# On Todd Trimble's notes on "topogeny"

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*Web:* `http://www.mathematics21.org`

This short article is written as a response on Todd Trimble's notes:

http://ncatlab.org/toddtrimble/published/topogeny

In this my article I am going to reprise these Todd's theorems which are new for me, converted into terminology and notation of my book [1] and of

http://www.mathematics21.org/binaries/rewrite-plan.pdf

Now this article is a partial draft. I am going to integrate materials of this article into my book.

## 1 Misc

The following theorem is a strenghtening suggested by Todd of my theorem:

**Theorem 1.** $\prod^{\mathfrak{F}} S = \bigcup \{\uparrow(K_0 \sqcap^{3} \ldots \sqcap^{3} K_n) \mid K_i \in \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\}$ for every nonempty set $S$ of filters on a meet-semilattice. [TODO: Strengthen my theorems requiring distributivity of lattice with this result which does not require it even to be a lattice.]

**Proof.** It follows from the fact that

$$\prod^{\mathfrak{F}} S = \prod^{\mathfrak{F}} \left\{K_0 \sqcap^{3} \ldots \sqcap^{3} K_n \mid K_i \in \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\right\}$$

and that $\{K_0 \sqcap^{3} \ldots \sqcap^{3} K_n \mid K_i \in \bigcup S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\}$ is a filter base. □

**Definition 2.** A complete lattice is *co-compact* iff $\prod S = 0$ for a set $S$ of elements of this lattice implies that there is its finite subset $T \subseteq S$ such that $\prod T = 0$. [TODO: Remove the requirement to have least.]

**Proposition 3.** The poset of filters on a meet-semilattice $\mathfrak{Z}$ with least element is co-compact.

**Proof.** If $0 \in \prod^{\mathfrak{F}} S$ then there are $K_i \in \bigcup S$ such that $0 \in \uparrow(K_0 \sqcap^{3} \ldots \sqcap^{3} K_n)$ that is $K_0 \sqcap^{3} \ldots \sqcap^{3} K_n = 0$ from which easily follows $\mathcal{F}_0 \sqcap^{\mathfrak{F}} \ldots \sqcap^{\mathfrak{F}} \mathcal{F}_n = 0$ for some $\mathcal{F}_i \in S$. □

**Proposition 4.** $\mathcal{X} [f] \prod S \Leftrightarrow \exists \mathcal{Y} \in S: \mathcal{X} [f] \mathcal{Y}$ if $S$ is a generalized filter base on $\operatorname{Dst} f$. [TODO: Pointfree funcoids.]

**Proof.** $\mathcal{X} [f] \prod S \Leftrightarrow \prod S \sqcap \langle f \rangle \mathcal{X} \neq 0 \Leftrightarrow \prod \langle \langle f \rangle \mathcal{X} \sqcap \rangle^{*} S \neq 0 \Leftrightarrow$ (by properties of generalized filter bases)$\Leftrightarrow \exists \mathcal{Y} \in \langle \langle f \rangle \mathcal{X} \sqcap \rangle^{*} S: \mathcal{Y} \neq 0 \Leftrightarrow \exists \mathcal{Y} \in S: \langle f \rangle \mathcal{X} \sqcap \mathcal{Y} \neq 0 \Leftrightarrow \exists \mathcal{Y} \in S: \mathcal{X} [f] \mathcal{Y}$. □

The following theorem was easy to prove, but I would not discovered it without Todd's help:

**Theorem 5.** A function $\varphi: \mathfrak{F}(A) \rightarrow \mathfrak{F}(B)$ preserves finite joins and filtered meets iff there exists a funcoid $f$ such that $\langle f \rangle = \varphi$. [TODO: Define filtered meets. Say about empty join 0.]

**Proof.** Backward implication is easy.

Let $\psi = \varphi|_{\mathscr{P}A}$. Then $\psi$ preserves bottom element and binary joins. Thus there exists a funcoid $f$ such that $\langle f \rangle^* = \psi$.

It remains to prove that $\langle f \rangle = \varphi$.

Really, $\langle f \rangle \mathcal{X} = \bigsqcap \langle \langle f \rangle^* \rangle^* \mathcal{X} = \bigsqcap \langle \psi \rangle^* \mathcal{X} = \bigsqcap \langle \varphi \rangle^* \mathcal{X} = \varphi \bigsqcap \mathcal{X} = \varphi \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(A)$. □

**Corollary 6.** Funcoids $f$ from $A$ to $B$ bijectively correspond by the formula $\langle f \rangle = \varphi$ to functions $\varphi: \mathfrak{F}(A) \to \mathfrak{F}(B)$ preserving finite joins and filtered meets.

