Identity Staroids

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1 Draft status

This text is a draft.

Read my book [1] before reading this. Well, most probably I will integrate materials from this article into my book.

2 Additional propositions

Proposition 1. $\left\{ \langle f \rangle_k X \mid X \in \operatorname{up}^{\left(\prod_{i \in n \setminus \{k\}} \mathfrak{A}_i ; \prod_{i \in n \setminus \{k\}} \mathfrak{Z}_i \right)} \mathcal{X} \right\}$ is a filter base on $\mathfrak{A}_k$ for every family $(\mathfrak{A}_i; \mathfrak{Z}_i)$ of filtrators where $i \in n$ for some index set n (provided that f is a multifuncoid of the form $\mathfrak{A}$ and $k \in n$ and every $\mathfrak{Z}_i$ for $i \in n \setminus \{k\}$ is a filter base and $\mathcal{X} \in \prod_{i \in n \setminus \{k\}} \mathfrak{A}_i$).

Proof. Let $\mathcal{K}, \mathcal{L} \in \{ \langle f \rangle_k X \mid X \in \operatorname{up} \mathcal{X} \}$. Then there exist $X, Y \in \operatorname{up} \mathcal{X}$ such that $\mathcal{K} = \langle f \rangle_k X$, $\mathcal{L} = \langle f \rangle_k Y$. We can take $Z \in \operatorname{up} \mathcal{X}$ such that $Z \sqsubseteq X, Y$. Then evidently $\langle f \rangle_k Z \sqsubseteq \mathcal{K}$ and $\langle f \rangle_k Z \sqsubseteq \mathcal{L}$ and $\langle f \rangle_k Z \in \{ \langle f \rangle_k X \mid X \in \operatorname{up} \mathcal{X} \}$. □

Proposition 2. $\langle \upuparrows f \rangle_k \mathcal{X} = \bigsqcap_{X \in \operatorname{up} \mathcal{X}} \langle f \rangle_k X$ for a filtrator $\left(\prod_{i \in n \setminus \{k\}} \mathfrak{F}_i ; \prod_{i \in n \setminus \{k\}} \mathfrak{P}_i \right)$ ($i \in n$ for some index set n) where every $\mathfrak{Z}_i$ is a boolean lattice, $k \in n$, and $\mathcal{X} \in \prod_{i \in n \setminus \{k\}} \mathfrak{F}_i$.

Proof. $\mathfrak{F}_k$ is separable by obvious 4.136??. $(\mathfrak{F}_k; \mathfrak{P}_k)$ is with separable core by theorem 4.112??.

$\mathcal{Y} \not\asymp \langle \upuparrows f \rangle_i \mathcal{X} \Leftrightarrow \mathcal{X} \cup \{(i; \mathcal{Y})\} \in \operatorname{GR} [\upuparrows f] \Leftrightarrow \mathcal{X} \cup \{(i; \mathcal{Y})\} \in \upuparrows \operatorname{GR} [f] \Leftrightarrow \operatorname{up}(\mathcal{X} \cup \{(i; \mathcal{Y})\}) \subseteq \operatorname{GR} [f] \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} \colon X \cup \{(i; Y)\} \in \operatorname{GR} [f] \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} \colon Y \not\asymp \langle f \rangle_i X \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} \colon Y \sqcap \langle f \rangle_i X \neq 0 \Leftrightarrow \forall Y \in \operatorname{up} \mathcal{Y} \colon 0 \notin \{ Y \sqcap \langle f \rangle_i X \mid X \in \operatorname{up} \mathcal{X} \} \Leftrightarrow \forall Y \in \operatorname{up} \mathcal{Y} \colon 0 \notin \langle Y \sqcap \rangle \{ \langle f \rangle_i X \mid X \in \operatorname{up} \mathcal{X} \} \Leftrightarrow$ (by properties of generalized filter bases) $\Leftrightarrow \forall Y \in \operatorname{up} \mathcal{Y} \colon \bigsqcap \langle Y \sqcap \rangle \{ \langle f \rangle_i X \mid X \in \operatorname{up} \mathcal{X} \} \neq 0 \Leftrightarrow \forall Y \in \operatorname{up} \mathcal{Y} \colon Y \sqcap \bigsqcap \{ \langle f \rangle_i X \mid X \in \operatorname{up} \mathcal{X} \} \neq 0 \Leftrightarrow \forall Y \in \operatorname{up} \mathcal{Y} \colon Y \not\asymp \bigsqcap_{X \in \operatorname{up} \mathcal{X}} \langle f \rangle_i X \Leftrightarrow \mathcal{Y} \not\asymp \bigsqcap_{X \in \operatorname{up} \mathcal{X}} \langle f \rangle_i X$; so $\langle \upuparrows f \rangle_i \mathcal{X} = \bigsqcap_{X \in \operatorname{up} \mathcal{X}} \langle f \rangle_i X$. □

3 On pseudofuncoids

Definition 3. *Pseudofuncoid* from a set A to a set B is a relation f between filters on A and B such that:

$$\neg (I\, f\, 0), \quad \mathcal{I} \sqcup \mathcal{J}\, f\, \mathcal{K} \Leftrightarrow \mathcal{I}\, f\, \mathcal{K} \vee \mathcal{J}\, f\, \mathcal{K} \quad \text{(for every } \mathcal{I}, \mathcal{J} \in \mathfrak{F}(A),\ \mathcal{K} \in \mathfrak{F}(B)),$$
$$\neg (0\, f\, I), \quad \mathcal{K}\, f\, \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K}\, f\, \mathcal{I} \vee \mathcal{K}\, f\, \mathcal{J} \quad \text{(for every } \mathcal{I}, \mathcal{J} \in \mathfrak{F}(B),\ \mathcal{K} \in \mathfrak{F}(A)).$$

Obvious 4. Pseudofuncoid is just a staroid of the form $(\mathfrak{F}(A); \mathfrak{F}(B))$.

Obvious 5. $[f]$ is a pseudofuncoid for every funcoid f.

Example 6. If A and B are infinite sets, then there exist two different pseudofuncoids f and g from A to B such that $f \cap (\mathfrak{P} \times \mathfrak{P}) = g \cap (\mathfrak{P} \times \mathfrak{P}) = [c] \cap (\mathfrak{P} \times \mathfrak{P})$ for some funcoid c.

Remark 7. Considering a pseudofuncoid f as a staroid, we get $f \cap (\mathfrak{P} \times \mathfrak{P}) = \downdownarrows f$.

Proof. Take

$$f = \left\{ (\mathcal{X}; \mathcal{Y}) \mid \mathcal{X} \in \mathfrak{F}(A), \mathcal{Y} \in \mathfrak{F}(B), \bigcap \mathcal{X} \text{ and } \bigcap \mathcal{Y} \text{ are infinite} \right\}$$

and

$$g = f \cup \{ (\mathcal{X}; \mathcal{Y}) \mid \mathcal{X} \in \mathfrak{F}(A), \mathcal{Y} \in \mathfrak{F}(B), \mathcal{X} \sqsupseteq a, \mathcal{Y} \sqsupseteq b \}$$

where a and b are nontrivial ultrafilters on A and B correspondingly, c is the funcoid defined by the relation

$$[c]^* = \delta = \{ (X; Y) \mid X \in \mathscr{P}A, Y \in \mathscr{P}B, X \text{ and } Y \text{ are infinite} \}.$$

First prove that f is a pseudofuncoid. The formulas $\neg(I\, f\, 0)$ and $\neg(0\, f\, I)$ are obvious. We have $\mathcal{I} \sqcup \mathcal{J}\, f\, \mathcal{K} \Leftrightarrow \bigcap\, (\mathcal{I} \sqcup \mathcal{J})$ and $\bigcap\, \mathcal{Y}$ are infinite$\Leftrightarrow \bigcap\, \mathcal{I} \cup \bigcap\, \mathcal{J}$ and $\bigcap\, \mathcal{Y}$ are infinite$\Leftrightarrow$($\bigcap\, \mathcal{I}$ or $\bigcap\, \mathcal{J}$ is infinite) $\wedge \bigcap\, \mathcal{Y}$ is infinite$\Leftrightarrow$($\bigcap\, \mathcal{I}$ and $\bigcap\, \mathcal{Y}$ are infinite) $\vee$ ($\bigcap\, \mathcal{J}$ and $\bigcap\, \mathcal{Y}$ are infinite) $\Leftrightarrow \mathcal{I}\, f\, \mathcal{K} \vee \mathcal{J}\, f\, \mathcal{K}$. Similarly $\mathcal{K}\, f\, \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K}\, f\, \mathcal{I} \vee \mathcal{K}\, f\, \mathcal{J}$. So f is a pseudofuncoid.

