Set Theoretic Filters*

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Abstract

Considered basic properties of set-theoretic filters, the lattice of filters.

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1 Draft status

It is a very preliminary draft.

This text has suffered translation from an other language with simultaneous rewriting in an other file format and other changes, and it not yet properly edited.

2 Some concepts and definitions

Let $\mho$ is some set and $\mathscr{P}\mho$ is the set of its subsets.

I will denote the complement of a set A as $\overline{A}$ or as $\neg A$. For a set A we have $\overline{A} = \mho \setminus A$ but if A is a set of sets then $\overline{A} = \mathscr{P}\mho \setminus A$. (It should be clear from context to which set complement is meant.)

I will denote $\langle f \rangle X = \{ f x \mid x \in X \}$.

Definition 1. A *chain* is a subset of a partially ordered set any whose two elements are comparable.

3 Lattice of filters

Definition 2. Let's call *filter* (on the lattice $\mathscr{P}\mho$) object $\mathcal{A}$ defined by a nonempty set[1] up $\mathcal{A}$ (from the word *upper*) such that[2] for any $A, B \in \mathscr{P}\mho$

1. $A, B \in \operatorname{up} \mathcal{A} \Rightarrow A \cap B \in \operatorname{up} \mathcal{A}$;

*. This document has been written using the GNU $\TeX_{\text{MACS}}$ text editor (see www.texmacs.org).

1. This set may contain $\emptyset$, unlike definition of filter in some other works.

2. $A \in \operatorname{up} \mathcal{A} \wedge B \supseteq A \Rightarrow B \in \operatorname{up} \mathcal{A}$.

Let's denote the set of all filters as $\mathscr{F}$.

Obvious 3. The above axioms can be replaced with one axiom:

$$A, B \in \operatorname{up} \mathcal{A} \Leftrightarrow A \cap B \in \operatorname{up} \mathcal{A}.$$

Definition 4. For a filter $\mathcal{A}$ we will denote $\operatorname{pu} \mathcal{A} \stackrel{\text{def}}{=} \langle \neg \rangle \overline{\operatorname{up} \mathcal{A}}$.

Obvious 5. $\operatorname{pu} \mathcal{A} = \overline{\langle \neg \rangle \operatorname{up} \mathcal{A}}$.

So up and pu are dual in some sense. It's simple to see that the definition of filters can be equivalently reformulated as follows:

Definition 6. Let's call *filter* (on the lattice $\mathscr{P}\mho$) object $\mathcal{A}$ defined by a set $\operatorname{pu} \mathcal{A}$ of nonempty sets (from the word *upper*) such that for any $A, B \in \mathscr{P}\mho$

$$A \cup B \in \operatorname{pu} \mathcal{A} \Leftrightarrow A \in \operatorname{pu} \mathcal{A} \vee B \in \operatorname{pu} \mathcal{A}.$$

Obvious 7. This axiom is equivalent to the following two axioms:

1. $A \cup B \in \operatorname{pu} \mathcal{A} \Rightarrow A \in \operatorname{pu} \mathcal{A} \vee B \in \operatorname{pu} \mathcal{A}$;
2. $A \in \operatorname{pu} \mathcal{A} \wedge B \supseteq A \Rightarrow B \in \operatorname{pu} \mathcal{A}$.

3.1 Subfilters

Definition 8. $\mathcal{A} \subseteq \mathcal{B} \stackrel{\text{def}}{=} \operatorname{up} \mathcal{A} \supseteq \operatorname{up} \mathcal{B}$.[3]

Obvious 9. $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \operatorname{pu} \mathcal{A} \subseteq \operatorname{pu} \mathcal{B}$.

Definition 10. $\mathcal{A} \in \mathscr{F}$, $B \in \mho$, and $\mathcal{A} = \{X \in \mathscr{P}\mho \mid X \supseteq B\}$, then we will count $\mathcal{A} = B$.[4]

Obviously this definition does not contradict to the previous one.

Obvious 11. $\operatorname{up} \emptyset = \mathscr{P}\mho$, $\operatorname{up} \mho = \{\mho\}$, $\operatorname{pu} \emptyset = \emptyset$, $\operatorname{pu} \mho = \mathscr{P}\mho \setminus \{\emptyset\}$.

Obvious 12. $\mathscr{F}$ is a POS (partially ordered set) with minimal and maximal elements correspondingly $\emptyset$ and $\mho$.

Lemma 13. For any $A \in \mathscr{P}\mho$, $\mathcal{F} \in \mathscr{F}$ is true $A \cap \mathcal{F} \neq \emptyset \Leftrightarrow \overline{A} \nsupseteq \mathcal{F}$.

