Dst}\, f_i$;
4°. (*associativity law*)

$$\mathrm{StarComp}(\mathrm{StarComp}(m, f), g) = \mathrm{StarComp}(m, \lambda i \in \mathrm{arity}\, m : g_i \circ f_i).$$

The meaning of the set M is an extension of C having as morphisms things with arbitrary (possibly infinite) indexed set Obj_m of objects, not just two objects as morphisms of C have only source and destination.

Definition 1713. I will call Obj_m the *form* of the star-morphism m.

(Having fixed a precategory with star-morphisms) I will denote $\mathrm{StarHom}(P)$ the set of star-morphisms of the form P.

Proposition 1714. The sets $\mathrm{StarHom}(P)$ are disjoint (for different P).

Proof. If two star-morphisms have different forms, they are clearly not equal. □

Definition 1715. A *category with star-morphisms* is a precategory with star-morphisms whose base is a category and the following equality (*the law of composition with identity*) holds for every star-morphism m:

$$\mathrm{StarComp}(m, \lambda i \in \mathrm{arity}\, m : 1_{\mathrm{Obj}_m\, i}) = m.$$

Definition 1716. A *partially ordered precategory with star-morphisms* is a category with star-morphisms, whose base precategory is a partially ordered precategory and every set $\mathrm{StarHom}(X)$ is partially ordered for every X, such that:

$$m_0 \sqsubseteq m_1 \wedge f_0 \sqsubseteq f_1 \Rightarrow \mathrm{StarComp}(m_0, f_0) \sqsubseteq \mathrm{StarComp}(m_1, f_1)$$

for every $m_0, m_1 \in M$ such that $\mathrm{Obj}_{m_0} = \mathrm{Obj}_{m_1}$ and indexed families f_0 and f_1 of morphisms such that

$$\forall i \in \operatorname{arity} m : \operatorname{Src} f_0 i = \operatorname{Src} f_1 i = \mathrm{Obj}_{m_0} i = \mathrm{Obj}_{m_1} i;$$
$$\forall i \in \operatorname{arity} m : \operatorname{Dst} f_0 i = \operatorname{Dst} f_1 i.$$

Definition 1717. A *partially ordered category with star-morphisms* is a category with star-morphisms which is also a partially ordered precategory with star-morphisms.

Definition 1718. A *quasi-invertible* precategory with star-morphisms is a partially ordered precategory with star-morphisms whose base precategory is a quasi-invertible precategory, such that for every index set n, star-morphisms a and b of arity n, and an n-indexed family f of morphisms of the base precategory it holds

$$b \not\asymp \operatorname{StarComp}(a, f) \Leftrightarrow a \not\asymp \operatorname{StarComp}(b, f^\dagger).$$

(Here $f^\dagger = \lambda i \in \operatorname{dom} f : (f_i)^\dagger$.)

Definition 1719. A *quasi-invertible* category with star-morphisms is a quasi-invertible precategory with star-morphisms which is a category with star-morphisms.

Each category with star-morphisms gives rise to a category (*abrupt category*, see a remark below why I call it “abrupt”), as described below. Below for simplicity I assume that the set M and the set of our indexed families of functions are disjoint. The general case (when they are not necessarily disjoint) may be easily elaborated by the reader.

- Objects are indexed (by $\operatorname{arity} m$ for some $m \in M$) families of objects of the category C and an (arbitrarily chosen) object None not in this set.
- There are the following disjoint sets of morphisms:
  1°. indexed (by $\operatorname{arity} m$ for some $m \in M$) families of morphisms of C;
  2°. elements of M;
  3°. the identity morphism 1_{None} on None.
- Source and destination of morphisms are defined by the formulas:
  – $\operatorname{Src} f = \lambda i \in \operatorname{dom} f : \operatorname{Src} f_i$;
  – $\operatorname{Dst} f = \lambda i \in \operatorname{dom} f : \operatorname{Dst} f_i$;
  – $\operatorname{Src} m = \mathrm{None}$;
  – $\operatorname{Dst} m = \mathrm{Obj}_m$.
- Compositions of morphisms are defined by the formulas:
  – $g \circ f = \lambda i \in \operatorname{dom} f : g_i \circ f_i$ for our indexed families f and g of morphisms;
  – $f \circ m = \operatorname{StarComp}(m, f)$ for $m \in M$ and a composable indexed family f;
  – $m \circ 1_{\mathrm{None}} = m$ for $m \in M$;
  – $1_{\mathrm{None}} \circ 1_{\mathrm{None}} = 1_{\mathrm{None}}$.
- Identity morphisms for an object X are:
  – $\lambda i \in X : 1_{X_i}$ if $X \neq \mathrm{None}$;
  – 1_{None} if $X = \mathrm{None}$.

Proof. We need to prove it is really a category.
We need to prove:

1°. Composition is associative.
2°. Composition with identities complies with the identity law.

Really:

1°. $(h \circ g) \circ f = \lambda i \in \operatorname{dom} f : (h_i \circ g_i) \circ f_i = \lambda i \in \operatorname{dom} f : h_i \circ (g_i \circ f_i) = h \circ (g \circ f)$;

$$g \circ (f \circ m) = \operatorname{StarComp}(\operatorname{StarComp}(m, f), g) = \\ \operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : g_i \circ f_i) = \operatorname{StarComp}(m, g \circ f) = (g \circ f) \circ m;$$

$f \circ (m \circ 1_{\mathrm{None}}) = f \circ m = (f \circ m) \circ 1_{\mathrm{None}}$.

2°. $m \circ 1_{\mathrm{None}} = m$; $1_{\operatorname{Dst} m} \circ m = \operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : 1_{\operatorname{Obj}_m i}) = m$.

□

REMARK 1720. I call the above defined category *abrupt category* because (excluding identity morphisms) it allows composition with an $m \in M$ only on the left (not on the right) so that the morphism m is "abrupt" on the right.

By $[\![x_0, \ldots, x_{n-1}]\!]$ I denote an n-tuple.

DEFINITION 1721. Precategory with star morphisms *induced* by a dagger precategory C is:

- The base category is C.
- Star-morphisms are morphisms of C.
- $\operatorname{arity} f = \{0, 1\}$.
- $\operatorname{Obj}_m = [\![\operatorname{Src} m, \operatorname{Dst} m]\!]$.
- $\operatorname{StarComp}(m, [\![f, g]\!]) = g \circ m \circ f^\dagger$.

Let prove it is really a precategory with star-morphisms.

PROOF. We need to prove the associativity law:

$$\operatorname{StarComp}(\operatorname{StarComp}(m, [\![f, g]\!]), [\![p, q]\!]) = \operatorname{StarComp}(m, [\![p \circ f, q \circ g]\!]).$$

Really,

$$\operatorname{StarComp}(\operatorname{StarComp}(m, [\![f, g]\!]), [\![p, q]\!]) = \operatorname{StarComp}(g \circ m \circ f^\dagger, [\![p, q]\!]) = \\ q \circ g \circ m \circ f^\dagger \circ p^\dagger = q \circ g \circ m \circ (p \circ f)^\dagger = \operatorname{StarComp}(m, [\![p \circ f, q \circ g]\!]).$$

□

DEFINITION 1722. Category with star morphisms *induced* by a dagger category C is the above defined precategory with star-morphisms.

That it is a category (the law of composition with identity) is trivial.

REMARK 1723. We can carry definitions (such as below defined cross-composition product) from categories with star-morphisms into plain dagger categories. This allows us to research properties of cross-composition product of indexed families of morphisms for categories with star-morphisms without separately considering the special case of dagger categories and just binary star-composition product.

21.9.1. Abrupt of quasi-invertible categories with star-morphisms.

DEFINITION 1724. The abrupt partially ordered precategory of a partially ordered precategory with star-morphisms is the abrupt precategory with the following order of morphisms:

- Indexed (by $\operatorname{arity} m$ for some $m \in M$) families of morphisms of C are ordered as function spaces of posets.
- Star-morphisms (which are morphisms $\mathrm{None} \to \operatorname{Obj}_m$ for some $m \in M$) are ordered in the same order as in the precategory with star-morphisms.
- Morphisms $\mathrm{None} \to \mathrm{None}$ which are only the identity morphism ordered by the unique order on this one-element set.

We need to prove it is a partially ordered precategory.

PROOF. It trivially follows from the definition of partially ordered precategory with star-morphisms. $\square$

21.10. Product of an arbitrary number of funcoids

In this section it will be defined a product of an arbitrary (possibly infinite) indexed family of funcoids.

21.10.1. Mapping a morphism into a pointfree funcoid.

DEFINITION 1725. Let's define the pointfree funcoid χf for every morphism f of a quasi-invertible category:

$$\langle \chi f \rangle a = f \circ a \quad \text{and} \quad \left\langle (\chi f)^{-1} \right\rangle b = f^{\dagger} \circ b.$$

We need to prove it is really a pointfree funcoid.

PROOF. $b \not\asymp \langle \chi f \rangle a \Leftrightarrow b \not\asymp f \circ a \Leftrightarrow a \not\asymp f^{\dagger} \circ b \Leftrightarrow a \not\asymp \left\langle (\chi f)^{-1} \right\rangle b$. $\square$

REMARK 1726. $\langle \chi f \rangle = (f \circ -)$ is the Hom-functor $\operatorname{Hom}(f, -)$ and we can apply Yoneda lemma to it. (See any category theory book for definitions of these terms.)

OBVIOUS 1727. $\langle \chi (g \circ f) \rangle a = g \circ f \circ a$ for composable morphisms f and g or a quasi-invertible category.

21.10.2. General cross-composition product.

DEFINITION 1728. Let fix a quasi-invertible category with with star-morphisms. If f is an indexed family of morphisms from its base category, then the pointfree funcoid $\prod^{(C)} f$ (*cross-composition product* of f) from $\operatorname{StarHom}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)$ to $\operatorname{StarHom}(\lambda i \in \operatorname{dom} f : \operatorname{Dst} f_i)$ is defined by the formulas (for all star-morphisms a and b of these forms):

$$\left\langle \prod^{(C)} f \right\rangle a = \operatorname{StarComp}(a, f) \quad \text{and} \quad \left\langle \left(\prod^{(C)} f \right)^{-1} \right\rangle b = \operatorname{StarComp}(b, f^{\dagger}).$$

It is really a pointfree funcoid by the definition of quasi-invertible category with star-morphisms.

THEOREM 1729. $\left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) = \prod^{(C)}_{i \in n} (g_i \circ f_i)$ for every n-indexed families f and g of composable morphisms of a quasi-invertible category with star-morphisms.

PROOF. $\left\langle \prod^{(C)}_{i \in n} (g_i \circ f_i) \right\rangle a = \operatorname{StarComp}(a, \lambda i \in n : g_i \circ f_i) = \operatorname{StarComp}(\operatorname{StarComp}(a, f), g)$ and

$$\left\langle \left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) \right\rangle a = \left\langle \prod^{(C)} g \right\rangle \left\langle \prod^{(C)} f \right\rangle a = \operatorname{StarComp}(\operatorname{StarComp}(a, f), g).$$

The rest follows from symmetry. $\square$

COROLLARY 1730. $\left(\prod^{(C)} f_{k-1} \right) \circ \ldots \circ \left(\prod^{(C)} f_0 \right) = \prod^{(C)}_{i \in n} (f_{k-1} \circ \ldots \circ f_0)$ for every n-indexed families $f_0, \ldots, f_{n-1}$ of composable morphisms of a quasi-invertible category with star-morphisms.

PROOF. By math induction. $\square$

21.10.3. Star composition of binary relations. First define *star composition* for an n-ary relation a and an n-indexed family f of binary relations as an n-ary relation complying with the formulas:

$$\mathrm{Obj}_{\mathrm{StarComp}(a,f)} = \{*\}^n;$$
$$L \in \mathrm{StarComp}(a,f) \Leftrightarrow \exists y \in a \forall i \in n : y_i \; f_i \; L_i$$

where $*$ is a unique object of the group of small binary relations considered as a category.

PROPOSITION 1731. $b \not\asymp \mathrm{StarComp}(a,f) \Leftrightarrow \exists x \in a, y \in b \forall j \in n : x_j \; f_j \; y_j$.

PROOF.

$$b \not\asymp \mathrm{StarComp}(a,f) \Leftrightarrow \exists y : (y \in b \wedge y \in \mathrm{StarComp}(a,f)) \Leftrightarrow$$
$$\exists y : (y \in b \wedge \exists x \in a \forall j \in n : x_j \; f_j \; y_j) \Leftrightarrow \exists x \in a, y \in b \forall j \in n : x_j \; f_j \; y_j.$$

□

THEOREM 1732. The group of small binary relations considered as a category together with the set of of all small n-ary relations (for every small n) and the above defined star-composition form a quasi-invertible category with star-morphisms.

PROOF. We need to prove:

1°. $\mathrm{StarComp}(\mathrm{StarComp}(m,f),g) = \mathrm{StarComp}(m, \lambda i \in n : g_i \circ f_i)$;
2°. $\mathrm{StarComp}(m, \lambda i \in \operatorname{arity} m : 1_{\mathrm{Obj}_m i}) = m$;
3°. $b \not\asymp \mathrm{StarComp}(a,f) \Leftrightarrow a \not\asymp \mathrm{StarComp}(b, f^\dagger)$ (the rest is obvious).

Really,

1°. $L \in \mathrm{StarComp}(a,f) \Leftrightarrow \exists y \in a \forall i \in n : y_i \; f_i \; L_i$.
Define the relation $R(f)$ by the formula $x \; R(f) \; y \Leftrightarrow \forall i \in n : x_i \; f_i \; y_i$. Obviously

$$R(\lambda i \in n : g_i \circ f_i) = R(g) \circ R(f).$$

$L \in \mathrm{StarComp}(a,f) \Leftrightarrow \exists y \in a : y \; R(f) \; L$.

$$L \in \mathrm{StarComp}(\mathrm{StarComp}(a,f),g) \Leftrightarrow \exists p \in \mathrm{StarComp}(a,f) : p \; R(g) \; L \Leftrightarrow$$
$$\exists p, y \in a : (y \; R(f) \; p \wedge p \; R(g) \; L) \Leftrightarrow \exists y \in a : y \; (R(g) \circ R(f)) \; L \Leftrightarrow$$
$$\exists y \in a : y \; R(\lambda i \in n : g_i \circ f_i) \; L \Leftrightarrow L \in \mathrm{StarComp}(a, \lambda i \in n : g_i \circ f_i)$$

because $p \in \mathrm{StarComp}(a,f) \Leftrightarrow \exists y \in a : y \; R(f) \; p$.
2°. Obvious.
3°. It follows from the proposition above.

□

OBVIOUS 1733. $\mathrm{StarComp}(a \cup b, f) = \mathrm{StarComp}(a,f) \cup \mathrm{StarComp}(b,f)$ for n-ary relations a, b and an n-indexed family f of binary relations.

THEOREM 1734. $\left\langle \prod^{(C)} f \right\rangle \prod a = \prod_{i \in n} \langle f_i \rangle^* a_i$ for every family $f = f_{i \in n}$ of binary relations and $a = a_{i \in n}$ where a_i is a small set (for each $i \in n$).

PROOF.

$$\begin{aligned}
L \in \left\langle \prod^{(C)} f \right\rangle \prod a &\Leftrightarrow \\
L \in \operatorname{StarComp}\left(\prod a, f\right) &\Leftrightarrow \\
\exists y \in \prod a \forall i \in n : y_i \; f_i \; L_i &\Leftrightarrow \\
\exists y \in \prod a \forall i \in n : \{y_i\} \not\asymp \left\langle f_i^{-1} \right\rangle^* \{L_i\} &\Leftrightarrow \\
\forall i \in n \exists y \in a_i : \{y\} \not\asymp \left\langle f_i^{-1} \right\rangle^* \{L_i\} &\Leftrightarrow \\
\forall i \in n : a_i \not\asymp \left\langle f_i^{-1} \right\rangle^* \{L_i\} &\Leftrightarrow \\
\forall i \in n : \{L_i\} \not\asymp \langle f_i \rangle^* a_i &\Leftrightarrow \\
\forall i \in n : L_i \in \langle f_i \rangle^* a_i &\Leftrightarrow \\
L \in \prod_{i \in n} \langle f_i \rangle^* a_i&.
\end{aligned}$$

□

21.10.4. Star composition of Rel-morphisms. Define *star composition* for an n-ary anchored relation a and an n-indexed family f of **Rel**-morphisms as an n-ary anchored relation complying with the formulas:

$$\operatorname{Obj}_{\operatorname{StarComp}(a,f)} = \lambda i \in \operatorname{arity} a : \operatorname{Dst} f_i;$$
$$\operatorname{arity} \operatorname{StarComp}(a, f) = \operatorname{arity} a;$$
$$L \in \operatorname{GR} \operatorname{StarComp}(a, f) \Leftrightarrow L \in \operatorname{StarComp}(\operatorname{GR} a, \operatorname{GR} \circ f).$$

(Here I denote $\operatorname{GR}(A, B, f) = f$ for every **Rel**-morphism f.)

PROPOSITION 1735.

$$b \not\asymp \operatorname{StarComp}(a, f) \Leftrightarrow \exists x \in a, y \in b \forall j \in n : x_j \operatorname{GR}(f_j) \, y_j.$$

PROOF. From the previous section. □

THEOREM 1736. Relations with above defined compositions form a quasi-invertible category with star-morphisms.

PROOF. We need to prove:

1°. $\operatorname{StarComp}(\operatorname{StarComp}(m, f), g) = \operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : g_i \circ f_i)$;
2°. $\operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : 1_{\operatorname{Obj}_m i}) = m$;
3°. $b \not\asymp \operatorname{StarComp}(a, f) \Leftrightarrow a \not\asymp \operatorname{StarComp}(b, f^\dagger)$

(the rest is obvious).

It follows from the previous section. □

PROPOSITION 1737. $\operatorname{StarComp}(a \sqcup b, f) = \operatorname{StarComp}(a, f) \sqcup \operatorname{StarComp}(b, f)$ for an n-ary anchored relations a, b and an n-indexed family f of **Rel**-morphisms.

PROOF. It follows from the previous section. □

THEOREM 1738. Cross-composition product of a family of **Rel**-morphisms is a principal funcoid.

PROOF. By the proposition and symmetry $\prod^{(C)} f$ is a pointfree funcoid. Obviously it is a funcoid $\prod_{i \in n} \operatorname{Src} f_i \to \prod_{i \in n} \operatorname{Dst} f_i$. Its completeness (and dually co-completeness) is obvious. □

21.10.5. Cross-composition product of funcoids. Let a be a an anchored relation of the form $\mathfrak{A}$ and $\operatorname{dom}\mathfrak{A} = n$.

Let every f_i (for all $i \in n$) be a pointfree funcoid with $\operatorname{Src} f_i = \mathfrak{A}_i$.

The star-composition of a with f is an anchored relation of the form $\lambda i \in \operatorname{dom}\mathfrak{A} : \operatorname{Dst} f_i$ defined by the formula

$$L \in \operatorname{GR}\operatorname{StarComp}(a, f) \Leftrightarrow (\lambda i \in n : \langle f_i^{-1} \rangle L_i) \in \operatorname{GR} a.$$

THEOREM 1739. Let $\operatorname{Src} f_i$ be separable starrish join-semilattice and $\operatorname{Dst} f_i$ be a starrish join-semilattice for every $i \in n$ for a set n. Let $\operatorname{form} a = \prod_{i \in n} (\operatorname{Src} f_i)$.

1°. If a is a prestaroid then $\operatorname{StarComp}(a, f)$ is a prestaroid.
2°. If a is a staroid and $\operatorname{Src} f_i$ are strongly separable then $\operatorname{StarComp}(a, f)$ is a staroid.
3°. If a is a completary staroid and then $\operatorname{StarComp}(a, f)$ is a completary staroid.

PROOF. We have $\langle f_i^{-1} \rangle (X \sqcup Y) = \langle f_i^{-1} \rangle X \sqcup \langle f_i^{-1} \rangle Y$ by theorem 1501.

1°. Let $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} (\operatorname{form} f_i)$ for some $k \in n$ and $X, Y \in \operatorname{form} f_k$. Then

$$\begin{aligned}
X \sqcup Y \in \langle \operatorname{StarComp}(a, f) \rangle_k^* L &\Leftrightarrow \\
\left(\lambda i \in \operatorname{dom} f : \langle f_i^{-1} \rangle \left(\left\{ \begin{array}{ll} X \sqcup Y & \text{if } i = k \\ L_i & \text{if } i \neq k \end{array} \right. \right)_i \right) \in \operatorname{GR} a &\Leftrightarrow \\
\left(\lambda i \in \operatorname{dom} f : \left(\left\{ \begin{array}{ll} \langle f_i^{-1} \rangle X \sqcup \langle f_i^{-1} \rangle Y & \text{if } i = k \\ \langle f_i^{-1} \rangle L_i & \text{if } i \neq k \end{array} \right. \right)_i \right) \in \operatorname{GR} a &\Leftrightarrow \\
\langle f_i^{-1} \rangle X \sqcup \langle f_i^{-1} \rangle Y \in \langle a \rangle_k^* (\lambda i \in (\operatorname{dom} f) \setminus \{k\} : \langle f_i^{-1} \rangle L_i) &\Leftrightarrow \\
\langle f_i^{-1} \rangle X \in \langle a \rangle_k^* (\lambda i \in n \setminus \{k\} : \langle f_i^{-1} \rangle L_i) \vee \langle f_i^{-1} \rangle Y \in \langle a \rangle_k^* (\lambda i \in n \setminus \{k\} : \langle f_i^{-1} \rangle L_i) &\Leftrightarrow \\
\left(\lambda i \in \operatorname{dom} f : \left(\left\{ \begin{array}{ll} \langle f_i^{-1} \rangle X & \text{if } i = k \\ \langle f_i^{-1} \rangle L_i & \text{if } i \neq k \end{array} \right. \right)_i \right) \in \operatorname{GR} a \vee & \\
\left(\lambda i \in \operatorname{dom} f : \left(\left\{ \begin{array}{ll} \langle f_i^{-1} \rangle Y & \text{if } i = k \\ \langle f_i^{-1} \rangle L_i & \text{if } i \neq k \end{array} \right. \right)_i \right) \in \operatorname{GR} a &\Leftrightarrow \\
\left(\lambda i \in \operatorname{dom} f : \langle f_i^{-1} \rangle \left(\left\{ \begin{array}{ll} X & \text{if } i = k \\ L_i & \text{if } i \neq k \end{array} \right. \right)_i \right) \in \operatorname{GR} a \vee & \\
\left(\lambda i \in \operatorname{dom} f : \langle f_i^{-1} \rangle \left(\left\{ \begin{array}{ll} Y & \text{if } i = k \\ L_i & \text{if } i \neq k \end{array} \right. \right)_i \right) \in \operatorname{GR} a &\Leftrightarrow \\
X \in \langle \operatorname{StarComp}(a, f) \rangle_k^* L \vee Y \in \langle \operatorname{StarComp}(a, f) \rangle_k^* L. &
\end{aligned}$$

Thus $\operatorname{StarComp}(a, f)$ is a pre-staroid.

2°. $\langle f_i^{-1} \rangle$ are monotone functions by the proposition 1500. Thus $\langle f_i^{-1} \rangle X_i \sqsubseteq \langle f_i^{-1} \rangle Y_i$ if $X, Y \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} (\operatorname{form} f_i)$ and $X \sqsubseteq Y$. So if a is a staroid and $X \in \operatorname{GR}\operatorname{StarComp}(a, f)$ then $(\lambda i \in \operatorname{dom} f : \langle f_i^{-1} \rangle X_i) \in \operatorname{GR} a$ then $(\lambda i \in \operatorname{dom} f : \langle f_i^{-1} \rangle Y_i) \in \operatorname{GR} a$ that is $Y \in \operatorname{GR}\operatorname{StarComp}(a, f)$.

3°.

$$\begin{aligned}
L_0 \sqcup L_1 \in \operatorname{GR}\operatorname{StarComp}(a, f) &\Leftrightarrow \\
(\lambda i \in n : \langle f_i^{-1} \rangle (L_0 \sqcup L_1) i) \in \operatorname{GR} a &\Leftrightarrow \\
(\lambda i \in n : \langle f_i^{-1} \rangle L_0 i \sqcup \langle f_i^{-1} \rangle L_1 i) \in \operatorname{GR} a &\Leftrightarrow \\
\exists c \in \{0, 1\} : (\lambda i \in n : \langle f_i^{-1} \rangle L_{c(i)} i) \in \operatorname{GR} a &\Leftrightarrow \\
\exists c \in \{0, 1\} : (\lambda i \in n : L_{c(i)} i) \in \operatorname{GR}\operatorname{StarComp}(a, f). &
\end{aligned}$$

□

CONJECTURE 1740. $b \not\asymp^{\mathrm{Anch}(\mathfrak{A})} \operatorname{StarComp}(a, f) \Leftrightarrow \forall A \in \operatorname{GR} a, B \in \operatorname{GR} b, i \in n : A_i \; [f_i] \; B_i$ for anchored relations a and b on powersets.

It's consequence:

CONJECTURE 1741. $b \not\asymp^{\mathrm{Anch}(\mathfrak{A})} \operatorname{StarComp}(a, f) \Leftrightarrow a \not\asymp^{\mathrm{Anch}(\mathfrak{A})} \operatorname{StarComp}(b, f^\dagger)$ for anchored relations a and b on powersets.

CONJECTURE 1742. $b \not\asymp^{\mathsf{Strd}(\mathfrak{A})} \operatorname{StarComp}(a, f) \Leftrightarrow a \not\asymp^{\mathsf{Strd}(\mathfrak{A})} \operatorname{StarComp}(b, f^\dagger)$ for pre-staroids a and b on powersets.

PROPOSITION 1743. Anchored relations with objects being posets with above defined star-morphisms is a category with star morphisms.

PROOF. We need to prove:

1°. $\operatorname{StarComp}(\operatorname{StarComp}(m, f), g) = \operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : g_i \circ f_i)$;
2°. $\operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : 1_{\mathrm{Obj}_m i}) = m$.