Let now prove that g is a pseudofuncoid. The formulas $\neg(I\, g\, 0)$ and $\neg(0\, g\, I)$ are obvious. Let $\mathcal{I} \sqcup \mathcal{J}\, g\, \mathcal{K}$. Then either $\mathcal{I} \sqcup \mathcal{J}\, f\, \mathcal{K}$ and then $\mathcal{I} \sqcup \mathcal{J}\, g\, \mathcal{K}$ or $\mathcal{I} \sqcup \mathcal{J} \sqsupseteq a$ and then $\mathcal{I} \sqsupseteq a \vee \mathcal{J} \sqsupseteq a$ thus having $\mathcal{I}\, g\, \mathcal{K} \vee \mathcal{J}\, g\, \mathcal{K}$. So $\mathcal{I} \sqcup \mathcal{J}\, g\, \mathcal{K} \Rightarrow \mathcal{I}\, g\, \mathcal{K} \vee \mathcal{J}\, g\, \mathcal{K}$. The reverse implication is obvious. We have $\mathcal{I} \sqcup \mathcal{J}\, g\, \mathcal{K} \Leftrightarrow \mathcal{I}\, g\, \mathcal{K} \vee \mathcal{J}\, g\, \mathcal{K}$ and similarly $\mathcal{K}\, g\, \mathcal{I} \sqcup \mathcal{J} \Leftrightarrow \mathcal{K}\, g\, \mathcal{I} \vee \mathcal{K}\, g\, \mathcal{J}$. So g is a pseudofuncoid.

Obviously $f \neq g$ ($a\, g\, b$ but not $a\, f\, b$).

It remains to prove $f \cap (\mathfrak{P} \times \mathfrak{P}) = g \cap (\mathfrak{P} \times \mathfrak{P}) = [c] \cap (\mathfrak{P} \times \mathfrak{P})$. Really, $f \cap (\mathfrak{P} \times \mathfrak{P}) = [c] \cap (\mathfrak{P} \times \mathfrak{P})$ is obvious. If $(\uparrow^A X; \uparrow^B Y) \in g \cap (\mathfrak{P} \times \mathfrak{P})$ then either $(\uparrow^A X; \uparrow^B Y) \in f \cap (\mathfrak{P} \times \mathfrak{P})$ or $X \in \operatorname{up} a$, $Y \in \operatorname{up} b$, so X and Y are infinite and thus $(\uparrow^A X; \uparrow^B Y) \in f \cap (\mathfrak{P} \times \mathfrak{P})$. So $g \cap (\mathfrak{P} \times \mathfrak{P}) = f \cap (\mathfrak{P} \times \mathfrak{P})$. □

Remark 8. The above counter-example shows that pseudofuncoids (and more generally, any staroids on filters) are "second class" objects, they are not full-fledged because they don't bijectively correspond to funcoids and the elegant funcoids theory does not apply to them.

From the above it follows that staroids on filters do not correspond (by restriction) to staroids on principal filters (or staroids on sets).

4 Complete staroids and multifuncoids

4.1 Complete free stars

Definition 9. Let $\mathfrak{A}$ be a poset. *Complete free stars* on $\mathfrak{A}$ are such $S \in \mathscr{P}\mathfrak{A}$ that the least element (if it exists) is not in S and for every $T \in \mathscr{P}\mathfrak{A}$

$$\forall Z \in \mathfrak{A} \colon (\forall X \in T \colon Z \sqsupseteq X \Rightarrow Z \in S) \Leftrightarrow T \cap S \neq \emptyset.$$

Obvious 10. Every complete free star is a free star.

Proposition 11. $S \in \mathscr{P}\mathfrak{A}$ where $\mathfrak{A}$ is a poset is a complete free star iff all the following:

1. The least element (if it exists) is not in S.
2. $\forall Z \in \mathfrak{A} \colon (\forall X \in T \colon Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$.
3. S is an upper set.

Proof.

⇒. (1) and (2) are obvious. S is an upper set because S is a free star.

⇐. We need to prove that

$$\forall Z \in \mathfrak{A} \colon (\forall X \in T \colon Z \sqsupseteq X \Rightarrow Z \in S) \Leftarrow T \cap S \neq \emptyset.$$

Let $X' \in T \cap S$. Then $\forall X \in T \colon Z \sqsupseteq X \Rightarrow Z \sqsupseteq X' \Rightarrow Z \in S$ because S is an upper set. □

Proposition 12. Let S be a complete lattice. $S \in \mathscr{P}\mathfrak{A}$ is a complete free star iff all the following:

1. The least element (if it exists) is not in S.

2. $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$ for every $T \in \mathscr{P} S$.
3. S is an upper set.

Proof.

⇒. We need to prove only $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$. Let $\bigsqcup T \in S$. Because S is an upper set, we have $\forall X \in T: Z \sqsupseteq X \Rightarrow Z \sqsupseteq \bigsqcup T \Rightarrow Z \in S$ from which we conclude $T \cap S \neq \emptyset$.

⇐. We need to prove only $\forall Z \in \mathfrak{A}: (\forall X \in T: Z \sqsupseteq X \Rightarrow Z \in S) \Rightarrow T \cap S \neq \emptyset$.
Really, if $\forall Z \in \mathfrak{A}: (\forall X \in T: Z \sqsupseteq X \Rightarrow Z \in S)$ then $\bigsqcup T \in S$ and thus $\bigsqcup T \in S \Rightarrow T \cap S \neq \emptyset$. □

Proposition 13. Let $\mathfrak{A}$ be a complete lattice. $S \in \mathscr{P}\mathfrak{A}$ is a complete free star iff the least element (if it exists) is not in S and for every $T \in \mathscr{P}\mathfrak{A}$

$$\bigsqcup T \in S \Leftrightarrow T \cap S \neq \emptyset.$$

Proof.

⇒. We need to prove only $\bigsqcup T \in S \Leftarrow T \cap S \neq \emptyset$ what follows from that S is an upper set.

⇐. We need to prove only that S is an upper set. To prove this we can use the fact that S is a free star. □

4.1.1 Completely starrish posets

Definition 14. I will call a poset *completely starrish* when the full star $\star a$ is a free star for every element a of this poset.

Obvious 15. Every completely starrish poset is starrish.

Proposition 16. Every complete join infinite distributive lattice is starrish.

Proof. Let $\mathfrak{A}$ be a join infinite distributive lattice, $a \in \mathfrak{A}$. Obviously $0 \notin \star a$ (if 0 exists); obviously $\star a$ is an upper set. If $\bigsqcup T \in \star a$, then $(\bigsqcup T) \sqcap a$ is non-least that is $\bigsqcup \langle a \sqcap \rangle T$ is non-least what is equivalent to $a \sqcap x$ being non-least for some $x \in T$ that is $x \in \star a$. □

Theorem 17. If $\mathfrak{A}$ is a completely starrish complete lattice lattice then

$$\operatorname{atoms} \bigsqcup T = \bigcup \langle \operatorname{atoms} \rangle T.$$

for every $T \in \mathscr{P}\mathfrak{A}$.

Proof. For every atom c we have: $c \in \operatorname{atoms} \bigsqcup T \Leftrightarrow c \not\asymp \bigsqcup T \Leftrightarrow \bigsqcup T \in \star c \Leftrightarrow \exists X \in T: X \in \star c \Leftrightarrow \exists X \in T: X \not\asymp c \Leftrightarrow \exists X \in T: c \in \operatorname{atoms} X \Leftrightarrow c \in \bigcup \langle \operatorname{atoms} \rangle T$. □

4.2 More on free stars and complete free stars

Obvious 18. $\partial \mathcal{F} = \downdownarrows \star \mathcal{F}$ for an element $\mathcal{F}$ of down-aligned finitely meet closed filtrator.

Corollary 19. $\partial \mathcal{F} = \downdownarrows \star \mathcal{F}$ for every filter $\mathcal{F}$ on a poset.

Proposition 20. $\star \mathcal{F} = \upuparrows \partial \mathcal{F}$ for an element $\mathcal{F}$ of a filtrator with separable core.

Proof. $\mathcal{X} \in \upuparrows \partial \mathcal{F} \Leftrightarrow \operatorname{up} \mathcal{X} \subseteq \partial \mathcal{F} \Leftrightarrow \forall X \in \mathcal{X}: X \not\asymp \mathcal{F} \Leftrightarrow \mathcal{X} \not\asymp \mathcal{F} \Leftrightarrow \mathcal{X} \in \star \mathcal{F}$. □

Corollary 21. $\star \mathcal{F} = \upuparrows \partial \mathcal{F}$ for every filter $\mathcal{F}$ on a distributive lattice with least element.

Proposition 22. For a semifiltered, star-separable, down-aligned filtrator $(\mathfrak{A}; \mathfrak{Z})$ with finitely meet closed and separable core where $\mathfrak{Z}$ is a complete boolean lattice and both $\mathfrak{Z}$ and $\mathfrak{A}$ are atomistic lattices the following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:

1. $\mathcal{F} \in \mathfrak{Z}$.