Proof.

$$A \cap \mathcal{F} \neq \emptyset \Leftrightarrow \forall K \in \operatorname{up} \mathcal{F} \colon A \cap K \neq \emptyset \Leftrightarrow \forall K \in \operatorname{up} \mathcal{F} \colon \overline{A} \nsupseteq K \Leftrightarrow \overline{A} \nsupseteq \mathcal{F}. \quad \square$$

Theorem 14. For any filter $\mathcal{A}$

1. $\operatorname{up} \mathcal{A} = \{K \in \mathscr{P}\mho \mid K \supseteq \mathcal{A}\}$;
2. $\operatorname{pu} \mathcal{A} = \{K \in \mathscr{P}\mho \mid K \cap \mathcal{A} \neq \emptyset\}$.

Proof. By items:

1. $B \in \operatorname{up} \mathcal{A} \Leftrightarrow \operatorname{up} B \subseteq \operatorname{up} \mathcal{A} \Leftrightarrow B \supseteq \mathcal{A}$.

2. Traditionally filter $\mathcal{A}$ is considered equal with up $\mathcal{A}$ (and the notation up $\mathcal{A}$ is not introduced). But this could cause a contradiction with the further. From the further is seen that filters are convenient to be considered as abstract objects without a definite structure.

3. This order is the reverse of the conventional order on the lattice of filters.

4. Filters with such the property are called *principal* filters, but equating principal filters with a set removes the necessity for a special term for such filters.

2. From the previous item taking in account the last lemma. □

Corollary 15. $\operatorname{up}\mathcal{A} \subseteq \operatorname{pu}\mathcal{A}$ for any filter $\mathcal{A} \neq \emptyset$.

Proof. Obvious. □

3.2 Operations $\bigcup$ and $\bigcap$

Theorem 16. $\mathcal{A} = \bigcap^{\mathscr{F}} \operatorname{up}\mathcal{A}$ for any filter $\mathcal{A}$.

Proof. $\mathcal{A} \subseteq \bigcap^{\mathscr{F}} \operatorname{up}\mathcal{A}$ because $\operatorname{up}\mathcal{A} = \{K \in \mathscr{P}\mho \mid K \supseteq \mathcal{A}\}$. Reversely, if $\mathcal{B} \in \mathscr{F}$ then $\operatorname{up}\mathcal{A} \subseteq \{K \in \mathscr{P}\mho \mid K \supseteq \mathcal{B}\} \Rightarrow \mathcal{A} \supseteq \mathcal{B}$. □

Theorem 17. Let $A, B \in \mathscr{P}\mho$, $S \in \mathscr{P}\mathscr{P}\mho$. Then

1. $\bigcup^{\mathscr{F}} S = \bigcup S$;
2. $A \cap^{\mathscr{F}} B = A \cap B$.

Proof. By items:

1. It is enough to prove that $\forall K \in S: K \subseteq \mathcal{F} \Rightarrow \mathcal{F} \supseteq \bigcup S$ for any $\mathcal{F} \in \mathscr{F}$. It is really so: If $C \in \operatorname{up}\mathcal{F}$ then $C \supseteq \mathcal{F}$, $\forall K \in S: K \subseteq C$, $C \supseteq \bigcup S$, $C \in \operatorname{up}\bigcup S$, consequently $\mathcal{F} \supseteq \bigcup S$.
2. It's enough to prove that $\mathcal{F} \subseteq A, B \Rightarrow \mathcal{F} \subseteq A \cap B$ for any $\mathcal{F} \in \mathscr{F}$. Let $A, B \in \operatorname{up}\mathcal{F}$; then $A \cap B \in \operatorname{up}\mathcal{F}$; $\mathcal{F} \subseteq A \cap B$. □

This theorem allows in many cases skip indices with the operations $\cap$, $\cup$, $\bigcap$, and $\bigcup$, and we will skip.

Theorem 18. $\mathscr{F}$ is a full lattice and for $S \in \mathscr{P}\mathscr{F}$, $\mathcal{F}_0, \ldots, \mathcal{F}_{n-1} \in \mathscr{F}$ $(n \in \mathbb{N})$

1. $\operatorname{pu}\bigcup S = \bigcup \langle\operatorname{pu}\rangle S$;
2. $\operatorname{up}\bigcup S = \bigcap \langle\operatorname{up}\rangle S$;
3. $\operatorname{up}\bigcap^{\mathscr{F}} S = \{K_0 \cap \ldots \cap K_n \mid K_i \in \langle\operatorname{up}\rangle S \text{ where } i = 0, \ldots, n \text{ for } n \in \mathbb{N}\}$, particularly

$$\operatorname{up}(\mathcal{F}_0 \cap \ldots \cap \mathcal{F}_n) = \{K_0 \cap \ldots \cap K_n \mid K_i \in \mathcal{F}_i, i = 0, \ldots, n-1\}.$$

Proof. That $\mathscr{F}$ is a full lattice follows from proofs of the items 2 and 3 and the definition of filter.

