Free stars

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Abstract

Free stars are certain subsets of a poset, first defined by me in [1]. Free stars are closely related with filters, namely free stars on a boolean lattice bijectively correspond to filters on this lattice. This article introduces consideration of free stars. Some questions are raised. Free stars are useful in my further (yet unpublished) research (in general topology).

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1 Notation

Partial order is denoted as $\sqsubseteq$.

Meets and joins are denoted as $\sqcap$, $\sqcup$, $\bigsqcap$, $\bigsqcup$.

I don't use in this article the concept of filter objects [1], but instead just define the order on a lattice of filters reverse to set-theoretic inclusion: $\mathcal{A} \sqsubseteq \mathcal{B} \Leftrightarrow \mathcal{A} \supseteq \mathcal{B}$.

2 Definition

I recall the definition from [1]:

Definition 1. Let $\mathfrak{A}$ be a poset. *Free stars* on $\mathfrak{A}$ are such $S \in \mathscr{P}\mathfrak{A}$ that the least element (if it exists) is not in S and for every $X, Y \in \mathfrak{A}$

$$\forall Z \in \mathfrak{A}: (Z \sqsupseteq X \wedge Z \sqsupseteq Y \Rightarrow Z \in S) \Leftrightarrow X \in S \vee Y \in S.$$

Proposition 2. $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset is a free star iff all of the following:

1. The least element (if it exists) is not in S.
2. $\forall Z \in \mathfrak{A}: (Z \sqsupseteq X \wedge Z \sqsupseteq Y \Rightarrow Z \in S) \Rightarrow X \in S \vee Y \in S$ for every $X, Y \in \mathfrak{A}$.
3. S is an upper set.

Proof. See [1]. □

Proposition 3. Let $\mathfrak{A}$ be a join-semilattice. $S \in \mathscr{P}\mathfrak{A}$ is a free star iff all of the following:

1. The least element (if it exists) is not in S.
2. $X \sqcup Y \in S \Rightarrow X \in S \vee Y \in S$ for every $X, Y \in \mathfrak{A}$.
3. S is an upper set.

Proof. See [1]. □

Proposition 4. Let $\mathfrak{A}$ be a join-semilattice. $S \in \mathscr{P}\mathfrak{A}$ is a free star iff the least element (if it exists) is not in S and for every $X, Y \in \mathfrak{A}$

$$X \sqcup Y \in S \Leftrightarrow X \in S \vee Y \in S.$$

Proof. See [1]. □

See [1] for definition and properties of full stars and core stars. There it is also described a bijection between free stars on a boolean lattice and filters on that lattice (theorems 43, 44, and 45 in [1]).

3 Order of free stars

The set of free stars is ordered by the partial order defined by the formula $S \sqsubseteq T \Leftrightarrow S \subseteq T$.

Obvious 5. The minimal free star is $\emptyset$.

Proposition 6. The maximal free star is $\mathfrak{A} \setminus \{0\}$ (or $\mathfrak{A}$ if 0 doesn't exist).

Proof. Let's denote our star S (that is $S = \mathfrak{A} \setminus \{0\}$ or $S = \mathfrak{A}$ dependently on existence of 0).

It is enough to prove that S is a free star.

That the least element is not in S is obvious. That S is an upper set is obvious.

The only thing remained to prove is

$$\forall Z \in \mathfrak{A}: (Z \sqsupseteq X \wedge Z \sqsupseteq Y \Rightarrow Z \in S) \Rightarrow X \in S \vee Y \in S.$$

If X or Y are non-least, it is obvious. Then only case remained to consider is $X = Y = 0$. In this case our formula takes the form:

$\forall Z \in \mathfrak{A}: Z \in S \Rightarrow X \in S \vee Y \in S$ what is true because $\forall Z \in \mathfrak{A}: Z \in S$ is false ($0 \notin Z$). □

Proposition 7. If S is a set of free stars on a join-semilattice, then $\bigcup S$ is a free star.

Proof. Let S is a set of free stars on a join-semilattice.

Obviously $\bigcup S$ does not contain the least element (if it exists) and $\bigcup S$ is an upper set.

It is remained to prove that $X \sqcup Y \in \bigcup S \Rightarrow X \in \bigcup S \vee Y \in \bigcup S$.

Let $X \sqcup Y \in \bigcup S$. Then there exists $T \in S$ such that $X \sqcup Y \in T$. Hence $X \in T \vee Y \in T$ and thus $X \in \bigcup S \vee Y \in \bigcup S$. □

Conjecture 8. This proposition cannot be strengthened for arbitrary posets instead of join-semilattices.

Next two corollaries from the above proposition:

Corollary 9. $\bigsqcup S = \bigcup S$ if S is a set of free stars on a join-semilattice.

Corollary 10. The set of free stars on a join-semilattice is a complete lattice.

4 Informal open problems

For which kinds of posets the corresponding poset of free stars is:

1. a distributive lattice?
2. a co-brouwerian lattice?
3. an atomic poset?
4. an atomistic poset?
5. a separable poset? (see [1] for a definition)
6. an atomically separable poset? (see [1] for a definition)

Answers to these questions for the poset of all subsets of a fixed set are obviously all true, due the mentioned above bijective correspondence with the set of filters on that poset and results in [1].

Can these results be significantly strengthened?

Bibliography

[1] Victor Porton. Filters on posets and generalizations. *International Journal of Pure and Applied Mathematics*, 74(1):55–119, 2012.