Upgrading a Multifuncoid*

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Abstract

I define the concepts of **multifuncoid** (and **completary multifuncoid**) and **upgrading**. Then I conjecture that upgrading of certain multifuncoids are multifuncoids (and that upgrading certain completary multifuncoids are completary multifuncoids). I have proved the conjectures for $n \leqslant 2$.

The main conjecture from this article is now proved in the article "Multidimensional Funcoids"

This short article is the first my public writing where I introduce the concept of **multidimensional funcoid** which I am investigating now.

Refer to this Web site for the theory which I now attempt to generalize.

1 Background

1.1 About some posets

Let $\mathfrak{A}$ is a poset that is a set partially ordered by a relation $\geq$.

If $\mathfrak{A}$ is a join-semilattice, I will denote $a \sqcup b$ join of its elements. (Dually for a meet-semilattice I will denote $a \sqcap b$ meet of its elements.)

If $\mathfrak{A} = \mathfrak{A}_{i \in n}$ is a family of posets, then I will denote $\prod \mathfrak{A}$ the product order on $\mathfrak{A}$ that is we have for every $a, b \in \prod \mathfrak{A}$

$$a \geq b \Leftrightarrow \forall i \in n : a_i \geq b_i.$$

Note that if every $\mathfrak{A}_i$ is a join-semilattice then $\prod \mathfrak{A}$ is also a join-semilattice and

$$a \sqcup b = \lambda i \in n : a_i \sqcup b_i.$$

I will denote $\mathfrak{A}^n = \prod_{i \in n} \mathfrak{A}$ for every poset $\mathfrak{A}$ and an index set n.

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1.2 Filtrators and upgrading

Definition 1 *A **filtrator** is a pair* $(\mathfrak{A}; \mathfrak{Z})$ *of a poset* $\mathfrak{A}$ *and its subset* $\mathfrak{Z}$*.*

See [2] for a detailed study of filtrators.

Having fixed a filtrator, we define:

Definition 2 $\operatorname{up} x = \{ Y \in \mathfrak{Z} \quad | \quad Y \geqslant x \}$ *for every* $X \in \mathfrak{A}$*.*

Definition 3 $E^* K = \{ L \in \mathfrak{A} \quad | \quad \operatorname{up} L \subseteq K \}$ *(**upgrading** the set* K*) for every* $K \in \mathscr{P} \mathfrak{Z}$*.*

1.3 Multifuncoids

Definition 4 *A **free star** on a join-semilattice* $\mathfrak{A}$ *with least element* 0 *is a set* S *such that* $0 \notin S$ *and*

$$\forall A, B \in \mathfrak{A} : (A \sqcup B \in S \Leftrightarrow A \in S \vee B \in S) .$$

I will denote the set of free stars on $\mathfrak{A}$ as $\mathfrak{A} \operatorname{Star}$.

Let n be a set. As an example, n may be an ordinal, n may be a natural number, considered as a set by the formula $n = \{0, \ldots, n-1\}$. Let $\mathfrak{A} = \mathfrak{A}_{i \in n}$ is a family of posets indexed by the set n.

Definition 5 *Let* $f \in \mathscr{P} \prod \mathfrak{A}$*,* $i \in \operatorname{dom} \mathfrak{A}$*,* $L \in \prod \mathfrak{A}|_{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}}$*.*

$$(\operatorname{val} f)_i L = \{ X \in \mathfrak{A}_i \quad | \quad L \cup \{(i; X)\} \in f \} .$$

*("*val*" is an abbreviation of the word "value".)*

Proposition 1 f *can be restored knowing* $(\operatorname{val} f)_i$ *for some* $i \in n$*.*

Proof $f = \{ K \in \prod \mathfrak{A} \quad | \quad K \in f \} =$
$\left\{ L \cup \{(i; X)\} \quad | \quad L \in \prod \mathfrak{A}|_{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}}, X \in \mathfrak{A}_i, L \cup \{(i; X)\} \in f \right\} =$
$\left\{ L \cup \{(i; X)\} \quad | \quad L \in \prod \mathfrak{A}|_{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}}, X \in (\operatorname{val} f)_i L \right\}$. □