(the rest is obvious). Really,

$$\begin{aligned} & L \in \operatorname{GR} \operatorname{StarComp}(\operatorname{StarComp}(m, f), g) \Leftrightarrow \\ & (\lambda i \in \operatorname{arity} m : \langle g_i^{-1} \rangle L_i) \in \operatorname{GR} \operatorname{StarComp}(m, f) \Leftrightarrow \\ & (\lambda i \in n : \langle f_i^{-1} \rangle (\lambda j \in n : \langle g_j^{-1} \rangle L_j)_i) \in \operatorname{GR} m \Leftrightarrow \\ & (\lambda i \in \operatorname{arity} m : \langle f_i^{-1} \rangle \langle g_i^{-1} \rangle L_i) \in \operatorname{GR} m \Leftrightarrow \\ & (\lambda i \in \operatorname{arity} m : \langle (g_i \circ f_i)^{-1} \rangle L_i) \in \operatorname{GR} m \Leftrightarrow \\ & L \in \operatorname{GR} \operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : g_i \circ f_i); \end{aligned}$$

and

$$\begin{aligned} & L \in \operatorname{GR} \operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : 1_{\mathrm{Obj}_m i}) \Leftrightarrow \\ & (\lambda i \in n : \langle 1_{\mathrm{Obj}_m i} \rangle L_i) \in \operatorname{GR} m \Leftrightarrow \\ & (\lambda i \in \operatorname{arity} m : \langle 1_{\mathrm{Obj}_m i} \rangle L_i) \in \operatorname{GR} m \Leftrightarrow \\ & (\lambda i \in \operatorname{arity} m : L_i) \in \operatorname{GR} m \Leftrightarrow L \in \operatorname{GR} m. \end{aligned}$$

□

CONJECTURE 1744. $\operatorname{StarComp}(a \sqcup b, f) = \operatorname{StarComp}(a, f) \sqcup \operatorname{StarComp}(b, f)$ for anchored relations a, b of a form $\mathfrak{A}$, where every $\mathfrak{A}_i$ is a distributive lattice, and an indexed family f of pointfree funcoids with $\operatorname{Src} f_i = \mathfrak{A}_i$.

21.10.6. Cross-composition product of funcoids through atoms. Let a be a an anchored relation of the form $\mathfrak{A}$ and $\operatorname{dom} \mathfrak{A} = n$.

Let every f_i (for all $i \in n$) be a pointfree funcoid with $\operatorname{Src} f_i = \mathfrak{A}_i$.

The *atomary star-composition* of a with f is an anchored relation of the form $\lambda i \in \operatorname{dom} \mathfrak{A} : \operatorname{Dst} f_i$ defined by the formula

$$L \in \operatorname{GR} \operatorname{StarComp}^{(a)}(a, f) \Leftrightarrow \exists y \in \operatorname{GR} a \cap \prod_{i \in n} \operatorname{atoms}^{\mathfrak{A}_i} \forall i \in n : y_i \; [f_i] \; L_i.$$

THEOREM 1745. Let $\operatorname{Dst} f_i$ be a starrish join-semilattice for every $i \in n$.

1°. If a is a prestaroid then $\operatorname{StarComp}^{(a)}(a, f)$ is a staroid.
2°. If a is a completary staroid and then $\operatorname{StarComp}^{(a)}(a, f)$ is a completary staroid.

PROOF.

1°. First prove that $\operatorname{StarComp}^{(a)}(a,f)$ is a prestaroid. We need to prove that $(\operatorname{val}\operatorname{StarComp}^{(a)}(a,f))_j L$ (for every $j\in n$) is a free star, that is

$$\left\{\frac{X\in(\operatorname{form}f)_j}{L\cup\{(j,X)\}\in\operatorname{GR}\operatorname{StarComp}^{(a)}(a,f)}\right\}$$

is a free star, that is the following is a free star

$$\left\{\frac{X\in(\operatorname{form}f)_j}{R(X)}\right\}$$

where $R(X)\Leftrightarrow\exists y\in\prod_{i\in n}\operatorname{atoms}^{\mathfrak{A}_i}:(\forall i\in n\setminus\{j\}:y_i\;[f_i]\;L_i\wedge y_j\;[f_j]\;X\wedge y\in\operatorname{GR}a)$.

$$\begin{aligned}
&R(X)\Leftrightarrow\\
&\exists y\in\prod_{i\in n}\operatorname{atoms}^{\mathfrak{A}_i}:(\forall i\in n\setminus\{j\}:y_i\;[f_i]\;L_i\wedge y_j\;[f_j]\;X\wedge y_j\in(\operatorname{val}a)_j(y|_{n\setminus\{j\}}))\Leftrightarrow\\
&\exists y\in\prod_{i\in n\setminus\{j\}}\operatorname{atoms}^{\mathfrak{A}_i},y'\in\operatorname{atoms}^{\mathfrak{A}_j}:\begin{pmatrix}\forall i\in n\setminus\{j\}:y_i\;[f_i]\;L_i\wedge\\ y'\;[f_j]\;X\wedge y'\in(\operatorname{val}a)_j(y|_{n\setminus\{j\}})\end{pmatrix}\Leftrightarrow\\
&\exists y\in\prod_{i\in n\setminus\{j\}}\operatorname{atoms}^{\mathfrak{A}_i}\forall i\in n\setminus\{j\}:y_i\;[f_i]\;L_i\wedge\\
&\exists y'\in\operatorname{atoms}^{\mathfrak{A}_j}:(y'\;[f_j]\;X\wedge y'\in(\operatorname{val}a)_j(y|_{n\setminus\{j\}})).
\end{aligned}$$

If $\exists y\in\prod_{i\in n\setminus\{j\}}\operatorname{atoms}^{\mathfrak{A}_i}\forall i\in n\setminus\{j\}:y_i\;[f_i]\;L_i$ is false our statement is obvious. We can assume it is true.

So it is enough to prove that

$$\left\{\frac{X\in(\operatorname{form}f)_j}{\exists y\in\prod_{i\in n\setminus\{j\}}\operatorname{atoms}^{\mathfrak{A}_i},y'\in\operatorname{atoms}^{\mathfrak{A}_j}:(y'\;[f_j]\;X\wedge y'\in(\operatorname{val}a)_j(y|_{n\setminus\{j\}}))}\right\}$$

is a free star. That is

$$Q=\left\{\frac{X\in(\operatorname{form}f)_j}{\exists y\in\prod_{i\in n\setminus\{j\}}\operatorname{atoms}^{\mathfrak{A}_i},y'\in(\operatorname{atoms}^{\mathfrak{A}_j})\cap(\operatorname{val}a)_j(y|_{n\setminus\{j\}}):y'\;[f_j]\;X}\right\}$$

is a free star. $\bot^{(\operatorname{form}f)_j}\notin Q$ is obvious. That Q is an upper set is obvious. It remains to prove that $X_0\sqcup X_1\in Q\Rightarrow X_0\in Q\vee X_1\in Q$ for every $X_0,X_1\in(\operatorname{form}f)_j$. Let $X_0\sqcup X_1\in Q$. Then there exist $y\in\prod_{i\in n\setminus\{j\}}\operatorname{atoms}^{\mathfrak{A}_i}$, $y'\in(\operatorname{atoms}^{\mathfrak{A}_j})\cap(\operatorname{val}a)_j(y|_{n\setminus\{j\}})$ such that $y'\;[f_j]\;X_0\sqcup X_1$. Consequently (proposition 1502) $y'\;[f_j]\;X_0\vee y'\;[f_j]\;X_1$. But then $X_0\in Q\vee X_1\in Q$.

To finish the proof we need to show that $\operatorname{GR}\operatorname{StarComp}^(a)(a,f)$ is an upper set, but this is obvious.

2°. Let a be a completary staroid. Let $L_0\sqcup L_1\in\operatorname{GR}\operatorname{StarComp}^{(a)}(a,f)$ that is $\exists y\in\prod_{i\in n}\operatorname{atoms}\mathfrak{A}_i:(\forall i\in n:y_i\;[f_i]\;L_0 i\sqcup L_1 i\wedge y\in\operatorname{GR}a)$ that is $\exists c\in\{0,1\}^n,y\in\prod_{i\in n}\operatorname{atoms}^{\mathfrak{A}_i}:\left(\forall i\in n:y_i\;[f_i]\;L_{c(i)}i\wedge y\in\operatorname{GR}a\right)$ (taken into account that $\operatorname{Dst}f_i$ is starrish) that is $\exists c\in\{0,1\}^n:(\lambda i\in n:L_{c(i)}i)\in\operatorname{GR}\operatorname{StarComp}^{(a)}(a,f)$. So $\operatorname{StarComp}^{(a)}(a,f)$ is a completary staroid.

□

LEMMA 1746. $b\not\asymp^{\operatorname{Anch}(\mathfrak{B})}\operatorname{StarComp}^{(a)}(a,f)\Leftrightarrow\forall A\in\operatorname{GR}a,B\in\operatorname{GR}b,i\in n:A_i\;[f_i]\;B_i$ for anchored relations a and b, provided that $\operatorname{Src}f_i$ are atomic posets.

PROOF.

$$\begin{aligned}
b \not\asymp^{\mathrm{Anch}(\mathfrak{B})} \mathrm{StarComp}^{(a)}(a,f) \Leftrightarrow \\
\exists x \in \mathrm{Anch}(\mathfrak{B}) \setminus \{\bot\} : (x \sqsubseteq b \wedge x \sqsubseteq \mathrm{StarComp}^{(a)}(a,f)) \Leftrightarrow \\
\exists x \in \mathrm{Anch}(\mathfrak{B}) \setminus \{\bot\} : (x \sqsubseteq b \wedge \forall B \in \mathrm{GR}\, x : B \in \mathrm{GR}\,\mathrm{StarComp}^{(a)}(a,f)) \Leftrightarrow \\
\exists x \in \mathrm{Anch}(\mathfrak{B}) \setminus \{\bot\} : \\
\left(x \sqsubseteq b \wedge \forall B \in \mathrm{GR}\, x \exists A \in \prod_{i \in \mathrm{dom}\,\mathfrak{B}} \mathrm{atoms}^{\mathfrak{B}_i} : (\forall i \in n : A_i \, [f_i] \, B_i \wedge A \in \mathrm{GR}\, a)\right) \Leftrightarrow \\
\exists x \in \mathrm{Anch}(\mathfrak{B}) \setminus \{\bot\} : (x \sqsubseteq b \wedge \forall B \in \mathrm{GR}\, x, A \in \mathrm{GR}\, a, i \in n : A_i \, [f_i] \, B_i) \Leftrightarrow \\
\exists x \in \mathrm{Anch}(\mathfrak{B}) : (x \sqsubseteq b \wedge \forall B \in \mathrm{GR}\, x, A \in \mathrm{GR}\, a, i \in n : A_i \, [f_i] \, B_i) \Leftrightarrow \\
\forall B \in \mathrm{GR}\, b, A \in \mathrm{GR}\, a, i \in n : A_i \, [f_i] \, B_i.
\end{aligned}$$

□

DEFINITION 1747. I will denote the cross-composition product for the star-composition $\mathrm{StarComp}^{(a)}$ as $\prod^{(a)}$.

THEOREM 1748. $a \left[\prod^{(a)} f\right] b \Leftrightarrow \forall A \in \mathrm{GR}\, a, B \in \mathrm{GR}\, b, i \in n : A_i \, [f_i] \, B_i$ for anchored relations a and b, provided that $\mathrm{Src}\, f_i$ and $\mathrm{Dst}\, f_i$ are atomic posets.

PROOF. From the lemma. □

CONJECTURE 1749. $b \not\asymp^{\mathsf{Strd}(\mathfrak{B})} \mathrm{StarComp}(a,f) \Leftrightarrow a \not\asymp^{\mathsf{Strd}(\mathfrak{A})} \mathrm{StarComp}(b,f^\dagger)$ for staroids a and b on indexed families $\mathfrak{A}$ and $\mathfrak{B}$ of filters on powersets.

THEOREM 1750. Anchored relations with objects being atomic posets and above defined compositions form a quasi-invertible precategory with star-morphisms.

REMARK 1751. It seems that this precategory with star-morphisms isn't a category with star-morphisms.

PROOF. We need to prove:

1°. $\mathrm{StarComp}^{(a)}(\mathrm{StarComp}^{(a)}(m,f),g) = \mathrm{StarComp}^{(a)}(m, \lambda i \in \mathrm{arity}\, m : g_i \circ f_i)$;
2°. $b \not\asymp \mathrm{StarComp}^{(a)}(a,f) \Leftrightarrow a \not\asymp \mathrm{StarComp}^{(a)}(b,f^\dagger)$

(the rest is obvious).

Really, let a be a star morphism and $\mathfrak{A}_i = (\mathrm{Obj}_a)i$ for every $i \in \mathrm{arity}\, a$;

1°. $L \in \mathrm{GR}\,\mathrm{StarComp}^{(a)}(a,f) \Leftrightarrow \exists y \in \mathrm{GR}\, a \cap \prod_{i \in n} \mathrm{atoms}^{\mathfrak{A}_i}\, \forall i \in n : y_i \, [f_i] \, L_i$.
Define the relation $R(f)$ by the formula $x \; R(f) \; y \Leftrightarrow \forall i \in n : x_i \, [f_i] \, y_i$. Obviously

$$R(\lambda i \in n : g_i \circ f_i) = R(g) \circ R(f).$$

$L \in \operatorname{GR}\operatorname{StarComp}^{(a)}(a, f) \Leftrightarrow \exists y \in \operatorname{GR} a \cap \prod_{i\in n} \operatorname{atoms}^{\mathfrak{A}_i} : y\ R(f)\ L.$

$$\begin{aligned}
L \in \operatorname{GR}\operatorname{StarComp}^{(a)}(\operatorname{StarComp}(a, f), g) \Leftrightarrow\\
\exists p \in \operatorname{GR}\operatorname{StarComp}^{(a)}(a, f) \cap \prod_{i\in n} \operatorname{atoms}^{(\operatorname{Dst} f)_i} : p\ R(g)\ L \Leftrightarrow\\
\exists p \in \prod_{i\in n} \operatorname{atoms}^{(\operatorname{Dst} f)_i}, y \in \operatorname{GR} a \cap \prod_{i\in n} \operatorname{atoms}^{(\operatorname{Src} f)_i} : (y\ R(f)\ p \wedge p\ R(g)\ L) \Leftrightarrow\\
\exists y \in \operatorname{GR} a \cap \prod_{i\in n} \operatorname{atoms}^{(\operatorname{Src} f)_i} : y\ (R(g) \circ R(f))\ L \Leftrightarrow\\
\exists y \in \operatorname{GR} a \cap \prod_{i\in n} \operatorname{atoms}^{(\operatorname{Src} f)_i} : y\ R(\lambda i \in n : g_i \circ f_i)\ L \Leftrightarrow\\
\exists y \in \operatorname{GR} a \cap \prod_{i\in n} \operatorname{atoms}^{(\operatorname{Src} f)_i}\ y \forall i \in n : y_i\ [g_i \circ f_i]\ L_i \Leftrightarrow\\
L \in \operatorname{GR}\operatorname{StarComp}^{(a)}(a, \lambda i \in n : g_i \circ f_i)
\end{aligned}$$

because $p \in \operatorname{GR}\operatorname{StarComp}^{(a)}(a, f) \Leftrightarrow \exists y \in \operatorname{GR} a \cap \prod_{i\in n} \operatorname{atoms}^{(\operatorname{Src} f)_i}\ y : y\ R(f)\ p$.

2°. It follows from the lemma above.

□

THEOREM 1752. $\left\langle \prod^{(a)} f \right\rangle \prod^{\mathsf{Strd}} a = \prod^{\mathsf{Strd}}_{i\in n} \langle f_i \rangle a_i$ for every family $f = f_{i\in n}$ of pointfree funcoids between atomic posets and $a = a_{i\in n}$ where $a_i \in \operatorname{Src} f_i$.

PROOF.

$$\begin{aligned}
L \in \operatorname{GR}\left\langle \prod^{(a)} f \right\rangle \prod^{\mathsf{Strd}} a \Leftrightarrow\\
L \in \operatorname{GR}\operatorname{StarComp}^{(a)}\left(\prod^{\mathsf{Strd}} a, f \right) \Leftrightarrow\\
\exists y \in \prod_{i\in\operatorname{dom}\mathfrak{A}} \operatorname{atoms}^{\mathfrak{A}_i}\ \forall i \in n : (y_i\ [f_i]\ L_i \wedge y_i \not\asymp a_i) \Leftrightarrow\\
\forall i \in n \exists y \in \operatorname{atoms}^{\mathfrak{A}_i} : (y\ [f_i]\ L_i \wedge y \not\asymp a_i) \Leftrightarrow\\
\forall i \in n : a_i\ [f_i]\ L_i \Leftrightarrow\\
\forall i \in n : L_i \not\asymp \langle f_i \rangle a_i \Leftrightarrow\\
L \in \operatorname{GR} \prod^{\mathsf{Strd}}_{i\in n} \langle f_i \rangle a_i.
\end{aligned}$$

□

CONJECTURE 1753. $\operatorname{StarComp}^{(a)}(a \sqcup b, f) = \operatorname{StarComp}^{(a)}(a, f) \sqcup \operatorname{StarComp}^{(a)}(b, f)$ for anchored relations a, b of a form $\mathfrak{A}$, where every $\mathfrak{A}_i$ is a distributive lattice, and an indexed family f of pointfree funcoids with $\operatorname{Src} f_i = \mathfrak{A}_i$.

21.10.7. Simple product of pointfree funcoids.

DEFINITION 1754. Let f be an indexed family of pointfree funcoids with every $\operatorname{Src} f_i$ and $\operatorname{Dst} f_i$ (for all $i \in \operatorname{dom} f$) being a poset with least element. *Simple product* of f is

$$\prod^{(S)} f = \left(\lambda x \in \prod_{i\in\operatorname{dom} f} \operatorname{Src} f_i : \lambda i \in \operatorname{dom} f : \langle f_i \rangle x_i, \lambda y \in \prod_{i\in\operatorname{dom} f} \operatorname{Dst} f_i : \lambda i \in \operatorname{dom} f : \left\langle f_i^{-1} \right\rangle y_i \right).$$

PROPOSITION 1755. Simple product is a pointfree funcoid

$$\prod^{(S)} f \in \mathsf{pFCD}\left(\prod_{i\in\operatorname{dom} f} \operatorname{Src} f_i, \prod_{i\in\operatorname{dom} f} \operatorname{Dst} f_i\right).$$

PROOF. Let $x \in \prod_{i\in\operatorname{dom} f} \operatorname{Src} f_i$ and $y \in \prod_{i\in\operatorname{dom} f} \operatorname{Dst} f_i$. Then (take into account that $\operatorname{Src} f_i$ and $\operatorname{Dst} f_i$ are posets with least elements)

$$\begin{aligned}
y \not\asymp \left(\lambda x \in \prod_{i\in\operatorname{dom} f} \operatorname{Src} f_i : \lambda i \in \operatorname{dom} f : \langle f_i\rangle x_i\right) x \Leftrightarrow \\
y \not\asymp \lambda i \in \operatorname{dom} f : \langle f_i\rangle x_i \Leftrightarrow \\
\exists i \in \operatorname{dom} f : y_i \not\asymp \langle f_i\rangle x_i \Leftrightarrow \\
\exists i \in \operatorname{dom} f : x_i \not\asymp \left\langle f_i^{-1}\right\rangle y_i \Leftrightarrow \\
x \not\asymp \lambda i \in \operatorname{dom} f : \left\langle f_i^{-1}\right\rangle y_i \Leftrightarrow \\
x \not\asymp \left(\lambda y \in \prod_{i\in\operatorname{dom} f} \operatorname{Dst} f_i : \lambda i \in \operatorname{dom} f : \left\langle f_i^{-1}\right\rangle y_i\right) y.
\end{aligned}$$

□

OBVIOUS 1756. $\left\langle \prod^{(S)} f\right\rangle x = \lambda i \in \operatorname{dom} f : \langle f_i\rangle x_i$ for $x \in \prod \operatorname{Src} f_i$.

OBVIOUS 1757. $\left(\left\langle \prod^{(S)} f\right\rangle x\right)_i = \langle f_i\rangle x_i$ for $x \in \prod \operatorname{Src} f_i$.

PROPOSITION 1758. f_i can be restored if we know $\prod^{(S)} f$ if f_i is a family of pointfree funcoids between posets with least elements.

PROOF. Let's restore the value of $\langle f_i\rangle x$ where $i \in \operatorname{dom} f$ and $x \in \operatorname{Src} f_i$.

Let $x'_i = x$ and $x'_j = \bot$ for $j \neq i$.

Then $\langle f_i\rangle x = \langle f_i\rangle x'_i = \left(\left\langle \prod^{(S)} f\right\rangle x'\right)_i$.

We have restored the value of $\langle f_i\rangle$. Restoring the value of $\left\langle f_i^{-1}\right\rangle$ is similar. □

REMARK 1759. In the above proposition it is not required that f_i are non-zero.

PROPOSITION 1760. $\left(\prod^{(S)} g\right) \circ \left(\prod^{(S)} f\right) = \prod^{(S)}_{i\in n}(g_i \circ f_i)$ for n-indexed families f and g of composable pointfree funcoids between posets with least elements.

PROOF.

$$\left\langle \prod^{(S)}_{i\in n}(g_i \circ f_i)\right\rangle x = \lambda i \in \operatorname{dom} f : \langle g_i \circ f_i\rangle x_i = \lambda i \in \operatorname{dom} f : \langle g_i\rangle\langle f_i\rangle x_i =$$

$$\left\langle \prod^{(S)} g\right\rangle \lambda i \in \operatorname{dom} f : \langle f_i\rangle x_i = \left\langle \prod^{(S)} g\right\rangle\left\langle \prod^{(S)} f\right\rangle x = \left\langle \left(\prod^{(S)} g\right) \circ \left(\prod^{(S)} f\right)\right\rangle x.$$

Thus $\left\langle \prod^{(S)}_{i\in n}(g_i \circ f_i)\right\rangle = \left\langle \left(\prod^{(S)} g\right) \circ \left(\prod^{(S)} f\right)\right\rangle$.

$\left\langle \left(\prod^{(S)}_{i\in n}(g_i \circ f_i)\right)^{-1}\right\rangle = \left\langle \left(\left(\prod^{(S)} g\right) \circ \left(\prod^{(S)} f\right)\right)^{-1}\right\rangle$ is similar. □

COROLLARY 1761. $\left(\prod^{(S)} f_{k-1}\right) \circ \ldots \circ \left(\prod^{(S)} f_0\right) = \prod^{(S)}_{i\in n}(f_{k-1} \circ \ldots \circ f_0)$ for every n-indexed families $f_0, \ldots, f_{n-1}$ of composable pointfree funcoids between posets with least elements.

21.11. Multireloids

DEFINITION 1762. I will call a *multireloid* of the form $A = A_{i\in n}$, where every each A_i is a set, a pair (f, A) where f is a filter on the set $\prod A$.

DEFINITION 1763. I will denote $\operatorname{Obj}(f, A) = A$ and $\operatorname{GR}(f, A) = f$ for every multireloid (f, A).

I will denote $\mathsf{RLD}(A)$ the set of multireloids of the form A.

The multireloid $\uparrow^{\mathsf{RLD}(A)} F$ for a relation F is defined by the formulas:

$$\operatorname{Obj} \uparrow^{\mathsf{RLD}(A)} F = A \quad \text{and} \quad \operatorname{GR} \uparrow^{\mathsf{RLD}(A)} F = \uparrow^{\prod A} F.$$

For an anchored relation f I define $\operatorname{Obj} \uparrow f = \operatorname{form} f$ and $\operatorname{GR} \uparrow f = \uparrow^{\prod \operatorname{form} f} \operatorname{GR} f$.

Let a be a multireloid of the form A and $\operatorname{dom} A = n$.

Let every f_i be a reloid with $\operatorname{Src} f_i = A_i$.

The star-composition of a with f is a multireloid of the form $\lambda i \in \operatorname{dom} A : \operatorname{Dst} f_i$ defined by the formulas:

$$\operatorname{arity} \operatorname{StarComp}(a, f) = n;$$
$$\operatorname{GR} \operatorname{StarComp}(a, f) = \prod^{\mathsf{RLD}(A)} \left\{ \frac{\operatorname{GR} \operatorname{StarComp}(A, F)}{A \in \operatorname{GR} a, F \in \prod_{i\in n} \operatorname{GR} f_i} \right\};$$
$$\operatorname{Obj}_m \operatorname{StarComp}(a, f) = \lambda i \in n : \operatorname{Dst} f_i.$$

THEOREM 1764. Multireloids with above defined compositions form a quasi-invertible category with star-morphisms.

PROOF. We need to prove:

1°. $\operatorname{StarComp}(\operatorname{StarComp}(m, f), g) = \operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : g_i \circ f_i)$;
2°. $\operatorname{StarComp}(m, \lambda i \in \operatorname{arity} m : 1_{\operatorname{Obj}_m i}) = m$;
3°. $b \not\asymp \operatorname{StarComp}(a, f) \Leftrightarrow a \not\asymp \operatorname{StarComp}(b, f^\dagger)$

(the rest is obvious).

Really,

1°. Using properties of generalized filter bases,

$$\begin{aligned}
&\operatorname{StarComp}(\operatorname{StarComp}(a, f), g) = \\
&\prod^{\mathsf{RLD}} \left\{ \frac{\operatorname{StarComp}(B, G)}{B \in \operatorname{GR} \operatorname{StarComp}(a, f), G \in \prod_{i\in n} \operatorname{GR} g_i} \right\} = \\
&\prod^{\mathsf{RLD}} \left\{ \frac{\operatorname{StarComp}(\operatorname{StarComp}(A, F), G)}{A \in \operatorname{GR} a, F \in \prod_{i\in n} \operatorname{GR} f_i, G \in \prod_{i\in n} \operatorname{GR} g_i} \right\} = \\
&\prod^{\mathsf{RLD}} \left\{ \frac{\operatorname{StarComp}(A, G \circ F)}{A \in \operatorname{GR} a, F \in \prod_{i\in n} \operatorname{GR} f_i, G \in \prod_{i\in n} \operatorname{GR} g_i} \right\} = \\
&\prod^{\mathsf{RLD}} \left\{ \frac{\operatorname{StarComp}(A, H)}{A \in \operatorname{GR} a, H \in \prod_{i\in n} \operatorname{GR}(g_i \circ f_i)} \right\} = \\
&\operatorname{StarComp}(a, \lambda i \in \operatorname{arity} n : g_i \circ f_i).
\end{aligned}$$

2°.

$$\begin{aligned}
&\mathrm{StarComp}(m, \lambda i \in \operatorname{arity} m : 1_{\mathrm{Obj}_m i}) = \\
&\bigsqcap^{\mathsf{RLD}(A)} \left\{ \frac{\mathrm{StarComp}(A, H)}{A \in \mathrm{GR}\, m, H \in \prod_{i \in \operatorname{arity} m} \mathrm{GR}\, 1_{\mathrm{Obj}_m i}} \right\} = \\
&\bigsqcap^{\mathsf{RLD}(A)} \left\{ \frac{\mathrm{StarComp}(A, \lambda i \in \operatorname{arity} m : H_i)}{A \in \mathrm{GR}\, m, H \in \prod_{i \in \operatorname{arity} m} \mathrm{GR}\, 1_{\mathrm{Obj}_m i}} \right\} = \\
&\bigsqcap^{\mathsf{RLD}(A)} \left\{ \frac{\mathrm{StarComp}(A, \lambda i \in \operatorname{arity} m : 1_{X_i})}{A \in \mathrm{GR}\, m, X \in \prod_{i \in \operatorname{arity} m} \mathrm{Obj}_m i} \right\} = \\
&\bigsqcap^{\mathsf{RLD}(A)} \left\{ \frac{(A \cap \prod X)}{A \in \mathrm{GR}\, m, X \in \prod_{i \in \operatorname{arity} m} \mathrm{Obj}_m i} \right\} = \\
&\bigsqcap^{\mathsf{RLD}(A)} \left\{ \frac{A}{A \in \mathrm{GR}\, m} \right\} = m.
\end{aligned}$$

3°. Using properties of generalized filter bases,

$$\begin{aligned}
&b \not\asymp \mathrm{StarComp}(a, f) \Leftrightarrow \\
&\forall A \in \mathrm{GR}\, a, B \in \mathrm{GR}\, b, F \in \prod_{i \in n} \mathrm{GR}\, f_i : B \not\asymp \mathrm{StarComp}(A, F) \Leftrightarrow \\
&\forall A \in \mathrm{GR}\, a, B \in \mathrm{GR}\, b, F \in \prod_{i \in n} \mathrm{GR}\, f_i : B \not\asymp \left\langle \prod^{(C)} F \right\rangle A \Leftrightarrow \\
&\forall A \in \mathrm{GR}\, a, B \in \mathrm{GR}\, b, F \in \prod_{i \in n} \mathrm{GR}\, f_i : A \not\asymp \left\langle \left(\prod^{(C)} F \right)^{-1} \right\rangle B \Leftrightarrow \\
&\forall A \in \mathrm{GR}\, a, B \in \mathrm{GR}\, b, F \in \prod_{i \in n} \mathrm{GR}\, f_i : A \not\asymp \mathrm{StarComp}(B, F^\dagger) \Leftrightarrow \\
&a \not\asymp \mathrm{StarComp}(b, f^\dagger).
\end{aligned}$$

□

Definition 1765. Let f be a multireloid of the form A. Then for $i \in \operatorname{dom} A$

$$\Pr_i^{\mathsf{RLD}} f = \bigsqcap^{\mathscr{F}} \left\langle \Pr_i \right\rangle^* \mathrm{GR}\, f.$$

Proposition 1766. $\operatorname{up} \Pr_i^{\mathsf{RLD}} f = \langle \Pr_i \rangle^* \mathrm{GR}\, f$ for every multireloid f and $i \in \operatorname{arity} f$.