# 2 Representing funcoids as binary relations

**Definition 7.** The binary relation $\xi^{\circledast} \in \mathscr{P}(\mathfrak{F}(\operatorname{Src} \xi) \times \mathfrak{F}(\operatorname{Dst} \xi))$ for a funcoid $\xi$ is defined by the formula $\mathcal{A}\, \xi^{\circledast} \mathcal{B} \Leftrightarrow \mathcal{B} \sqsupseteq \langle \xi \rangle \mathcal{A}$.

**Definition 8.** The binary relation $\xi^* \in \mathscr{P}(\mathscr{P} \operatorname{Src} \xi \times \mathscr{P} \operatorname{Dst} \xi)$ for a funcoid $\xi$ is defined by the formula

$$A\, \xi^* B \Leftrightarrow B \sqsupseteq \langle \xi \rangle A \Leftrightarrow B \in \operatorname{up} \langle \xi \rangle A.$$

**Proposition 9.** Funcoid $\xi$ can be restored from

1. the value of $\xi^{\circledast}$;
2. the value of $\xi^*$.

**Proof.**

1. The value of $\langle \xi \rangle$ can be restored from $\xi^{\circledast}$.
2. The value of $\langle \xi \rangle^*$ can be restored from $\xi^*$. □

**Theorem 10.** Let $\nu$ and $\xi$ be composable funcoids. Then:

1. $\xi^{\circledast} \circ \nu^{\circledast} = (\xi \circ \nu)^{\circledast}$;
2. $\xi^* \circ \nu^* = (\xi \circ \nu)^*$.

**Proof.**

1. $\mathcal{A}\, [\xi^{\circledast} \circ \nu^{\circledast}]\, \mathcal{C} \Leftrightarrow \exists \mathcal{B}: (\mathcal{A}\, \nu^{\circledast} \mathcal{B} \wedge \mathcal{B}\, \xi^{\circledast} \mathcal{C}) \Leftrightarrow \exists \mathcal{B} \in \mathfrak{F}(\operatorname{Dst} \nu): (\mathcal{B} \sqsupseteq \langle \nu \rangle \mathcal{A} \wedge \mathcal{C} \sqsupseteq \langle \xi \rangle \mathcal{B}) \Leftrightarrow \mathcal{C} \sqsupseteq \langle \xi \rangle \langle \nu \rangle \mathcal{A} \Leftrightarrow \mathcal{C} \sqsupseteq \langle \xi \circ \nu \rangle \mathcal{A} \Leftrightarrow \mathcal{A}\, [(\xi \circ \nu)^{\circledast}]\, \mathcal{C}$.
2. $A\, [\xi^* \circ \nu^*]\, C \Leftrightarrow \exists B: (A\, \nu^* B \wedge B\, \xi^* C) \Leftrightarrow \exists B: (B \in \operatorname{up} \langle \nu \rangle A \wedge C \in \operatorname{up} \langle \xi \rangle B) \Leftrightarrow \exists B \in \operatorname{up} \langle \nu \rangle A: C \in \operatorname{up} \langle \xi \rangle B$.

   $A\, [(\xi \circ \nu)^*]\, C \Leftrightarrow C \in \operatorname{up} \langle \xi \circ \nu \rangle B \Leftrightarrow C \in \operatorname{up} \langle \xi \rangle \langle \nu \rangle B$.

   It remains to prove

   $$\exists B \in \operatorname{up} \langle \nu \rangle A: C \in \operatorname{up} \langle \xi \rangle B \Leftrightarrow C \in \operatorname{up} \langle \xi \rangle \langle \nu \rangle A.$$

   $\exists B \in \operatorname{up} \langle \nu \rangle A: C \in \operatorname{up} \langle \xi \rangle B \Rightarrow C \in \operatorname{up} \langle \xi \rangle \langle \nu \rangle A$ is obvious.

   Let $C \in \operatorname{up} \langle \xi \rangle \langle \nu \rangle A$. Then $C \in \operatorname{up} \bigsqcap \langle \langle \xi \rangle \rangle^* \operatorname{up} \langle \nu \rangle A$; so by properties of generalized filter bases, $\exists P \in \langle \langle \xi \rangle \rangle^* \operatorname{up} \langle \nu \rangle A: C \in \operatorname{up} P$; $\exists B \in \operatorname{up} \langle \nu \rangle A: C \in \operatorname{up} \langle \xi \rangle B$. □

**Remark 11.** The above theorem is interesting by the fact that composition of funcoids is represented as relational composition of binary relations.

# Bibliography

**[1]** Victor Porton. *Algebraic General Topology. Volume 1*. 2014.