Definition 6 *Let* $\mathfrak{A}$ *is a family of join-semilattices. A **pre-multidimensional funcoid** (or **pre-multifuncoid** for short) of the form* $\mathfrak{A}$ *is an* $f \in \mathscr{P} \prod \mathfrak{A}$ *such that we have that:* $(\operatorname{val} f)_i L$ *is a free star for every* $i \in \operatorname{dom} \mathfrak{A}$*,* $L \in \prod \mathfrak{A}|_{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}}$*.*

Definition 7 *A **multidimensional funcoid** (or **multifuncoid** for short) is a pre-multifuncoid which is an upper set.*

Proposition 2 *If* $L \in \prod \mathfrak{A}$ *and* $L_i = 0^{\mathfrak{A}_i}$ *for some* i *then* $L \notin f$ *if* f *is a pre-multifuncoid.*

Proof Let $K = L|_{\operatorname{dom} \mathfrak{A} \setminus \{i\}}$. We have $0 \notin (\operatorname{val} f)_i K$; $K \cup \{(i; 0)\} \notin f$; $L \notin f$. □

Definition 8 ***Infinitary pre-multifuncoid*** *is such an n-ary multifuncoid that n is infinite;* ***finitary pre-multifuncoid*** *is such an n-ary multifuncoid that n is finite.*

Definition 9 *Let $\mathfrak{A}$ is a family of join-semilattices. A* ***completary multifuncoid*** *of the form $\mathfrak{A}$ is an $f \in \mathscr{P} \prod_{i \in \operatorname{dom} \mathfrak{A}} \mathfrak{A}_i$ such that*

1. *$L_0 \sqcup L_1 \in f \Leftrightarrow \exists c \in \{0,1\}^n : \left(\lambda i \in n : L_{c(i)} i\right) \in f$ for every $L_0, L_1 \in \prod \mathfrak{A}$.*

2. *If $L \in \prod \mathfrak{A}$ and $L_i = 0^{\mathfrak{A}_i}$ for some i then $\neg f L$.*

Proposition 3 *A completary multifuncoid is a multifuncoid.*

Proof Let f is a completary multifuncoid.

Let $K \in \prod_{i \in (\operatorname{dom} \mathfrak{A}) \setminus \{i\}} \mathfrak{A}_i$. Let $L_0 = K \cup \{(i; X_0)\}$, $L_1 = K \cup \{(i; X_1)\}$ for some $X_0, X_1 \in \mathfrak{A}_i$. Then $X_0 \sqcup X_1 \in (\operatorname{val} f)_i K \Leftrightarrow L_0 \sqcup L_1 \in f \Leftrightarrow \exists k \in \{0,1\} : K \cup \{(i; X_k)\} \in f \Leftrightarrow K \cup \{(i; X_0)\} \in f \vee K \cup \{(i; X_1)\} \in f \Leftrightarrow X_0 \in (\operatorname{val} f)_i K \vee X_1 \in (\operatorname{val} f)_i K$.

So $(\operatorname{val} f)_i K$ is a free star (taken in account that $K_i = 0^{\mathfrak{A}_i} \Rightarrow f \notin K$).

It remained to prove that f is an upper set. Let $L_0 \leq L_1$ for some $L_0, L_1 \in \prod \mathfrak{A}$ and $L_0 \in f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n : L_{c(i)} i = \lambda i \in n : L_0 i = L_0 \in f$ and thus $L_1 = L_0 \sqcup L_1 \in f$. □