1. Because POSes $\mathscr{F}$ and $\mathscr{P}\mho$ are homomorphic regarding pu then it's enough to prove that exist such $\mathcal{F} \in \mathscr{F}$ that $\operatorname{pu}\mathcal{F} = \bigcup \langle\operatorname{pu}\rangle S$. But trueness for the right part of the axioms for pu is obvious.
2. Let's denote the right part of the equality as R. Analogously it is enough to prove that exists such $\mathcal{F} \in \mathscr{F}$ that $\operatorname{up}\mathcal{F} = R$. Let's prove it: Let $A, B \in R$, $C \in \operatorname{up}A$. Then

$$\forall K \in \langle\operatorname{up}\rangle S: A, B \in K;$$

   Consequently $\forall K \in \langle\operatorname{up}\rangle S: (A \cap B \in K \wedge C \in K)$. So for R are fulfilled axioms for up of a filter.
3. First consider the general case. Let's denote the right part of the equality to prove as R. Let $A, B \in R$, $C \in \operatorname{up}A$. Then after choosing corresponding indices,

$$A = A_0 \cap \ldots \cap A_{m-1}, \quad B = B_l \cap \ldots \cap B_{n-1},$$

   where $l, m, n \in \mathbb{N}$, $0 < l < m < n$, $A_i \in S_i$, $B_j \in S_j$, $S_k \in \langle\operatorname{up}\rangle S$ when $i \in \{0, \ldots, m-1\}$, $j \in \{l, \ldots, n-1\}$, $k \in \{0, \ldots, n-1\}$.

$$A \cap B = A_0 \cap \ldots \cap A_{k-1} \cap (A_k \cap B_k) \cap \ldots \cap (A_{m-1} \cap B_{m-1}) \cap B_m \cap \ldots \cap B_{n-1},$$

consequently $A\cap B\in R$. $C=A\cup C=(A_0\cup C)\cap\ldots\cap(A_{m-1}\cup C)\in R$.

So $\exists\mathcal{R}\in\mathscr{F}:\operatorname{up}\mathcal{R}=R$.

On the one hand if $\mathcal{A}\in S$ then $\operatorname{up}\mathcal{A}\in\langle\operatorname{up}\rangle S$ from which as easy to see $\operatorname{up}\mathcal{A}\subseteq R$, $\mathcal{A}\supseteq\mathcal{R}$. On the other hand if $\mathcal{B}\in\mathscr{F}$ and $\forall\mathcal{A}\in S:\mathcal{A}\supseteq\mathcal{B}$ then $\forall\mathcal{A}\in S:\operatorname{up}\mathcal{B}\supseteq\operatorname{up}\mathcal{A}$; from what $\operatorname{up}\mathcal{B}\supseteq T$, $\operatorname{up}\mathcal{B}\ni\bigcap T$ for any $T=\{T_1,\ldots,T_r\}$ where $T_i\in\bigcup\langle\operatorname{up}\rangle S$; consequently $\operatorname{up}\mathcal{B}\supseteq R$, $\mathcal{B}\subseteq\mathcal{R}$. Comparing this we get what is to prove.

The formula for the finite case can be inferred from the formula for the general case taking in account that $\forall\mathcal{F}\in\mathscr{F}:\mho\in\operatorname{up}\mathcal{F}$. □

Theorem 19. If $S\in\mathscr{P}\mathscr{F}$, $\mathcal{A},\mathcal{B},\mathcal{C}\in\mathscr{F}$ then

1. $\mathcal{A}\cup\bigcap^{\mathscr{F}}S=\bigcap^{\mathscr{F}}\langle\mathcal{A}\cup\rangle S$;
2. $\mathcal{A}\cap(\mathcal{B}\cup\mathcal{C})=\mathcal{A}\cap\mathcal{B}\cup\mathcal{A}\cap\mathcal{C}$.