2. $\partial\mathcal{F}$ is a complete free star on $\mathfrak{Z}$.
3. $\star\mathcal{F}$ is a complete free star on $\mathfrak{F}$.

Proof.

(1)⇒(2). That $\partial\mathcal{F}$ does not contain the least element is obvious. That $\partial\mathcal{F}$ is an upper set is obvious. So it remains to apply theorem 4.53??.

(2)⇒(3). That $\star\mathcal{F}$ does not contain the least element is obvious. That $\star\mathcal{F}$ is an upper set is obvious. So it remains to apply theorem 4.53??.

(3)⇒(1). Apply theorem 4.53??. □

Corollary 23. For a filter $\mathcal{F}\in\mathfrak{F}$ on a complete atomic boolean lattice the following conditions are equivalent:

1. $\mathcal{F}\in\mathfrak{P}$.
2. $\partial\mathcal{F}$ is a complete free star on $\mathfrak{P}$.
3. $\star\mathcal{F}$ is a complete free star on $\mathfrak{F}$.

Theorem 24. Let $\mathfrak{Z}$ be a boolean lattice. For any set $S\in\mathscr{P}\mathfrak{P}$ there exists a principal filter $\mathcal{A}$ such that $\partial\mathcal{A}=S$ iff S is a complete free star (on $\mathfrak{P}$).

Proof.

⇒. From the previous theorem.

⇐. $0^{\mathfrak{P}}\notin S$ and $\bigsqcup T\in S\Leftrightarrow T\cap S\neq\emptyset\Leftrightarrow\exists X\in T: X\in S$. Take $\mathcal{A}=\{\overline{X}\mid X\in\mathfrak{P}\setminus S\}$. We will prove that $\mathcal{A}$ is a principal filter. That $\mathcal{A}$ is a filter follows from properties of free stars. It remains to show that $\mathcal{A}$ is a principal filter. It follows from the following equivalence:

$\bigsqcap^{\mathfrak{P}}\mathcal{A}\in\mathcal{A}\Leftrightarrow\overline{\bigsqcup^{\mathfrak{P}}\langle\neg\rangle\mathcal{A}}\in\mathcal{A}\Leftrightarrow\bigsqcup^{\mathfrak{P}}\langle\neg\rangle\mathcal{A}\notin S\Leftrightarrow\neg\exists X\in\langle\neg\rangle\mathcal{A}: X\in S\Leftrightarrow\forall X\in\langle\neg\rangle\mathcal{A}: X\notin S\Leftrightarrow\forall X\in\mathcal{A}: X\in\mathcal{A}\Leftrightarrow 1$. □

Proposition 25.

1. If S is a free star on $\mathfrak{A}$ then $\downdownarrows S$ is a free star on $\mathfrak{Z}$, provided that $\mathfrak{Z}$ is a join-semilattice and the filtrator $(\mathfrak{A};\mathfrak{Z})$ is down-aligned and with finitely join-closed core.
2. If S is a free star on $\mathfrak{P}$ then $\upuparrows S$ is a free star on $\mathfrak{F}$, provided that $\mathfrak{Z}$ is a boolean lattice.

Proof.

1. $X\sqcup^{\mathfrak{Z}}Y\in\downdownarrows S\Leftrightarrow X\sqcup^{\mathfrak{Z}}Y\in S\Leftrightarrow X\sqcup^{\mathfrak{A}}Y\in S\Leftrightarrow X\in S\vee Y\in S\Leftrightarrow X\in\downdownarrows S\vee Y\in\downdownarrows S$ for every $X,Y\in\mathfrak{Z}$; $0\notin\downdownarrows S$ is obvious.
2. There exists a filter $\mathcal{F}$ such that $S=\partial\mathcal{F}$. For every filters $\mathcal{X},\mathcal{Y}\in\mathfrak{F}$

   $\mathcal{X}\sqcup^{\mathfrak{A}}\mathcal{Y}\in\upuparrows S\Leftrightarrow\operatorname{up}(\mathcal{X}\sqcup^{\mathfrak{A}}\mathcal{Y})\subseteq S\Leftrightarrow\forall K\in\operatorname{up}(\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}): K\in\partial\mathcal{F}\Leftrightarrow\forall K\in\operatorname{up}(\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}): K\not\asymp\mathcal{F}\Leftrightarrow\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}\not\asymp\mathcal{F}\Leftrightarrow\mathcal{X}\sqcup^{\mathfrak{F}}\mathcal{Y}\in\star\mathcal{F}\Leftrightarrow\mathcal{X}\in\star\mathcal{F}\vee\mathcal{Y}\in\star\mathcal{F}\Leftrightarrow\mathcal{X}\not\asymp\mathcal{F}\vee\mathcal{Y}\not\asymp\mathcal{F}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X}: X\not\asymp\mathcal{F}\vee\forall Y\in\operatorname{up}\mathcal{Y}: Y\not\asymp\mathcal{F}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X}: X\in\partial\mathcal{F}\vee\forall Y\in\operatorname{up}\mathcal{Y}: Y\in\partial\mathcal{F}\Leftrightarrow\operatorname{up}\mathcal{X}\subseteq S\vee\operatorname{up}\mathcal{Y}\subseteq S\Leftrightarrow\mathcal{X}\in\upuparrows S\vee\mathcal{Y}\in\upuparrows S$;

   $0\in\upuparrows S\Leftrightarrow\operatorname{up}0\subseteq S\Leftrightarrow 0\in S$ what is false. □

Corollary 26. If S is a free star on $\mathfrak{F}$ then $\downdownarrows S$ is a free star on $\mathfrak{P}$, provided that $\mathfrak{P}$ is a join-semilattice.

Proposition 27.

1. If S is a complete free star on $\mathfrak{A}$ then $\downdownarrows S$ is a complete free star on $\mathfrak{Z}$, provided that $\mathfrak{Z}$ is a complete lattice and the filtrator $(\mathfrak{A};\mathfrak{Z})$ is down-aligned and with join-closed core.
2. If S is a complete free star on $\mathfrak{P}$ then $\upuparrows S$ is a complete free star on $\mathfrak{F}$, provided that $\mathfrak{Z}$ is a boolean lattice.

Proof.

1. $\bigsqcup^{\mathfrak{Z}} T \in \Downarrow S \Leftrightarrow \bigsqcup^{\mathfrak{Z}} T \in S \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \in S \Leftrightarrow T \cap S \neq \emptyset \Leftrightarrow T \cap \Downarrow S \neq \emptyset$ for every $T \in \mathscr{P}\mathfrak{Z}$; $0 \notin \Downarrow S$ is obvious.

2. There exists a principal filter $\mathcal{F}$ such that $S = \partial \mathcal{F}$.

   $\bigsqcup^{\mathfrak{A}} T \in \Uparrow S \Leftrightarrow \operatorname{up} \bigsqcup^{\mathfrak{A}} T \in S \Leftrightarrow \forall K \in \operatorname{up} \bigsqcup^{\mathfrak{A}} T\colon K \in \partial \mathcal{F} \Leftrightarrow \forall K \in \operatorname{up} \bigsqcup^{\mathfrak{A}} T\colon K \not\asymp \mathcal{F} \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \not\asymp \mathcal{F} \Leftrightarrow \bigsqcup^{\mathfrak{A}} T \in \star \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T\colon \mathcal{K} \in \star \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T\colon \mathcal{K} \not\asymp \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \forall K \in \operatorname{up} \mathcal{K}\colon K \not\asymp \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T \forall K \in \operatorname{up} \mathcal{K}\colon K \in \partial \mathcal{F} \Leftrightarrow \exists \mathcal{K} \in T\colon \operatorname{up} \mathcal{K} \subseteq S \Leftrightarrow \exists \mathcal{K} \in T\colon \mathcal{K} \in \Uparrow S \Leftrightarrow T \cap \Uparrow S \neq \emptyset$.

   $0 \in \Uparrow S \Leftrightarrow \operatorname{up} 0 \subseteq S \Leftrightarrow 0 \in S$ what is false. □

Corollary 28. If S is a complete free star on $\mathfrak{F}$ then $\Downarrow S$ is a complete free star on $\mathfrak{P}$, provided that $\mathfrak{Z}$ is a complete lattice.

5 Complete staroids and multifuncoids

Definition 29. Consider an indexed family $(\mathfrak{A}_i; \mathfrak{Z}_i)$ of filtrators. A pre-staroid f of the form $\prod \mathfrak{Z}$ is *complete* in argument $k \in \operatorname{arity} f$ when $(\operatorname{val} f)_k\, L$ is a complete free star for every $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} \mathfrak{Z}_i$.