Proof. It's enough to show that $\langle \Pr_i \rangle^* \mathrm{GR}\, f$ is a filter.

That $\langle \Pr_i \rangle^* \mathrm{GR}\, f$ is an upper set is obvious.

Let $X, Y \in \langle \Pr_i \rangle^* \mathrm{GR}\, f$. Then there exist $F, G \in \mathrm{GR}\, f$ such that $X = \Pr_i F$, $Y = \Pr_i G$. Then $X \cap Y \supseteq \Pr_i (F \cap G) \in \langle \Pr_i \rangle^* \mathrm{GR}\, f$. Thus $X \cap Y \in \langle \Pr_i \rangle^* \mathrm{GR}\, f$. □

Definition 1767. $\prod^{\mathsf{RLD}} \mathcal{X} = \bigsqcap^{\mathsf{RLD}(\lambda i \in \operatorname{dom} \mathcal{X} : \mathrm{Base}(\mathcal{X}_i))}_{X \in \operatorname{up} \prod \mathcal{X}} \prod X$ for every indexed family $\mathcal{X}$ of filters on powersets.

Proposition 1768. $\Pr_k^{\mathsf{RLD}} \prod^{\mathsf{RLD}} x = x_k$ for every indexed family x of proper filters.

Proof. $\operatorname{up} \Pr_k^{\mathsf{RLD}} \prod^{\mathsf{RLD}} x = \langle \Pr_k \rangle^* \prod^{\mathsf{RLD}} x = \operatorname{up} x_k$. □

CONJECTURE 1769. $\operatorname{GR}\operatorname{StarComp}(a \sqcup b, f) = \operatorname{GR}\operatorname{StarComp}(a, f) \sqcup \operatorname{GR}\operatorname{StarComp}(b, f)$ if f is a reloid and a, b are multireloids of the same form, composable with f.

THEOREM 1770. $\prod^{\mathsf{RLD}} A = \bigsqcup \left\{ \frac{\prod^{\mathsf{RLD}} a}{a \in \prod_{i \in \operatorname{dom} A} \operatorname{atoms} A_i} \right\}$ for every indexed family A of filters on powersets.

PROOF. Obviously $\prod^{\mathsf{RLD}} A \sqsupseteq \bigsqcup \left\{ \frac{\prod^{\mathsf{RLD}} a}{a \in \prod_{i \in \operatorname{dom} A} \operatorname{atoms} A_i} \right\}$.

Reversely, let $K \in \operatorname{GR} \bigsqcup \left\{ \frac{\prod^{\mathsf{RLD}} a}{a \in \prod_{i \in \operatorname{dom} A} \operatorname{atoms} A_i} \right\}$.

Consequently $K \in \operatorname{GR} \prod^{\mathsf{RLD}} a$ for every $a \in \prod_{i \in \operatorname{dom} A} \operatorname{atoms} A_i$; $K \supseteq \prod X$ and thus $K \supseteq \bigcup_{a \in \prod_{i \in \operatorname{dom} a} \operatorname{atoms} A_i} \prod X_a$ for some $X_a \in \prod_{i \in \operatorname{dom} a} \operatorname{atoms} A_i$.

But $\bigcup_{a \in \prod_{i \in \operatorname{dom} a} \operatorname{atoms} A_i} \prod X_a = \prod_{i \in \operatorname{dom} A} \bigcup_{a \in \operatorname{atoms} A_i} \langle \Pr_i \rangle^* X_a \supseteq \prod_{j \in \operatorname{dom} A} Z_j$ for some $Z_j \in \operatorname{up} A_j$ because $\langle \Pr_i \rangle^* X \in \operatorname{up} a_i$ and our lattice is atomistic. So $K \in \operatorname{GR} \prod^{\mathsf{RLD}} A$. □

THEOREM 1771. Let a, b be indexed families of filters on powersets of the same form $\mathfrak{A}$. Then

$$\prod^{\mathsf{RLD}} a \sqcap \prod^{\mathsf{RLD}} b = \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} \mathfrak{A}} (a_i \sqcap b_i).$$

PROOF.

$$\begin{aligned}
&\operatorname{up}\left(\prod^{\mathsf{RLD}} a \sqcap \prod^{\mathsf{RLD}} b \right) = \\
&\bigsqcap^{\mathsf{RLD}(\mathfrak{A})} \left\{ \frac{P \cap Q}{P \in \operatorname{GR} \prod^{\mathsf{RLD}} a, Q \in \prod^{\mathsf{RLD}} b} \right\} = \\
&\bigsqcap^{\mathsf{RLD}(\mathfrak{A})} \left\{ \frac{\prod p \cap \prod q}{p \in \operatorname{up} \prod a, q \in \operatorname{up} \prod b} \right\} = \\
&\bigsqcap^{\mathsf{RLD}(\mathfrak{A})} \left\{ \frac{\prod_{i \in \operatorname{dom} \mathfrak{A}} (p_i \cap q_i)}{p \in \prod \operatorname{up} a, q \in \prod \operatorname{up} b} \right\} = \\
&\bigsqcap^{\mathsf{RLD}(\mathfrak{A})} \left\{ \frac{\prod r}{r \in \operatorname{up} \prod_{i \in \operatorname{dom} \mathfrak{A}} (a_i \sqcap b_i)} \right\} = \\
&\operatorname{up} \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} \mathfrak{A}} (a_i \sqcap b_i).
\end{aligned}$$

□

THEOREM 1772. If $S \in \mathscr{P} \prod_{i \in \operatorname{dom} \mathfrak{Z}} \mathscr{F}(\mathfrak{Z}_i)$ where $\mathfrak{Z}$ is an indexed family of sets, then

$$\bigsqcap_{a \in S} \prod^{\mathsf{RLD}} a = \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} \mathfrak{Z}} \bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \Pr_i S.$$

PROOF. If $S = \emptyset$ then $\bigsqcap_{a \in S} \prod^{\mathsf{RLD}} a = \bigsqcap \emptyset = \top^{\mathsf{RLD}(\mathfrak{Z})}$ and

$$\prod^{\mathsf{RLD}}_{i \in \operatorname{dom} \mathfrak{Z}} \bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \Pr_i S = \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} \mathfrak{Z}} \bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \emptyset = \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} \mathfrak{Z}} \top^{\mathscr{F}(\mathfrak{Z}_i)} = \top^{\mathsf{RLD}(\mathfrak{Z})},$$

thus $\prod_{a\in S}\prod^{\mathsf{RLD}} a = \prod^{\mathsf{RLD}}_{i\in\operatorname{dom}\mathfrak{Z}} \bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \Pr_i S$.

Let $S \neq \emptyset$.

$\bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \Pr_i S \sqsubseteq \bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)}\{a_i\} = a_i$ for every $a \in S$ because $a_i \in \Pr_i S$. Thus $\prod^{\mathsf{RLD}}_{i\in\operatorname{dom}\mathfrak{Z}} \bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \Pr_i S \sqsubseteq \prod^{\mathsf{RLD}} a$;

$$\bigsqcap_{a\in S}\prod^{\mathsf{RLD}} a \sqsupseteq \prod^{\mathsf{RLD}}_{i\in\operatorname{dom}\mathfrak{Z}} \bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \Pr_i S.$$

Now suppose $F \in \operatorname{GR}\prod^{\mathsf{RLD}}_{i\in\operatorname{dom}\mathfrak{Z}} \bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \Pr_i S$. Then there exists $X \in \operatorname{up}\prod_{i\in\operatorname{dom}\mathfrak{Z}} \bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \Pr_i S$ such that $F \supseteq \prod X$. It is enough to prove that there exist $a \in S$ such that $F \in \operatorname{GR}\prod^{\mathsf{RLD}} a$. For this it is enough $\prod X \in \operatorname{GR}\prod^{\mathsf{RLD}} a$.

Really, $X_i \in \operatorname{up}\bigsqcap^{\mathscr{F}(\mathfrak{Z}_i)} \Pr_i S$ thus $X_i \in \operatorname{up} a_i$ for every $A \in S$ because $\Pr_i S \supseteq \{a_i\}$.

Thus $\prod X \in \operatorname{GR}\prod^{\mathsf{RLD}} a$. □

Definition 1773. I call a multireloid *principal* iff its graph is a principal filter.

Definition 1774. I call a multireloid *convex* iff it is a join of reloidal products.

Theorem 1775. $\operatorname{StarComp}(a \sqcup b, f) = \operatorname{StarComp}(a, f) \sqcup \operatorname{StarComp}(b, f)$ for multireloids a, b and an indexed family f of reloids with $\operatorname{Src} f_i = (\operatorname{form} a)_i = (\operatorname{form} b)_i$.

Proof.

$$\begin{aligned}
&\operatorname{GR}(\operatorname{StarComp}(a, f) \sqcup \operatorname{StarComp}(b, f)) = \\
&\bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}(\operatorname{form} a)} \operatorname{StarComp}(A, F)}{A \in \operatorname{GR} a, F \in \prod_{i\in n}\operatorname{GR} f_i}\right\} \sqcup \bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}(\operatorname{form} b)} \operatorname{StarComp}(B, F)}{B \in \operatorname{GR} b, F \in \prod_{i\in n}\operatorname{GR} f_i}\right\} = \\
&\bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}(\operatorname{form} a)} \operatorname{StarComp}(A, F) \sqcup \uparrow^{\mathsf{RLD}(\operatorname{form} b)} \operatorname{StarComp}(B, F)}{A \in \operatorname{GR} a, B \in \operatorname{GR} b, F \in \prod_{i\in n}\operatorname{GR} f_i}\right\} = \\
&\bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}(\operatorname{form} a)} (\operatorname{StarComp}(A, F) \cup \operatorname{StarComp}(B, F))}{A \in \operatorname{GR} a, B \in \operatorname{GR} b, F \in \prod_{i\in n}\operatorname{GR} f_i}\right\} = \\
&\bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}(\operatorname{form} a)} \operatorname{StarComp}(A \cup B, F)}{A \in \operatorname{GR} a, B \in \operatorname{GR} b, F \in \prod_{i\in n}\operatorname{GR} f_i}\right\} = \\
&\bigsqcap\left\{\frac{\uparrow^{\mathsf{RLD}(\operatorname{form} a)} \operatorname{StarComp}(C, F)}{C \in \operatorname{GR}(a \sqcup b), F \in \prod_{i\in n}\operatorname{GR} f_i}\right\} = \\
&\operatorname{GR}\operatorname{StarComp}(a \sqcup b, f).
\end{aligned}$$

□

21.11.1. Starred reloidal product. Tychonoff product of topological spaces inspired me the following definition, which seems possibly useful just like Tychonoff product:

Definition 1776. Let a be an n-indexed (n is an arbitrary index set) family of filters on sets. $\prod^{\mathsf{RLD}*} a$ (*starred reloidal product*) is the reloid of the form $\prod_{i\in n}\operatorname{Base}(a_i)$ induced by the filter base

$$\left\{\frac{\prod_{i\in n}\left(\begin{cases} A_i & \text{if } i \in m \\ \operatorname{Base}(a_i) & \text{if } i \in n \setminus m \end{cases}\right)}{m \text{ is a finite subset of } n, A \in \prod(a|_m)}\right\}.$$

OBVIOUS 1777. It is really a filter base.

OBVIOUS 1778. $\prod^{\mathsf{RLD}*} a \sqsupseteq \prod^{\mathsf{RLD}} a$.

PROPOSITION 1779. $\prod^{\mathsf{RLD}*} a = \prod^{\mathsf{RLD}} a$ if n is finite.

PROOF. Take $m = n$ to show that $\prod^{\mathsf{RLD}*} a \sqsubseteq \prod^{\mathsf{RLD}} a$. □

PROPOSITION 1780. $\prod^{\mathsf{RLD}*} a = \bot^{\mathsf{RLD}(\lambda i \in n:\mathrm{Base}(a_i))}$ if a_i is the non-proper filter for some $i \in n$.

PROOF. Take $A_i = \bot$ and $m = \{i\}$. Then $\prod_{i\in n}\left(\begin{cases} A_i & \text{if } i \in m \\ \mathrm{Base}(a_i) & \text{if } i \in n \setminus m \end{cases}\right) = \bot$. □

EXAMPLE 1781. There exists an indexed family a of principal filters such that $\prod^{\mathsf{RLD}*} a$ is non-principal.

PROOF. Let n be infinite and $\mathrm{Base}(a_i)$ is a set of at least two elements. Let each a_i be a trivial ultrafilter.

Every $\prod_{i\in n}\left(\begin{cases} A_i & \text{if } i \in m \\ \mathrm{Base}(a_i) & \text{if } i \in n \setminus m \end{cases}\right)$ has at least 2^n elements.

There are elements $\operatorname{up}\prod^{\mathsf{RLD}} a$ with cardinality 1. They can't be elements of $\operatorname{up}\prod^{\mathsf{RLD}*} a$ because of cardinality issues. □

COROLLARY 1782. There exists an indexed family a of principal filters such that $\prod^{\mathsf{RLD}*} a \neq \prod^{\mathsf{RLD}} a$.

PROOF. Because $\prod^{\mathsf{RLD}} a$ is principal. □

PROPOSITION 1783. $\Pr^{\mathsf{RLD}}_k \prod^{\mathsf{RLD}*} x = x_k$ for every indexed family x of proper filters.

PROOF. $\Pr^{\mathsf{RLD}}_k \prod^{\mathsf{RLD}*} x = \langle \Pr_k \rangle^* \mathrm{GR} \prod^{\mathsf{RLD}*} x = x_k$. □

THEOREM 1784. $\Pr^{\mathsf{RLD}}_i f \sqsubseteq \mathcal{A}_i$ for all $i \in n$ iff $f \sqsubseteq \prod^{\mathsf{RLD}*} \mathcal{A}$ (for every reloid f of arity n and n-indexed family $\mathcal{A}$ of filters on sets).

PROOF. $f \sqsubseteq \prod^{\mathsf{RLD}*} \mathcal{A} \Rightarrow \Pr^{\mathsf{RLD}}_i f \sqsubseteq \Pr^{\mathsf{RLD}}_i \prod^{\mathsf{RLD}*} \mathcal{A} \sqsubseteq \mathcal{A}_i$.

Let now $\Pr^{\mathsf{RLD}}_i f \sqsubseteq \mathcal{A}_i$.

$f \sqsubseteq \prod\left(\begin{cases} \Pr^{\mathsf{RLD}}_i f & \text{if } i \in m \\ \mathrm{Base}(\operatorname{form} f)_i & \text{if } i \notin m \end{cases}\right)$ for finite $m \subseteq n$, as it can be easily be proved by induction.

It follows $f \sqsubseteq \prod^{\mathsf{RLD}*} \mathcal{A}$. □

21.12. Subatomic product of funcoids

DEFINITION 1785. Let f be an indexed family of funcoids. Then $\prod^{(A)} f$ (*subatomic product*) is a funcoid $\prod_{i\in\operatorname{dom} f} \operatorname{Src} f_i \to \prod_{i\in\operatorname{dom} f} \operatorname{Dst} f_i$ such that for every $a \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i\in\operatorname{dom} f:\operatorname{Src} f_i)}$, $b \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i\in\operatorname{dom} f:\operatorname{Dst} f_i)}$

$$a \left[\prod^{(A)} f\right] b \Leftrightarrow \forall i \in \operatorname{dom} f : \Pr^{\mathsf{RLD}}_i a \, [f_i] \Pr^{\mathsf{RLD}}_i b.$$

PROPOSITION 1786. The funcoid $\prod^{(A)} f$ exists.

Proof. To prove that $\prod^{(A)} f$ exists we need to prove (for every $a \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f:\operatorname{Src} f_i)}$, $b \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f:\operatorname{Dst} f_i)}$)

$$\forall X \in \operatorname{GR} a, Y \in \operatorname{GR} b$$
$$\exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f:\operatorname{Src} f_i)} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f:\operatorname{Dst} f_i)} Y : x \left[\prod^{(A)} f\right] y \Rightarrow a \left[\prod^{(A)} f\right] b.$$

Let

$$\forall X \in \operatorname{GR} a, Y \in \operatorname{GR} b$$
$$\exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f:\operatorname{Src} f_i)} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f:\operatorname{Dst} f_i)} Y : x \left[\prod^{(A)} f\right] y.$$

Then

$$\forall X \in \operatorname{GR} a, Y \in \operatorname{GR} b$$
$$\exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f:\operatorname{Src} f_i)} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f:\operatorname{Dst} f_i)} Y$$
$$\forall i \in \operatorname{dom} f : \operatorname*{Pr}^{\mathsf{RLD}}_i x \, [f_i] \operatorname*{Pr}^{\mathsf{RLD}}_i y.$$

Then because $\Pr^{\mathsf{RLD}}_i x \in \operatorname{atoms} \uparrow^{\operatorname{Src} f_i} \Pr_i X$ and likewise for y:

$$\forall X \in \operatorname{GR} a, Y \in \operatorname{GR} b \forall i \in \operatorname{dom} f$$
$$\exists x \in \operatorname{atoms} \uparrow^{\operatorname{Src} f_i} \operatorname*{Pr}_i X, y \in \operatorname{atoms} \uparrow^{\operatorname{Dst} f_i} \operatorname*{Pr}_i Y : x \, [f_i] \, y.$$

Thus $\forall X \in \operatorname{GR} a, Y \in \operatorname{GR} b \forall i \in \operatorname{dom} f : \uparrow^{\operatorname{Src} f_i} \Pr_i X \, [f_i] \uparrow^{\operatorname{Dst} f_i} \Pr_i Y$;

$\forall X \in \operatorname{GR} a, Y \in \operatorname{GR} b \forall i \in \operatorname{dom} f : \Pr_i X \, [f_i]^* \Pr_i Y$.

Then $\forall X \in \langle \Pr_i \rangle^* \operatorname{GR} a, Y \in \langle \Pr_i \rangle^* \operatorname{GR} b : X \, [f_i]^* \, Y$.

Thus $\Pr^{\mathsf{RLD}}_i a \, [f_i] \Pr^{\mathsf{RLD}}_i b$. So

$$\forall i \in \operatorname{dom} f : \operatorname*{Pr}^{\mathsf{RLD}}_i a \, [f_i] \operatorname*{Pr}^{\mathsf{RLD}}_i b$$

and thus $a \left[\prod^{(A)} f\right] b$. □

Remark 1787. It seems that the proof of the above theorem can be simplified using cross-composition product.

Theorem 1788. $\prod^{(A)}_{i \in n} (g_i \circ f_i) = \prod^{(A)} g \circ \prod^{(A)} f$ for indexed (by an index set n) families f and g of funcoids such that $\forall i \in n : \operatorname{Dst} f_i = \operatorname{Src} g_i$.

PROOF. Let a, b be ultrafilters on $\prod_{i\in n}\operatorname{Src} f_i$ and $\prod_{i\in n}\operatorname{Dst} g_i$ correspondingly,

$$\begin{aligned}
a\left[\prod_{i\in n}^{(A)}(g_i\circ f_i)\right]b &\Leftrightarrow\\
\forall i\in\operatorname{dom} f:\left\langle\Pr_i\right\rangle^* a\,[g_i\circ f_i]\left\langle\Pr_i\right\rangle^* b &\Leftrightarrow\\
\forall i\in\operatorname{dom} f\exists C\in\operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f_i)}:\left(\left\langle\Pr_i\right\rangle^* a\,[f_i]\,C\wedge C\,[g_i]\left\langle\Pr_i\right\rangle^* b\right) &\Leftrightarrow\\
\forall i\in\operatorname{dom} f\exists c\in\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in n:\operatorname{Dst} f)}:\left(\left\langle\Pr_i\right\rangle^* a\,[f_i]\left\langle\Pr_i\right\rangle^* c\wedge\left\langle\Pr_i\right\rangle^* c\,[g_i]\left\langle\Pr_i\right\rangle^* b\right) &\Leftarrow\\
\exists c\in\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in n:\operatorname{Dst} f)}\forall i\in\operatorname{dom} f:\left(\left\langle\Pr_i\right\rangle^* a\,[f_i]\left\langle\Pr_i\right\rangle^* c\wedge\left\langle\Pr_i\right\rangle^* c\,[g_i]\left\langle\Pr_i\right\rangle^* b\right) &\Leftrightarrow\\
\exists c\in\operatorname{atoms}^{\mathsf{RLD}(\lambda i\in n:\operatorname{Dst} f)}:\left(a\left[\prod^{(A)} f\right]c\wedge c\left[\prod^{(A)} g\right]b\right) &\Leftrightarrow\\
a\left[\prod^{(A)} g\circ\prod^{(A)} f\right]b.
\end{aligned}$$

But

$$\forall i\in\operatorname{dom} f\exists C\in\operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f_i)}:\left(\left\langle\Pr_i\right\rangle^* a\,[f_i]\,C\wedge C\,[g_i]\left\langle\Pr_i\right\rangle^* b\right)$$

implies

$$\exists C\in\prod_{i\in n}\operatorname{atoms}^{\mathscr{F}(\operatorname{Dst} f_i)}\forall i\in\operatorname{dom} f:\left(\left\langle\Pr_i\right\rangle^* a\,[f_i]\,C_i\wedge C_i\,[g_i]\left\langle\Pr_i\right\rangle^* b\right).$$

Take $c\in\operatorname{atoms}\prod^{\mathsf{RLD}} C$. Then

$$\forall i\in\operatorname{dom} f:\left(\left\langle\Pr_i\right\rangle^* a\,[f_i]\Pr_i c\wedge\Pr_i c\,[g_i]\left\langle\Pr_i\right\rangle^* b\right)$$

that is

$$\forall i\in\operatorname{dom} f:\left(\left\langle\Pr_i\right\rangle^* a\,[f_i]\left\langle\Pr_i\right\rangle^* c\wedge\left\langle\Pr_i\right\rangle^*\,[g_i]\left\langle\Pr_i\right\rangle^* b\right)$$

We have $a\left[\prod_{i\in n}^{(A)}(g_i\circ f_i)\right]b\Leftrightarrow a\left[\prod^{(A)} g\circ\prod^{(A)} f\right]b$. □

COROLLARY 1789. $\left(\prod^{(A)} f_{k-1}\right)\circ\ldots\circ\left(\prod^{(A)} f_0\right)=\prod_{i\in n}^{(A)}(f_{k-1}\circ\ldots\circ f_0)$ for every n-indexed families $f_0,\ldots,f_{n-1}$ of composable funcoids.

PROPOSITION 1790. $\prod^{\mathsf{RLD}} a\left[\prod^{(A)} f\right]\prod^{\mathsf{RLD}} b\Leftrightarrow\forall i\in\operatorname{dom} f: a_i\,[f_i]\,b_i$ for an indexed family f of funcoids and indexed families a and b of filters where $a_i\in\mathscr{F}(\operatorname{Src} f_i)$, $b_i\in\mathscr{F}(\operatorname{Dst} f_i)$ for every $i\in\operatorname{dom} f$.

PROOF. If $a_i=\bot$ or $b_i=\bot$ for some i our theorem is obvious. We will take $a_i\neq\bot$ and $b_i\neq\bot$, thus there exist

$$x\in\operatorname{atoms}\prod^{\mathsf{RLD}} a,\quad y\in\operatorname{atoms}\prod^{\mathsf{RLD}} b.$$

$$\prod^{\mathsf{RLD}} a \left[\prod^{(A)} f\right] \prod^{\mathsf{RLD}} b \Leftrightarrow$$
$$\exists x \in \operatorname{atoms} \prod^{\mathsf{RLD}} a, y \in \operatorname{atoms} \prod^{\mathsf{RLD}} b : x \left[\prod^{(A)} f\right] y \Leftrightarrow$$
$$\exists x \in \operatorname{atoms} \prod^{\mathsf{RLD}} a, y \in \operatorname{atoms} \prod^{\mathsf{RLD}} b \forall i \in \operatorname{dom} f : \left\langle \Pr_i \right\rangle^* x \, [f_i] \left\langle \Pr_i \right\rangle^* y \Leftrightarrow$$
$$\forall i \in \operatorname{dom} f \exists x \in \operatorname{atoms} a_i, y \in \operatorname{atoms} b_i : x \, [f_i] \, y \Leftrightarrow$$
$$\forall i \in \operatorname{dom} f : a_i \, [f_i] \, b_i.$$

□

THEOREM 1791. $\left\langle \prod^{(A)} f \right\rangle x = \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} f} \langle f_i \rangle \Pr^{\mathsf{RLD}}_i x$ for an indexed family f of funcoids and $x \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)}$ for every $n \in \operatorname{dom} f$.

PROOF. For every ultrafilter $y \in \mathscr{F}\left(\prod_{i \in \operatorname{dom} f} \operatorname{Dst} f_i\right)$ we have:

$$y \not\asymp \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} f} \langle f_i \rangle \Pr^{\mathsf{RLD}}_i x \Leftrightarrow$$
$$\forall i \in \operatorname{dom} f : \Pr^{\mathsf{RLD}}_i y \not\asymp \langle f_i \rangle \Pr^{\mathsf{RLD}}_i x \Leftrightarrow$$
$$\forall i \in \operatorname{dom} f : \Pr^{\mathsf{RLD}}_i x \, [f_i] \Pr^{\mathsf{RLD}}_i y \Leftrightarrow$$
$$x \left[\prod^{(A)} f\right] y \Leftrightarrow$$
$$y \not\asymp \left\langle \prod^{(A)} f \right\rangle x.$$

Thus $\left\langle \prod^{(A)} f \right\rangle x = \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} f} \langle f_i \rangle \Pr^{\mathsf{RLD}}_i x$. □

COROLLARY 1792. $\langle f \times^{(A)} g \rangle x = \langle f \rangle (\operatorname{dom} x) \times^{\mathsf{RLD}} \langle g \rangle (\operatorname{im} x)$ for atomic x.

