Rewrite plan for my research monograph

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This is my plan to rewrite my draft research monograph “Algebraic General Topology. Volume 1”.


Say that the book intended mainly to represent the current status of my research.

Add thanks to Todd Trimble and Andreas Blass.

Shall I denote sets like $\frac{x \in A}{P(x)}$ instead of $\{x \in A\mid P(x)\}$?

Continuity of functions between metric spaces as generalized continuity.

1 Implications tuples

In the old version of my book there are multiple situations where there are both theorems of the form $B \Rightarrow C$ and $A \Rightarrow C$ where the $A \Rightarrow C$ is proved using that $A \Rightarrow B$ and $B \Rightarrow C$. It is accompanied with verbal explanation, that in these theorems $C$ is the most important component of the theorems and $B \Rightarrow C$ is a boring generalization of the “main” theorem $A \Rightarrow C$.

I decided to rewrite every occurrence of this case as single theorem $A \Rightarrow B \Rightarrow C$ with multiple implications. (There may be more than two implications, in general it is a $P_1 \Rightarrow \ldots \Rightarrow P_n$ implication.)

To describe such implications verbally, I define implications tuples.

Definition 1. An implications tuple is a tuple $(P_1, \ldots, P_n)$ such that $P_i \Rightarrow \ldots \Rightarrow P_n$.

Obvious 2. $(P_1, \ldots, P_n)$ is an implications tuple iff $P_i \Rightarrow P_j$ for every $i < j$ (where $i, j \in \{1, \ldots, n\}$).

The following is an example of a theorem using an implication tuple:

Example 3. The following is an implications tuple:

1. $A$;
2. $B$;
3. $C$.

This example means just that $A \Rightarrow B \Rightarrow C$.

I prefer here a verbal description instead of symbolic implications $A \Rightarrow B \Rightarrow C$, because $A$, $B$, $C$ may be long English phrases and they may not fit into the formula layout.

2 Theory of filters

I have “discovered” that there are four sets (including the set of filters itself) isomorphic to the set of filters on any poset.


As there are several isomorphic sets, it makes sense to describe it more generally than the special case of the set of filters.

To describe this I redefine primary filtrator (earlier defined as a filtrator $(\mathfrak{F}; \mathfrak{Z})$ where $\mathfrak{F}$ is the set of filters on a poset $\mathfrak{Z}$) in another (non-equivalent) way. [TODO: Are filtered filtrators the same as primary filtrators?]
Definition 4. A primary filtrator is such a filtrator \((\mathcal{A}; \mathcal{F})\) that \(\mathcal{A}\) is isomorphic to the set of filters on the poset \(\mathcal{F}\).

Definition 5. A primary filtrator over a poset \(\mathcal{F}\) is a primary filtrator of the form \((\mathcal{A}; \mathcal{F})\).

Define \(\mathcal{F}\) as the set of elements of an arbitrary primary filtrator. Call \(\mathcal{F}\) as filter objects.

Theorem 6. For every poset \(\mathcal{F}\) there exists a primary filtrator over \(\mathcal{F}\).

Proof. See [1].

Remark 7. The concept of a primary filtrator over a poset \(\mathcal{F}\) is a new incarnation of my old concept of filter objects (see [1]). I refused this idea (in the old version of the book) but now I returned it back. The main idea of filter objects is that the set of principal filters is equated with the base set. And now again \(\mathcal{F} \subseteq \mathcal{A}\) what essentially means that principal filters are element of the base set. However, this has a drawback. If \(S \in \mathcal{P}\mathcal{F}\) we need explicitly differentiate between \(\prod^{\mathcal{F}} S\) and \(\prod^{\mathcal{A}} S\).

What I called “complete free stars” could be better called principal free stars. The same is true for ideals and mixers. I should write explicit characterizations of principality for all four kinds of “filtrator objects”.

Define two isomorphic filtrators \((\mathcal{A}; \mathcal{F})\) and \((\mathcal{A}'; \mathcal{F}')\) for a shifted filtrator \((\mathcal{A}; \mathcal{F}; \uparrow)\). (BTW, define isomorphic filtrators.)

\(\mathcal{F}(\mathcal{A}) \rightarrow \mathcal{F}(\mathcal{P}\mathcal{A})\) to eliminate ambiguities. Filter in a set vs filter on a poset, not to make terminology confusion. (Say that it is not a standard terminology.)

3 2-staroids

Definition 8. 2-staroid is a binary relation \(\delta\) between two posets such that \(\{Y \in \mathcal{B} \mid \exists X \in \mathcal{A}: X \delta Y\}\) and \(\{X \in \mathcal{A} \mid \exists Y \in \mathcal{B}: X \delta Y\}\) are free stars. [TODO: Say that 2-staroids are a special case of staroids.]

Definition 9. \(\uparrow \downarrow \delta\) is the binary relation defined by the formula
\[\mathcal{X}(\uparrow \downarrow \delta) \mathcal{Y} \Leftrightarrow \forall X \in \text{up}\mathcal{X}, Y \in \text{up}\mathcal{Y}: X \delta Y.\]

Obvious 10. Provided that \(\mathcal{A}\) and \(\mathcal{B}\) are join-semilattice with least elements, \(\delta \in \mathcal{P}(\mathcal{A} \times \mathcal{B})\) is a 2-staroid iff
\[
\neg(I \delta 0), \quad I \sqcup J \delta K \Leftrightarrow I \delta K \lor J \delta K \quad (\text{for every } I, J \in \mathcal{A}, K \in \mathcal{B}),
\]
\[
\neg(0 \delta I), \quad K \delta I \sqcup J \Leftrightarrow K \delta I \lor K \delta J \quad (\text{for every } I, J \in \mathcal{B}, K \in \mathcal{A}).
\]

The following theorem was generalized below (for arbitrary distributive lattices rather than as boolean lattices):

Theorem 11. Fix distributive lattices \(\mathcal{A}\) and \(\mathcal{B}\). Let \(\lambda f \in \text{FCD}(\mathcal{A}; \mathcal{B})\): \(\langle f \rangle^+\) and \(\lambda f \in \text{FCD}(\mathcal{A}; \mathcal{B})\): \([f]^+\).