Definition 30. Consider an indexed family $(\mathfrak{A}_i; \mathfrak{Z}_i)$ of filtrators and pre-multifuncoid f is of the form $\prod \mathfrak{A}$. Then f is *complete* in argument $k \in \operatorname{arity} f$ iff $\langle f \rangle_k L \in \mathfrak{Z}_k$ for every family $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} \mathfrak{Z}_i$.

Proposition 31. Consider an indexed family $(\mathfrak{F}_i; \mathfrak{Z}_i)$ of primary filtrators over boolean lattices. Let f be a pre-multifuncoid of the form $\mathfrak{A}$ and $k \in \operatorname{arity} f$. The following are equivalent:

1. Pre-multifuncoid f is complete in argument k.
2. Pre-staroid $\Downarrow [f]$ is complete in argument k.

Proof. $L \in \operatorname{GR}[f] \Leftrightarrow L_i \not\asymp \langle f \rangle_i L|_{(\operatorname{dom} L) \setminus \{i\}}$;

$(\operatorname{val} \Downarrow [f])_k L = \partial \langle f \rangle_k L$ by the theorem ??17.81.

So $(\operatorname{val} \Downarrow [f])_k\, L$ is a complete free star iff $\langle f \rangle_k\, L \in \mathfrak{Z}_k$ (proposition 22) for every $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} \mathfrak{Z}_i$. □

Example 32. Consider funcoid $f = \operatorname{id}^{\mathsf{FCD}(U)}$. It is obviously complete in each its two arguments. Then $[f]$ is not complete in each of its two arguments because $(\mathcal{X}; \mathcal{Y}) \in [f] \Leftrightarrow \mathcal{X} \not\asymp \mathcal{Y}$ what does not generate a complete free star if one of the arguments (say $\mathcal{X}$) is a fixed nonprincipal filter.

Theorem 33. Consider a semifiltered, star-separable, down-aligned filtrator $(\mathfrak{A}; \mathfrak{Z})$ with finitely meet closed and separable core where $\mathfrak{Z}$ is a complete boolean lattice and both $\mathfrak{Z}$ and $\mathfrak{A}$ are atomistic lattices.

Let f be a multifuncoid of the aforementioned form. Let $k, l \in \operatorname{arity} f$ and $k \neq l$. The following are equivalent:

1. f is complete in the argument k.
2. $\langle f \rangle_l (L \cup \{(k; \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l (L \cup \{(k; x)\})$ for every $X \in \mathscr{P}\mathfrak{Z}_k$, $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k, l\}} \mathfrak{Z}_i$.
3. $\langle f \rangle_l (L \cup \{(k; \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l (L \cup \{(k; x)\})$ for every $X \in \mathscr{P}\mathfrak{A}_k$, $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k, l\}} \mathfrak{Z}_i$.

Proof.

(3)⇒(2). Obvious.

(2)⇒(1). Let $Y \in \mathfrak{Z}$.

$\bigsqcup X \not\asymp \langle f \rangle_k (L \cup \{(l; Y)\}) \Leftrightarrow Y \not\asymp \langle f \rangle_l (L \cup \{(k; \bigsqcup X)\}) \Leftrightarrow Y \not\asymp \bigsqcup_{x \in X} \langle f \rangle_l (L \cup \{(k; x)\}) \Leftrightarrow$ (proposition 4.144??) $\Leftrightarrow \exists x \in X\colon Y \not\asymp \langle f \rangle_l (L \cup \{(k; x)\}) \Leftrightarrow \exists x \in X\colon x \not\asymp \langle f \rangle_k (L \cup (l; Y))$.

It is equivalent (proposition 22 and the fact that $[f]$ is an upper set) to $\langle f\rangle_k(L\cup\{(l;Y)\})$ being a principal filter and thus $(\operatorname{val}[f])_l L$ being a complete free star.

(1)⇒(3). $Y\not\asymp\langle f\rangle_l(L\cup\{(k;\bigsqcup X)\})\Leftrightarrow\bigsqcup X\not\asymp\langle f\rangle_k(L\cup\{(l;Y)\})\Leftrightarrow\exists x\in X: x\not\asymp\langle f\rangle_k(L\cup\{(l;Y)\})\Leftrightarrow\exists x\in X: Y\not\asymp\langle f\rangle_l(L\cup\{(k;x)\})\Leftrightarrow Y\not\asymp\bigsqcup_{x\in X}\langle f\rangle_l(L\cup\{(k;x)\})$ for every principal Y. □

6 Identity staroids and multifuncoids

6.1 Identity relations

Denote $\operatorname{id}_{A[n]}=\{(\lambda i\in n: x)\mid x\in A\}=\{n\times\{x\}\mid x\in A\}$ the n-ary identity relation on a set A (for each index set n).

Proposition 34. $\prod X\not\asymp\operatorname{id}_{A[n]}\Leftrightarrow\bigcap_{i\in n}X_i\cap A\neq\emptyset$.

Proof. $\prod X\not\asymp\operatorname{id}_{A[n]}\Leftrightarrow\exists t\in A: n\times\{t\}\in\prod X\Leftrightarrow\exists t\in A\forall i\in n: t\in X_i\Leftrightarrow\bigcap_{i\in n}X_i\cap A\neq\emptyset$. □

6.2 Universal definitions of identity staroids

Consider a filtrator $(\mathfrak{A};\mathfrak{Z})$ and $\mathcal{A}\in\mathfrak{A}$.

I will define below *small identity staroids* $\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ and *big identity staroids* $\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. That they are really staroids and even completary staroids (under certain conditions) is proved below.

Definition 35. Consider a filtrator $(\mathfrak{A};\mathfrak{Z})$. Let $\mathfrak{Z}$ be a complete lattice. Let $\mathcal{A}\in\mathfrak{A}$, let n be an index set.

$\operatorname{form}\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}=\mathfrak{Z}^n$; $L\in\operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow\bigsqcap^{\mathfrak{Z}}_{i\in n}L_i\in\partial\mathcal{A}$.

Obvious 36. $X\in\operatorname{GR}\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow\forall A\in\operatorname{up}\mathcal{A}:\bigsqcap^{\mathfrak{Z}}_{i\in n}X_i\sqcap A\neq 0$ if our filtrator is with separable core.

Definition 37. The subset X of a poset $\mathfrak{A}$ *has a nontrivial lower bound* (I denote this predicate as $\operatorname{MEET}(X)$) iff there is nonleast $a\in\mathfrak{A}$ such that $\forall x\in X: a\sqsubseteq x$.

Definition 38. Staroid $\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ (for any $\mathcal{A}\in\mathfrak{A}$ where $\mathfrak{A}$ is a poset) is defined by the formulas:

$\operatorname{form}\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}=\mathfrak{A}^n$; $\mathcal{L}\in\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow\operatorname{MEET}(\{\mathcal{L}_i\mid i\in n\}\cup\{\mathcal{A}\})$.

Obvious 39. If $\mathfrak{A}$ is complete lattice, then $\mathcal{L}\in\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow\bigsqcap\mathcal{L}\not\asymp\mathcal{A}$.

Obvious 40. If $\mathfrak{A}$ is complete lattice and a is an atom, then $\mathcal{L}\in\operatorname{GR}\operatorname{ID}^{\mathsf{Strd}}_{a[n]}\Leftrightarrow\bigsqcap\mathcal{L}\sqsupseteq a$.

Obvious 41. If $\mathfrak{A}$ is a complete lattice then there exists a multifuncoid $\Lambda\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ such that $\left\langle\Lambda\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}\right\rangle_k L=\bigsqcap_{i\in n}L_i\sqcap\mathcal{A}$ for every $k\in n$, $L\in\mathfrak{A}^{n\setminus\{k\}}$.

Proposition 42. If $(\mathfrak{A};\mathfrak{Z})$ is a meet-closed filtrator and $\mathfrak{Z}$ is a complete lattice and $\mathfrak{A}$ is a meet-semilattice. There exists a multifuncoid $\Lambda\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ such that $\left\langle\Lambda\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\right\rangle_k L=\bigsqcap^{\mathfrak{Z}}_{i\in n}L_i\sqcap^{\mathfrak{A}}\mathcal{A}$ for every $k\in n$, $L\in\mathfrak{Z}^{n\setminus\{k\}}$.

Proof. We need to prove that $L\cup\{(k;X)\}\in\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}\Leftrightarrow\bigsqcap^{\mathfrak{Z}}_{i\in n}L_i\sqcap^{\mathfrak{A}}\mathcal{A}\not\asymp^{\mathfrak{A}}X$. But

$$\bigsqcap^{\mathfrak{Z}}_{i\in n}L_i\sqcap^{\mathfrak{A}}\mathcal{A}\not\asymp^{\mathfrak{A}}X\Leftrightarrow\bigsqcap^{\mathfrak{Z}}_{i\in n}L_i\sqcap^{\mathfrak{A}}X\not\asymp^{\mathfrak{A}}\mathcal{A}\Leftrightarrow\bigsqcap^{\mathfrak{Z}}_{i\in n}(L\cup\{(k;X)\})_i\not\asymp^{\mathfrak{A}}\mathcal{A}\Leftrightarrow L\cup\{(k;X)\}\in\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$ □

6.3 Identities are staroids

Proposition 43. Let $\mathfrak{A}$ be a complete distributive lattice and $\mathcal{A}\in\mathfrak{A}$. Then $\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a staroid.