21.13. On products and projections

CONJECTURE 1793. For principal funcoids $\prod^{(C)}$and $\prod^{(A)}$ coincide with the conventional product of binary relations.

21.13.1. Staroidal product. Let f be a staroid, whose form components are boolean lattices.

DEFINITION 1794. *Staroidal projection* of a staroid f is the filter $\Pr^{\mathsf{Strd}}_k f$ corresponding to the free star

$$(\operatorname{val} f)_k (\lambda i \in (\operatorname{arity} f) \setminus \{k\} : \top^{(\operatorname{form} f)_i}).$$

PROPOSITION 1795. $\Pr_k \operatorname{GR} \prod^{\mathsf{Strd}} x = \star x_k$ if x is an indexed family of proper filters, and $k \in \operatorname{dom} x$.

PROOF.

$$\Pr_k \mathrm{GR} \prod^{\mathsf{Strd}} x =$$

$$\Pr_k \left\{ \frac{L \in \operatorname{form} x}{\forall i \in \operatorname{dom} x : x_i \not\asymp L_i} \right\} =$$

(used the fact that x_i are proper filters)

$$\left\{ \frac{l \in (\operatorname{form} x)_k}{x_k \not\asymp l} \right\} = \star x_k.$$

□

PROPOSITION 1796. $\Pr_k^{\mathsf{Strd}} \prod^{\mathsf{Strd}} x = x_k$ if x is an indexed family of proper filters, and $k \in \operatorname{dom} x$.

PROOF.

$$\partial \Pr_k^{\mathsf{Strd}} \prod^{\mathsf{Strd}} x =$$

$$\left(\operatorname{val} \prod^{\mathsf{Strd}} x \right)_k (\lambda i \in (\operatorname{dom} x) \setminus \{k\} : \top^{(\operatorname{form} x)_i}) =$$

$$\left\{ \frac{X \in \left(\operatorname{form} \prod^{\mathsf{Strd}} x \right)_k}{(\lambda i \in (\operatorname{dom} x) \setminus \{k\} : \top^{(\operatorname{form} x)_i}) \cup \{(k, X)\} \in \mathrm{GR} \prod^{\mathsf{Strd}} x} \right\} =$$

$$\left\{ \frac{X \in \operatorname{Base} x_k}{(\forall i \in (\operatorname{dom} x) \setminus \{k\} : \top^{(\operatorname{form} x)_i} \not\asymp x_i) \wedge X \not\asymp x_k} \right\} =$$

$$\left\{ \frac{X \in \operatorname{Base} x_k}{X \not\asymp x_k} \right\} = \partial x_k.$$

Consequently $\Pr_k^{\mathsf{Strd}} \prod^{\mathsf{Strd}} x = x_k$. □

21.13.2. Cross-composition product of pointfree funcoids.

DEFINITION 1797. *Zero* pointfree funcoid $\bot^{\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})}$ from a poset $\mathfrak{A}$ to to a poset $\mathfrak{B}$ is the least pointfree funcoid in the set $\mathsf{pFCD}(\mathfrak{A}, \mathfrak{B})$.

PROPOSITION 1798. A pointfree funcoid f is zero iff $[f] = \emptyset$.

PROOF. Direct implication is obvious.

Let now $[f] = \emptyset$. Then $\langle f \rangle x \asymp y$ for every $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$ and thus $\langle f \rangle x \asymp \langle f \rangle x$. It is possible only when $\langle f \rangle x = \bot^{\operatorname{Dst} f}$. □

COROLLARY 1799. A pointfree funcoid is zero iff its reverse is zero.

PROPOSITION 1800. Values x_i (for every $i \in \operatorname{dom} x$) can be restored from the value of $\prod^{(C)} x$ provided that x is an indexed family of non-zero pointfree funcoids, $\operatorname{Src} f_i$ (for every $i \in n$) is an atomic lattice and every $\operatorname{Dst} f_i$ is an atomic poset with greatest element.

PROOF. $\left\langle \prod^{(C)} x \right\rangle \prod^{\mathsf{Strd}} p = \prod^{\mathsf{Strd}}_{i \in n} \langle x_i \rangle p_i$ by theorem 1752.

Since x_i is non-zero there exist p such that $\langle x_i \rangle p_i$ is non-least. Take $k \in n$, $p'_i = p_i$ for $i \neq k$ and $p'_k = q$ for an arbitrary value q; then (using the staroidal projections from the previous subsection)

$$\langle x_k \rangle q = \Pr_k^{\mathsf{Strd}} \prod_{i \in n}^{\mathsf{Strd}} \langle x_i \rangle p'_i = \Pr_k^{\mathsf{Strd}} \left\langle \prod^{(C)} x \right\rangle \prod^{\mathsf{Strd}} p'.$$

So the value of x can be restored from $\prod^{(C)} x$ by this formula. □

21.13.3. Subatomic product.

Proposition 1801. Values x_i (for every $i \in \operatorname{dom} x$) can be restored from the value of $\prod^{(A)} x$ provided that x is an indexed family of non-zero funcoids.

Proof. Fix $k \in \operatorname{dom} f$. Let for some filters $\mathcal{X}$ and $\mathcal{Y}$

$$a = \begin{cases} \top^{\mathscr{F}(\mathrm{Base}(x))} & \text{if } i \neq k; \\ \mathcal{X} & \text{if } i = k \end{cases} \quad \text{and} \quad b = \begin{cases} \top^{\mathscr{F}(\mathrm{Base}(y))} & \text{if } i \neq k; \\ \mathcal{Y} & \text{if } i = k. \end{cases}$$

Then $\mathcal{X}\ [x_k]\ \mathcal{Y} \Leftrightarrow a_k\ [x_k]\ b_k \Leftrightarrow \forall i \in \operatorname{dom} f : a_i\ [x_i]\ b_i \Leftrightarrow \prod^{\mathsf{RLD}} a \left[\prod^{(A)} x\right] \prod^{\mathsf{RLD}} b$. So we have restored x_k from $\prod^{(A)} x$. □

Definition 1802. For every funcoid $f : \prod A \to \prod B$ (where A and B are indexed families of typed sets) consider the funcoid $\operatorname{Pr}^{(A)}_k f$ defined by the formula

$$X \left[\operatorname*{Pr}^{(A)}_{k} f\right]^* Y \Leftrightarrow \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} A} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} X & \text{if } i = k \end{cases}\right) [f] \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} B} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases}\right).$$

Proposition 1803. $\operatorname{Pr}^{(A)}_k f$ is really a funcoid.

Proof. $\neg\left(\bot \left[\operatorname{Pr}^{(A)}_k f\right]^* Y\right)$ is obvious.

$$\begin{aligned} & I \sqcup J \left[\operatorname*{Pr}^{(A)}_{k} f\right]^* Y \Leftrightarrow \\ & \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} A} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} (I \sqcup J) & \text{if } i = k \end{cases}\right) [f] \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} B} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases}\right) \Leftrightarrow \\ & \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} A} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} I \sqcup \uparrow^{A_i} J & \text{if } i = k \end{cases}\right) [f] \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} B} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases}\right) \Leftrightarrow \\ & \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} A} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} I & \text{if } i = k \end{cases}\right) \sqcup \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} A} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} J & \text{if } i = k \end{cases}\right) [f] \\ & \qquad \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} B} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases}\right) \Leftrightarrow \\ & \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} A} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} I & \text{if } i = k \end{cases}\right) [f] \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} B} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases}\right) \vee \\ & \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} A} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} J & \text{if } i = k \end{cases}\right) [f] \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} B} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases}\right) \Leftrightarrow \\ & I \left[\operatorname*{Pr}^{(A)}_{k} f\right]^* Y \vee J \left[\operatorname*{Pr}^{(A)}_{k} f\right]^* Y. \end{aligned}$$

The rest follows from symmetry. □

Proposition 1804. For every funcoid $f : \prod A \to \prod B$ (where A and B are indexed families of typed sets) the funcoid $\operatorname{Pr}^{(A)}_k f$ conforms to the formula

$$\mathcal{X} \left[\operatorname*{Pr}^{(A)}_{k} f\right] \mathcal{Y} \Leftrightarrow \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} A} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \mathcal{X} & \text{if } i = k \end{cases}\right) [f] \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} B} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \mathcal{Y} & \text{if } i = k \end{cases}\right).$$

Proof.

$$\mathcal{X} \left[\mathop{\mathrm{Pr}}_{k}^{(A)} f\right] \mathcal{Y} \Leftrightarrow$$

$$\forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : X \left[\mathop{\mathrm{Pr}}_{k}^{(A)} f\right]^{*} Y \Leftrightarrow$$

$$\forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} :$$

$$\prod_{i\in\operatorname{dom} A}^{\mathsf{RLD}} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \uparrow^{A_i} X & \text{if } i = k \end{cases}\right) [f] \prod_{i\in\operatorname{dom} B}^{\mathsf{RLD}} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \uparrow^{B_i} Y & \text{if } i = k \end{cases}\right) \Leftrightarrow$$

$$\forall X \in \operatorname{up} \prod_{i\in\operatorname{dom} A}^{\mathsf{RLD}} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \mathcal{X} & \text{if } i = k \end{cases}\right), Y \in \operatorname{up} \prod_{i\in\operatorname{dom} B}^{\mathsf{RLD}} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \mathcal{Y} & \text{if } i = k \end{cases}\right) :$$

$$X\, [f]^{*}\, Y \Leftrightarrow$$

$$\prod_{i\in\operatorname{dom} A}^{\mathsf{RLD}} \left(\begin{cases} \top^{\mathscr{F}(A_i)} & \text{if } i \neq k; \\ \mathcal{X} & \text{if } i = k \end{cases}\right) [f] \prod_{i\in\operatorname{dom} B}^{\mathsf{RLD}} \left(\begin{cases} \top^{\mathscr{F}(B_i)} & \text{if } i \neq k; \\ \mathcal{Y} & \text{if } i = k \end{cases}\right).$$

□

Remark 1805. Reloidal product above can be replaced with starred reloidal product, because of finite number of non-maximal multipliers in the products.

Obvious 1806. $\mathrm{Pr}_k^{(A)} \prod^{(A)} x = x_k$ provided that x is an indexed family of non-zero funcoids.

21.13.4. Other.

Definition 1807. *Displaced product* $\prod^{(DP)} f = \downarrow\downarrow \prod^{(C)} f$ for every indexed family of pointfree funcoids, where downgrading is defined for the filtrator

$$\left(\mathsf{FCD}(\mathrm{StarHom}(\mathrm{Src}\circ f), \mathrm{StarHom}(\mathrm{Dst}\circ f)), \mathbf{Rel}\left(\prod(\mathrm{Src}\circ f), \prod(\mathrm{Dst}\circ f)\right)\right).$$

Remark 1808. Displaced product is a funcoid (not just a pointfree funcoid).

Conjecture 1809. Values x_i (for every $i \in \operatorname{dom} x$) can be restored from the value of $\prod^{(DP)} x$ provided that x is an indexed family of non-zero funcoids.

Definition 1810. Let $f \in \mathscr{P}\left(Z^{\coprod Y}\right)$ where Z is a set and Y is a function.

$$\mathop{\mathrm{Pr}}_{k}^{(D)} f = \mathop{\mathrm{Pr}}_{k}\left\{\frac{\operatorname{curry} z}{z \in f}\right\}.$$

Proposition 1811. $\mathrm{Pr}_k^{(D)} \prod^{(D)} F = F_k$ for every indexed family F of non-empty relations.

Proof. Obvious. □

Corollary 1812. $\operatorname{GR} \mathrm{Pr}_k^{(D)} \prod^{(D)} F = \operatorname{GR} F_k$ and $\operatorname{form} \mathrm{Pr}_k^{(D)} \prod^{(D)} F = \operatorname{form} F_k$ for every indexed family F of non-empty anchored relations.

21.14. Relationships between cross-composition and subatomic products

Proposition 1813. $a \left[f \times^{(C)} g\right] b \Leftrightarrow \operatorname{dom} a\, [f]\, \operatorname{dom} b \wedge \operatorname{im} a\, [g]\, \operatorname{im} b$ for funcoids f and g and atomic funcoids $a \in \mathsf{FCD}(\mathrm{Src}\, f, \mathrm{Src}\, g)$ and $b \in \mathsf{FCD}(\mathrm{Dst}\, f, \mathrm{Dst}\, g)$.

PROOF.

$$\begin{aligned} a \left[f \times^{(C)} g\right] b \Leftrightarrow \\ a \circ f^{-1} \not\asymp g^{-1} \circ b \Leftrightarrow \\ (\operatorname{dom} a \times^{\mathsf{FCD}} \operatorname{im} a) \circ f^{-1} \not\asymp g^{-1} \circ (\operatorname{dom} b \times^{\mathsf{FCD}} \operatorname{im} b) \Leftrightarrow \\ \langle f\rangle \operatorname{dom} a \times^{\mathsf{FCD}} \operatorname{im} a \not\asymp \operatorname{dom} b \times^{\mathsf{FCD}} \left\langle g^{-1}\right\rangle \operatorname{im} b \Leftrightarrow \\ \langle f\rangle \operatorname{dom} a \not\asymp \operatorname{dom} b \wedge \operatorname{im} a \not\asymp \left\langle g^{-1}\right\rangle \operatorname{im} b \Leftrightarrow \\ \operatorname{dom} a\ [f] \operatorname{dom} b \wedge \operatorname{im} a\ [g] \operatorname{im} b. \end{aligned}$$

□

PROPOSITION 1814. $\mathcal{X} \left[\prod^{(A)} f\right] \mathcal{Y} \Leftrightarrow \forall i \in \operatorname{dom} f : \operatorname{Pr}_i^{\mathsf{RLD}} \mathcal{X}\ [f_i]\ \operatorname{Pr}_i^{\mathsf{RLD}} \mathcal{Y}$ for every indexed family f of funcoids and $\mathcal{X} \in \mathsf{RLD}(\operatorname{Src} \circ f)$, $\mathcal{Y} \in \mathsf{RLD}(\operatorname{Dst} \circ f)$.

PROOF.

$$\begin{aligned} \mathcal{X} \left[\prod^{(A)} f\right] \mathcal{Y} \Leftrightarrow \\ \exists a \in \operatorname{atoms} \mathcal{X}, b \in \operatorname{atoms} \mathcal{Y} : a \left[\prod^{(A)} f\right] b \Leftrightarrow \\ \exists a \in \operatorname{atoms} \mathcal{X}, b \in \operatorname{atoms} \mathcal{Y} \forall i \in \operatorname{dom} f : \underset{i}{\overset{\mathsf{RLD}}{\operatorname{Pr}}}\ a\ [f_i]\ \underset{i}{\overset{\mathsf{RLD}}{\operatorname{Pr}}}\ b \Leftrightarrow \\ \forall i \in \operatorname{dom} f \exists x \in \operatorname{atoms} \underset{i}{\overset{\mathsf{RLD}}{\operatorname{Pr}}} \mathcal{X}, y \in \operatorname{atoms} \underset{i}{\overset{\mathsf{RLD}}{\operatorname{Pr}}} \mathcal{Y} : x_i\ [f_i]\ y_i \Leftrightarrow \\ \forall i \in \operatorname{dom} f : \underset{i}{\overset{\mathsf{RLD}}{\operatorname{Pr}}} \mathcal{X}\ [f_i]\ \underset{i}{\overset{\mathsf{RLD}}{\operatorname{Pr}}} \mathcal{Y}. \end{aligned}$$

□

COROLLARY 1815. $\mathcal{X} \left[f \times^{(A)} g\right] \mathcal{Y} \Leftrightarrow \operatorname{dom} \mathcal{X}\ [f]\ \operatorname{dom} \mathcal{Y} \wedge \operatorname{im} \mathcal{X}\ [g]\ \operatorname{im} \mathcal{Y}$ for funcoids f, g and reloids $\mathcal{X} \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Src} g)$, and $\mathcal{Y} \in \mathsf{RLD}(\operatorname{Dst} f, \operatorname{Dst} g)$.

LEMMA 1816. For every $A \in \mathbf{Rel}(X, Y)$ (for every sets X, Y) we have:

$$\left\{\frac{(\operatorname{dom} a, \operatorname{im} a)}{a \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} A}\right\} = \left\{\frac{(\operatorname{dom} a, \operatorname{im} a)}{a \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} A}\right\}.$$

PROOF. Let $x \in \left\{\frac{(\operatorname{dom} a, \operatorname{im} a)}{a \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} A}\right\}$. Take $x_0 = \operatorname{dom} a$ and $x_1 = \operatorname{im} a$ where $a \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} A$.

Then $x_0 = \operatorname{dom}(\mathsf{FCD})a$ and $x_1 = \operatorname{im}(\mathsf{FCD})a$ and obviously $(\mathsf{FCD})a \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} A$. So $x \in \left\{\frac{(\operatorname{dom} a, \operatorname{im} a)}{a \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} A}\right\}$.

Let now $x \in \left\{\frac{(\operatorname{dom} a, \operatorname{im} a)}{a \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} A}\right\}$. Take $x_0 = \operatorname{dom} a$ and $x_1 = \operatorname{im} a$ where $a \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} A$.

$x_0 \left[\uparrow^{\mathsf{FCD}} A\right] x_1 \Leftrightarrow x_0 \left[(\mathsf{FCD}) \uparrow^{\mathsf{RLD}} A\right] x_1 \Leftrightarrow x_0 \times^{\mathsf{RLD}} x_1 \not\asymp \uparrow^{\mathsf{RLD}} A$. Thus there exists atomic reloid x' such that $x' \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} A$ and $\operatorname{dom} x' = x_0$, $\operatorname{im} x' = x_1$.

So $x \in \left\{\frac{(\operatorname{dom} a', \operatorname{im} a')}{a' \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} A}\right\}$. □

THEOREM 1817. $\uparrow^{\mathsf{FCD}} A \left[f \times^{(C)} g\right] \uparrow^{\mathsf{FCD}} B \Leftrightarrow \uparrow^{\mathsf{RLD}} A \left[f \times^{(A)} g\right] \uparrow^{\mathsf{RLD}} B$ for funcoids f, g, and $\mathbf{Rld}$-morphisms $A : \operatorname{Src} f \to \operatorname{Src} g$, and $B : \operatorname{Dst} f \to \operatorname{Dst} g$.

PROOF.

$$\begin{aligned}
&\uparrow^{\mathsf{FCD}} A \left[f \times^{(C)} g\right] \uparrow^{\mathsf{FCD}} B \Leftrightarrow \\
&\exists a \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} A, b \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} B : a \left[f \times^{(C)} g\right] b \Leftrightarrow \\
&\exists a \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} A, b \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} B : (\operatorname{dom} a \; [f] \; \operatorname{dom} b \wedge \operatorname{im} a \; [g] \; \operatorname{im} b) \Rightarrow \\
&\exists a_0 \in \operatorname{atoms} \operatorname{dom} \uparrow^{\mathsf{FCD}} A, a_1 \in \operatorname{atoms} \operatorname{im} \uparrow^{\mathsf{FCD}} A, \\
&b_0 \in \operatorname{atoms} \operatorname{dom} \uparrow^{\mathsf{FCD}} B, b_1 \in \operatorname{atoms} \operatorname{im} \uparrow^{\mathsf{FCD}} B : (a_0 \; [f] \; b_0 \wedge a_1 \; [g] \; b_1).
\end{aligned}$$

On the other hand:

$$\begin{aligned}
&\exists a_0 \in \operatorname{atoms} \operatorname{dom} \uparrow^{\mathsf{FCD}} A, a_1 \in \operatorname{atoms} \operatorname{im} \uparrow^{\mathsf{FCD}} A, \\
&b_0 \in \operatorname{atoms} \operatorname{dom} \uparrow^{\mathsf{FCD}} B, b_1 \in \operatorname{atoms} \operatorname{im} \uparrow^{\mathsf{FCD}} B : (a_0 \; [f] \; b_0 \wedge a_1 \; [g] \; b_1) \Rightarrow \\
&\exists a_0 \in \operatorname{atoms} \operatorname{dom} \uparrow^{\mathsf{FCD}} A, a_1 \in \operatorname{atoms} \operatorname{im} \uparrow^{\mathsf{FCD}} A, \\
&b_0 \in \operatorname{atoms} \operatorname{dom} \uparrow^{\mathsf{FCD}} B, b_1 \in \operatorname{atoms} \operatorname{im} \uparrow^{\mathsf{FCD}} B : (a_0 \times^{\mathsf{FCD}} b_0 \not\asymp f \wedge a_1 \times^{\mathsf{FCD}} b_1 \not\asymp g) \Rightarrow \\
&\exists a \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} A, b \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} B : (\operatorname{dom} a \; [f] \; \operatorname{dom} b \wedge \operatorname{im} a \; [g] \; \operatorname{im} b).
\end{aligned}$$

Also using the lemma we have

$$\begin{aligned}
&\exists a \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} A, b \in \operatorname{atoms} \uparrow^{\mathsf{FCD}} B : (\operatorname{dom} a \; [f] \; \operatorname{dom} b \wedge \operatorname{im} a \; [g] \; \operatorname{im} b) \Leftrightarrow \\
&\exists a \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} A, b \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} B : (\operatorname{dom} a \; [f] \; \operatorname{dom} b \wedge \operatorname{im} a \; [g] \; \operatorname{im} b).
\end{aligned}$$

So

$$\begin{aligned}
&\uparrow^{\mathsf{FCD}} A \left[f \times^{(C)} g\right] \uparrow^{\mathsf{FCD}} B \Leftrightarrow \\
&\exists a \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} A, b \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} B : (\operatorname{dom} a \; [f] \; \operatorname{dom} b \wedge \operatorname{im} a \; [g] \; \operatorname{im} b) \Leftrightarrow \\
&\exists a \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} A, b \in \operatorname{atoms} \uparrow^{\mathsf{RLD}} B : a \left[f \times^{(A)} g\right] b \Leftrightarrow \\
&\uparrow^{\mathsf{RLD}} A \left[f \times^{(A)} g\right] \uparrow^{\mathsf{RLD}} B.
\end{aligned}$$

□

COROLLARY 1818. $f \times^{(A)} g = \upuparrows \downdownarrows (f \times^{(C)} g)$ where downgrading is taken on the filtrator

$$\left(\mathsf{pFCD}(\mathsf{FCD}(\operatorname{Src} \circ f), \mathsf{FCD}(\operatorname{Dst} \circ f)), \mathsf{FCD}\left(\mathscr{P} \prod(\operatorname{Src} \circ f), \mathscr{P} \prod(\operatorname{Dst} \circ f)\right)\right)$$

and upgrading is taken on the filtrator

$$\left(\mathsf{pFCD}(\mathsf{RLD}(\operatorname{Src} \circ f), \mathsf{RLD}(\operatorname{Dst} \circ f)), \mathsf{FCD}\left(\mathscr{P} \prod(\operatorname{Src} \circ f), \mathscr{P} \prod(\operatorname{Dst} \circ f)\right)\right).$$

where we equate n-ary relations with corresponding principal multifuncoids and principal multireloids, when appropriate.

PROOF. Leave as an exercise for the reader. □

CONJECTURE 1819. $\uparrow^{\mathsf{FCD}} A \left[\prod^{(C)} f\right] \uparrow^{\mathsf{FCD}} B \Leftrightarrow \uparrow^{\mathsf{RLD}} A \left[\prod^{(A)} f\right] \uparrow^{\mathsf{RLD}} B$ for every indexed family f of funcoids and $A \in \mathscr{P} \prod_{i \in \operatorname{dom} f} \operatorname{Src} f_i$, $B \in \mathscr{P} \prod_{i \in \operatorname{dom} f} \operatorname{Dst} f_i$.

THEOREM 1820. For every filters a_0, a_1, b_0, b_1 we have

$$a_0 \times^{\mathsf{FCD}} b_0 \left[f \times^{(C)} g\right] a_1 \times^{\mathsf{FCD}} b_1 \Leftrightarrow a_0 \times^{\mathsf{RLD}} b_0 \left[f \times^{(A)} g\right] a_1 \times^{\mathsf{RLD}} b_1.$$

PROOF.

$$a_0 \times^{\mathsf{RLD}} b_0 \left[f \times^{(A)} g\right] a_1 \times^{\mathsf{RLD}} b_1 \Leftrightarrow$$
$$\forall A_0 \in a_0, B_0 \in b_0, A_1 \in a_1, B_1 \in b_1 : A_0 \times B_0 \left[f \times^{(A)} g\right]^* A_1 \times B_1.$$

$$A_0 \times B_0 \left[f \times^{(A)} g\right]^* A_1 \times B_1 \Leftrightarrow A_0 \times B_0 \left[f \times^{(C)} g\right]^* A_1 \times B_1 \Leftrightarrow A_0 \left[f\right]^* A_1 \wedge B_0 \left[g\right]^* B_1.$$

(Here by $A_0 \times B_0 \left[f \times^{(C)} g\right]^* A_1 \times B_1$ I mean $\uparrow^{\mathsf{FCD}(\mathrm{Base}\, a, \mathrm{Base}\, b)} (A_0 \times B_0) \left[f \times^{(C)} g\right]^* \uparrow^{\mathsf{FCD}(\mathrm{Base}\, a, \mathrm{Base}\, b)} (A_1 \times B_1)$.)

Thus it is equivalent to $a_0 \; [f] \; a_1 \wedge b_0 \; [g] \; b_1$ that is $a_0 \times^{\mathsf{FCD}} b_0 \left[f \times^{(C)} g\right]^* a_1 \times^{\mathsf{FCD}} b_1$.

(It was used the corollary 1629.) □

Can the above theorem be generalized for the infinitary case?

21.15. Cross-inner and cross-outer product

Let f be an indexed family of funcoids.

DEFINITION 1821. $\prod^{\mathrm{in}}_{i \in \mathrm{dom}\, f} f = \prod^{(C)}_{i \in \mathrm{dom}\, f} (\mathsf{RLD})_{\mathrm{in}} f_i$ (*cross-inner product*).

DEFINITION 1822. $\prod^{\mathrm{out}}_{i \in \mathrm{dom}\, f} f = \prod^{(C)}_{i \in \mathrm{dom}\, f} (\mathsf{RLD})_{\mathrm{out}} f_i$ (*cross-outer product*).

PROPOSITION 1823. $\prod^{\mathrm{in}}_{i \in \mathrm{dom}\, f} f$ and $\prod^{\mathrm{out}}_{i \in \mathrm{dom}\, f} f$ are funcoids (not just pointfree funcoids).