Proposition 4 *Every finitary pre-multifuncoid is completary.*

Proof $\exists c \in \{0,1\}^n : (\lambda i \in n : L_{c(i)} i) \in f \Leftrightarrow$
$\exists c \in \{0,1\}^{n-1} : (\{(n-1; L_0(n-1))\} \cup \{(i; L_{c(i)} i) \mid i \in n-1\} \in f \vee \{(n-1; L_1(n-1))\} \cup \{(i; L_{c(i)} i) \mid i \in n-1\} \in f) \Leftrightarrow \exists c \in \{0,1\}^{n-1} : \{(n-1; L_0(n-1) \sqcup L_1(n-1))\} \cup \{(i; L_{c(i)} i) \mid i \in n-1\} \in f \Leftrightarrow \ldots \Leftrightarrow \{(i; L_0 i \sqcup L_1 i) \mid i \in n\} \in f$. □

Theorem 1 *For finite n the following are the same:*

1. *pre-multifuncoids;*

2. *multifuncoids;*

3. *completary multifuncoids.*

Proof f is a finitary pre-multifuncoid $\Rightarrow$ f is a finitary completary multifuncoid.

f is a finitary completary multifuncoid $\Rightarrow$ f is a finitary multifuncoid.

f is a finitary multifuncoid $\Rightarrow$ f is a finitary pre-multifuncoid. □

As it will be clear from below, (finitary) multifuncoids are a generalization of funcoids [1].

I will denote $\mathfrak{A}\mathsf{FCD}$ the set of multifuncoids for a finite family $\mathfrak{A}$ of join-semilattices.

2 Open problems

Conjecture 1 *Let $\mho$ be a set, $\mathfrak{F}$ be the set of filters on $\mho$ ordered reverse to set-theoretic inclusion, $\mathfrak{P}$ be the set of principal filters on $\mho$, let n be an index set. Consider the filtrator $(\mathfrak{F}^n; \mathfrak{P}^n)$. If f is a multifuncoid of the form $\mathfrak{P}^n$, then $E^* f$ is a multifuncoid of the form $\mathfrak{F}^n$.*

A similar conjecture about completary multifuncoids:

Conjecture 2 *Let $\mho$ be a set, $\mathfrak{F}$ be the set of filters on $\mho$ ordered reverse to set-theoretic inclusion, $\mathfrak{P}$ be the set of principal filters on $\mho$, let n be an index set. Consider the filtrator $(\mathfrak{F}^n; \mathfrak{P}^n)$. If f is a completary multifuncoid of the form $\mathfrak{P}^n$, then $E^* f$ is a completary multifuncoid of the form $\mathfrak{F}^n$.*

A weaker conjecture:

Conjecture 3 *Let $\mho$ be a set, $\mathfrak{F}$ be the set of filters on $\mho$ ordered reverse to set-theoretic inclusion, $\mathfrak{P}$ be the set of principal filters on $\mho$, let n be an index set. Consider the filtrator $(\mathfrak{F}^n; \mathfrak{P}^n)$. If f is a completary multifuncoid of the form $\mathfrak{P}^n$, then $E^* f$ is a multifuncoid of the form $\mathfrak{F}^n$.*

For finite n all three conjectures are equivalent.

For $n = 0$ the conjecture is trivial. For $n = 1$ it can be proved using the theory of filters [2]. For $n = 2$ we can prove it using the theory of funcoids [1]. For $\operatorname{card} n \geqslant 3$ (finite and infinite) the problem is open.

The full proofs for $\operatorname{card} n \leqslant 2$ are presented below.

If a conjecture will be proved true, we may generalize it for a wider set of filtrators.

3 Preliminary Results

3.1 Isomorphic filtrators

We will use the concept of isomorphic filtrators in the below proofs.

An **isomorphism** from a filtrator $(\mathfrak{A}_0; \mathfrak{Z}_0)$ to a filtrator $(\mathfrak{A}_1; \mathfrak{Z}_1)$ is an order embedding φ from $\mathfrak{A}_0$ to $\mathfrak{A}_1$ such that the image of $\mathfrak{Z}_0$ under φ is exactly $\mathfrak{Z}_1$.

Two filtrators are isomorphic when there exist an isomorphism from one to the other.

It is trivial that neither the property of being a multifuncoid, nor the result of upgrading does change under an isomorphism.