Proof. That $L \notin \operatorname{GR} \operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $L_k = 0$ for some $k \in n$ is obvious. It remains to prove

$$L \cup \{(k; X \sqcup Y)\} \in \operatorname{GR} \operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow L \cup \{(k; X)\} \in \operatorname{GR} \operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \vee L \cup \{(k; Y)\} \in \operatorname{GR} \operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

It is equivalent to

$$\prod_{i \in n \setminus \{k\}} L_i \sqcap (X \sqcup Y) \not\asymp \mathcal{A} \Leftrightarrow \prod_{i \in n \setminus \{k\}} L_i \sqcap X \not\asymp \mathcal{A} \vee \prod_{i \in n \setminus \{k\}} L_i \sqcap Y \not\asymp \mathcal{A}.$$

Really, $\prod_{i \in n \setminus \{k\}} L_i \sqcap (X \sqcup Y) \not\asymp \mathcal{A} \Leftrightarrow \left(\prod_{i \in n \setminus \{k\}} L_i \sqcap X\right) \sqcup \left(\prod_{i \in n \setminus \{k\}} L_i \sqcap Y\right) \not\asymp \mathcal{A} \Leftrightarrow \prod_{i \in n \setminus \{k\}} L_i \sqcap X \not\asymp \mathcal{A} \vee \prod_{i \in n \setminus \{k\}} L_i \sqcap Y \not\asymp \mathcal{A}$. □

Proposition 44. Let $(\mathfrak{A}; \mathfrak{Z})$ be a starrish filtrator over a complete meet infinite distributive lattice and $\mathcal{A} \in \mathfrak{A}$. Then $\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a staroid.

Proof. That $L \notin \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $L_k = 0$ for some $k \in n$ is obvious. It remains to prove

$$L \cup \{(k; X \sqcup Y)\} \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow L \cup \{(k; X)\} \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \vee L \cup \{(k; Y)\} \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}.$$

It is equivalent to

$$\prod^{\mathfrak{Z}}_{i \in n \setminus \{k\}} L_i \sqcap (X \sqcup Y) \not\asymp \mathcal{A} \Leftrightarrow \prod^{\mathfrak{Z}}_{i \in n \setminus \{k\}} L_i \sqcap X \not\asymp \mathcal{A} \vee \prod^{\mathfrak{Z}}_{i \in n \setminus \{k\}} L_i \sqcap Y \not\asymp \mathcal{A}.$$

Really, $\prod^{\mathfrak{Z}}_{i \in n \setminus \{k\}} L_i \sqcap (X \sqcup Y) \not\asymp \mathcal{A} \Leftrightarrow \left(\prod^{\mathfrak{Z}}_{i \in n \setminus \{k\}} L_i \sqcap X\right) \sqcup \left(\prod^{\mathfrak{Z}}_{i \in n \setminus \{k\}} L_i \sqcap Y\right) \not\asymp \mathcal{A} \Leftrightarrow \prod^{\mathfrak{Z}}_{i \in n \setminus \{k\}} L_i \sqcap X \not\asymp \mathcal{A} \vee \prod^{\mathfrak{Z}}_{i \in n \setminus \{k\}} L_i \sqcap Y \not\asymp \mathcal{A}$. □

Proposition 45. Let $(\mathfrak{A}; \mathfrak{Z})$ be a distributive lattice filtrator with least element and finitely join-closed core which is a join semilattice. $\operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid for every $\mathcal{A} \in \mathfrak{A}$.

Proof. $\partial \mathcal{A}$ is a free star by theorem ??4.47.

$L_0 \sqcup L_i \in \operatorname{GR} \operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \forall i \in n: (L_0 \sqcup L_i)\, i \in \partial \mathcal{A} \Leftrightarrow \forall i \in n: L_0\, i \sqcup L_1\, i \in \partial \mathcal{A} \Leftrightarrow \forall i \in n: (L_0 i \in \partial \mathcal{A} \vee L_1 i \in \partial \mathcal{A}) \Leftrightarrow \exists c \in \{0, 1\}^n \forall i \in n: L_{c(i)} i \in \partial \mathcal{A} \Leftrightarrow \exists c \in \{0, 1\}^n: (\lambda i \in n: L_{c(i)} i) \in \operatorname{GR} \operatorname{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. □

Lemma 46. $X \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \operatorname{Cor}' \prod^{\mathfrak{A}}_{i \in n} X_i \not\asymp \mathcal{A}$ for a join-closed filtrator $(\mathfrak{A}; \mathfrak{Z})$ such that both $\mathfrak{A}$ and $\mathfrak{Z}$ are complete lattices, provided that $\mathcal{A} \in \mathfrak{A}$.

Proof. $X \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \prod^{\mathfrak{Z}}_{i \in n} X_i \not\asymp \mathcal{A} \Leftrightarrow \operatorname{Cor}' \prod^{\mathfrak{A}}_{i \in n} X_i \not\asymp \mathcal{A}$. □

Conjecture 47. $\operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid for every set-theoretic filter $\mathcal{A}$.

Proposition 48. Let each $(\mathfrak{A}_i; \mathfrak{Z}_i)$ for $i \in n$ (where n is an index set) is a finitely join-closed filtrator, such that each $\mathfrak{A}_i$ and each $\mathfrak{Z}_i$ are join-semilattices. If f is a completary staroid of the form $\mathfrak{A}$ then $\downdownarrows f$ is a completary staroid of the form $\mathfrak{Z}$. [TODO: Move this proposition (and note its corollary).]

Proof. $L_0 \sqcup^{\mathfrak{Z}} L_1 \in \operatorname{GR} \downdownarrows f \Leftrightarrow L_0 \sqcup^{\mathfrak{Z}} L_1 \in \operatorname{GR} f \Leftrightarrow L_0 \sqcup^{\mathfrak{A}} L_1 \in \operatorname{GR} f \Leftrightarrow \exists c \in \{0, 1\}^n: (\lambda i \in n: L_{c(i)} i) \in \operatorname{GR} f \Leftrightarrow \exists c \in \{0, 1\}^n: (\lambda i \in n: L_{c(i)} i) \in \operatorname{GR} \downdownarrows f$ for every $L_0, L_1 \in \prod \mathfrak{Z}$. □

Conjecture 49. $\upuparrows \operatorname{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ is a completary staroid if $\mathcal{A}$ is a filter on a set and n is an index set.

6.4 Special case of sets and filters

Proposition 50. $\uparrow^{\mathfrak{Z}^n} X \in \operatorname{GR} \operatorname{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall A \in a: \prod X \not\asymp \operatorname{id}_{A[n]}$ for every filter a on a powerset and index set n.

Proof. $\forall A \in a: \prod X \not\asymp \operatorname{id}_{A[n]} \Leftrightarrow \forall A \in a: \bigcap_{i \in n} X_i \cap A \neq \emptyset \Leftrightarrow \forall A \in a: \prod^{\mathfrak{P}}_{i \in n} \uparrow^{\mathfrak{Z}} X_i \not\asymp \uparrow A \Leftrightarrow \prod^{\mathfrak{P}}_{i \in n} (\uparrow^{\mathfrak{Z}^n} X_i) \not\asymp a \Leftrightarrow \prod^{\mathfrak{P}}_{i \in n} (\uparrow^{\mathfrak{Z}^n} X)_i \not\asymp a \Leftrightarrow \uparrow^{\mathfrak{Z}^n} X \in \operatorname{GR} \operatorname{id}_{a[n]}$. □

Proposition 51. $Y \in \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \forall A \in \mathcal{A}: Y \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]}$ for every filter $\mathcal{A}$ on a powerset and $Y \in \mathfrak{P}^n$.

Proof. Take $Y = \uparrow^{3^n} X$.

$\forall A \in \mathcal{A}: Y \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \forall A \in \mathcal{A}: \uparrow^{3^n} X \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \forall A \in \mathcal{A}: \prod X \not\asymp \mathrm{id}_{A[n]} \Leftrightarrow \uparrow^{3^n} X \in \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow Y \in \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. □

Proposition 52. $\uparrow^{3^n} X \in \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall A \in a \exists t \in A \forall i \in n: t \in X_i$.

Proof. $\uparrow^{3^n} X \in \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \exists A \in a \exists t \in A: n \times \{t\} \in \prod X \Leftrightarrow \forall A \in a \exists t \in A \forall i \in n: t \in X_i$. □

6.5 Relationships between big and small identity staroids

Definition 53. $a^n_{\mathsf{Strd}} = \prod^{\mathsf{Strd}}_{i \in n} a$ for every element a of a poset and an index set n.