PROOF. They are both morphisms $\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f : \mathrm{Src}\, f_i) \to \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f : \mathrm{Src}\, f_i)$ for the category of multireloids with star-morphisms, that is $\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f : \mathrm{Src}\, f_i)$ is the set of filters on the cartesian product $\prod_{i \in \mathrm{dom}\, f} \mathrm{Src}\, f_i$ and likewise for $\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f : \mathrm{Src}\, f_i)$. □

OBVIOUS 1824. For every funcoids f and g

1°. $f \times^{\mathrm{in}} g = (\mathsf{RLD})_{\mathrm{in}} f \times^{(C)} (\mathsf{RLD})_{\mathrm{in}} g$;
2°. $f \times^{\mathrm{out}} g = (\mathsf{RLD})_{\mathrm{out}} f \times^{(C)} (\mathsf{RLD})_{\mathrm{out}} g$.

COROLLARY 1825.

1°. $\langle f \times^{\mathrm{in}} g \rangle a = (\mathsf{RLD})_{\mathrm{in}} g \circ a \circ (\mathsf{RLD})_{\mathrm{in}} f^{-1}$;
2°. $\langle f \times^{\mathrm{out}} g \rangle a = (\mathsf{RLD})_{\mathrm{out}} g \circ a \circ (\mathsf{RLD})_{\mathrm{out}} f^{-1}$

COROLLARY 1826. For every funcoids f and g and filters a and b on suitable sets:

1°. $a \; [f \times^{\mathrm{in}} g] \; b \Leftrightarrow b \not\asymp (\mathsf{RLD})_{\mathrm{in}} g \circ a \circ (\mathsf{RLD})_{\mathrm{in}} f^{-1} \Leftrightarrow b \circ (\mathsf{RLD})_{\mathrm{in}} f \not\asymp (\mathsf{RLD})_{\mathrm{in}} g \circ a$;
2°. $a \; [f \times^{\mathrm{out}} g] \; b \Leftrightarrow b \not\asymp (\mathsf{RLD})_{\mathrm{out}} g \circ a \circ (\mathsf{RLD})_{\mathrm{out}} f^{-1} \Leftrightarrow b \circ (\mathsf{RLD})_{\mathrm{out}} f \not\asymp (\mathsf{RLD})_{\mathrm{out}} g \circ a$.

PROPOSITION 1827. Knowing that every f_i is nonzero, we can restore the values of f_i from the value of $\prod^{\mathrm{in}}_{i \in \mathrm{dom}\, f} f$.

PROOF. It follows that every $(\mathsf{RLD})_{\mathrm{in}} f_i$ is nonzero, thus we can restore each $(\mathsf{RLD})_{\mathrm{in}} f_i$ from $\prod^{(C)}_{i \in \mathrm{dom}\, f} (\mathsf{RLD})_{\mathrm{in}} f_i = \prod^{\mathrm{in}}_{i \in \mathrm{dom}\, f} f$ and then we know $f_i = (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f_i$. □

EXAMPLE 1828. The values of f and g cannot be restored from $f \times^{\mathrm{out}} g$ for some nonzero funcoids f and g.

PROOF. Obviously $\mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \neq \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{R})}$, but $\mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \times^{\mathrm{out}} \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})} = (\mathsf{RLD})_{\mathrm{out}} \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \times^{(C)} (\mathsf{RLD})_{\mathrm{out}} \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})} = \bot \times^{(C)} \bot = (\mathsf{RLD})_{\mathrm{out}} \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{R})} \times^{(C)} (\mathsf{RLD})_{\mathrm{out}} \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{R})} = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{R})} \times^{\mathrm{out}} \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{R})}$.

That is the product $f \times^{\mathrm{out}} g$ is the same if we take $f = g = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$ and if we take $f = g = \mathrm{id}^{\mathsf{FCD}}_{\Omega(\mathbb{R})}$. □

QUESTION 1829. Which of the following are pairwise equal (for a. two funcoids, b. any (possibly infinite) number of funcoids)?

1°. subatomic product;
2°. displaced product;
3°. cross-inner product.

21.16. Coordinate-wise continuity

THEOREM 1830. Let μ and ν be indexed (by some index set n) families of endomorphisms for a quasi-invertible dagger category with star-morphisms, and $f_i \in \mathrm{Hom}(\mathrm{Ob}\,\mu_i, \mathrm{Ob}\,\nu_i)$ for every $i \in n$. Then:

1°. $\forall i \in n : f_i \in \mathrm{C}(\mu_i, \nu_i) \Rightarrow \prod^{(C)} f \in \mathrm{C}\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right)$;
2°. $\forall i \in n : f_i \in \mathrm{C}'(\mu_i, \nu_i) \Rightarrow \prod^{(C)} f \in \mathrm{C}'\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right)$;
3°. $\forall i \in n : f_i \in \mathrm{C}''(\mu_i, \nu_i) \Rightarrow \prod^{(C)} f \in \mathrm{C}''\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right)$.

PROOF. Using the corollary 1730:

$$\forall i \in n : f_i \in \mathrm{C}(\mu_i, \nu_i) \Leftrightarrow \forall i \in n : f_i \circ \mu_i \sqsubseteq \nu_i \circ f_i \Rightarrow \prod^{(C)}_{i \in n} (f_i \circ \mu_i) \sqsubseteq \prod^{(C)}_{i \in n} (\nu_i \circ f_i) \Leftrightarrow$$
$$\left(\prod^{(C)} f\right) \circ \left(\prod^{(C)} \mu\right) \sqsubseteq \left(\prod^{(C)} \nu\right) \circ \left(\prod^{(C)} f\right) \Leftrightarrow \prod^{(C)} f \in \mathrm{C}\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right).$$

$$\forall i \in n : f_i \in \mathrm{C}'(\mu_i, \nu_i) \Leftrightarrow \forall i \in n : \mu_i \sqsubseteq f_i^\dagger \circ \nu_i \circ f_i \Rightarrow \prod^{(C)} \mu \sqsubseteq \prod^{(C)}_{i \in n} (f_i^\dagger \circ \nu_i \circ f_i) \Leftrightarrow$$
$$\prod^{(C)} \mu \sqsubseteq \left(\prod^{(C)}_{i \in n} f_i^\dagger\right) \circ \left(\prod^{(C)}_{i \in n} \nu_i\right) \circ \left(\prod^{(C)}_{i \in n} f_i\right) \Leftrightarrow$$
$$\prod^{(C)} \mu \sqsubseteq \left(\prod^{(C)}_{i \in n} f_i\right)^\dagger \circ \left(\prod^{(C)}_{i \in n} \nu_i\right) \circ \left(\prod^{(C)}_{i \in n} f_i\right) \Leftrightarrow \prod^{(C)} f \in \mathrm{C}'\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right).$$

$$\forall i \in n : f_i \in \mathrm{C}''(\mu_i, \nu_i) \Leftrightarrow \forall i \in n : f_i \circ \mu_i \circ f_i^\dagger \sqsubseteq \nu_i \Rightarrow$$
$$\prod^{(C)}_{i \in n} (f_i \circ \mu_i \circ f_i^\dagger) \sqsubseteq \prod^{(C)}_{i \in n} \nu_i \Leftrightarrow \prod^{(C)}_{i \in n} f_i \circ \prod^{(C)}_{i \in n} \mu_i \circ \prod^{(C)}_{i \in n} f_i^\dagger \sqsubseteq \prod^{(C)}_{i \in n} \nu_i \Leftrightarrow$$
$$\prod^{(C)}_{i \in n} f_i \circ \prod^{(C)}_{i \in n} \mu_i \circ \left(\prod^{(C)}_{i \in n} f_i\right)^\dagger \sqsubseteq \prod^{(C)}_{i \in n} \nu_i \Leftrightarrow \prod^{(C)}_{i \in n} f_i \in \mathrm{C}''\left(\prod^{(C)} \mu, \prod^{(C)} \nu\right).$$

□

THEOREM 1831. Let μ and ν be indexed (by some index set n) families of endofuncoids, and $f_i \in \mathsf{FCD}(\operatorname{Ob}\mu_i, \operatorname{Ob}\nu_i)$ for every $i \in n$. Then:

1°. $\forall i \in n : f_i \in \mathrm{C}(\mu_i, \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}\left(\prod^{(A)} \mu, \prod^{(A)} \nu\right)$;
2°. $\forall i \in n : f_i \in \mathrm{C}'(\mu_i, \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}'\left(\prod^{(A)} \mu, \prod^{(A)} \nu\right)$;
3°. $\forall i \in n : f_i \in \mathrm{C}''(\mu_i, \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}''\left(\prod^{(A)} \mu, \prod^{(A)} \nu\right)$.

PROOF. Similar to the previous theorem. □

THEOREM 1832. Let μ and ν be indexed (by some index set n) families of point-free endofuncoids between posets with least elements, and $f_i \in \mathsf{pFCD}(\operatorname{Ob}\mu_i, \operatorname{Ob}\nu_i)$ for every $i \in n$. Then:

1°. $\forall i \in n : f_i \in \mathrm{C}(\mu_i, \nu_i) \Rightarrow \prod^{(S)} f \in \mathrm{C}\left(\prod^{(S)} \mu, \prod^{(S)} \nu\right)$;
2°. $\forall i \in n : f_i \in \mathrm{C}'(\mu_i, \nu_i) \Rightarrow \prod^{(S)} f \in \mathrm{C}'\left(\prod^{(S)} \mu, \prod^{(S)} \nu\right)$;
3°. $\forall i \in n : f_i \in \mathrm{C}''(\mu_i, \nu_i) \Rightarrow \prod^{(S)} f \in \mathrm{C}''\left(\prod^{(S)} \mu, \prod^{(S)} \nu\right)$.

PROOF. Similar to the previous theorem. □

21.17. Upgrading and downgrading multifuncoids

LEMMA 1833. $\left\{\frac{\langle f\rangle_k^* X}{X \in \operatorname{up} \prod_{i\in n\setminus\{k\}}^{\mathfrak{Z}_i} \mathcal{X}}\right\}$ is a filter base on $\mathfrak{A}_k$ for every family $(\mathfrak{A}_i, \mathfrak{Z}_i)$ of primary filtrators where $i \in n$ for some index set n (provided that f is a multifuncoid of the form $\mathfrak{Z}$ and $k \in n$ and $\mathcal{X} \in \prod_{i\in n\setminus\{k\}} \mathfrak{A}_i$).

PROOF. Let $\mathcal{K}, \mathcal{L} \in \left\{\frac{\langle f\rangle_k^* X}{X \in \operatorname{up}\mathcal{X}}\right\}$. Then there exist $X, Y \in \operatorname{up}\mathcal{X}$ such that $\mathcal{K} = \langle f\rangle_k^* X$, $\mathcal{L} = \langle f\rangle_k^* Y$. We can take $Z \in \operatorname{up}\mathcal{X}$ such that $Z \sqsubseteq X, Y$. Then evidently $\langle f\rangle_k^* Z \sqsubseteq \mathcal{K}$ and $\langle f\rangle_k^* Z \sqsubseteq \mathcal{L}$ and $\langle f\rangle_k^* Z \in \left\{\frac{\langle f\rangle_k^* X}{X \in \operatorname{up}\mathcal{X}}\right\}$. □

DEFINITION 1834. *Square* mult is a mult whose base and core are the same.

DEFINITION 1835. $\mathcal{L} \in [f] \Leftrightarrow \forall L \in \operatorname{up}\mathcal{L} : L \in [f]^*$ for every mult f.

DEFINITION 1836. $\langle f\rangle\mathcal{X} = \prod_{X\in\operatorname{up}\mathcal{X}} \langle f\rangle^* X$ for every mult f whose base is a complete lattice.

DEFINITION 1837. Let f be a mult whose base is a complete lattice. *Upgrading* of this mult is square mult $\upuparrows f$ with base $\upuparrows f = \operatorname{core} \upuparrows f = \operatorname{base} f$ and $\langle \upuparrows f\rangle^* \mathcal{X} = \langle f\rangle\mathcal{X}$ for every $\mathcal{X} \in \prod \operatorname{base} f$.

LEMMA 1838. $\mathcal{L}_i \not\asymp \langle \upuparrows f\rangle^* \mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}} \Leftrightarrow \forall L \in \operatorname{up}\mathcal{L} : L_i \not\asymp \langle f\rangle^* L|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}$, if every $((\operatorname{base} f)_i, (\operatorname{core} f)_i)$ is a primary filtrator over a meet-semilattice with least element.

PROOF.

$$\begin{aligned}
\mathcal{L}_i \not\asymp \langle \upuparrows f\rangle^* \mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}} &\Leftrightarrow \\
\mathcal{L}_i \not\asymp \langle f\rangle \mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}} &\Leftrightarrow \\
\mathcal{L}_i \not\asymp \bigsqcap_{X\in\operatorname{up}\mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}} \langle f\rangle^* X &\Leftrightarrow \\
\mathcal{L}_i \sqcap \bigsqcap_{X\in\operatorname{up}\mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}} \langle f\rangle^* X \neq \bot &\Leftrightarrow \\
\bigsqcap_{X\in\operatorname{up}\mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}} \langle \mathcal{L}_i\sqcap\rangle^* \langle f\rangle^* X \neq \bot &\Leftrightarrow \\
\bigsqcap\left\{ \frac{\mathcal{L}_i \sqcap \langle f\rangle^* X}{X \in \operatorname{up}\mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}} \right\} \neq \bot &\Leftrightarrow \quad (*) \\
\bot \notin \left\{ \frac{\mathcal{L}_i \sqcap \langle f\rangle^* X}{X \in \operatorname{up}\mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}} \right\} &\Leftrightarrow \\
\forall X \in \operatorname{up}\mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}} : \mathcal{L}_i \sqcap \langle f\rangle^* X \neq \bot &\Leftrightarrow \quad (**) \\
\forall L \in \operatorname{up}\mathcal{L} : \langle f\rangle^* L|_{\operatorname{dom}\mathcal{L}} \sqcap L_i \neq \bot &\Leftrightarrow \\
\forall L \in \operatorname{up}\mathcal{L} : L_i \not\asymp \langle f\rangle^* L|_{\operatorname{dom}\mathcal{L}}&.
\end{aligned}$$

(*) because $\left\{ \frac{\mathcal{L}_i\sqcap\langle f\rangle^* X}{X\in\operatorname{up}\mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}} \right\}$ is a filter base (by lemma 1833) of the filter $\bigsqcap\left\{ \frac{\mathcal{L}_i\sqcap\langle f\rangle^* X}{X\in\operatorname{up}\mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}} \right\}$.

(**) by theorem 537. □

PROPOSITION 1839. $\upuparrows f$ is a square multifuncoid, if every $((\operatorname{base} f)_i, (\operatorname{core} f)_i)$ is a primary filtrator over a bounded meet-semilattice.

PROOF. Our filtrators are with complete base by corollary 518.

$\mathcal{L}_i \not\asymp \langle \upuparrows f\rangle^* \mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}} \Leftrightarrow \forall L \in \operatorname{up}\mathcal{L} : L_i \not\asymp \langle f\rangle^* L|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}$ by the lemma.

Similarly $\mathcal{L}_j \not\asymp \langle \upuparrows f\rangle^* \mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{j\}} \Leftrightarrow \forall L \in \operatorname{up}\mathcal{L} : L_j \not\asymp \langle f\rangle^* L|_{(\operatorname{dom}\mathcal{L})\setminus\{j\}}$. So $\mathcal{L}_i \not\asymp \langle \upuparrows f\rangle^* \mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}} \Leftrightarrow \mathcal{L}_j \not\asymp \langle \upuparrows f\rangle^* \mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{j\}}$ because $L_i \not\asymp \langle f\rangle^* L|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}} \Leftrightarrow L_j \not\asymp \langle f\rangle^* L|_{(\operatorname{dom}\mathcal{L})\setminus\{j\}}$. □

PROPOSITION 1840. $[\upuparrows f]^* = [f]$ if every $((\operatorname{base} f)_i, (\operatorname{core} f)_i)$ is a primary filtrator over a bounded meet-semilattice.

PROOF. Our filtrators are with complete base by corollary 518.

$$\begin{aligned}
\mathcal{L} \in [\upuparrows f]^* &\Leftrightarrow \\
\mathcal{L}_i \not\asymp \langle \upuparrows f\rangle^* \mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}} &\Leftrightarrow \quad \text{(by the lemma)} \\
\forall L \in \operatorname{up}\mathcal{L} : L_i \not\asymp \langle f\rangle^* L|_{(\operatorname{dom} L)\setminus\{i\}} &\Leftrightarrow \\
\forall L \in \operatorname{up}\mathcal{L} : L \in [f]^* &\Leftrightarrow \\
\mathcal{L} \in [f]&.
\end{aligned}$$

□

PROPOSITION 1841. $\mathcal{L} \in [f] \Leftrightarrow \mathcal{L}_i \not\asymp \langle f\rangle \mathcal{L}|_{(\operatorname{dom}\mathcal{L})\setminus\{i\}}$ if every $((\operatorname{base} f)_i, (\operatorname{core} f)_i)$ is a primary filtrator over a bounded meet-semilattice.

PROOF. Our filtrators are with complete base by corollary 518.

The theorem holds because $\upuparrows f$ is a multifuncoid and $[f] = [\upuparrows f]^*$ and $\langle f\rangle = \langle \upuparrows f\rangle^*$. □

PROPOSITION 1842. $\Lambda \upuparrows g = \upuparrows \Lambda g$ for every prestaroid g on boolean lattices.

PROOF. Our filtrators are with separable core by theorem 537.

$$\begin{aligned}
Y \in \langle \Lambda \upuparrows g\rangle_i^* \mathcal{L} &\Leftrightarrow \\
\mathcal{L} \cup \{(i, Y)\} \in \operatorname{GR} \upuparrows g &\Leftrightarrow \\
\operatorname{up}(\mathcal{L} \cup \{(i, Y)\}) \subseteq \operatorname{GR} g &\Leftrightarrow \\
\forall K \in \operatorname{up}(\mathcal{L} \cup \{(i, Y)\}) : K \in \operatorname{GR} g &\Leftrightarrow \\
\forall X \in \operatorname{up} \mathcal{L}, P \in \operatorname{up} Y : X \cup \{(i, P)\} \in \operatorname{GR} g &\Leftrightarrow \\
\forall X \in \operatorname{up} \mathcal{L}, P \in \operatorname{up} Y : P \not\asymp (\operatorname{val} g)_i X &\Leftrightarrow \\
\forall X \in \operatorname{up} \mathcal{L} : Y \not\asymp (\operatorname{val} g)_i X &\Leftrightarrow \\
\forall X \in \operatorname{up} \mathcal{L} : Y \in (\operatorname{val} g)_i X &\Leftrightarrow \\
\forall X \in \operatorname{up} \mathcal{L} : X \cup \{(i, Y)\} \in \operatorname{GR} g &\Leftrightarrow \\
\forall X \in \operatorname{up} \mathcal{L} : Y \in \langle \Lambda g\rangle^* X &\Leftrightarrow \\
\forall X \in \operatorname{up} \mathcal{L} : Y \sqcap \langle \Lambda g\rangle^* X \neq \bot &\Leftrightarrow \\
\bot \notin \left\{ \frac{Y \sqcap \langle \Lambda g\rangle^* X}{X \in \operatorname{up} \mathcal{L}} \right\} &\Leftrightarrow \quad (*) \\
\bigsqcap \left\{ \frac{Y \sqcap \langle \Lambda g\rangle^* X}{X \in \operatorname{up} \mathcal{L}} \right\} \neq \bot &\Leftrightarrow \\
\bigsqcap_{X \in \operatorname{up} \mathcal{L}} \langle Y \sqcap\rangle^* \langle \Lambda g\rangle^* X \neq \bot &\Leftrightarrow \\
Y \not\asymp \bigsqcap_{X \in \operatorname{up} \mathcal{L}} \langle \Lambda g\rangle^* X &\Leftrightarrow \\
Y \in \bigsqcap_{X \in \operatorname{up} \mathcal{L}} \langle \Lambda g\rangle^* X &\Leftrightarrow \\
Y \in \langle \Lambda g\rangle_i \mathcal{L} &\Leftrightarrow \\
Y \in \langle \upuparrows \Lambda g\rangle_i^* \mathcal{L}&.
\end{aligned}$$

(*) because $\left\{ \frac{Y \sqcap \langle \Lambda g\rangle^* X}{X \in \operatorname{up} \mathcal{L}} \right\}$ is a filter base (by the lemma 1833) of $\bigsqcap \left\{ \frac{Y \sqcap \langle \Lambda g\rangle^* X}{X \in \operatorname{up} \mathcal{L}} \right\}$. □

DEFINITION 1843. Fix an indexed family $(\mathfrak{A}_i, \mathfrak{Z}_i)$ of filtrators. *Downgrading* of a square mult f of the form $(\mathfrak{A}_i, \mathfrak{A}_i)$ is the mult $\downdownarrows f$ of the form $(\mathfrak{A}_i, \mathfrak{Z}_i)$ defined by the formula $\langle \downdownarrows f\rangle_i^* = \langle f\rangle_i^*|_{\mathfrak{Z}_i}$ for every i.

OBVIOUS 1844. Downgrading of a square multifuncoid is a multifuncoid.

OBVIOUS 1845. $\downdownarrows \upuparrows f = f$ for every mult f of the form $(\mathfrak{A}_i, \mathfrak{Z}_i)$.

PROPOSITION 1846. Let f be a square mult whose base is a complete lattice. Then $\upuparrows \downdownarrows f = f$.

PROOF. $\langle \upuparrows \downdownarrows f\rangle^* \mathcal{X} = \bigsqcap_{X \in \operatorname{up} \mathcal{X}} \langle \downdownarrows f\rangle^* X = \bigsqcap_{X \in \operatorname{up} \mathcal{X}} \langle f\rangle^* X = \langle f\rangle^* \mathcal{X}$ for every $\mathcal{X} \in \prod_{i \in \operatorname{arity} f} (\operatorname{base} f)_i$. □

21.18. On pseudofuncoids

DEFINITION 1847. *Pseudofuncoid* from a set A to a set B is a relation f between filters on A and B such that:

$$\neg(\mathcal{I} \mathrel{f} \bot), \quad \mathcal{I} \sqcup \mathcal{J} \mathrel{f} \mathcal{K} \Leftrightarrow \mathcal{I} \mathrel{f} \mathcal{K} \vee \mathcal{J} \mathrel{f} \mathcal{K} \quad (\text{for every } \mathcal{I}, \mathcal{J} \in \mathscr{F}(A), \mathcal{K} \in \mathscr{F}(B)),$$
$$\neg(\bot \mathrel{f} \mathcal{I}), \quad \mathcal{K} \mathrel{f} \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K} \mathrel{f} \mathcal{I} \vee \mathcal{K} \mathrel{f} \mathcal{J} \quad (\text{for every } \mathcal{I}, \mathcal{J} \in \mathscr{F}(B), \mathcal{K} \in \mathscr{F}(A)).$$

OBVIOUS 1848. Pseudofuncoid is just a staroid of the form $(\mathscr{F}(A), \mathscr{F}(B))$.

OBVIOUS 1849. $[f]$ is a pseudofuncoid for every funcoid f.

EXAMPLE 1850. If A and B are infinite sets, then there exist two different pseudofuncoids f and g from A to B such that $f \cap (\mathscr{T} A \times \mathscr{T} B) = g \cap (\mathscr{T} A \times \mathscr{T} B) = [c] \cap (\mathscr{T} A \times \mathscr{T} B)$ for some funcoid c.

REMARK 1851. Considering a pseudofuncoid f as a staroid, we get $f \cap (\mathscr{T} A \times \mathscr{T} B) = \Downarrow f$.

PROOF. Take

$$f = \left\{ \frac{(\mathcal{X}, \mathcal{Y})}{\mathcal{X} \in \mathscr{F}(A), \mathcal{Y} \in \mathscr{F}(B), \bigcap \mathcal{X} \text{ and } \bigcap \mathcal{Y} \text{ are infinite}} \right\}$$

and

$$g = f \cup \left\{ \frac{(\mathcal{X}, \mathcal{Y})}{\mathcal{X} \in \mathscr{F}(A), \mathcal{Y} \in \mathscr{F}(B), \mathcal{X} \sqsupseteq a, \mathcal{Y} \sqsupseteq b} \right\}$$

where a and b are nontrivial ultrafilters on A and B correspondingly, c is the funcoid defined by the relation

$$[c]^* = \delta = \left\{ \frac{(X, Y)}{X \in \mathscr{P} A, Y \in \mathscr{P} B, X \text{ and } Y \text{ are infinite}} \right\}.$$

First prove that f is a pseudofuncoid. The formulas $\neg(\mathcal{I} \mathrel{f} \bot)$ and $\neg(\bot \mathrel{f} \mathcal{I})$ are obvious. We have

$$\mathcal{I} \sqcup \mathcal{J} \mathrel{f} \mathcal{K} \Leftrightarrow \bigcap(\mathcal{I} \sqcup \mathcal{J}) \text{ and } \bigcap \mathcal{Y} \text{ are infinite} \Leftrightarrow$$
$$\bigcap \mathcal{I} \cup \bigcap \mathcal{J} \text{ and } \bigcap \mathcal{Y} \text{ are infinite} \Leftrightarrow \left(\bigcap \mathcal{I} \text{ or } \bigcap \mathcal{J} \text{ is infinite}\right) \wedge \bigcap \mathcal{Y} \text{ is infinite} \Leftrightarrow$$
$$\left(\bigcap \mathcal{I} \text{ and } \bigcap \mathcal{Y} \text{ are infinite}\right) \vee \left(\bigcap \mathcal{J} \text{ and } \bigcap \mathcal{Y} \text{ are infinite}\right) \Leftrightarrow$$
$$\mathcal{I} \mathrel{f} \mathcal{K} \vee \mathcal{J} \mathrel{f} \mathcal{K}.$$

Similarly $\mathcal{K} \mathrel{f} \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K} \mathrel{f} \mathcal{I} \vee \mathcal{K} \mathrel{f} \mathcal{J}$. So f is a pseudofuncoid.

Let now prove that g is a pseudofuncoid. The formulas $\neg(\mathcal{I} \mathrel{g} \bot)$ and $\neg(\bot \mathrel{g} \mathcal{I})$ are obvious. Let $\mathcal{I} \sqcup \mathcal{J} \mathrel{g} \mathcal{K}$. Then either $\mathcal{I} \sqcup \mathcal{J} \mathrel{f} \mathcal{K}$ and then $\mathcal{I} \sqcup \mathcal{J} \mathrel{g} \mathcal{K}$ or $\mathcal{I} \sqcup \mathcal{J} \sqsupseteq a$ and then $\mathcal{I} \sqsupseteq a \vee \mathcal{J} \sqsupseteq a$ thus having $\mathcal{I} \mathrel{g} \mathcal{K} \vee \mathcal{J} \mathrel{g} \mathcal{K}$. So $\mathcal{I} \sqcup \mathcal{J} \mathrel{g} \mathcal{K} \Rightarrow \mathcal{I} \mathrel{g} \mathcal{K} \vee \mathcal{J} \mathrel{g} \mathcal{K}$. The reverse implication is obvious. We have $\mathcal{I} \sqcup \mathcal{J} \mathrel{g} \mathcal{K} \Leftrightarrow \mathcal{I} \mathrel{g} \mathcal{K} \vee \mathcal{J} \mathrel{g} \mathcal{K}$ and similarly $\mathcal{K} \mathrel{g} \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K} \mathrel{g} \mathcal{I} \vee \mathcal{K} \mathrel{g} \mathcal{J}$. So g is a pseudofuncoid.