Proof. By items:

1. Taking in account the previous theorem, we have:

$$\begin{aligned}\operatorname{up}\Big(\mathcal{A}\cup\bigcap^{\mathscr{F}}S\Big) &= \operatorname{up}\mathcal{A}\cap\operatorname{up}\bigcap^{\mathscr{F}}S\\ &= \operatorname{up}\mathcal{A}\cap\{K_0\cap\ldots\cap K_n\mid K_i\in\langle\operatorname{up}\rangle S\text{ where }i=0,\ldots,n\text{ for }n\in\mathbb{N}\}\\ &= \{\operatorname{up}\mathcal{A}\cap K_0\cap\ldots\cap K_n\mid K_i\in\langle\operatorname{up}\rangle S\text{ where }i=0,\ldots,n\text{ for }n\in\mathbb{N}\}\\ &= \{K_0\cap\ldots\cap K_n\mid K_i\in\{\operatorname{up}\mathcal{A}\cap\operatorname{up}\mathcal{X}\mid\mathcal{X}\in S\},i=0,\ldots,n\text{ for }n\in\mathbb{N}\}\\ &= \{K_0\cap\ldots\cap K_n\mid K_i\in\{\operatorname{up}(\mathcal{A}\cup\mathcal{X})\mid\mathcal{X}\in S\}\text{ where }i=0,\ldots,n\text{ for }n\in\mathbb{N}\}\\ &= \operatorname{up}\bigcap^{\mathscr{F}}\{\mathcal{A}\cup\mathcal{X}\mid\mathcal{X}\in S\}.\end{aligned}$$

2. Obviously $\mathcal{A}\cap(\mathcal{B}\cup\mathcal{C})\supseteq\mathcal{A}\cap\mathcal{B}\cup\mathcal{A}\cap\mathcal{C}$.

   If $K\in\operatorname{up}(\mathcal{A}\cap\mathcal{B}\cup\mathcal{A}\cap\mathcal{C})$, then $K\in\operatorname{up}(\mathcal{A}\cap\mathcal{B})$, $K\in\operatorname{up}(\mathcal{A}\cap\mathcal{C})$, exist $A_0,A_1\in\operatorname{up}\mathcal{A}$, $B\in\operatorname{up}\mathcal{B}$, $C\in\operatorname{up}\mathcal{C}$ such that $K=A_0\cap B=A_1\cap C$; $K\supseteq A\cap B$, $A\cap C$ where $A=A_0\cap A_1\in\operatorname{up}\mathcal{A}$; $K\supseteq A\cap(B\cup C)\supseteq\mathcal{A}\cap(\mathcal{B}\cup\mathcal{C})$. □

Theorem 20. $A\cap\bigcup S=\bigcup\langle A\cap\rangle S$ for any $A\in\mathscr{P}\mho$, $S\in\mathscr{P}\mathscr{F}$.

Proof. Obviously $A\cap\bigcup S\supseteq\bigcup\langle A\cap\rangle S$. Let $K\in\operatorname{up}\bigcup\langle A\cap\rangle S$. Then $\forall\mathcal{F}\in S:K\in\operatorname{up}(A\cap\mathcal{F})$, $\forall\mathcal{F}\in S:K\cap A\in\langle A\cap\rangle\operatorname{up}(A\cap\mathcal{F})$. But $\langle A\cap\rangle\operatorname{up}(A\cap\mathcal{F})=\langle A\cap\rangle\operatorname{up}\mathcal{F}$, consequently $\forall\mathcal{F}\in S:K\cap A\in\langle A\cap\rangle\operatorname{up}\mathcal{F}$, $\forall\mathcal{F}\in S\exists L\in\operatorname{up}\mathcal{F}:K\cap A=L\cap A$, $\exists L\in\operatorname{up}\bigcup S:K\cap A=L\cap A$, $K\cap A\in\langle A\cap\rangle\operatorname{up}\bigcup S$, $K\cap A\in\operatorname{up}(A\cap\bigcup S)$, $K\in\operatorname{up}(A\cap\bigcup S)$. □

Definition 21. I will call *separable lattice*[5] such a lattice $\mathfrak{A}$ that:

1. It has the minimal element $0\in\mathfrak{A}$.
2. $\forall a,b\in\mathfrak{A}:(\{c\in\mathfrak{A}\mid c\cap a\neq0\}=\{c\in\mathfrak{A}\mid c\cap b\neq0\}\Rightarrow a=b)$.

Theorem 22. For a lattice $\mathfrak{A}$ with a minimal element 0 the following statements are equivalent:

1. $\mathfrak{A}$ is separable.
2. $\forall a,b\in\mathfrak{A}:(\{c\in\mathfrak{A}\mid c\cap a\neq0\}\subseteq\{c\in\mathfrak{A}\mid c\cap b\neq0\}\Rightarrow a\subseteq b)$.
3. $\forall a,b\in\mathfrak{A}:(a\subset b\Rightarrow\{c\in\mathfrak{A}\mid c\cap a\neq0\}\subset\{c\in\mathfrak{A}\mid c\cap b\neq0\})$.

Proof.

(1) ⇒ (3). Let $a<b$. Then $a\neq b$ from what

$$\{c\in\mathfrak{A}\mid c\cap a\neq0\}\neq\{c\in\mathfrak{A}\mid c\cap b\neq0\},$$

and consequently $\{c\in\mathfrak{A}\mid c\cap a\neq0\}\subset\{c\in\mathfrak{A}\mid c\cap b\neq0\}$.