3.2 The proof of the conjecture for $\operatorname{card} n \leqslant 2$

The below constitutes a proof of my conjecture for $n \in \{0, 1, 2\}$ as well as n being any set of cardinality $\operatorname{card} n \leqslant 2$, because a particular index set does not matter, just it's cardinality.

Let $\mathfrak{Z} = \mathfrak{P}^n$ and $\mathfrak{A} = \mathfrak{F}^n$.

3.2.1 The proof for $n = 0$

In this case a multifuncoid f of the form $\mathfrak{Z} = \mathfrak{P}^0 = \{()\}$ is an element of the set $\mathscr{P}\mathfrak{B}^0 = \{\{()\}\}$ that is $f = \{()\}$. Obviously f is an upper set. Then

$$E^* f = \left\{ L \in \mathfrak{F}^0 \quad \middle| \quad \operatorname{up} L \subseteq f \right\} = \{() \quad | \quad \operatorname{up}() \subseteq \{()\}\} = \{() \quad | \quad \{()\} \subseteq \{()\}\} = \{()\}\,.$$

For $i = \operatorname{dom} \mathfrak{F}^0$ we have $(\operatorname{val} f)_i L$ is a free star just because i doesn't exist. Obviously $E^* f$ is an upper set.

So $E^* f$ is a multifuncoid of the form $\mathfrak{F}^0$.

3.2.2 The proof for $n = 1$

We will use notation from [2].

Lemma 1 *The upgrading (regarding the filtrator $(\mathfrak{F}; \mathfrak{P})$) of every free star on $\mathfrak{P}$ is a free star on $\mathfrak{F}$.*

Proof Let f is a free star on $\mathfrak{P}$. Then (theorem 45 in [2]) there exist a $g \in \mathfrak{F}$ such that $\partial g = f$.

$E^* f = \{L \in \mathfrak{F} \quad | \quad \operatorname{up} L \subseteq \partial g\} = \{L \in \mathfrak{F} \quad | \quad L \not\asymp g\}$. It remained to prove that $\{L \in \mathfrak{F} \quad | \quad L \not\asymp g\}$ is a free star.

Obviously $0^{\mathfrak{F}} \notin \{L \in \mathfrak{F} \quad | \quad L \not\asymp g\}$.

For every $A, B \in \{L \in \mathfrak{F} \quad | \quad L \not\asymp g\}$ we have $A \sqcup B \in \{L \in \mathfrak{F} \quad | \quad L \not\asymp g\} \Leftrightarrow (A \sqcup B) \sqcap g \neq 0^{\mathfrak{F}} \Leftrightarrow (A \sqcap g) \sqcup (B \sqcap g) \neq 0^{\mathfrak{F}} \Leftrightarrow A \sqcap g \neq 0^{\mathfrak{F}} \vee B \sqcap g \neq 0^{\mathfrak{F}} \Leftrightarrow A \in \{L \in \mathfrak{F} \quad | \quad L \not\asymp g\} \vee B \in \{L \in \mathfrak{F} \quad | \quad L \not\asymp g\}$.

The proof is finished. □

Let Q be the set of all multifuncoids of the form $\mathfrak{A}^1$ where $\mathfrak{A}$ is a join-semilattice with least element. Then $f \in Q$ if and only if $(\operatorname{val} f)_0 \emptyset$ is a free star.

But $(\operatorname{val} f)_0 \emptyset = \{X \in \mathfrak{A} \quad | \quad \{(0; X)\} \in f\} = \{X \in \mathfrak{A} \quad | \quad f0 = X\} = f0$.

So it uis easy to show that the filtrator of the form $\left(\mathfrak{F}^1 \mathrm{FCD}; \mathfrak{P}^1 \mathrm{FCD}\right)$ is isomorphic to the filtrator $(\mathfrak{F}\,\mathrm{Star}; \mathfrak{P}\,\mathrm{Star})$.

Thus by the lemma upgrading a multifuncoid of the form $\mathfrak{P}^1$ is a multifuncoid of the form $\mathfrak{F}^1$.