Proposition 54. $\Uparrow \mathrm{id}^{\mathsf{Strd}}_{a[n]} \sqsubseteq \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \sqsubseteq a^n_{\mathsf{Strd}}$ for every filter a (on any distributive lattice) and an index set n.

Proof.

$\mathrm{GR} \Uparrow \mathrm{id}^{\mathsf{Strd}}_{a[n]} \subseteq \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$. $\mathcal{L} \in \mathrm{GR} \Uparrow \mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \mathrm{up}\,\mathcal{L} \subseteq \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall L \in \mathrm{up}\,\mathcal{L}: L \in \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow$ (proposition 4.112??)$\Leftrightarrow \forall L \in \mathrm{up}\,\mathcal{L} \forall A \in \mathrm{up}\,a: \bigsqcap^3_{i \in n} L_i \not\asymp A \Leftrightarrow \forall L \in \mathrm{up}\,\mathcal{L} \forall A \in \mathrm{up}\,a: \bigsqcap^3_{i \in n} L_i \sqcap A \neq 0 \Rightarrow \bigcup_{i \in n} \mathcal{L}_i \cup a$ has finite intersection property$\Leftrightarrow \mathcal{L} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$.

$\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]} \subseteq \mathrm{GR}\,a^n_{\mathsf{Strd}}$. $\mathcal{L} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \mathrm{MEET}(\{\mathcal{L}_i \mid i \in n\} \cup \{a\}) \Rightarrow \forall i \in a: \mathcal{L}_i \not\asymp a \Leftrightarrow \mathcal{L} \in \mathrm{GR}\,a^a_{\mathsf{Strd}}$. □

Proposition 55. $\Uparrow \mathrm{id}^{\mathsf{Strd}}_{a[a]} \sqsubset \mathrm{ID}^{\mathsf{Strd}}_{a[a]} = a^a_{\mathsf{Strd}}$ for every nontrivial ultrafilter a on a set.

Proof.

$\mathrm{GR} \Uparrow \mathrm{id}^{\mathsf{Strd}}_{a[a]} \neq \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[a]}$. Let $\mathcal{L}_i = \uparrow^{\mathrm{Base}(a)} i$. Then trivially $\mathcal{L} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[a]}$. But to disprove $\mathcal{L} \in \mathrm{GR} \Uparrow \mathrm{id}^{\mathsf{Strd}}_{a[a]}$ it's enough to show $L \notin \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[a]}$ for some $L \in \mathrm{up}\,\mathcal{L}$. Really, take $L_i = \mathcal{L}_i = \uparrow^{\mathrm{Base}(a)} i$. Then $L \in \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{a[a]} \Leftrightarrow \forall A \in a \exists t \in A \forall i \in a: t \in i$ what is clearly false (we can always take $i \in a$ such that $t \notin i$ for any point t).

$\mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[a]} = \mathrm{GR}\,a^a_{\mathsf{Strd}}$. $\mathcal{L} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[a]} \Leftrightarrow \forall i \in n: \mathcal{L}_i \sqsupseteq a \Leftrightarrow \forall i \in a: \mathcal{L}_i \not\asymp a \Leftrightarrow \mathcal{L} \in \mathrm{GR}\,a^a_{\mathsf{Strd}}$. □

Corollary 56. a^a_{Strd} isn't an atom when a is a nontrivial ultrafilter.

Corollary 57. Staroidal product of an infinite indexed family of ultrafilters may be non-atomic.

Proposition 58. $\mathrm{id}^{\mathsf{Strd}}_{a[n]}$ is determined by the value of $\Uparrow \mathrm{id}^{\mathsf{Strd}}_{a[n]}$. Moreover $\mathrm{id}^{\mathsf{Strd}}_{a[n]} = \Downarrow \Uparrow \mathrm{id}^{\mathsf{Strd}}_{a[n]}$.

Proof. Use general properties of upgrading and downgrading (proposition 17.63??). □

Lemma 59. $\mathcal{L} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ iff $\bigcup_{i \in n} \mathcal{L}_i \cup a$ has finite intersection property (for primary filtrators).

Proof. $\mathcal{L} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \bigsqcap_{i \in n} \mathcal{L} \sqcap a \neq 0^{\mathfrak{F}} \Leftrightarrow \forall X \in \bigsqcap_{i \in n} \mathcal{L} \sqcap a: X \neq \emptyset$ what is equivalent of $\bigcup_{i \in n} \mathcal{L}_i \cup a$ having finite intersection property. □

Proposition 60. $\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ is determined by the value of $\Downarrow \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$, moreover $\mathrm{ID}^{\mathsf{Strd}}_{a[n]} = \Uparrow \Downarrow \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ (for primary filtrators).

Proof. $\mathcal{L} \in \Uparrow \Downarrow \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \mathrm{up}\,\mathcal{L} \subseteq \Downarrow \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \mathrm{up}\,\mathcal{L} \subseteq \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall L \in \mathrm{up}\,\mathcal{L}: L \in \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall L \in \mathrm{up}\,\mathcal{L}: \bigsqcap_{i \in n} L_i \sqcap a \neq 0^{\mathfrak{F}} \Leftrightarrow \bigcup_{i \in n} \mathcal{L}_i \cup a$ has finite intersection property$\Leftrightarrow$(lemma)$\Leftrightarrow \mathcal{L} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]}$. □

Proposition 61. $\mathrm{id}^{\mathsf{Strd}}_{a[n]} \sqsubseteq \downarrow\downarrow \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ for every filter a and an index set n.

Proof. $\mathrm{id}^{\mathsf{Strd}}_{a[n]} = \downarrow\downarrow \uparrow\uparrow \mathrm{id}^{\mathsf{Strd}}_{a[n]} \sqsubseteq \downarrow\downarrow \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$. □

Proposition 62. $\mathrm{id}^{\mathsf{Strd}}_{a[a]} \sqsubset \downarrow\downarrow \mathrm{ID}^{\mathsf{Strd}}_{a[a]}$ for every nontrivial ultrafilter a.

Proof. Suppose $\mathrm{id}^{\mathsf{Strd}}_{a[a]} = \downarrow\downarrow \mathrm{ID}^{\mathsf{Strd}}_{a[a]}$. Then $\mathrm{ID}^{\mathsf{Strd}}_{a[a]} = \uparrow\uparrow \downarrow\downarrow \mathrm{ID}^{\mathsf{Strd}}_{a[a]} = \uparrow\uparrow \mathrm{id}^{\mathsf{Strd}}_{a[a]}$ what contradicts to the above. □

Obvious 63. $\mathcal{L} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow a \sqcap \prod_{i\in n} \mathcal{L}_i \neq 0^{\mathfrak{F}}$ if a is an element of a complete lattice.

Obvious 64. $\mathcal{L} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{a[n]} \Leftrightarrow \forall i \in n: \mathcal{L}_i \sqsupseteq a \Leftrightarrow \forall i \in n: \mathcal{L}_i \not\asymp a$ if a is an ultrafilter on $\mathfrak{A}$.

6.6 Identity staroids on principal filters

For principal filter $\uparrow A$ (where A is a set) the above definitions coincide with n-ary identity relation, as formulated in the below propositions:

Proposition 65. $\uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]}$.

Proof. $L \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \prod L \not\asymp \mathrm{id}_{A[n]} \Leftrightarrow \exists t \in A \forall i \in n: t \in L_i \Leftrightarrow \bigcap_{i\in n} L_i \cap A \neq \emptyset \Leftrightarrow L \in \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]}$. Thus $\uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]}$. □

Corollary 66. $\mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]}$ is a principal staroid.

Problem 67. Is $\mathrm{ID}^{\mathsf{Strd}}_{A[n]}$ principal for every principal filter A on a set and index set n?

Proposition 68. $\uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \sqsubseteq \downarrow\downarrow \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$ for every set A.

Proof. $L \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow L \in \mathrm{GR}\,\mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]} \Leftrightarrow \uparrow A \not\asymp \prod^{\mathfrak{A}}_{i\in n} L_i \Leftarrow \uparrow A \not\asymp \prod^{\mathfrak{F}}_{i\in n} L_i \Leftrightarrow L \in \downarrow\downarrow \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$. □

Proposition 69. $\uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \sqsubset \downarrow\downarrow \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$ for some set A and index set n.

Proof. $L \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \prod^{\mathfrak{F}}_{i\in n} L_i \not\asymp \uparrow A$ what is not implied by $\prod^{\mathfrak{A}}_{i\in n} L_i \not\asymp \uparrow A$ that is $L \in \downarrow\downarrow \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$. (For a counter example take $n = \mathbb{N}$, $L_i = (0; 1/i)$, $A = \mathbb{R}$.) □

Proposition 70. $\uparrow\uparrow \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \uparrow\uparrow \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]}$.