(2) ⇒ (1). Let $\{c\in\mathfrak{A}\mid c\cap a\neq0\}=\{c\in\mathfrak{A}\mid c\cap b\neq0\}$. Then $a\subseteq b\wedge b\subseteq a$ and consequently $a=b$.

5. See `http://planetmath.org/?op=getobj;from=requests;id=685` about the term "separable lattice".

(3) ⇒ (2). Let $a \nsubseteq b$. Then $a \supset a \cap b$,

$$\{c \in \mathfrak{A} \mid c \cap a \neq 0\} \supset \{c \in \mathfrak{A} \mid c \cap a \cap b \neq 0\}.$$

So exists $c \in \mathfrak{A}$ such that $c \cap a \neq 0 \wedge c \cap a \cap b = 0$. From this $(c \cap a) \cap a \neq 0$, $(c \cap a) \cap b = 0$;

$$\{c \in \mathfrak{A} \mid c \cap a \neq 0\} \nsubseteq \{c \in \mathfrak{A} \mid c \cap b \neq 0\}.$$ □

[??TODO: Remove the below theorem (and above?) as it is a subcase of pu?]

Theorem 23. The lattice $\mathscr{F}$ is separable.

Proof. Let $\mathcal{A} \subset \mathcal{B}$ for some $\mathcal{A}, \mathcal{B} \in \mathscr{F}$. Then $\operatorname{pu} \mathcal{A} \subset \operatorname{pu} \mathcal{B}$; exists K such that $\operatorname{pu} \mathcal{A} \not\ni K \in \operatorname{pu} \mathcal{B}$; $K \cap \mathcal{B} \neq \emptyset$, $(K \cap \mathcal{B}) \cap \mathcal{A} = \emptyset$, $K \cap \mathcal{B} \subseteq \mathcal{B}$. □

Definition 24. *Generalized filter base* is a set $S \in \mathscr{P}\mathscr{F} \setminus \{\emptyset\}$ such that

$$\forall \mathcal{A}, \mathcal{B} \in S \exists \mathcal{C} \in S : \mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}.$$

Obvious 25. A chain of filters is a generalized filter base.

Definition 26. A *filter base* is such generalized filter base S that $S \subseteq \mathscr{P}\mho$.

Definition 27. If S is a generalized filter base and $\mathcal{A} = \bigcap^{\mathscr{F}} S$, then we will call S a generalized base of filter $\mathcal{A}$.

Proposition 28. Let S is a generalized filter base. If $A_1, \ldots, A_n \in S$ $(n \in \mathbb{N})$, then

$$\exists \mathcal{C} \in S : \mathcal{C} \subseteq A_1 \cap \ldots \cap A_n.$$

Proof. Can be easily proved by induction. □

Theorem 29. If S is a generalized base of filter $\mathcal{F}$ then for any set K

$$K \in \operatorname{up} \mathcal{F} \Leftrightarrow \exists \mathcal{L} \in S : \mathcal{L} \subseteq K.$$

Proof.

⇐. Because $\mathcal{F} = \bigcap^{\mathscr{F}} S$.

⇒. Let $K \in \operatorname{up} \mathcal{F}$. Then exist $X_1, \ldots, X_n \in \bigcup \langle \operatorname{up} \rangle S$ such that $X_1 \cup \ldots \cup X_n = K$. Replacing every X_i with such $\mathcal{X}_i \in S$ that $X \in \operatorname{up} \mathcal{X}$ (this is obviously possible to do), we get a finite set $T_0 \subseteq S$ such that $\bigcap T_0 \subseteq K$. From this exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \bigcap T_0 \subseteq K$. □

Corollary 30. If S is a generalized base of filter $\mathcal{F}$ then $\emptyset \in S \Leftrightarrow \mathcal{F} = \emptyset$.

Proof. Substitute $\emptyset$ as K. □

3.3 Some criteria

Theorem 31. For any $\mathcal{F} \in \mathscr{F}$

$$\mathcal{F} \in \mathscr{P}\mho \Leftrightarrow \forall S \in \mathscr{P}\mathscr{P}\mho : \left(\bigcup S \in \operatorname{pu} \mathcal{F} \Rightarrow S \cap \operatorname{pu} \mathcal{F} \neq \emptyset \right).$$

Proof.