Obviously $f \neq g$ ($a \mathrel{g} b$ but not $a \mathrel{f} b$).

It remains to prove $f \cap (\mathscr{T} A \times \mathscr{T} B) = g \cap (\mathscr{T} A \times \mathscr{T} B) = [c] \cap (\mathscr{T} A \times \mathscr{T} B)$. Really, $f \cap (\mathscr{T} A \times \mathscr{T} B) = [c] \cap (\mathscr{T} A \times \mathscr{T} B)$ is obvious. If $(\uparrow^A X, \uparrow^B Y) \in g \cap (\mathscr{T} A \times \mathscr{T} B)$ then either $(\uparrow^A X, \uparrow^B Y) \in f \cap (\mathscr{T} A \times \mathscr{T} B)$ or $X \in \operatorname{up} a$, $Y \in \operatorname{up} b$, so X and Y are infinite and thus $(\uparrow^A X, \uparrow^B Y) \in f \cap (\mathscr{T} A \times \mathscr{T} B)$. So $g \cap (\mathscr{T} A \times \mathscr{T} B) = f \cap (\mathscr{T} A \times \mathscr{T} B)$. □

REMARK 1852. The above counter-example shows that pseudofuncoids (and more generally, any staroids on filters) are "second class" objects, they are not full-fledged because they don't bijectively correspond to funcoids and the elegant funcoids theory does not apply to them.

From the above it follows that staroids on filters do not correspond (by restriction) to staroids on principal filters (or staroids on sets).

21.18.1. More on free stars and principal free stars.

PROPOSITION 1853. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator.
4°. $\partial\mathcal{F} = \downdownarrows \star\mathcal{F}$ for every $\mathcal{F} \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Obvious.
3°⇒4°. $X \in \partial\mathcal{F} \Leftrightarrow X \not\asymp^{\mathfrak{A}} \mathcal{F} \Leftrightarrow X \in \star\mathcal{F} \Leftrightarrow X \in \downdownarrows \star\mathcal{F}$ for every $X \in \mathfrak{Z}$.

□

PROPOSITION 1854. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a meet-semilattice with least element.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator with separable core.
4°. $\star\mathcal{F} = \upuparrows \partial\mathcal{F}$ for every $\mathcal{F} \in \mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. Theorem 537.
3°⇒4°. $\mathcal{X} \in \upuparrows \partial\mathcal{F} \Leftrightarrow \operatorname{up}\mathcal{X} \subseteq \partial\mathcal{F} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X} : X \not\asymp \mathcal{F} \Leftrightarrow \mathcal{X} \not\asymp \mathcal{F} \Leftrightarrow \mathcal{X} \in \star\mathcal{F}$.

□

PROPOSITION 1855. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a complete boolean lattice.
3°. $(\mathfrak{A}, \mathfrak{Z})$ is a down-aligned, with join-closed, binarily meet-closed and separable core which is a complete boolean lattice.
4°. The following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:
 (a) $\mathcal{F} \in \mathfrak{Z}$.
 (b) $\partial\mathcal{F}$ is a principal free star on $\mathfrak{Z}$.
 (c) $\star\mathcal{F}$ is a principal free star on $\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.
2°⇒3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is with with join-closed core by theorem 534, binarily meet-closed core by corollary 536, with separable core by theorem 537.
3°⇒4°.
 4°a⇒4°b. That $\partial\mathcal{F}$ does not contain the least element is obvious. That $\partial\mathcal{F}$ is an upper set is obvious. So it remains to apply theorem 583.
 4°b⇒4°c. That $\star\mathcal{F}$ does not contain the least element is obvious. That $\star\mathcal{F}$ is an upper set is obvious. So it remains to apply theorem 583.
 4°c⇒4°a. Apply theorem 583.

□

PROPOSITION 1856. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a join-semilattice.
3°. The filtrator $(\mathfrak{A}, \mathfrak{Z})$ is weakly down-aligned and with binarily join-closed core and $\mathfrak{Z}$ is a join-semilattice.
4°. If S is a free star on $\mathfrak{A}$ then $\downdownarrows S$ is a free star on $\mathfrak{Z}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. It is weakly down-aligned by obvious 511 and with join-closed core by theorem 534.

3°⇒4°. For every $X, Y \in \mathfrak{Z}$ we have

$$X \sqcup^{\mathfrak{Z}} Y \in \downdownarrows S \Leftrightarrow X \sqcup^{\mathfrak{Z}} Y \in S \Leftrightarrow X \sqcup^{\mathfrak{A}} Y \in S \Leftrightarrow X \in S \vee Y \in S \Leftrightarrow X \in \downdownarrows S \vee Y \in \downdownarrows S;$$

Suppose there is least element $\bot^{\mathfrak{Z}} \in \downdownarrows S$. Then $\bot^{\mathfrak{A}} = \bot^{\mathfrak{Z}} \in S$ what is impossible. □

PROPOSITION 1857. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is a powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a boolean lattice.
3°. If S is a free star on $\mathfrak{Z}$ then $\upuparrows S$ is a free star on $\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. There exists a filter $\mathcal{F}$ such that $S = \partial \mathcal{F}$. For every filters $\mathcal{X}, \mathcal{Y} \in \mathfrak{A}$

$$\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \in \upuparrows S \Leftrightarrow \operatorname{up}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y}) \subseteq S \Leftrightarrow \forall K \in \operatorname{up}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y}) : K \in \partial \mathcal{F} \Leftrightarrow$$
$$\forall K \in \operatorname{up}(\mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y}) : K \not\asymp \mathcal{F} \Leftrightarrow \mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \not\asymp \mathcal{F} \Leftrightarrow \mathcal{X} \sqcup^{\mathfrak{A}} \mathcal{Y} \in \star \mathcal{F} \Leftrightarrow \mathcal{X} \in \star \mathcal{F} \vee \mathcal{Y} \in \star \mathcal{F} \Leftrightarrow$$
$$\mathcal{X} \not\asymp \mathcal{F} \vee \mathcal{Y} \not\asymp \mathcal{F} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X} : X \not\asymp \mathcal{F} \vee \forall Y \in \operatorname{up} \mathcal{Y} : Y \not\asymp \mathcal{F} \Leftrightarrow$$
$$\forall X \in \operatorname{up} \mathcal{X} : X \in \partial \mathcal{F} \vee \forall Y \in \operatorname{up} \mathcal{Y} : Y \in \partial \mathcal{F} \Leftrightarrow$$
$$\operatorname{up} \mathcal{X} \subseteq S \vee \operatorname{up} \mathcal{Y} \subseteq S \Leftrightarrow \mathcal{X} \in \upuparrows S \vee \mathcal{Y} \in \upuparrows S;$$

$\bot \in \upuparrows S \Leftrightarrow \operatorname{up} \bot \subseteq S \Leftrightarrow \bot \in S$ what is false.

□

PROPOSITION 1858. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is primary filtrator over a complete lattice.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is down-aligned filtrator with join-closed core over a complete lattice.
3°. If S is a principal free star on $\mathfrak{A}$ then $\downdownarrows S$ is a principal free star on $\mathfrak{Z}$.

PROOF.

1°⇒2°. It is down-aligned by obvious 506 and with join-closed core by theorem 534.

2°⇒3°. $\bigsqcup^{\mathfrak{Z}} T \in \downdownarrows S \Leftrightarrow \bigsqcup^{\mathfrak{Z}} T \in S \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \in S \Leftrightarrow T \cap S \neq \emptyset \Leftrightarrow T \cap \downdownarrows S \neq \emptyset$ for every $T \in \mathscr{P}\mathfrak{Z}$; $\bot \notin \downdownarrows S$ is obvious.

□

PROPOSITION 1859. The following is an implications tuple:

1°. $(\mathfrak{A}, \mathfrak{Z})$ is powerset filtrator.
2°. $(\mathfrak{A}, \mathfrak{Z})$ is primary filtrator over a boolean lattice.
3°. If S is a principal free star on $\mathfrak{Z}$ then $\upuparrows S$ is a principal free star on $\mathfrak{A}$.

PROOF.

1°⇒2°. Obvious.

2°⇒3°. There exists a principal filter $\mathcal{F}$ such that $S = \partial\mathcal{F}$.

$$\bigsqcup^{\mathfrak{A}} T \in \upuparrows S \Leftrightarrow \operatorname{up} \bigsqcup^{\mathfrak{A}} T \subseteq S \Leftrightarrow \forall K \in \operatorname{up} \bigsqcup^{\mathfrak{A}} T : K \in \partial\mathcal{F} \Leftrightarrow$$
$$\forall K \in \operatorname{up} \bigsqcup^{\mathfrak{A}} T : K \not\asymp \mathcal{F} \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \not\asymp \mathcal{F} \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \in \star\mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T : \mathcal{K} \in \star\mathcal{F} \Leftrightarrow$$
$$\exists \mathcal{K} \in T : \mathcal{K} \not\asymp \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \forall K \in \operatorname{up}\mathcal{K} : K \not\asymp \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \forall K \in \operatorname{up}\mathcal{K} : K \in \partial\mathcal{F} \Leftrightarrow$$
$$\exists \mathcal{K} \in T : \operatorname{up}\mathcal{K} \subseteq S \Leftrightarrow \exists \mathcal{K} \in T : \mathcal{K} \in \upuparrows S \Leftrightarrow T \cap \upuparrows S \neq \emptyset.$$

$$\bot \in \upuparrows S \Leftrightarrow \operatorname{up} \bot \subseteq S \Leftrightarrow \bot \in S \text{ what is false.}$$

□

21.18.2. Complete staroids and multifuncoids.

DEFINITION 1860. Consider an indexed family $\mathfrak{Z}$ of posets. A pre-staroid f of the form $\mathfrak{Z}$ is *complete* in argument $k \in \operatorname{arity} f$ when $(\operatorname{val} f)_k L$ is a principal free star for every $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} \mathfrak{Z}_i$.

DEFINITION 1861. Consider an indexed family $(\mathfrak{A}_i, \mathfrak{Z}_i)$ of filtrators and multifuncoid f is of the form $(\mathfrak{A}, \mathfrak{Z})$. Then f is *complete* in argument $k \in \operatorname{arity} f$ iff $\langle f \rangle_k^* L \in \mathfrak{Z}_k$ for every family $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} \mathfrak{Z}_i$.

PROPOSITION 1862. Consider an indexed family $(\mathfrak{A}_i, \mathfrak{Z}_i)$ of primary filtrators over complete boolean lattices. Let f be a multifuncoid of the form $(\mathfrak{A}, \mathfrak{Z})$ and $k \in \operatorname{arity} f$. The following are equivalent:

1°. Multifuncoid f is complete in argument k.
2°. Pre-staroid $\downdownarrows [f]^*$ is complete in argument k.

PROOF. Let $L \in \prod \mathfrak{Z}$. We have $L \in \operatorname{GR} [f]^* \Leftrightarrow L_i \not\asymp \langle f \rangle_i^* L|_{(\operatorname{dom} L) \setminus \{i\}}$;
$\left(\operatorname{val} [f]^*\right)_k L = \partial \langle f \rangle_k^* L$ by the definition.
So $\left(\operatorname{val} [f]^*\right)_k L$ is a principal free star iff $\langle f \rangle_k^* L \in \mathfrak{Z}_k$ (proposition 1855) for every $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} \mathfrak{Z}_i$. □

EXAMPLE 1863. Consider funcoid $f = 1_U^{\mathsf{FCD}}$. It is obviously complete in each its two arguments. Then $[f]^*$ is not complete in each of its two arguments because $(\mathcal{X}, \mathcal{Y}) \in [f]^* \Leftrightarrow \mathcal{X} \not\asymp \mathcal{Y}$ what does not generate a principal free star if one of the arguments (say $\mathcal{X}$) is a fixed nonprincipal filter.

THEOREM 1864. Consider an indexed family $(\mathfrak{A}, \mathfrak{Z})$ of filtrators which are down-aligned, separable, with join-closed, binarily meet-closed and with separable core which is a complete boolean lattice.
Let f be a multifuncoid of the aforementioned form. Let $k, l \in \operatorname{arity} f$ and $k \neq l$. The following are equivalent:

1°. f is complete in the argument k.
2°. $\langle f \rangle_l^* (L \cup \{(k, \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l^* (L \cup \{(k, x)\})$ for every $X \in \mathscr{P}\mathfrak{Z}_k$, $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k,l\}} \mathfrak{Z}_i$.
3°. $\langle f \rangle_l^* (L \cup \{(k, \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l^* (L \cup \{(k, x)\})$ for every $X \in \mathscr{P}\mathfrak{A}_k$, $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k,l\}} \mathfrak{Z}_i$.

PROOF.

3°⇒2°. Obvious.

2°⇒1°. Let $Y \in \mathfrak{Z}$.

$$\bigsqcup X \not\asymp \langle f\rangle_k^*(L \cup \{(l,Y)\}) \Leftrightarrow Y \not\asymp \langle f\rangle_l^*\left(L \cup \left\{\left(k, \bigsqcup X\right)\right\}\right) \Leftrightarrow$$
$$Y \not\asymp \bigsqcup_{x\in X} \langle f\rangle_l^*(L \cup \{(k,x)\}) \Leftrightarrow \text{(proposition 583)} \Leftrightarrow$$
$$\exists x \in X : Y \not\asymp \langle f\rangle_l^*(L \cup \{(k,x)\}) \Leftrightarrow \exists x \in X : x \not\asymp \langle f\rangle_k^*(L \cup (l,Y)).$$

It is equivalent (proposition 1855 and the fact that $[f]^*$ is an upper set) to $\langle f\rangle_k^*(L \cup \{(l,Y)\})$ being a principal filter and thus $(\operatorname{val} f)_k L$ being a principal free star.

1°⇒3°.

$$Y \not\asymp \langle f\rangle_l^*\left(L \cup \left\{\left(k, \bigsqcup X\right)\right\}\right) \Leftrightarrow \bigsqcup X \not\asymp \langle f\rangle_k(L \cup \{(l,Y)\}) \Leftrightarrow$$
$$\exists x \in X : x \not\asymp \langle f\rangle_k^*(L \cup \{(l,Y)\}) \Leftrightarrow \exists x \in X : Y \not\asymp \langle f\rangle_l^*(L \cup \{(k,x)\}) \Leftrightarrow$$
$$Y \not\asymp \bigsqcup_{x\in X} \langle f\rangle_l^*(L \cup \{(k,x)\})$$

for every principal Y. Thus $\langle f\rangle_l^*(L \cup \{(k, \bigsqcup X)\}) = \bigsqcup_{x\in X}\langle f\rangle_l^*(L \cup \{(k,x)\})$ by separability.

□

21.19. Identity staroids and multifuncoids

21.19.1. Identity relations. Denote $\mathrm{id}_{A[n]} = \left\{\frac{\lambda i\in n:x}{x\in A}\right\} = \left\{\frac{n\times\{x\}}{x\in A}\right\}$ the n-ary identity relation on a set A (for each index set n).

Proposition 1865. $\prod X \not\asymp \mathrm{id}_{A[n]} \Leftrightarrow \bigcap_{i\in n} X_i \cap A \neq \emptyset$ for every indexed family X of sets.

Proof.

$$\prod X \not\asymp \mathrm{id}_{A[n]} \Leftrightarrow \exists t \in A : n\times\{t\} \in \prod X \Leftrightarrow \exists t \in A \forall i \in n : t \in X_i \Leftrightarrow \bigcap_{i\in n} X_i \cap A \neq \emptyset.$$

□

21.19.2. General definitions of identity staroids. Consider a filtrator $(\mathfrak{A}, \mathfrak{Z})$ and $\mathcal{A} \in \mathfrak{A}$.

I will define below *small identity staroids* $\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ and *big identity staroids* $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. That they are really staroids and even completary staroids (under certain conditions) is proved below.

Definition 1866. Consider a filtrator $(\mathfrak{A}, \mathfrak{Z})$. Let $\mathfrak{Z}$ be a complete lattice. Let $\mathcal{A} \in \mathfrak{A}$, let n be an index set.

$$\operatorname{form} \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \mathfrak{Z}^n; \quad L \in \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \prod_{i\in n}^{\mathfrak{Z}} L_i \in \partial\mathcal{A}.$$

Obvious 1867. $X \in \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A} : \bigsqcap^{\mathfrak{Z}}_{i\in n} X_i \sqcap A \neq 0$ if our filtrator is with separable core.

Definition 1868. The subset X of a poset $\mathfrak{A}$ *has a nontrivial lower bound* (I denote this predicate as $\operatorname{MEET}(X)$) iff there is nonleast $a \in \mathfrak{A}$ such that $\forall x \in X : a \sqsubseteq x$.

Definition 1869. Staroid $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ (for any $\mathcal{A} \in \mathfrak{A}$ where $\mathfrak{A}$ is a poset) is defined by the formulas:

$$\operatorname{form} \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \mathfrak{A}^n; \quad \mathcal{L} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \operatorname{MEET}\left(\left\{ \frac{\mathcal{L}_i}{i \in n} \right\} \cup \{\mathcal{A}\} \right).$$

Obvious 1870. If $\mathfrak{A}$ is complete lattice, then $\mathcal{L} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \bigsqcap \mathcal{L} \not\asymp \mathcal{A}$.

Obvious 1871. If $\mathfrak{A}$ is complete lattice and a is an atom, then $\mathcal{L} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \bigsqcap \mathcal{L} \sqsupseteq a$.

Obvious 1872. If $\mathfrak{A}$ is a complete lattice then there exists a multifuncoid $\Lambda \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ such that $\langle \Lambda \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \rangle_k L = \bigsqcap_{i \in n} L_i \sqcap \mathcal{A}$ for every $k \in n$, $L \in \mathfrak{A}^{n \setminus \{k\}}$.

Proposition 1873. Let $(\mathfrak{A}, \mathfrak{Z})$ be a meet-closed filtrator and $\mathfrak{Z}$ be a complete lattice and $\mathfrak{A}$ be a meet-semilattice. There exists a multifuncoid $\Lambda \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ such that $\langle \Lambda \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \rangle_k L = \bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \sqcap^{\mathfrak{A}} \mathcal{A}$ for every $k \in n$, $L \in \mathfrak{Z}^{n \setminus \{k\}}$.

Proof. We need to prove that $L \cup \{(k, X)\} \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \sqcap^{\mathfrak{A}} \mathcal{A} \not\asymp^{\mathfrak{A}} X$. But

$$\bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \sqcap^{\mathfrak{A}} \mathcal{A} \not\asymp^{\mathfrak{A}} X \Leftrightarrow \bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \sqcap^{\mathfrak{A}} X \not\asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \bigsqcap^{\mathfrak{Z}}_{i \in n} (L \cup \{(k, X)\})_i \not\asymp^{\mathfrak{A}} \mathcal{A} \Leftrightarrow L \cup \{(k, X)\} \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

□

21.19.3. Identities are staroids.

Proposition 1874. Let $\mathfrak{A}$ be a complete meet infinite distributive lattice and $\mathcal{A} \in \mathfrak{A}$. Then $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a staroid.

Proof. That $L \notin \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $L_k = \bot$ for some $k \in n$ is obvious. It remains to prove

$$L \cup \{(k, X \sqcup Y)\} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow L \cup \{(k, X)\} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \vee L \cup \{(k, Y)\} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

It is equivalent to

$$\bigsqcap_{i \in n \setminus \{k\}} L_i \sqcap (X \sqcup Y) \not\asymp \mathcal{A} \Leftrightarrow \bigsqcap_{i \in n \setminus \{k\}} L_i \sqcap X \not\asymp \mathcal{A} \vee \bigsqcap_{i \in n \setminus \{k\}} L_i \sqcap Y \not\asymp \mathcal{A}.$$

Really,

$$\bigsqcap_{i \in n \setminus \{k\}} L_i \sqcap (X \sqcup Y) \not\asymp \mathcal{A} \Leftrightarrow \bigsqcap_{i \in n \setminus \{k\}} ((L_i \sqcap X) \sqcup (L_i \sqcap Y)) \not\asymp \mathcal{A} \Leftrightarrow \left(\bigsqcap_{i \in n \setminus \{k\}} L_i \sqcap X \right) \sqcup \left(\bigsqcap_{i \in n \setminus \{k\}} L_i \sqcap Y \right) \not\asymp \mathcal{A} \Leftrightarrow \bigsqcap_{i \in n \setminus \{k\}} L_i \sqcap X \not\asymp \mathcal{A} \vee \bigsqcap_{i \in n \setminus \{k\}} L_i \sqcap Y \not\asymp \mathcal{A}.$$

□

Proposition 1875. Let $(\mathfrak{A}, \mathfrak{Z})$ be a starrish filtrator over a complete meet infinite distributive lattice and $\mathcal{A} \in \mathfrak{A}$. Then $\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a staroid.

PROOF. That $L \notin \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $L_k = \bot$ for some $k \in n$ is obvious. It remains to prove

$$L\cup\{(k, X\sqcup Y)\} \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow L\cup\{(k, X)\} \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \vee L\cup\{(k, Y)\} \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

It is equivalent to

$$\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i \sqcap (X \sqcup Y) \not\asymp \mathcal{A} \Leftrightarrow \prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i \sqcap X \not\asymp \mathcal{A} \vee \prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i \sqcap Y \not\asymp \mathcal{A}.$$

Really,

$$\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i \sqcap (X \sqcup Y) \not\asymp \mathcal{A} \Leftrightarrow \prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} ((L_i \sqcap X) \sqcup (L_i \sqcap Y)) \Leftrightarrow$$
$$\left(\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i \sqcap X\right) \sqcup \left(\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i \sqcap Y\right) \not\asymp \mathcal{A} \Leftrightarrow$$
$$\prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i \sqcap X \not\asymp \mathcal{A} \vee \prod^{\mathfrak{Z}}_{i\in n\setminus\{k\}} L_i \sqcap Y \not\asymp \mathcal{A}.$$

□

PROPOSITION 1876. Let $(\mathfrak{A}, \mathfrak{Z})$ be a primary filtrator over a boolean lattice. $\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid for every $\mathcal{A} \in \mathfrak{A}$.

PROOF. $\star\mathcal{A}$ is a free star by theorem 614.

$$L_0 \sqcup L_1 \in \operatorname{GR} \operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \forall i \in n : (L_0 \sqcup L_1)i \in \star\mathcal{A} \Leftrightarrow \forall i \in n : L_0 i \sqcup L_1 i \in \star\mathcal{A} \Leftrightarrow$$
$$\forall i \in n : (L_0 i \in \star\mathcal{A} \vee L_1 i \in \star\mathcal{A}) \Leftrightarrow \exists c \in \{0, 1\}^n \forall i \in n : L_{c(i)} i \in \star\mathcal{A} \Leftrightarrow$$
$$\exists c \in \{0, 1\}^n : (\lambda i \in n : L_{c(i)} i) \in \operatorname{GR} \operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

□

LEMMA 1877. $X \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \operatorname{Cor}' \prod^{\mathfrak{A}}_{i\in n} X_i \not\asymp \mathcal{A}$ for a join-closed filtrator $(\mathfrak{A}, \mathfrak{Z})$ such that both $\mathfrak{A}$ and $\mathfrak{Z}$ are complete lattices, provided that $\mathcal{A} \in \mathfrak{A}$.

PROOF. $X \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \prod^{\mathfrak{Z}}_{i\in n} X_i \not\asymp \mathcal{A} \Leftrightarrow \operatorname{Cor}' \prod^{\mathfrak{A}}_{i\in n} X_i \not\asymp \mathcal{A}$ (theorem 602). □

CONJECTURE 1878. $\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid for every set-theoretic filter $\mathcal{A}$.

CONJECTURE 1879. $\upuparrows \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid if $\mathcal{A}$ is a filter on a set and n is an index set.

21.19.4. Special case of sets and filters.

PROPOSITION 1880. $\uparrow^{\mathfrak{Z}^n} X \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall A \in a : \prod X \not\asymp \operatorname{id}_{A[n]}$ for every filter a on a powerset and index set n.

PROOF.

$$\forall A \in a : \prod X \not\asymp \operatorname{id}_{A[n]} \Leftrightarrow \forall A \in a : \bigcap_{i\in n} X_i \cap A \neq \emptyset \Leftrightarrow \forall A \in a : \prod^{\mathfrak{Z}}_{i\in n} X_i \not\asymp A \Leftrightarrow$$

$$\forall A \in a : \prod^{\mathfrak{Z}}_{i\in n} X_i \not\asymp^{\mathfrak{A}} A \Leftrightarrow \prod^{\mathfrak{Z}}_{i\in n} \uparrow^{\mathfrak{Z}} X_i \not\asymp^{\mathfrak{A}} a \Leftrightarrow \prod^{\mathfrak{Z}}_{i\in n} (\uparrow^{\mathfrak{Z}^n} X)_i \not\asymp^{\mathfrak{A}} a \Leftrightarrow \uparrow^{\mathfrak{Z}^n} X \in \operatorname{GR} \operatorname{id}_{a[n]}.$$

□

PROPOSITION 1881. $Y \in \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A} : Y \in \operatorname{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]}$ for every filter $\mathcal{A}$ on a powerset and $Y \in \mathfrak{Z}^n$.

PROOF. Take $Y = \uparrow^{\mathfrak{Z}^n} X$.

$$\forall A \in \operatorname{up} \mathcal{A} : Y \in \operatorname{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A} : \uparrow^{\mathfrak{Z}^n} X \in \operatorname{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow$$
$$\forall A \in \operatorname{up} \mathcal{A} : \prod X \not\asymp \mathrm{id}_{A[n]} \Leftrightarrow \uparrow^{\mathfrak{Z}^n} X \in \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow Y \in \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

□

PROPOSITION 1882. $\uparrow^{\mathfrak{Z}^n} X \in \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall A \in a \exists t \in A \forall i \in n : t \in X_i$.

PROOF.

$$\uparrow^{\mathfrak{Z}^n} X \in \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \exists A \in a \exists t \in A : n \times \{t\} \in \prod X \Leftrightarrow \forall A \in a \exists t \in A \forall i \in n : t \in X_i.$$

□

21.19.5. Relationships between big and small identity staroids.

DEFINITION 1883. $a^n_{\mathsf{Strd}} = \prod^{\mathsf{Strd}}_{i \in n} a$ for every element a of a poset and an index set n.

LEMMA 1884. $\mathcal{L} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ iff $\bigcup_{i \in n} \operatorname{up} \mathcal{L}_i \cup \{a\}$ has finite intersection property (for primary filtrators over meet semilattices with greatest element).

PROOF. The lattice $\mathfrak{A}$ is complete by corollary 518. $\mathcal{L} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \bigsqcap_{i \in n} \operatorname{up} \mathcal{L} \sqcap a \neq \bot^{\mathscr{F}} \Leftrightarrow \forall X \in \bigsqcap_{i \in n} \operatorname{up} \mathcal{L} \sqcap a : X \neq \bot$ what is equivalent of $\bigcup_{i \in n} \mathcal{L}_i \cup \{a\}$ having finite intersection property. □

PROPOSITION 1885. $\upuparrows \mathrm{id}^{\mathsf{Strd}}_{a[n]} \sqsubseteq \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \sqsubseteq a^n_{\mathsf{Strd}}$ for every filter a (on any distributive lattice with least element) and an index set n.