3.2.3 The proof for $n = 2$

An f is a (finitary) multifuncoid of the form $\mathfrak{A} \times \mathfrak{B}$ (for $\mathfrak{A}$, $\mathfrak{B}$ being join-semilattices with least elements) iff all the following:

1. $(\operatorname{val} f)_0 L$ is a free star for every $L = \{(1; Y)\}$ where $Y \in \mathfrak{B}$;

2. $(\operatorname{val} f)_1 L$ is a free star for every $L = \{(0; X)\}$ where $X \in \mathfrak{A}$;

what is equal to the following:

1. $\{X \in \mathfrak{A} \quad | \quad X \mathrel{f} Y\}$ is a free star for every $Y \in \mathfrak{B}$;

2. $\{Y \in \mathfrak{B} \quad | \quad X \ f \ Y\}$ is a free star for every $X \in \mathfrak{A}$;

what is equal to the following:

1. $(I \sqcup J) \ f \ Y \Leftrightarrow I \ f \ Y \vee J \ f \ Y$ and not $0 \ f \ Y$ for every $Y \in \mathfrak{B}$, $I, J \in \mathfrak{A}$;

2. $X \ f \ (I \sqcup J) \Leftrightarrow X \ f \ I \vee X \ f \ J$ and not $X \ f \ 0$ for every $X \in \mathfrak{A}$, $I, J \in \mathfrak{B}$.

By the way, it implies that $f \mapsto [f]^*$ is a bijection from the set of funcoids from $\mho_0$ to $\mho_1$ into the set of multifuncoids of the form $\mathscr{P}\mho_0 \times \mathscr{P}\mho_1$, for every sets $\mho_0$ and $\mho_1$.

Now suppose f is a multifuncoid of the form $\mathfrak{P}^2$. Then:

1. $(I \sqcup J) \ f \ Y \Leftrightarrow I \ f \ Y \vee J \ f \ Y$ and not $0 \ f \ Y$ for every $Y, I, J \in \mathfrak{P}$;

2. $X \ f \ (I \sqcup J) \Leftrightarrow X \ f \ I \vee X \ f \ J$ and not $X \ f \ 0$. for every $X, I, J \in \mathfrak{P}$.

Thus multifuncoids of the form $\mathfrak{P}^2$ are essentially equivalent to funcoids from $\mathfrak{P}$ to $\mathfrak{P}$ ([1]), formally: there exist a funcoid f' such that $[f']^* = f$.

$E^* f = \left\{ L \in \mathfrak{F}^2 \quad | \quad \operatorname{up} L \subseteq f \right\} = \{(L_0; L_1) \quad | \quad L_0, L_1 \in \mathfrak{F}, \forall g_0 \in \operatorname{up} L_0, g_1 \in \operatorname{up} L_1 : (g_0; g_1) \in f\} = \left\{ (L_0; L_1) \quad | \quad L_0, L_1 \in \mathfrak{F}, \forall g_0 \in \operatorname{up} L_0, g_1 \in \operatorname{up} L_1 : g_0 \, [f']^* \, g_1 \right\} = \{(L_0; L_1) \quad | \quad L_0, L_1 \in \mathfrak{F}, L_0 \, [f'] \, L_1\} = [f']$.

Thus:

1. $(I \sqcup J) \ (E^* f) \ Y \Leftrightarrow I \ (E^* f) \ Y \vee J \ (E^* f) \ Y$ and not $0 \ (E^* f) \ Y$ for every $Y, I, J \in \mathfrak{F}$;

2. $X \ (E^* f) \ (I \sqcup J) \Leftrightarrow X \ (E^* f) \ I \vee X \ (E^* f) \ J$ and not $X \ (E^* f) \ 0$ for every $X, I, J \in \mathfrak{F}$.

that is $E^* f$ is a multifuncoid of the form $\mathfrak{F}^2$.

References

[1] Victor Porton. Funcoids and reloids. At `http://www.mathematics21.org/binaries/funcoids-reloids.pdf`.

[2] Victor Porton. Filters on posets and generalizations. *International Journal of Pure and Applied Mathematics*, 74(1):55–119, 2012.