Proof. $\uparrow\uparrow \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \uparrow\uparrow \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]}$ is obvious from the above. □

Proposition 71. $\uparrow\uparrow \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \sqsubseteq \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$.

Proof. $\mathcal{X} \in \mathrm{GR} \uparrow\uparrow \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \mathrm{up}\,\mathcal{X} \subseteq \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \forall Y \in \mathrm{up}\,\mathcal{X}: Y \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Leftrightarrow \forall Y \in \mathrm{up}\,\mathcal{X}: Y \in \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]} \Leftrightarrow \forall Y \in \mathrm{up}\,\mathcal{X}: \prod^{\mathfrak{F}}_{i\in n} Y_i \sqcap \uparrow A \neq 0 \Rightarrow \prod^{\mathfrak{A}}_{i\in n} \mathcal{X}_i \sqcap \uparrow A \neq 0 \Leftrightarrow \mathcal{X} \in \mathrm{GR}\,\mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$. □

Proposition 72. $\uparrow\uparrow \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \sqsubset \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$ for some set A.

Proof. We need to prove $\uparrow\uparrow \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \neq \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]}$ that is it's enough to prove (see the above proof) that $\forall Y \in \mathrm{up}\,\mathcal{X}: \prod^{\mathfrak{F}}_{i\in n} Y_i \sqcap \uparrow A \neq 0 \not\Leftarrow \prod^{\mathfrak{A}}_{i\in n} \mathcal{X}_i \sqcap \uparrow A \neq 0$. A counter-example follows:

$\forall Y \in \mathrm{up}\,\mathcal{X}: \prod^{\mathfrak{F}}_{i\in n} Y_i \sqcap \uparrow A \neq 0$ does not hold for $n = \mathbb{N}$, $\mathcal{X}_i = \uparrow(-1/i; 0)$ for $i \in n$, $A = (-\infty; 0)$. To show this, it's enough to prove $\prod^{\mathfrak{F}}_{i\in n} Y_i \sqcap \uparrow A \neq 0$ for $Y_i = \uparrow(-1/i; 0)$ but this is obvious since $\prod^{\mathfrak{F}}_{i\in n} Y_i = 0$.

On the other hand, $\prod^{\mathfrak{A}}_{i\in n} \mathcal{X}_i \sqcap \uparrow A \neq 0$ for the same $\mathcal{X}$ and A. □

The above theorems are summarized in the following diagram:

$$\begin{array}{ccc}
\downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]} & \sqsupseteq & \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]} \\
\big\downarrow \upuparrows \quad \big\uparrow \downdownarrows & & \big\downarrow \upuparrows \quad \big\uparrow \downdownarrows \\
\mathrm{ID}^{\mathsf{Strd}}_{\uparrow A[n]} & \sqsupseteq & \upuparrows \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} = \upuparrows \mathrm{id}^{\mathsf{Strd}}_{\uparrow A[n]}
\end{array}$$

Remark 73. $\sqsubseteq$ on the diagram means inequality which can become strict for some A and n.

6.7 Identity staroids represented as meets and joins

Proposition 74. $\mathrm{id}^{\mathsf{Strd}}_{a[n]} = \bigsqcap \{\uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \mid A \in a\}$ for every set-theoretic filter a where the meet may be taken on every of the following posets: anchored relations, staroids.

Proof. That $\mathrm{id}^{\mathsf{Strd}}_{a[n]} \sqsubseteq \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]}$ for every $A \in a$ is obvious.

Let $f \sqsubseteq \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]}$ for every $A \in a$. $L \in \mathrm{GR}\, f \Rightarrow L \in \mathrm{GR} \uparrow^{\mathsf{Strd}} \mathrm{id}_{A[n]} \Rightarrow \forall A \in a \colon \bigsqcap^{\mathfrak{A}}_{i \in n} L_i \not\asymp A \Rightarrow \bigsqcap^{\mathfrak{A}}_{i \in n} L_i \not\asymp a \Rightarrow L \in \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{a[n]}$. Thus $f \sqsubseteq \mathrm{id}^{\mathsf{Strd}}_{a[n]}$. □

Proposition 75. $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \bigsqcup \left\{\mathrm{ID}^{\mathsf{Strd}}_{a[n]} \mid a \in \operatorname{atoms} \mathcal{A}\right\} = \bigsqcup \{a^n_{\mathsf{Strd}} \mid a \in \operatorname{atoms} \mathcal{A}\}$ where the meet may be taken on every of the following posets: anchored relations, staroids, completary staroids, provided that $\mathcal{A}$ is a filter on a set.

Proof. $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \sqsupseteq \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ for every $a \in \operatorname{atoms} \mathcal{A}$ is obvious.

Let $f \sqsupseteq \mathrm{ID}^{\mathsf{Strd}}_{a[n]}$ for every $a \in \operatorname{atoms} \mathcal{A}$. Then $\forall L \in \mathrm{GR}\, \mathrm{ID}^{\mathsf{Strd}}_{a[n]} \colon L \in \mathrm{GR}\, f$ that is

$$\forall L \in \operatorname{form} f \colon (\mathrm{MEET}(\{L_i \mid i \in n\} \cup \{a\}) \Rightarrow L \in \mathrm{GR}\, f).$$

But $\exists a \in \operatorname{atoms} \mathcal{A} \colon \mathrm{MEET}(\{L_i \mid i \in n\} \cup \{a\}) \Leftrightarrow \exists a \in \operatorname{atoms} \mathcal{A} \colon \bigsqcap^{\mathfrak{A}}_{i \in n} L_i \not\asymp a \Leftarrow \bigsqcap^{\mathfrak{A}}_{i \in n} L_i \not\asymp \mathcal{A} \Leftrightarrow L \in \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$.

So $L \in \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Rightarrow L \in \mathrm{GR}\, f$. Thus $f \sqsupseteq \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$.

Then use the fact that $\mathrm{ID}^{\mathsf{Strd}}_{a[n]} = a^n_{\mathsf{Strd}}$. □

Proposition 76. $\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \bigsqcup \left\{\mathrm{id}^{\mathsf{Strd}}_{a[n]} \mid a \in \operatorname{atoms} \mathcal{A}\right\}$ where the meet may be taken on every of the following posets: anchored relations, staroids, provided that $\mathcal{A}$ is a filter on a set.

Proof. $\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \sqsupseteq \mathrm{id}^{\mathsf{Strd}}_{a[n]}$ for every $a \in \operatorname{atoms} \mathcal{A}$ is obvious.

Let $f \sqsupseteq \mathrm{id}^{\mathsf{Strd}}_{a[n]}$ for every $a \in \operatorname{atoms} \mathcal{A}$. Then $\forall L \in \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{a[n]} \colon L \in \mathrm{GR}\, f$ that is

$$\forall L \in \operatorname{form} f \colon \left(\bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \not\asymp a \Rightarrow L \in \mathrm{GR}\, f\right).$$

But $\exists a \in \operatorname{atoms} \mathcal{A} \colon \bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \not\asymp a \Leftarrow \bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \not\asymp \mathcal{A} \Leftrightarrow L \in \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$.

So $L \in \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Rightarrow L \in \mathrm{GR}\, f$. Thus $f \sqsupseteq \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. □

7 Finite case

Theorem 77. Let n be a finite set.

1. $\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $\mathfrak{A}$ and $\mathfrak{Z}$ are meet-semilattices and $(\mathfrak{A}; \mathfrak{Z})$ is a finitely meet-closed filtrator.

2. $\mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} = \upuparrows \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ if $(\mathfrak{A}; \mathfrak{Z})$ is a primary filtrator over a distributive lattice.

Proof.

1. $L \in \mathrm{GR} \downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow L \in \mathrm{GR}\, \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \mathrm{MEET}(\{L_i \mid i \in n\} \cup \{\mathcal{A}\}) \Leftrightarrow \bigsqcap^{\mathfrak{A}}_{i \in n} L_i \sqcap \mathcal{A} \neq 0 \Leftrightarrow$ (by finiteness)$\Leftrightarrow \bigsqcap^{\mathfrak{Z}}_{i \in n} L_i \sqcap \mathcal{A} \neq 0 \Leftrightarrow L \in \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ for every $L \in \prod \mathfrak{Z}$.