PROOF.

$\operatorname{GR} \upuparrows \mathrm{id}^{\mathsf{Strd}}_{a[n]} \subseteq \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$.

$$\mathcal{L} \in \operatorname{GR} \upuparrows \mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \operatorname{up} \mathcal{L} \subseteq \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall L \in \operatorname{up} \mathcal{L} : L \in \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow$$
$$(\text{theorem 537}) \Leftrightarrow \forall L \in \operatorname{up} \mathcal{L} \forall A \in \operatorname{up} a : \bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \not\asymp A \Leftrightarrow$$
$$\forall L \in \operatorname{up} \mathcal{L} \forall A \in \operatorname{up} a : \bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \sqcap A \neq \bot \Rightarrow$$
$$\bigcup_{i \in n} \operatorname{up} \mathcal{L}_i \cup \{a\} \text{ has finite intersection property} \Leftrightarrow \mathcal{L} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{a[n]}.$$

$\operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \subseteq \operatorname{GR} a^n_{\mathsf{Strd}}$. $\mathcal{L} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \operatorname{MEET}\left(\left\{\frac{\mathcal{L}_i}{i \in n}\right\} \cup \{a\}\right) \Rightarrow \forall i \in a : \mathcal{L}_i \not\asymp a \Leftrightarrow \mathcal{L} \in \operatorname{GR} a^a_{\mathsf{Strd}}$.

□

PROPOSITION 1886. $\upuparrows \mathrm{id}^{\mathsf{Strd}}_{a[a]} \sqsubset \mathrm{ID}^{\mathsf{Strd}}_{a[a]} = a^a_{\mathsf{Strd}}$ for every nontrivial ultrafilter a on a set.

PROOF. □

$\operatorname{GR} \upuparrows \mathrm{id}^{\mathsf{Strd}}_{a[a]} \neq \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{a[a]}$. Let $\mathcal{L}_i = \uparrow^{\operatorname{Base}(a)} i$. Then trivially $\mathcal{L} \in \operatorname{GR} \mathrm{ID}^{\mathsf{Strd}}_{a[a]}$. But to disprove $\mathcal{L} \in \operatorname{GR} \upuparrows \mathrm{id}^{\mathsf{Strd}}_{a[a]}$ it's enough to show $L \notin \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{a[a]}$ for some $L \in \operatorname{up} \mathcal{L}$. Really, take $L_i = \mathcal{L}_i = \uparrow^{\operatorname{Base}(a)} i$. Then $L \in \operatorname{GR} \mathrm{id}^{\mathsf{Strd}}_{a[a]} \Leftrightarrow \forall A \in a \exists t \in A \forall i \in a : t \in i$ what is clearly false (we can always take $i \in a$ such that $t \notin i$ for any point t).

$\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{a[a]} = \operatorname{GR} a^a_{\mathsf{Strd}}$. $\mathcal{L} \in \operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{a[a]} \Leftrightarrow \forall i \in a : \mathcal{L}_i \sqsupseteq a \Leftrightarrow \forall i \in a : \mathcal{L}_i \not\asymp a \Leftrightarrow \mathcal{L} \in \operatorname{GR} a^a_{\mathsf{Strd}}$.

Corollary 1887. a^a_{Strd} isn't an atom when a is a nontrivial ultrafilter.

Corollary 1888. Staroidal product of an infinite indexed family of ultrafilters may be non-atomic.

Proposition 1889. $\operatorname{id}^{\mathsf{Strd}}_{a[n]}$ is determined by the value of $\upuparrows \operatorname{id}^{\mathsf{Strd}}_{a[n]}$ (for every element a of a filtrator $(\mathfrak{A}, \mathfrak{Z})$ over a complete lattice $\mathfrak{Z}$). Moreover $\operatorname{id}^{\mathsf{Strd}}_{a[n]} = \downdownarrows\upuparrows \operatorname{id}^{\mathsf{Strd}}_{a[n]}$.

Proof. Use general properties of upgrading and downgrading (proposition 1653). □

Proposition 1890. $\operatorname{ID}^{\mathsf{Strd}}_{a[n]}$ is determined by the value of $\downdownarrows \operatorname{ID}^{\mathsf{Strd}}_{a[n]}$, moreover $\operatorname{ID}^{\mathsf{Strd}}_{a[n]} = \upuparrows\downdownarrows \operatorname{ID}^{\mathsf{Strd}}_{a[n]}$ (for filter a on a primary filtrator over a meet semilattice with greatest element).

Proof.

$$\mathcal{L} \in \upuparrows\downdownarrows \operatorname{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \operatorname{up}\mathcal{L} \subseteq \downdownarrows \operatorname{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \operatorname{up}\mathcal{L} \subseteq \operatorname{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow$$
$$\forall L \in \operatorname{up}\mathcal{L} : L \in \operatorname{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall L \in \operatorname{up}\mathcal{L} : \bigsqcap_{i\in n} L_i \sqcap a \neq \bot^{\mathscr{F}} \Leftrightarrow$$
$$\bigcup_{i\in n} \operatorname{up}\mathcal{L}_i \cup \{a\} \text{ has finite intersection property} \Leftrightarrow \text{(lemma)} \Leftrightarrow \mathcal{L} \in \operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{a[n]}.$$

□

Proposition 1891. $\operatorname{id}^{\mathsf{Strd}}_{a[n]} \sqsubseteq \downdownarrows \operatorname{ID}^{\mathsf{Strd}}_{a[n]}$ for every filter a and an index set n.

Proof. $\operatorname{id}^{\mathsf{Strd}}_{a[n]} = \downdownarrows\upuparrows \operatorname{id}^{\mathsf{Strd}}_{a[n]} \sqsubseteq \downdownarrows \operatorname{ID}^{\mathsf{Strd}}_{a[n]}$. □

Proposition 1892. $\operatorname{id}^{\mathsf{Strd}}_{a[a]} \sqsubset \downdownarrows \operatorname{ID}^{\mathsf{Strd}}_{a[a]}$ for every nontrivial ultrafilter a.

Proof. Suppose $\operatorname{id}^{\mathsf{Strd}}_{a[a]} = \downdownarrows \operatorname{ID}^{\mathsf{Strd}}_{a[a]}$. Then $\operatorname{ID}^{\mathsf{Strd}}_{a[a]} = \upuparrows\downdownarrows \operatorname{ID}^{\mathsf{Strd}}_{a[a]} = \upuparrows \operatorname{id}^{\mathsf{Strd}}_{a[a]}$ what contradicts to the above. □

Obvious 1893. $\mathcal{L} \in \operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow a \sqcap \bigsqcap_{i\in n} \mathcal{L}_i \neq \bot$ if a is an element of a complete lattice.

Obvious 1894. $\mathcal{L} \in \operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall i \in n : \mathcal{L}_i \sqsupseteq a \Leftrightarrow \forall i \in n : \mathcal{L}_i \not\asymp a$ if a is an ultrafilter on $\mathfrak{A}$.

21.19.6. Identity staroids on principal filters. For principal filter $\uparrow A$ (where A is a set) the above definitions coincide with n-ary identity relation, as formulated in the following propositions:

Proposition 1895. $\uparrow^{\mathsf{Strd}} \operatorname{id}_{A[n]} = \operatorname{id}^{\mathsf{Strd}}_{\uparrow A[n]}$.

Proof.

$$L \in \operatorname{GR} \uparrow^{\mathsf{Strd}} \operatorname{id}_{A[n]} \Leftrightarrow \prod L \not\asymp \operatorname{id}_{A[n]} \Leftrightarrow \exists t \in A \forall i \in n : t \in L_i \Leftrightarrow$$
$$\bigcap_{i\in n} L_i \cap A \neq \emptyset \Leftrightarrow L \in \operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{\uparrow A[n]}.$$

Thus $\uparrow^{\mathsf{Strd}} \operatorname{id}_{A[n]} = \operatorname{id}^{\mathsf{Strd}}_{\uparrow A[n]}$. □

Corollary 1896. $\operatorname{id}^{\mathsf{Strd}}_{\uparrow A[n]}$ is a principal staroid.

QUESTION 1897. Is $\mathrm{ID}^{\mathsf{Strd}}_{A[n]}$ principal for every principal filter A on a set and index set n?

PROPOSITION 1898. $\uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \sqsubseteq \downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$ for every set A.

PROOF.

$$L \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow L \in \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]} \Leftrightarrow \uparrow A \not\asymp \prod^{\mathfrak{Z}}_{i\in n} L_i \Rightarrow$$
$$\uparrow A \not\asymp \prod^{\mathfrak{A}}_{i\in n} L_i \Leftrightarrow L \in \downdownarrows \mathrm{GR}\, \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}.$$

□

PROPOSITION 1899. $\uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \sqsubset \downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$ for some set A and index set n.

PROOF. $L \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \prod^{\mathfrak{Z}}_{i\in n} L_i \not\asymp \uparrow A$ what is not implied by $\prod^{\mathfrak{A}}_{i\in n} L_i \not\asymp \uparrow A$ that is $L \in \downdownarrows \mathrm{GR}\, \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$. (For a counter example take $n = \mathbb{N}$, $L_i =]0; 1/i[$, $A = \mathbb{R}$.) □

PROPOSITION 1900. $\upuparrows \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \upuparrows \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]}$.

PROOF. $\upuparrows \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \upuparrows \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]}$ is obvious from the above. □

PROPOSITION 1901. $\upuparrows \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \sqsubseteq \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$.

PROOF.

$$\mathcal{X} \in \mathrm{GR} \upuparrows \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \mathrm{up}\, \mathcal{X} \subseteq \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow$$
$$\forall Y \in \mathrm{up}\, \mathcal{X} : Y \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \forall Y \in \mathrm{up}\, \mathcal{X} : Y \in \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]} \Leftrightarrow$$
$$\forall Y \in \mathrm{up}\, \mathcal{X} : \prod^{\mathfrak{Z}}_{i\in n} Y_i \sqcap \uparrow A \neq \bot \Rightarrow \prod^{\mathfrak{A}}_{i\in n} \mathcal{X}_i \sqcap \uparrow A \neq \bot \Leftrightarrow \mathcal{X} \in \mathrm{GR}\, \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}.$$

□

PROPOSITION 1902. $\upuparrows \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \sqsubset \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$ for some set A.

PROOF. We need to prove $\upuparrows \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \neq \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$ that is it's enough to prove (see the above proof) that $\forall Y \in \mathrm{up}\, \mathcal{X} : \prod^{\mathfrak{Z}}_{i\in n} Y_i \sqcap \uparrow A \neq \bot \not\Leftarrow \prod^{\mathfrak{A}}_{i\in n} \mathcal{X}_i \sqcap \uparrow A \neq \bot$. A counter-example follows:

$\forall Y \in \mathrm{up}\, \mathcal{X} : \prod^{\mathfrak{Z}}_{i\in n} Y_i \sqcap \uparrow A \neq \bot$ does not hold for $n = \mathbb{N}$, $\mathcal{X}_i = \uparrow] - 1/i; 0[$ for $i \in n$, $A =] - \infty; 0[$. To show this, it's enough to prove $\prod^{\mathfrak{Z}}_{i\in n} Y_i \sqcap \uparrow A = \bot$ for $Y_i = \uparrow] - 1/i; 0[$ but this is obvious since $\prod^{\mathfrak{Z}}_{i\in n} Y_i = \bot$.

On the other hand, $\prod^{\mathfrak{A}}_{i\in n} \mathcal{X}_i \sqcap \uparrow A \neq \bot$ for the same $\mathcal{X}$ and A. □

The above theorems are summarized in the diagram at figure 1:

$$\begin{tikzcd}
\downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]} \arrow[d, shift right, "\upuparrows"'] \arrow[r, phantom, "\sqsupseteq"] & \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]} \arrow[d, shift right, "\upuparrows"'] \\
\mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]} \arrow[u, shift right, "\downdownarrows"'] \arrow[r, phantom, "\sqsupseteq"] & \upuparrows \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \upuparrows \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]} \arrow[u, shift right, "\downdownarrows"']
\end{tikzcd}$$

FIGURE 13. Relationships of identity staroids for principal filters.

REMARK 1903. $\sqsubseteq$ on the diagram means inequality which can become strict for some A and n.

21.19.7. Identity staroids represented as meets and joins.

PROPOSITION 1904. $\mathrm{id}^{\mathsf{Strd}}_{a[n]} = \bigsqcap^{\mathrm{Anch}}_{A \in \mathrm{up}\, a} \mathrm{id}_{A[n]} = \bigsqcap^{\mathsf{Strd}}_{A \in \mathrm{up}\, a} \mathrm{id}_{A[n]}$ for every filter a on a powerset.

PROOF. Since $\mathrm{id}^{\mathsf{Strd}}_{a[n]}$ is a staroid (proposition 1875), it's enough to prove that $\mathrm{id}^{\mathsf{Strd}}_{a[n]}$ is the greatest lower bound of $\left\{ \frac{\uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]}}{A \in \mathrm{up}\, a} \right\}$.

That $\mathrm{id}^{\mathsf{Strd}}_{a[n]} \sqsubseteq \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]}$ for every $A \in \mathrm{up}\, a$ is obvious.

Let $f \sqsubseteq \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]}$ for every $A \in \mathrm{up}\, a$.

$$L \in \mathrm{GR}\, f \Rightarrow \forall A \in \mathrm{up}\, a : L \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow$$
$$\forall A \in \mathrm{up}\, a : \prod L \not\asymp \mathrm{id}_{A[n]} \Leftrightarrow \forall A \in \mathrm{up}\, a : \bigsqcap_{i \in n}^{\mathfrak{Z}} L_i \not\asymp A \Rightarrow$$
$$\forall A \in \mathrm{up}\, a : \bigsqcap_{i \in n}^{\mathfrak{A}} L_i \not\asymp A \Rightarrow \bigsqcap_{i \in n}^{\mathfrak{A}} L_i \not\asymp a \Rightarrow L \in \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{a[n]}.$$

Thus $f \sqsubseteq \mathrm{id}^{\mathsf{Strd}}_{a[n]}$. $\square$

PROPOSITION 1905. $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \bigsqcup_{a \in \mathrm{atoms}\, \mathcal{A}} \mathrm{ID}^{\mathsf{Strd}}_{a[n]} = \bigsqcup_{a \in \mathrm{atoms}\, \mathcal{A}} a^n_{\mathsf{Strd}}$ where the join may be taken on every of the following posets: anchored relations, staroids, completary staroids, provided that $\mathcal{A}$ is a filter on a set.

PROOF. $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid (proposition 1876). Thus, it's enough to prove that $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is the lowest upper bound of $\left\{ \frac{\mathrm{ID}^{\mathsf{Strd}}_{a[n]}}{a \in \mathrm{atoms}\, \mathcal{A}} \right\}$ (also use the fact that $\mathrm{ID}^{\mathsf{Strd}}_{a[n]} = a^n_{\mathsf{Strd}}$).

$\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \sqsupseteq \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ for every $a \in \mathrm{atoms}\, \mathcal{A}$ is obvious.

Let $f \sqsupseteq \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ for every $a \in \mathrm{atoms}\, \mathcal{A}$. Then $\forall L \in \mathrm{GR}\, \mathrm{ID}^{\mathsf{Strd}}_{a[n]} : L \in \mathrm{GR}\, f$ that is

$$\forall L \in \mathrm{form}\, f : \left(\mathrm{MEET}\left(\left\{ \frac{L_i}{i \in n} \right\} \cup \{a\} \right) \Rightarrow L \in \mathrm{GR}\, f \right).$$

But

$$\exists a \in \mathrm{atoms}\, \mathcal{A} : \mathrm{MEET}\left(\left\{ \frac{L_i}{i \in n} \right\} \cup \{a\} \right) \Leftrightarrow \exists a \in \mathrm{atoms}\, \mathcal{A} : \bigsqcap_{i \in n}^{\mathfrak{A}} L_i \not\asymp a \Leftarrow$$
$$\bigsqcap_{i \in n}^{\mathfrak{A}} L_i \not\asymp \mathcal{A} \Leftrightarrow L \in \mathrm{GR}\, \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

So $L \in \mathrm{GR}\, \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Rightarrow L \in \mathrm{GR}\, f$. Thus $f \sqsupseteq \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. $\square$

PROPOSITION 1906. $\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \bigsqcup_{a \in \mathrm{atoms}\, \mathcal{A}} \mathrm{id}^{\mathsf{Strd}}_{a[n]}$ where the meet may be taken on every of the following posets: anchored relations, staroids, provided that $\mathcal{A}$ is a filter on a set.

PROOF. Since $\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a staroid (proposition 1875), it's enough to prove the result for join on anchored relations.

$\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \sqsupseteq \mathrm{id}^{\mathsf{Strd}}_{a[n]}$ for every $a \in \mathrm{atoms}\, \mathcal{A}$ is obvious.

Let $f \sqsupseteq \mathrm{id}^{\mathsf{Strd}}_{a[n]}$ for every $a \in \operatorname{atoms} \mathcal{A}$. Then $\forall L \in \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{a[n]} : L \in \mathrm{GR}\, f$ that is

$$\forall L \in \operatorname{form} f : \left(\prod_{i \in n}^{\mathfrak{Z}} L_i \not\asymp a \Rightarrow L \in \mathrm{GR}\, f \right).$$

But $\exists a \in \operatorname{atoms} \mathcal{A} : \prod^{\mathfrak{Z}}_{i \in n} L_i \not\asymp a \Leftarrow \prod^{\mathfrak{Z}}_{i \in n} L_i \not\asymp \mathcal{A} \Leftrightarrow L \in \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$.

So $L \in \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Rightarrow L \in \mathrm{GR}\, f$. Thus $f \sqsupseteq \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. □

21.19.8. Finite case.

THEOREM 1907. Let n be a finite set.

1°. $\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $\mathfrak{A}$ and $\mathfrak{Z}$ are meet-semilattices and $(\mathfrak{A}, \mathfrak{Z})$ is a binarily meet-closed filtrator.

2°. $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \upuparrows \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator over a distributive lattice.

PROOF.

1°.

$$L \in \mathrm{GR} \downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow L \in \mathrm{GR}\, \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \mathrm{MEET}\left(\left\{ \frac{L_i}{i \in n} \right\} \cup \{\mathcal{A}\} \right) \Leftrightarrow$$
$$\prod_{i \in n}^{\mathfrak{A}} L_i \sqcap \mathcal{A} \neq 0 \Leftrightarrow (\text{by finiteness}) \Leftrightarrow \prod_{i \in n}^{\mathfrak{Z}} L_i \sqcap \mathcal{A} \neq 0 \Leftrightarrow L \in \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$$

for every $L \in \prod \mathfrak{Z}$.

2°.

$$L \in \mathrm{GR} \upuparrows \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \operatorname{up} L \subseteq \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \forall K \in \operatorname{up} L : K \in \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow$$
$$\forall K \in \operatorname{up} L : \prod_{i \in n}^{\mathfrak{Z}} K_i \in \partial \mathcal{A} \Leftrightarrow \forall K \in \operatorname{up} L : \prod_{i \in n}^{\mathfrak{Z}} K_i \not\asymp \mathcal{A} \Leftrightarrow$$
$$(\text{by finiteness and theorem 535}) \Leftrightarrow$$
$$\forall K \in \operatorname{up} L : \prod_{i \in n}^{\mathfrak{A}} K_i \not\asymp \mathcal{A} \Leftrightarrow \mathcal{A} \in \bigcap \langle \star \rangle^* \left\{ \frac{\prod^{\mathfrak{A}}_{i \in n} K_i}{K \in \operatorname{up} L} \right\} \Leftrightarrow$$
$$(\text{by the formula for finite meet of filters, theorem 523}) \Leftrightarrow$$
$$\mathcal{A} \in \bigcap \langle \star \rangle^* \operatorname{up} \prod_{i \in n}^{\mathfrak{A}} L_i \Leftrightarrow \forall K \in \operatorname{up} \prod_{i \in n}^{\mathfrak{A}} L_i : \mathcal{A} \in \star K \Leftrightarrow \forall K \in \operatorname{up} \prod_{i \in n}^{\mathfrak{A}} L_i : \mathcal{A} \not\asymp K \Leftrightarrow$$
$$(\text{by separability of core, theorem 537}) \Leftrightarrow$$
$$\prod_{i \in n}^{\mathfrak{A}} L_i \not\asymp \mathcal{A} \Leftrightarrow L \in \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

□

PROPOSITION 1908. Let $(\mathfrak{A}, \mathfrak{Z})$ be a binarily meet closed filtrator whose core is a meet-semilattice. $\downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ and $\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ are the same for finite n.

PROOF. Because $\prod^{\mathfrak{Z}}_{i \in \operatorname{dom} L} L_i = \prod^{\mathfrak{A}}_{i \in \operatorname{dom} L} L_i$ for finitary L. □

21.20. Counter-examples

EXAMPLE 1909. $\upuparrows \downdownarrows f \neq f$ for some staroid f whose form is an indexed family of filters on a set.

Proof. Let $f = \left\{ \frac{\mathcal{A} \in \mathscr{F}(\mho)}{\uparrow \operatorname{Cor} \mathcal{A} \not\asymp \Delta} \right\}$ for some infinite set $\mho$ where Δ is some non-principal filter on $\mho$.

$$A \sqcup B \in f \Leftrightarrow \uparrow^{\mho} \operatorname{Cor}(A \sqcup B) \not\asymp \Delta \Leftrightarrow \uparrow^{\mho} \operatorname{Cor} A \sqcup \uparrow^{\mho} \operatorname{Cor} B \not\asymp \Delta \Leftrightarrow \\ \uparrow^{\mho} \operatorname{Cor} A \sqcap \Delta \neq \bot^{\mathscr{F}(\mho)} \vee \uparrow^{\mho} \operatorname{Cor} B \sqcap \Delta \neq \bot^{\mathscr{F}(\mho)} \Leftrightarrow A \in f \vee B \in f.$$

Obviously $\bot^{\mathscr{F}(\mho)} \notin f$. So f is a free star. But free stars are essentially the same as 1-staroids.

$\downdownarrows f = \partial \Delta$. $\upuparrows \downdownarrows f = \left\{ \frac{Z \in \mathscr{F}}{\operatorname{up} Z \subseteq \partial \Delta} \right\} = \left\{ \frac{Z \in \mathscr{F}}{\forall K \in \operatorname{up} Z : K \not\asymp \Delta} \right\} = \left\{ \frac{Z \in \mathscr{F}}{Z \not\asymp \Delta} \right\} = \star \Delta \neq f$. □

For the below counter-examples we will define a staroid ϑ with arity $\vartheta = \mathbb{N}$ and $\operatorname{GR} \vartheta \in \mathscr{P}(\mathbb{N}^{\mathbb{N}})$ (based on a suggestion by Andreas Blass):

$$A \in \operatorname{GR} \vartheta \Leftrightarrow \sup_{i \in \mathbb{N}} \operatorname{card}(A_i \cap i) = \mathbb{N} \wedge \forall i \in \mathbb{N} : A_i \neq \emptyset.$$

Proposition 1910. ϑ is a staroid.

Proof. $(\operatorname{val} \vartheta)_i L = \mathscr{P}\mathbb{N} \setminus \{\emptyset\}$ for every $L \in (\mathscr{P}\mathbb{N})^{\mathbb{N} \setminus \{i\}}$ if

$$\sup_{j \in \mathbb{N} \setminus \{i\}} \operatorname{card}(A_j \cap j) = \mathbb{N} \wedge \forall j \in \mathbb{N} \setminus \{i\} : L_j \neq \emptyset.$$

Otherwise $(\operatorname{val} \vartheta)_i L = \emptyset$. Thus $(\operatorname{val} \vartheta)_i L$ is a free star. So ϑ is a staroid. (That ϑ is an upper set, is obvious.) □

Proposition 1911. ϑ is a completary staroid.

Proof.

$$A_0 \sqcup A_1 \in \operatorname{GR} \vartheta \Leftrightarrow A_0 \cup A_1 \in \operatorname{GR} \vartheta \Leftrightarrow \\ \sup_{i \in \mathbb{N}} \operatorname{card}((A_0 i \cup A_1 i) \cap i) = \mathbb{N} \wedge \forall i \in \mathbb{N} : A_0 i \cup A_1 i \neq \emptyset \Leftrightarrow \\ \sup_{i \in \mathbb{N}} \operatorname{card}((A_0 i \cap i) \cup (A_1 i \cap i)) = \mathbb{N} \wedge \forall i \in \mathbb{N} : A_0 i \cup A_1 i \neq \emptyset.$$

If $A_0 i = \emptyset$ then $A_0 i \cap i = \emptyset$ and thus $A_1 i \cap i \sqsupseteq A_0 i \cap i$. Thus we can select $c(i) \in \{0, 1\}$ in such a way that $\forall d \in \{0, 1\} : \operatorname{card}(A_{c(i)} i \cap i) \sqsupseteq \operatorname{card}(A_d i \cap i)$ and $A_{c(i)} i \neq \emptyset$. (Consider the case $A_0 i, A_1 i \neq \emptyset$ and the similar cases $A_0 i = \emptyset$ and $A_1 i = \emptyset$.)

So

$$A_0 \sqcup A_1 \in \operatorname{GR} \vartheta \Leftrightarrow \sup_{i \in \mathbb{N}} \operatorname{card}(A_{c(i)} i \cap i) = \mathbb{N} \wedge \forall i \in \mathbb{N} : A_{c(i)} i \neq \emptyset \Leftrightarrow \\ (\lambda i \in n : A_{c(i)} i) \in \operatorname{GR} \vartheta.$$

Thus ϑ is completary. □

Obvious 1912. ϑ is non-zero.

Example 1913. There is such a nonzero staroid f on powersets that $f \not\sqsupseteq \prod^{\mathsf{Strd}} a$ for every family $a = a_{i \in \mathbb{N}}$.

Proof. It's enough to prove $\vartheta \not\sqsupseteq \prod^{\mathsf{Strd}} a$.

Let $\uparrow^{\mathbb{N}} R_i = a_i$ if a_i is principal and $R_i = \mathbb{N} \setminus i$ if a_i is non-principal.

We have $\forall i \in \mathbb{N} : R_i \in a_i$.

We have $R \notin \operatorname{GR} \vartheta$ because $\sup_{i \in \mathbb{N}} \operatorname{card}(R_i \cap i) \neq \mathbb{N}$.

$R \in \prod^{\mathsf{Strd}} a$ because $\forall X \in a_i : X \cap R_i \neq \emptyset$.

So $\vartheta \not\sqsupseteq \prod^{\mathsf{Strd}} a$. □

REMARK 1914. At http://mathoverflow.net/questions/60925/special-infinitary-relations-and-ultrafilters there is a proof for arbitrary infinite form, not just for $\mathbb{N}$.