$$\begin{aligned}
&\forall S \in \mathscr{P}\mathscr{P}\mho : \left(\bigcup S \in \operatorname{pu} \mathcal{F} \Rightarrow S \cap \operatorname{pu} \mathcal{F} \neq \emptyset \right) \Leftrightarrow \\
&\forall S \in \mathscr{P}\mathscr{P}\mho : \left(\bigcup S \notin \operatorname{pu} \mathcal{F} \Leftarrow S \cap \operatorname{pu} \mathcal{F} = \emptyset \right) \Leftrightarrow \\
&\forall S \in \mathscr{P}\mathscr{P}\mho : \left(\overline{\bigcup S} \in \operatorname{up} \mathcal{F} \Leftarrow \langle \neg \rangle S \subseteq \operatorname{up} \mathcal{F} \right) \Leftrightarrow \\
&\forall S \in \mathscr{P}\mathscr{P}\mho : \left(\bigcap S \in \operatorname{up} \mathcal{F} \Leftarrow S \subseteq \operatorname{up} \mathcal{F} \right),
\end{aligned}$$

but

$$\mathcal{F} \in \mathscr{P}\mho \Rightarrow \forall S \in \mathscr{P}\mathscr{P}\mho : \left(\bigcap S \in \operatorname{up}\mathcal{F} \Leftarrow S \subseteq \operatorname{up}\mathcal{F} \right) \Rightarrow \bigcap \operatorname{up}\mathcal{F} \in \operatorname{up}\mathcal{F} \Rightarrow \mathcal{F} \in \mho. \quad \square$$

Theorem 32. For any $S \in \mathscr{P}\mathscr{F}$ the condition $\exists \mathcal{F} \in \mathscr{F} : S = \{\mathcal{K} \in \mathscr{F} \mid \mathcal{K} \cap \mathcal{F} \neq \emptyset\}$ is equivalent to conjunction of the following items:

1. $\emptyset \notin S$;
2. $\forall \mathcal{A}, \mathcal{B} \in \mathscr{F} : (\mathcal{A} \in S \wedge \mathcal{B} \supseteq \mathcal{A} \Rightarrow \mathcal{B} \in S)$;
3. $\forall \mathcal{A}, \mathcal{B} \in \mathscr{F} : (\mathcal{A} \cup \mathcal{B} \in S \Rightarrow \mathcal{A} \in S \vee \mathcal{B} \in S)$;
4. $T \subseteq S \Rightarrow \bigcap^{\mathscr{F}} T \in S$ for any chain of filters T.

Proof.

$\Rightarrow$. (1), (2), (3) are obvious. Let's prove (4). We have $T \subseteq S$ and need to prove that $\bigcap^{\mathscr{F}} T \cap \mathcal{F} \neq \emptyset$. Because $\langle \mathcal{F} \cap \rangle T$ is a chain and consequently a generalized filter base, $\emptyset \in \langle \mathcal{F} \cap \rangle T \Leftrightarrow \bigcap^{\mathscr{F}} \langle \mathcal{F} \cap \rangle T = \emptyset \Leftrightarrow \bigcap^{\mathscr{F}} T \cap \mathcal{F} = \emptyset$. The left to prove is $\emptyset \notin \langle \mathcal{F} \cap \rangle T$ what follows from $T \subseteq S$.

$\Leftarrow$. There exist $\mathcal{F} \in \mathscr{F}$ such that $\operatorname{pu}\mathcal{F} = S \cap \mathscr{P}\mho$. Let $\mathcal{X} \in \mathscr{F}$; then obviously exists a chain T of sets such that $\mathcal{X} = \bigcap^{\mathscr{F}} T$. Obviously $T \subseteq S \Leftrightarrow \mathcal{X} \in S$. Consequently

$$\mathcal{X} \in S \Leftrightarrow T \subseteq \operatorname{pu}\mathcal{F} \Leftrightarrow \emptyset \notin \langle \mathcal{F} \cap \rangle T \Leftrightarrow \bigcap^{\mathscr{F}} T \cap \mathcal{F} \neq 0 \Leftrightarrow \mathcal{X} \cap \mathcal{F} \neq \emptyset. \quad \square$$

Remark 33. The item (2) can be excluded with simultaneous replacing implication with equivalency in at least one of the items (3) or (4).

Proof. Obvious. $\square$

[TODO??: Can "chain" be replaced with "generalized filter base"?]

3.4 More interrelation of up and pu

Theorem 34. For any filter $\mathcal{A}$

1. $\operatorname{pu}\mathcal{A} = \{K \in \mathscr{P}\mho \mid \forall L \in \operatorname{up}\mathcal{A} : K \cap L \neq \emptyset\}$;
2. $\operatorname{up}\mathcal{A} = \{K \in \mathscr{P}\mho \mid \forall L \in \operatorname{pu}\mathcal{A} : K \cap L \neq \emptyset\}$.

Proof.