2. $L \in \mathrm{GR} \upuparrows \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \mathrm{up}\, L \subseteq \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \forall K \in \mathrm{up}\, L\colon K \in \mathrm{GR}\, \mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]} \Leftrightarrow \forall K \in \mathrm{up}\, L\colon \bigsqcap^{\mathfrak{Z}}_{i \in n} K_i \in \partial \mathcal{A} \Leftrightarrow \forall K \in \mathrm{up}\, L\colon \bigsqcap^{\mathfrak{Z}}_{i \in n} K_i \not\asymp \mathcal{A} \Leftrightarrow$ (by finiteness and theorem 4.44??)$\Leftrightarrow \forall K \in \mathrm{up}\, L\colon \bigsqcap^{\mathfrak{A}}_{i \in n} K_i \not\asymp \mathcal{A} \Leftrightarrow \mathcal{A} \in \bigcap \langle \star \rangle \{\bigsqcap^{\mathfrak{A}}_{i \in n} K_i \mid K \in \mathrm{up}\, L\} \Leftrightarrow$ (by the formula for finite meet of filters, theorem 4.111??)$\Leftrightarrow \mathcal{A} \in \bigcap \langle \star \rangle \bigsqcap^{\mathfrak{A}}_{i \in n} L_i \Leftrightarrow \forall K \in \bigsqcap^{\mathfrak{A}}_{i \in n} L_i\colon \mathcal{A} \in \star K \Leftrightarrow \forall K \in \bigsqcap^{\mathfrak{A}}_{i \in n} L_i\colon \mathcal{A} \not\asymp K \Leftrightarrow$ (by separability of core, theorem 4.112??)$\Leftrightarrow \bigsqcap^{\mathfrak{A}}_{i \in n} L_i \not\asymp \mathcal{A} \Leftrightarrow L \in \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$. □

Proposition 78. Let $(\mathfrak{A}; \mathfrak{Z})$ be a finitely meet closed filtrator. $\downdownarrows \mathrm{ID}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ and $\mathrm{id}^{\mathsf{Strd}}_{\mathcal{A}[n]}$ are the same for finite n.

Proof. Because $\bigsqcap^{\mathfrak{Z}}_{i \in \mathrm{dom}\, L} L_i = \bigsqcap^{\mathfrak{A}}_{i \in \mathrm{dom}\, L} L_i$ for finitary L. □

8 Counter-examples and conjectures

The following example shows that the theorem 33 can't be strenghtened:

Example 79. For some multifuncoid f on powersets complete in argument k the following formula is false:
$\langle f \rangle_l (L \cup \{(k; \bigsqcup X)\}) = \bigsqcup_{x \in X} \langle f \rangle_l (L \cup \{(k; x)\})$ for every $X \in \mathscr{P}\mathfrak{P}_k$, $L \in \prod_{i \in (\mathrm{arity}\, f) \setminus \{k, l\}} \mathfrak{F}_i$.

Proof. Consider multifuncoid $f = \Lambda \mathrm{id}^{\mathsf{Strd}}_{\uparrow U[3]}$ where U is an infinite set (of the form $\mathfrak{P}^3$) and $L = (Y)$ where Y is a nonprincipal filter on U.

$\langle f \rangle_0 (L \cup \{(k; \bigsqcup X)\}) = Y \sqcap \bigsqcup X$;

$\bigsqcup_{x \in X} \langle f \rangle_0 (L \cup \{(k; x)\}) = \bigsqcup_{x \in X} (Y \sqcap x)$.

It can be $Y \sqcap \bigsqcup X = \bigsqcup_{x \in X} (Y \sqcap x)$ only if Y is principal: Really: $Y \sqcap \bigsqcup X = \bigsqcup_{x \in X} (Y \sqcap x)$ implies $Y \not\asymp \bigsqcup X \Rightarrow \bigsqcup_{x \in X} (Y \sqcap x) \neq 0 \Rightarrow \exists x \in X\colon Y \not\asymp x$ and thus Y is principal. But we claimed above that it is nonprincipal. □

Example 80. There exists a staroid f and an indexed family X of principal filters (with arity $f = \mathrm{dom}\, X$ and $(\mathrm{form}\, f)_i = \mathrm{Base}(X_i)$ for every $i \in \mathrm{arity}\, f$), such that $f \sqsubseteq \prod^{\mathsf{Strd}} X$ and $Y \sqcap X \notin \mathrm{GR}\, f$ for some $Y \in \mathrm{GR}\, f$.

Remark 81. Such examples obviously do not exist if both f is a principal staroid and X and Y are indexed families of principal filters (because for powerset algebras staroidal product is equivalent to Cartesian product). This makes the above example inspired.

Proof. (Monroe Eskew) Let a be any (trivial or nontrivial) ultrafilter on an infinite set U. Let $A, B \in a$ be such that $A \cap B \subset A, B$. In other words, A, B are arbitrary nonempty sets such that $\emptyset \neq A \cap B \subset A, B$ and a be an ultrafilter on $A \cap B$.

Let f be the staroid whose graph consists of functions $p\colon U \to a$ such that either $p(n) \supseteq A$ for all but finitely many n or $p(n) \supseteq B$ for all but finitely many n. Let's prove f is really a staroid.

It's obvious $px \neq \emptyset$ for every $x \in U$. Let $k \in U$, $L \in a^{U \setminus \{k\}}$. It is enough (taking symmetry into account) to prove that

$$L \cup \{(k; x \sqcup y)\} \in \mathrm{GR}\, f \Leftrightarrow L \cup \{(k; x)\} \in \mathrm{GR}\, f \vee L \cup \{(k; y)\} \in \mathrm{GR}\, f. \quad (1)$$

Really, $L \cup \{(k; x \sqcup y)\} \in \mathrm{GR}\, f$ iff $x \sqcup y \in a$ and $L(n) \supseteq A$ for all but finitely many n or $L(n) \supseteq B$ for all but finitely many n; $L \cup \{(k; x)\} \in \mathrm{GR}\, f$ iff $x \in a$ and $L(n) \supseteq A$ for all but finitely many n or $L(n) \supseteq B$; and similarly for y.

But $x \sqcup y \in a \Leftrightarrow x \in a \vee y \in a$ because a is an ultrafilter. So, the formula (1) holds, and we have proved that f is really a staroid.

Take X be the constant function with value A and Y be the constant function with value B.

$\forall p \in \mathrm{GR}\, f : p \not\asymp X$ because $p_i \cap X_i \in a$; so $\mathrm{GR}\, f \subseteq \mathrm{GR} \prod^{\mathsf{Strd}} X$ that is $f \sqsubseteq \prod^{\mathsf{Strd}} X$.

Finally, $Y \sqcap X \notin \mathrm{GR}\, f$ because $X \sqcap Y = \lambda i \in U : A \cap B$. □

Some conjectures similar to the above example:

Conjecture 82. There exists a completary staroid f and an indexed family X of principal filters (with arity $f = \mathrm{dom}\, X$ and $(\mathrm{form}\, f)_i = \mathrm{Base}(X_i)$ for every $i \in \mathrm{arity}\, f$), such that $f \sqsubseteq \prod^{\mathsf{Strd}} X$ and $Y \sqcap X \notin \mathrm{GR}\, f$ for some $Y \in \mathrm{GR}\, f$.

Conjecture 83. There exists a staroid f and an indexed family x of ultrafilters (with arity $f = \mathrm{dom}\, x$ and $(\mathrm{form}\, f)_i = \mathrm{Base}(x_i)$ for every $i \in \mathrm{arity}\, f$), such that $f \sqsubseteq \prod^{\mathsf{Strd}} x$ and $Y \sqcap x \notin \mathrm{GR}\, f$ for some $Y \in \mathrm{GR}\, f$.

Other conjectures:

Conjecture 84. If staroid $0 \neq f \sqsubseteq a^n_{\mathsf{Strd}}$ for an ultrafilter a and an index set n, then $n \times \{a\} \in \mathrm{GR}\, f$. (Can it be generalized for arbitrary staroidal products?)

Conjecture 85. The following posets are atomic:

1. anchored relations on powersets;
2. staroids on powersets;
3. completary staroids on powersets.

Conjecture 86. The following posets are atomistic:

1. anchored relations on powersets;
2. staroids on powersets;
3. completary staroids on powersets.

The above conjectures seem difficult, because we know almost nothing about structure of atomic staroids.

Conjecture 87. A staroid on powersets is principal iff it is complete in every argument.

Conjecture 88. If a is an ultrafilter, then $\mathrm{id}^{\mathsf{Strd}}_{a[n]}$ is an atom of the lattice of:

1. anchored relations of the form $(\mathscr{P}\mathrm{Base}(a))^n$;
2. staroids of the form $(\mathscr{P}\mathrm{Base}(a))^n$;
3. completary staroids of the form $(\mathscr{P}\mathrm{Base}(a))^n$.

Conjecture 89. If a is an ultrafilter, then $\upuparrows \mathrm{id}^{\mathsf{Strd}}_{a[n]}$ is an atom of the lattice of:

1. anchored relations of the form $\mathfrak{F}(\mathrm{Base}(a))^n$;
2. staroids of the form $\mathfrak{F}(\mathrm{Base}(a))^n$;
3. completary staroids of the form $\mathfrak{F}(\mathrm{Base}(a))^n$.

Informal problem: Formulate and prove associativity of staroidal product.

Bibliography

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