CONJECTURE 1915. For every family $a = a_{i\in\mathbb{N}}$ of ultrafilters $\prod^{\mathsf{Strd}} a$ is not an atom nor of the poset of staroids neither of the poset of completary staroids of the form $\lambda i \in \mathbb{N} : \operatorname{Base}(a_i)$.

CONJECTURE 1916. There exists a non-completary staroid on powersets.

CONJECTURE 1917. There exists a prestaroid which is not a staroid.

CONJECTURE 1918. The set of staroids of the form A^B where A and B are sets is atomic.

CONJECTURE 1919. The set of staroids of the form A^B where A and B are sets is atomistic.

CONJECTURE 1920. The set of completary staroids of the form A^B where A and B are sets is atomic.

CONJECTURE 1921. The set of completary staroids of the form A^B where A and B are sets is atomistic.

EXAMPLE 1922. $\operatorname{StarComp}(a, f \sqcup g) \neq \operatorname{StarComp}(a, f) \sqcup \operatorname{StarComp}(a, g)$ in the category of binary relations with star-morphisms for some n-ary relation a and an n-indexed families f and g of functions.

PROOF. Let $n = \{0,1\}$. Let $\operatorname{GR} a = \{(0,1),(1,0)\}$ and $f = [\![\{(0,1)\},\{(1,0)\}]\!]$, $g = [\![\{(1,0)\},\{(0,1)\}]\!]$.

For every $\{0,1\}$-indexed family of μ of functions:

$$L \in \operatorname{StarComp}(a,\mu) \Leftrightarrow \exists y \in a : (y_0 \mathrel{\mu_0} L_0 \wedge y_1 \mathrel{\mu_1} L_1) \Leftrightarrow \exists y_0 \in \operatorname{dom}\mu_0, y_1 \in \operatorname{dom}\mu_1 : (y_0 \mathrel{\mu_0} L_0 \wedge y_1 \mathrel{\mu_1} L_1)$$

for every n-ary relation μ.

Consequently

$$L \in \operatorname{StarComp}(a,f) \Leftrightarrow L_0 = 1 \wedge L_1 = 0 \Leftrightarrow L = (1,0)$$

that is $\operatorname{StarComp}(a,f) = \{(1,0)\}$. Similarly

$\operatorname{StarComp}(a,g) = \{(0,1)\}$.

Also

$$L \in \operatorname{StarComp}(a, f \sqcup g) \Leftrightarrow \exists y_0, y_1 \in \{0,1\} : ((y_0 \mathrel{f_0} L_0 \vee y_0 \mathrel{g_0} L_0) \wedge (y_1 \mathrel{f_1} L_1 \vee y_1 \mathrel{g_1} L_1)).$$

Thus

$$\operatorname{StarComp}(a, f \sqcup g) = \{(0,1),(1,0),(0,0),(1,1)\}.$$

□

COROLLARY 1923. The above inequality is possible also for star-morphisms of funcoids and star-morphisms of reloids.

PROOF. Because finitary funcoids and reloids between finite sets are essentially the same as finitary relations and our proof above works for binary relations. □

The following example shows that the theorem 1864 can't be strengthened:

EXAMPLE 1924. For some multifuncoid f on powersets complete in argument k the following formula is false:

$\langle f\rangle_l(L \cup \{(k, \bigsqcup X)\}) = \bigsqcup_{x\in X} \langle f\rangle_l(L \cup \{(k,x)\})$ for every $X \in \mathscr{P}\mathfrak{Z}_k$, $L \in \prod_{i\in(\operatorname{arity} f)\setminus\{k,l\}} \mathscr{F}_i$.

Proof. Consider multifuncoid $f = \Lambda \operatorname{id}^{\mathsf{Strd}}_{\uparrow U[3]}$ where U is an infinite set (of the form $\mathfrak{Z}^3$) and $L = (Y)$ where Y is a nonprincipal filter on U.

$\langle f \rangle_0 (L \cup \{(k, \bigsqcup X)\}) = Y \sqcap \bigsqcup X$;

$\bigsqcup_{x \in X} \langle f \rangle_0 (L \cup \{(k, x)\}) = \bigsqcup_{x \in X} (Y \sqcap x)$.

It can be $Y \sqcap \bigsqcup X = \bigsqcup_{x \in X} (Y \sqcap x)$ only if Y is principal: Really: $Y \sqcap \bigsqcup X = \bigsqcup_{x \in X} (Y \sqcap x)$ implies $Y \not\asymp \bigsqcup X \Rightarrow \bigsqcup_{x \in X} (Y \sqcap x) \neq \bot \Rightarrow \exists x \in X : Y \not\asymp x$ and thus Y is principal. But we claimed above that it is nonprincipal. □

Example 1925. There exists a staroid f and an indexed family X of principal filters (with arity $f = \operatorname{dom} X$ and $(\operatorname{form} f)_i = \operatorname{Base}(X_i)$ for every $i \in \operatorname{arity} f$), such that $f \sqsubseteq \prod^{\mathsf{Strd}} X$ and $Y \sqcap X \notin \operatorname{GR} f$ for some $Y \in \operatorname{GR} f$.

Remark 1926. Such examples obviously do not exist if both f is a principal staroid and X and Y are indexed families of principal filters (because for powerset algebras staroidal product is equivalent to Cartesian product). This makes the above example inspired.

Proof. (Monroe Eskew) Let a be any (trivial or nontrivial) ultrafilter on an infinite set U. Let $A, B \in a$ be such that $A \cap B \subset A, B$. In other words, A, B are arbitrary nonempty sets such that $\emptyset \neq A \cap B \subset A, B$ and a be an ultrafilter on $A \cap B$.

Let f be the staroid whose graph consists of functions $p : U \to a$ such that either $p(n) \supseteq A$ for all but finitely many n or $p(n) \supseteq B$ for all but finitely many n. Let's prove f is really a staroid.

It's obvious $px \neq \emptyset$ for every $x \in U$. Let $k \in U$, $L \in a^{U \setminus \{k\}}$. It is enough (taking symmetry into account) to prove that

$$L \cup \{(k, x \sqcup y)\} \in \operatorname{GR} f \Leftrightarrow L \cup \{(k, x)\} \in \operatorname{GR} f \vee L \cup \{(k, y)\} \in \operatorname{GR} f. \tag{36}$$

Really, $L \cup \{(k, x \sqcup y)\} \in \operatorname{GR} f$ iff $x \sqcup y \in a$ and $L(n) \supseteq A$ for all but finitely many n or $L(n) \supseteq B$ for all but finitely many n; $L \cup \{(k, x)\} \in \operatorname{GR} f$ iff $x \in a$ and $L(n) \supseteq A$ for all but finitely many n or $L(n) \supseteq B$; and similarly for y.

But $x \sqcup y \in a \Leftrightarrow x \in a \vee y \in a$ because a is an ultrafilter. So, the formula (36) holds, and we have proved that f is really a staroid.

Take X be the constant function with value A and Y be the constant function with value B.

$\forall p \in \operatorname{GR} f : p \not\asymp X$ because $p_i \cap X_i \in a$; so $\operatorname{GR} f \subseteq \operatorname{GR} \prod^{\mathsf{Strd}} X$ that is $f \sqsubseteq \prod^{\mathsf{Strd}} X$.

Finally, $Y \sqcap X \notin \operatorname{GR} f$ because $X \sqcap Y = \lambda i \in U : A \cap B$. □

21.21. Conjectures

Remark 1927. Below I present special cases of possible theorems. The theorems may be generalized after the below special cases are proved.

Conjecture 1928. For every two funcoids f and g we have:

1°. $(\mathsf{RLD})_{\mathrm{in}} a \left[f \times^{(DP)} g \right] (\mathsf{RLD})_{\mathrm{in}} b \Leftrightarrow a \left[f \times^{(C)} g \right] b$ for every funcoids $a \in \mathsf{FCD}(\operatorname{Src} f, \operatorname{Src} g)$, $b \in \mathsf{FCD}(\operatorname{Dst} f, \operatorname{Dst} g)$;

2°. $(\mathsf{RLD})_{\mathrm{out}} a \left[f \times^{(DP)} g \right] (\mathsf{RLD})_{\mathrm{out}} b \Leftrightarrow a \left[f \times^{(C)} g \right] b$ for every funcoids $a \in \mathsf{FCD}(\operatorname{Src} f, \operatorname{Src} g)$, $b \in \mathsf{FCD}(\operatorname{Dst} f, \operatorname{Dst} g)$;

3°. $(\mathsf{FCD}) a \left[f \times^{(C)} g \right] (\mathsf{FCD}) b \Leftrightarrow a \left[f \times^{(DP)} g \right] b$ for every reloids $a \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Src} g)$, $b \in \mathsf{RLD}(\operatorname{Dst} f, \operatorname{Dst} g)$.

Conjecture 1929. For every two funcoids f and g we have:

1°. $(\mathsf{RLD})_{\mathrm{in}} a \left[f \times^{(A)} g \right] (\mathsf{RLD})_{\mathrm{in}} b \Leftrightarrow a \left[f \times^{(C)} g \right] b$ for every funcoids $a \in \mathsf{FCD}(\operatorname{Src} f, \operatorname{Src} g)$, $b \in \mathsf{FCD}(\operatorname{Dst} f, \operatorname{Dst} g)$;

2°. $(\mathsf{RLD})_{\mathrm{out}} a \left[f \times^{(A)} g \right] (\mathsf{RLD})_{\mathrm{out}} b \Leftrightarrow a \left[f \times^{(C)} g \right] b$ for every funcoids $a \in \mathsf{FCD}(\operatorname{Src} f, \operatorname{Src} g)$, $b \in \mathsf{FCD}(\operatorname{Dst} f, \operatorname{Dst} g)$;
3°. $(\mathsf{FCD}) a \left[f \times^{(C)} g \right] (\mathsf{FCD}) b \Leftrightarrow a \left[f \times^{(A)} g \right] b$ for every reloids $a \in \mathsf{RLD}(\operatorname{Src} f, \operatorname{Src} g)$, $b \in \mathsf{RLD}(\operatorname{Dst} f, \operatorname{Dst} g)$.

Conjecture 1930. $\prod^{\mathsf{Strd}} a \not\asymp \prod^{\mathsf{Strd}} b \Leftrightarrow b \in \prod^{\mathsf{Strd}} a \Leftrightarrow a \in \prod^{\mathsf{Strd}} b \Leftrightarrow \forall i \in n : a_i \not\asymp b_i$ for every n-indexed families a and b of filters on powersets.

Conjecture 1931. Let f be a staroid on powersets and $a \in \prod_{i \in \operatorname{arity} f} \operatorname{Src} f_i$, $b \in \prod_{i \in \operatorname{arity} f} \operatorname{Dst} f_i$. Then

$$\prod^{\mathsf{Strd}} a \left[\prod^{(C)} f \right] \prod^{\mathsf{Strd}} b \Leftrightarrow \forall i \in n : a_i \left[f_i \right] b_i.$$

Proposition 1932. The conjecture 1931 is a consequence of the conjecture 1930.

Proof.

$$\prod^{\mathsf{Strd}} a \left[\prod^{(C)} f \right] \prod^{\mathsf{Strd}} b \Leftrightarrow \prod^{\mathsf{Strd}} b \not\asymp \left\langle \prod^{(C)} f \right\rangle \prod^{\mathsf{Strd}} a \Leftrightarrow \prod^{\mathsf{Strd}} b \not\asymp \prod^{\mathsf{Strd}}_{i \in n} \langle f_i \rangle a_i \Leftrightarrow$$
$$\forall i \in n : b_i \not\asymp \langle f_i \rangle a_i \Leftrightarrow \forall i \in n : a_i \left[f_i \right] b_i.$$

□

Conjecture 1933. For every indexed families a and b of filters and an indexed family f of pointfree funcoids we have

$$\prod^{\mathsf{Strd}} a \left[\prod^{(C)} f \right] \prod^{\mathsf{Strd}} b \Leftrightarrow \prod^{\mathsf{RLD}} a \left[\prod^{(DP)} f \right] \prod^{\mathsf{RLD}} b.$$

Conjecture 1934. For every indexed families a and b of filters and an indexed family f of pointfree funcoids we have

$$\prod^{\mathsf{Strd}} a \left[\prod^{(C)} f \right] \prod^{\mathsf{Strd}} b \Leftrightarrow \prod^{\mathsf{RLD}} a \left[\prod^{(A)} f \right] \prod^{\mathsf{RLD}} b.$$

Strengthening of an above result:

Conjecture 1935. If a is a completary staroid and $\operatorname{Dst} f_i$ is a starrish poset for every $i \in n$ then $\operatorname{StarComp}(a, f)$ is a completary staroid.

Strengthening of above results:

Conjecture 1936.
1°. $\prod^{(D)} F$ is a prestaroid if every F_i is a prestaroid.
2°. $\prod^{(D)} F$ is a completary staroid if every F_i is a completary staroid.

Conjecture 1937. If f_1 and f_2 are funcoids, then there exists a pointfree funcoid $f_1 \times f_2$ such that

$$\langle f_1 \times f_2 \rangle x = \bigsqcup \left\{ \frac{\langle f_1 \rangle X \times^{\mathsf{FCD}} \langle f_2 \rangle X}{X \in \operatorname{atoms} x} \right\}$$

for every ultrafilter x.

Conjecture 1938. Let $(\mathfrak{A}, \mathfrak{Z}) = (\mathfrak{A}, \mathfrak{Z})_{i\in n}$ be a family of filtrators on boolean lattices.

A relation $\delta \in \mathscr{P} \prod \operatorname{atoms}^{\mathfrak{A}_i}$ such that for every $a \in \prod \operatorname{atoms}^{\mathfrak{A}_i}$

$$\forall A \in a : \delta \cap \prod_{i\in n} \operatorname{atoms} \uparrow^{\mathfrak{Z}_i} A_i \neq \emptyset \Rightarrow a \in \delta \tag{37}$$

can be continued till the function $\upuparrows f$ for a unique staroid f of the form $\lambda i \in n : \mathfrak{A}_i$. The funcoid f is completary.

Conjecture 1939. For every $\mathcal{X} \in \prod_{i\in n} \mathscr{F}(\mathfrak{A}_i)$

$$\mathcal{X} \in \operatorname{GR} \upuparrows f \Leftrightarrow \delta \cap \prod_{i\in n} \operatorname{atoms} \mathcal{X}_i \neq \emptyset. \tag{38}$$

Conjecture 1940. Let R be a set of staroids of the form $\lambda i \in n : \mathscr{F}(\mathfrak{A}_i)$ where every $\mathfrak{A}_i$ is a boolean lattice. If $x \in \prod_{i\in n} \operatorname{atoms}^{\mathscr{F}(\mathfrak{A}_i)}$ then $x \in \operatorname{GR} \upuparrows \bigsqcap R \Leftrightarrow \forall f \in R : x \in \upuparrows f$.

There exists a completary staroid f and an indexed family X of principal filters (with $\operatorname{arity} f = \operatorname{dom} X$ and $(\operatorname{form} f)_i = \operatorname{Base}(X_i)$ for every $i \in \operatorname{arity} f$), such that $f \sqsubseteq \prod^{\mathsf{Strd}} X$ and $Y \sqcap X \notin \operatorname{GR} f$ for some $Y \in \operatorname{GR} f$.

Conjecture 1941. There exists a staroid f and an indexed family x of ultrafilters (with $\operatorname{arity} f = \operatorname{dom} x$ and $(\operatorname{form} f)_i = \operatorname{Base}(x_i)$ for every $i \in \operatorname{arity} f$), such that $f \sqsubseteq \prod^{\mathsf{Strd}} x$ and $Y \sqcap x \notin \operatorname{GR} f$ for some $Y \in \operatorname{GR} f$.

Other conjectures:

Conjecture 1942. If staroid $\bot \neq f \sqsubseteq a^n_{\mathsf{Strd}}$ for an ultrafilter a and an index set n, then $n \times \{a\} \in \operatorname{GR} f$. (Can it be generalized for arbitrary staroidal products?)

Conjecture 1943. The following posets are atomic:

1°. anchored relations on powersets;
2°. staroids on powersets;
3°. completary staroids on powersets.

Conjecture 1944. The following posets are atomistic:

1°. anchored relations on powersets;
2°. staroids on powersets;
3°. completary staroids on powersets.

The above conjectures seem difficult, because we know almost nothing about structure of atomic staroids.

Conjecture 1945. A staroid on powersets is principal iff it is complete in every argument.

Conjecture 1946. If a is an ultrafilter, then $\operatorname{id}^{\mathsf{Strd}}_{a[n]}$ is an atom of the lattice of:

1°. anchored relations of the form $(\mathscr{P} \operatorname{Base}(a))^n$;
2°. staroids of the form $(\mathscr{P} \operatorname{Base}(a))^n$;
3°. completary staroids of the form $(\mathscr{P} \operatorname{Base}(a))^n$.

Conjecture 1947. If a is an ultrafilter, then $\upuparrows \operatorname{id}^{\mathsf{Strd}}_{a[n]}$ is an atom of the lattice of:

1°. anchored relations of the form $\mathscr{F}(\operatorname{Base}(a))^n$;
2°. staroids of the form $\mathscr{F}(\operatorname{Base}(a))^n$;
3°. completary staroids of the form $\mathscr{F}(\operatorname{Base}(a))^n$.

21.21.1. On finite unions of infinite Cartesian products. Let $\mathfrak{A}$ be an indexed family of sets.

Products are $\prod A$ for $A \in \prod \mathfrak{A}$.

Let the lattice Γ consists of all finite unions of products.

Let the lattice Γ^* be the lattice of complements of elements of the lattice Γ.

PROBLEM 1948. Is $\prod^{\mathsf{FCD}}$ a bijection from a. $\mathfrak{F}\Gamma$; b. $\mathfrak{F}\Gamma^*$ to:

1°. prestaroids on $\mathfrak{A}$;
2°. staroids on $\mathfrak{A}$;
3°. completary staroids on $\mathfrak{A}$?

If yes, is up^{Γ} defining the inverse bijection?

If not, characterize the image of the function $\prod^{\mathsf{FCD}}$ defined on a. $\mathfrak{F}\Gamma$; b. $\mathfrak{F}\Gamma^*$.

21.21.2. Informal questions. Do products of funcoids and reloids coincide with Tychonoff topology?

Limit and generalized limit for multiple arguments.

Is product of connected spaces connected?

Product of T_0-separable is T_0, of T_1 is T_1?

Relationships between multireloids and staroids.

Generalize the section "Specifying funcoids by functions or relations on atomic filters" from [29].

Generalize "Relationships between funcoids and reloids".

Explicitly describe the set of complemented funcoids.

Formulate and prove associativity of staroidal product.

What are necessary and sufficient conditions for $\mathrm{up}\, f$ to be a filter (for a funcoid f)? (See also proposition 1129.)

Part 5

Postface

CHAPTER 22

Postface

See this Web page for my research plans: http://www.mathematics21.org/agt-plans.html

I deem that now the most important research topics in Algebraic General Topology are:

- to solve the open problems mentioned in this work;
- research pointfree reloids (see below);
- define and research compactness of funcoids;
- research categories related with funcoids and reloids;
- research multifuncoids and staroids in more details;
- research generalized limit of compositions of functions;
- research more on complete pointfree funcoids.

All my research of funcoids and reloids is presented at

http://www.mathematics21.org/algebraic-general-topology.html

Please write to porton@narod.ru, if you discover anything new related with my theory.

22.1. Pointfree reloids

Let us define something (let call it *pointfree reloids*) corresponding to pointfree funcoids in the same way as reloids correspond to funcoids.

First note that $\mathsf{RLD}(A,B)$ are isomorphic to $\mathfrak{F}\mathscr{P}(\mathscr{P}X\times\mathscr{P}B)$. Then note that $\mathscr{P}(\mathscr{P}A\times\mathscr{P}B)$ are isomorphic both to $\mathsf{pFCD}(\mathscr{P}A,\mathscr{P}B)$ and to $\operatorname{atoms}^{\mathscr{P}A}\times\operatorname{atoms}^{\mathscr{P}B}$.

But $\mathsf{FCD}(A,B)$ is isomorphic to $\mathsf{pFCD}(\mathfrak{F}(A),\mathfrak{F}(B))$.

Thus both $\mathfrak{F}\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ and $\mathfrak{F}(\operatorname{atoms}^{\mathfrak{A}}\times\operatorname{atoms}^{\mathfrak{B}})$ correspond to $\mathsf{pFCD}(\mathfrak{F}(\mathfrak{A}),\mathfrak{F}(\mathfrak{B}))$ in the same way (replace $\mathscr{P}A\to\mathfrak{A}$, $\mathscr{P}B\to\mathfrak{B}$) as $\mathsf{RLD}(A,B)$ corresponds to $\mathsf{FCD}(A,B)$.

So we can name either $\mathfrak{F}\mathsf{pFCD}(\mathfrak{A},\mathfrak{B})$ or $\mathfrak{F}(\operatorname{atoms}^{\mathfrak{A}}\times\operatorname{atoms}^{\mathfrak{B}})$ as *pointfree reloids.*

Yes another possible way is to define pointfree reloids as the set of filters on the poset of Galois connections between two posets.

Note that there are three different definitions of pointfree reloids. They probably are not the same for arbitrary posets $\mathfrak{A}$ and $\mathfrak{B}$.

I have defined pointfree reloids, but have not yet started to research their properties.

Research convergence for pointfree funcoids (should be easy).

22.2. Formalizing this theory

Despite of all measures taken, it is possible that there are errors in this book. While special cases, such as filters of powersets or funcoids, are most likely correct, general cases (such as filters on posets or pointfree funcoids) may possibly contain wrong theorem conditions.

Thus it would be good to formalize the theory presented in this book in a proof assistant[1] such as Coq.

If you want to work on formalizing this theory, please let me know.

See also https://coq.inria.fr/bugs/show_bug.cgi?id=2957

[1]A *proof assistant* is a computer program which checks mathematical proofs written in a formal language understandable by computer.

APPENDIX A

Using logic of generalizations

A.1. Logic of generalization

In mathematics it is often encountered that a smaller set S naturally bijectively corresponds to a subset R of a larger set B. (In other words, there is specified an injection from S to B.) It is a widespread practice to equate S with R.

REMARK 1949. I denote the first set S from the first letter of the word “small” and the second set B from the first letter of the word “big”, because S is intuitively considered as smaller than B. (However we do not require $\operatorname{card} S < \operatorname{card} B$.)

The set B is considered as a generalization of the set S, for example: whole numbers generalizing natural numbers, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc.

But strictly speaking this equating may contradict to the axioms of ZF/ZFC because we are not insured against $S \cap B \neq \emptyset$ incidents. Not wonderful, as it is often labeled as “without proof”.

To work around of this (and formulate things exactly what could benefit computer proof assistants) we will replace the set B with a new set B' having a bijection $M : B \to B'$ such that $S \subseteq B'$. (I call this bijection M from the first letter of the word “move” which signifies the move from the old set B to a new set B').

The following is a formal but rather silly formalization of this situation in ZF. (A more natural formalization may be done in a smarter formalistic, such as dependent type theory.)

A.1.1. The formalistic. Let S and B be sets. Let E be an injection from S to B. Let $R = \operatorname{im} E$.

Let $t = \mathscr{P} \bigcup\bigcup S$.

Let $M(x) = \begin{cases} E^{-1}x & \text{if } x \in R; \\ (t, x) & \text{if } x \notin R. \end{cases}$

Recall that in standard ZF $(t, x) = \{t, \{t, x\}\}$ by definition.

THEOREM 1950. $(t, x) \notin S$.

PROOF. Suppose $(t, x) \in S$. Then $\{t, \{t, x\}\} \in S$. Consequently $\{t\} \in \bigcup S$; $\{t\} \subseteq \bigcup\bigcup S$; $\{t\} \in \mathscr{P} \bigcup\bigcup S$; $\{t\} \in t$ what contradicts to the axiom of foundation (aka axiom of regularity). □

DEFINITION 1951. Let $B' = \operatorname{im} M$.

THEOREM 1952. $S \subseteq B'$.

PROOF. Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$; $x \in \operatorname{im} M = B'$. □

OBVIOUS 1953. E is a bijection from S to R.

THEOREM 1954. M is a bijection from B to B'.

PROOF. Surjectivity of M is obvious. Let’s prove injectivity. Let $a, b \in B$ and $M(a) = M(b)$. Consider all cases: □

$a, b \in R$. $M(a) = E^{-1}a$; $M(b) = E^{-1}b$; $E^{-1}a = E^{-1}b$; thus $a = b$ because E^{-1} is a bijection.
$a \in R$, $b \notin R$. $M(a) = E^{-1}a$; $M(b) = (t, b)$; $M(a) \in S$; $M(b) \notin S$. Thus $M(a) \neq M(b)$.
$a \notin R$, $b \in R$. Analogous.
$a, b \notin R$. $M(a) = (t, a)$; $M(b) = (t, b)$. Thus $M(a) = M(b)$ implies $a = b$.

THEOREM 1955. $M \circ E = \mathrm{id}_S$.

PROOF. Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$. □

OBVIOUS 1956. $E = M^{-1}|_S$.

A.1.2. Existence of primary filtrator.

THEOREM 1957. For every poset $\mathfrak{Z}$ there exists a poset $\mathfrak{A} \supseteq \mathfrak{Z}$ such that $(\mathfrak{A}, \mathfrak{Z})$ is a primary filtrator.

PROOF. Take $S = \mathfrak{Z}$, $B = \mathfrak{F}$, $E = \uparrow$. By the above there exists an injection M defined on $\mathfrak{F}$ such that $M \circ \uparrow = \mathrm{id}_{\mathfrak{Z}}$.

Take $\mathfrak{A} = \operatorname{im} M$. Order ($\sqsubseteq'$) elements of $\mathfrak{A}$ in such a way that $M : \mathfrak{F}(\mathfrak{Z}) \to \mathfrak{A}$ become order isomorphism. If $x \in \mathfrak{Z}$ then $x = \mathrm{id}_{\mathfrak{Z}}\, x = M \uparrow x \in \operatorname{im} M = \mathfrak{A}$. Thus $\mathfrak{A} \supseteq \mathfrak{Z}$.

If $x \sqsubseteq y$ for elements x, y of $\mathfrak{Z}$, then $\uparrow x \sqsubseteq \uparrow y$ and thus $M \uparrow x \sqsubseteq' M \uparrow y$ that is $x \sqsubseteq' y$, so $\mathfrak{Z}$ is a subposet of $\mathfrak{A}$, that is $(\mathfrak{A}, \mathfrak{Z})$ is a filtrator.

It remains to prove that M is an isomorphism between filtrators $(\mathfrak{F}(\mathfrak{Z}), \mathfrak{P})$ and $(\mathfrak{A}, \mathfrak{Z})$. That M is an order isomorphism from $\mathfrak{F}(\mathfrak{Z})$ to $\mathfrak{A}$ is already known. It remains to prove that M maps $\mathfrak{P}$ to $\mathfrak{Z}$. We will instead prove that M^{-1} maps $\mathfrak{Z}$ to $\mathfrak{P}$. Really, $\uparrow x = M^{-1}x$ for every $x \in \mathfrak{Z}$. □

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