1. For any K we have

$$K \in \operatorname{pu}\mathcal{A} \Leftrightarrow K \cap \mathcal{A} \neq \emptyset \Leftrightarrow K \cap \bigcap^{\mathscr{F}} \operatorname{up}\mathcal{A} \neq \emptyset \Leftrightarrow \bigcap^{\mathscr{F}} \langle K \cap \rangle \operatorname{up}\mathcal{A} \neq \emptyset,$$

what taking account the collorary 30 is equivalent to $\forall L \in \operatorname{up}\mathcal{A} : K \cap L \neq \emptyset$.

2. If $K \in \operatorname{up}\mathcal{A}$, $L \in \operatorname{pu}\mathcal{A}$ then $K \supseteq \mathcal{A}$, $L \cap \mathcal{A} \neq \emptyset$, consequently $K \cap L \neq \emptyset$. Reversely $K \notin \operatorname{up}\mathcal{A} \Leftrightarrow \overline{K} \in \operatorname{pu}\mathcal{A}$, from what $\neg \forall L \in \operatorname{pu}\mathcal{A} : K \cap L \neq \emptyset$. $\square$

3.5 Difference and complement of filters

Let $\mathcal{A}, \mathcal{I} \in \mathscr{F}$.

Definition 35. Let's call a filter $\mathcal{A}$ *substractive* from the filter $\mathcal{I}$ iff exists a filter $\mathcal{C} = \mathcal{I} \setminus \mathcal{A}$ (*difference* of filters $\mathcal{I}$ and $\mathcal{A}$) such that $\mathcal{A} \cap \mathcal{C} = \emptyset$ and $\mathcal{I} \cup \mathcal{A} = \mathcal{A} \cup \mathcal{C}$.

There exists only one difference if any.

Definition 36. Let's call filter $\mathcal{A}$ complementive to the filter $\mathcal{I}$ iff there exists a filter $\overline{\mathcal{A}}$ such that $\mathcal{A} \cup \overline{\mathcal{A}} = \mathcal{I}$ and $\mathcal{A} \cap \overline{\mathcal{A}} = \emptyset$.

Obvious 37. $\mathcal{A}$ is complementive to $\mathcal{I}$ iff $\mathcal{A}$ is substractive from $\mathcal{I}$ and $\mathcal{A} \subseteq \mathcal{I}$.

Corollary 38. It exists only one complement if any.

Obvious 39. Sets are substractive from any filter.

Theorem 40. $\mathcal{A}$ is complementive to $\mathcal{I}$ iff $\exists B \in \mathscr{P}\mho: \mathcal{A} = B \cap \mathcal{I}$.

Proof. If $\mathcal{A}$ is complementive to $\mathcal{I}$ then $\mathcal{A} \cap \overline{\mathcal{A}} = \emptyset$, consequently exists $B \in \operatorname{up} \mathcal{A}$ such that $B \cap \overline{\mathcal{A}} = \emptyset$; we have $\mathcal{A} \subseteq B \cap \mathcal{I}$; but $(B \cap \mathcal{I}) \cap \overline{\mathcal{A}} = \emptyset$, consequently from oneness of complement $\mathcal{A} = B \cap \mathcal{I}$. Reversely, if $\mathcal{A} = B \cap \mathcal{I}$ then $\overline{B} \cap \mathcal{I}$ is the complement of $\mathcal{A}$ to $\mathcal{I}$. □

3.6 Atomic filters

Definition 41. Let's call *atomic filter*[6] an atom of the lattice $\mathscr{F}$. The set of all atomic subfilters of a filter $\mathcal{F}$ will be denoted as $\operatorname{atoms} \mathcal{F}$.

Theorem 42. A set is an atomic filter iff it is a one-element set.

Proof. In the reverse direction this theorem is obvious. If $A \in \mathscr{P}\mho$ isn't an atomic filter then exists such $\mathcal{X} \in \mathscr{F}$ that $\emptyset \neq \mathcal{X} \subset A$, that is $\emptyset \notin \operatorname{up} \mathcal{X} \supset \operatorname{up} A$, exists $K \in \operatorname{up} \mathcal{X}$ such that $\emptyset \neq K \not\supseteq A$; consequently (because $A \in \operatorname{up} \mathcal{X}$) $A \cap K \in \operatorname{up} \mathcal{X}$, $A \cap K \neq \emptyset$ and $A \cap K \subset A$; so A isn't atom of the lattice $\mathscr{P}\mho$. □

Definition 43. Let's call a *trivial* atomic filter a one-element set.

Theorem 44. Filter $\mathcal{X}$ is atomic iff $\operatorname{up} \mathcal{F} = \operatorname{pu} \mathcal{F}$.

Proof. If $\mathcal{X}$ is atomic then obviously $\operatorname{up} \mathcal{X} = \operatorname{pu} \mathcal{X}$. If $\operatorname{up} \mathcal{X} = \operatorname{pu} \mathcal{X}$ then $\mathcal{X} \neq \emptyset$ and consequently for any filter $\mathcal{F}$ have

$$\mathcal{F} \cap \mathcal{X} \neq \emptyset \Rightarrow \forall K \in \operatorname{up} \mathcal{F}: K \in \operatorname{pu} \mathcal{X} \Rightarrow \forall K \in \operatorname{up} \mathcal{F}: K \in \operatorname{up} \mathcal{X} \Rightarrow \mathcal{F} \supseteq \mathcal{X},$$

that is $\mathcal{X}$ is atomic filter. □

Theorem 45. $\mathscr{F}$ is an atomic lattice.

Proof. Let $\mathcal{F} \in \mathscr{F}$. Let choose (by Kuratowski's lemma) a chain S from $\emptyset$ to $\mathcal{F}$. Let $S' = S \setminus \{\emptyset\}$. $a = \bigcap^{\mathscr{F}} S' \neq \emptyset$ by the theorem 30. $a \in S$ because $\emptyset \subset a \subset \mathcal{X}$ for any $\mathcal{X} \in S$; consequently $a \in S'$. Obviously a is the minimal element of S'. Consequently there are no $\mathcal{Y} \in \mathscr{F}$ such that $\emptyset \subset \mathcal{Y} \subset a$. So a is an atomic filter. □

Theorem 46. $\mathscr{F}$ is an atomistic lattice.

Proof. Because $\mathscr{F}$ is atomic and separable. □

Corollary 47. For filters is true disjunct property of Wollman.

Proof. Because it is atomistic. □

Theorem 48. $\operatorname{atoms}(\mathcal{A} \cup \mathcal{B}) = \operatorname{atoms} \mathcal{A} \cup \operatorname{atoms} \mathcal{B}$ for any $\mathcal{A}, \mathcal{B} \in \mathscr{F}$.

Proof. $a \in \operatorname{atoms}(\mathcal{A} \cup \mathcal{B}) \Leftrightarrow a \cap (\mathcal{A} \cup \mathcal{B}) \neq \emptyset \Leftrightarrow a \cap \mathcal{A} \cup a \cap \mathcal{B} \neq \emptyset \Leftrightarrow a \cap \mathcal{A} \neq \emptyset \vee a \cap \mathcal{B} \neq \emptyset \Leftrightarrow a \in \operatorname{atoms} \mathcal{A} \vee a \in \operatorname{atoms} \mathcal{B}$ for any atomic filter a. □

6. Traditionally atomic filters are called *ultrafilters* or *maximal filters* but they are rather minimal than maximal in our order of the set of filters.

4 Pseudodifference of filters

[??Section skipped. Also skipped about infinitely small and set part of a filter.]

5 One more criterion

[??Currently present in the article about funcoids.]

6 Partitioning of a filter

Definition 49. A *tearing* of a filter $\mathcal{F}$ is such $S \in \mathscr{P}(\mathscr{F} \setminus \{\emptyset\})$ that $\bigcup S = \mathcal{F}$ and

$$\forall \mathcal{X}, \mathcal{Y} \in S: (\mathcal{X} \neq \mathcal{Y} \Rightarrow \mathcal{X} \cap \mathcal{Y} = \emptyset).$$

Definition 50. A *partition* of a filter $\mathcal{F}$ is such $S \in \mathscr{P}(\mathscr{F} \setminus \{\emptyset\})$ that $\bigcup S = \mathcal{F}$ and

$$\forall \mathcal{X} \in S, T \in \mathscr{P}S: (\mathcal{X} \notin T \Rightarrow \mathcal{X} \cap \bigcup T = \emptyset)$$

For sets the introduced concept of partition coincides with the conventional concept of partition of a set:

Theorem 51. S is a partition of a set F iff S is tearing of F and $S \subseteq \mathscr{P}\mho$.

Proof.

- $\Leftarrow$. Obvious.
- $\Rightarrow$. Let S is a partition of a set F. Then for any $\mathcal{X} \in S$ we have $\mathcal{X} \cap \bigcup (S \setminus \{X\}) = \emptyset$, and consequently $\mathcal{X} = \mathcal{F} \setminus \bigcup (S \setminus \{\mathcal{X}\}) \in \mathscr{P}\mho$. □

Theorem 52. If S is a partition of a filter, $T, U \in \mathscr{P}S$, then $T \cap U = \emptyset \Rightarrow \bigcup T \cap \bigcup U = \emptyset$.

Proof. [??Skipped in current version of the article.] □

Theorem 53. For any $T \in \mathscr{P}\mathscr{F}$ exists partition S of the filter $\bigcup T$ such that every element of S is a subfilter of some element of T.

Proof. [??Skipped in current version of the article.] □