1. \(\lambda f\) is a bijection from the set \(\text{FCD}(A; B)\) to the set of functions \(\alpha \in \mathcal{G}(\mathcal{B})^\mathcal{A}\) that obey the conditions (for every \(I, J \in \mathcal{P}\mathcal{A}\))
\[\alpha 0 = 0^\mathcal{B}(B), \quad \alpha(I \sqcup J) = \alpha I \sqcup \alpha J.\]
For such \(\alpha\) it holds (for every \(\mathcal{X} \in \mathcal{G}(A)\))
\[\langle \lambda f^{-1} \alpha \rangle \mathcal{X} = \bigcap \langle \alpha \rangle \mathcal{X}.\]

2. \(\lambda f\) is a bijection from the set \(\text{FCD}(A; B)\) to the set of 2-staroids (denoted \(\delta\)).
For such δ it holds (for every \( \mathcal{X} \in \mathfrak{F}(A) \), \( \mathcal{Y} \in \mathfrak{F}(B) \))

\[
\mathcal{X} [L_{R^{-1}} \delta] \mathcal{Y} \leftrightarrow \mathcal{X} (\mathcal{Y} \mathcal{X} \delta). \tag{3}
\]

**Proof.** Note that \( \mathfrak{F}(\mathfrak{B}) \) and \( \mathfrak{F}(\mathfrak{A}) \) are also distributive lattices.

2. Define \( \alpha \in \mathfrak{F}(\mathfrak{B}) \) by the formula

\[
\partial(\alpha \mathcal{X}) = \{ Y \in \mathfrak{B} \mid \exists X \in \mathfrak{A} \colon X \delta Y \}
\]

for every \( \mathcal{X} \in \mathfrak{A} \). (It is properly defined because \( \{ Y \in \mathfrak{B} \mid \exists X \in \mathfrak{A} \colon X \delta Y \} \) is a free star.) Similarly define

\[
\partial(\beta \mathcal{X}) = \{ X \in \mathfrak{A} \mid \exists Y \in \mathfrak{B} \colon X \delta Y \}.
\]

Let's continue \( \alpha \) and \( \beta \) to \( \alpha' \in \mathfrak{F}(\mathfrak{B})^{\delta(\mathfrak{A})} \) and \( \beta' \in \mathfrak{F}(\mathfrak{A})^{\delta(\mathfrak{B})} \) by the formulas:

\[
\alpha' \mathcal{X} = \bigcap \langle \alpha \rangle^* \uparrow \mathcal{X} \quad \text{and} \quad \beta' \mathcal{Y} = \bigcap \langle \beta \rangle^* \uparrow \mathcal{Y}.
\]

\( \mathcal{Y} \cap \alpha' \mathcal{X} \neq 0^{\mathfrak{B}}(B) \) \( \iff \mathcal{Y} \cap \bigcap \langle \alpha \rangle^* \uparrow \mathcal{X} \neq 0^{\mathfrak{B}}(B) \). Let's prove that

\[
W = \langle \mathcal{Y} \cap \rangle \langle \alpha \rangle^* \uparrow \mathcal{X}
\]

is a generalized filter base: To prove it is enough to show that \( \langle \alpha \rangle^* \uparrow \mathcal{X} \) is a generalized filter base. If \( A, B \in \langle \alpha \rangle^* \uparrow \mathcal{X} \) then exist \( X_1, X_2 \in \uparrow \mathcal{X} \) such that \( A = \alpha X_1, B = \alpha X_2 \).

Then \( \alpha(X_1 \cap X_2) \in \langle \alpha \rangle^* \uparrow \mathcal{X} \). So \( \langle \alpha \rangle^* \uparrow \mathcal{X} \) is a generalized filter base and thus \( W \) is a generalized filter base.

By properties of generalized filter bases, \( \bigcap \langle \mathcal{Y} \cap \rangle \langle \alpha \rangle^* \uparrow \mathcal{X} \neq 0^{\mathfrak{B}}(B) \) is equivalent to

\[
\forall \mathcal{X} \in \uparrow \mathcal{X} \colon \mathcal{Y} \cap \alpha \mathcal{X} \neq 0^{\mathfrak{B}}(B),
\]

what is equivalent to \( \forall \mathcal{X} \in \mathcal{X}, Y \in \mathcal{Y} : \uparrow^{\mathcal{Y}} \cap \alpha \mathcal{X} \neq 0^{\mathfrak{B}}(B) \implies \forall \mathcal{X} \in \mathcal{X}, Y \in \mathcal{Y} : Y \in \partial(\alpha \mathcal{X}) \implies \forall \mathcal{X} \in \mathcal{X}, Y \in \mathcal{Y} : \mathcal{X} \delta Y \implies \mathcal{X} (\mathcal{Y} \mathcal{X} \delta). \)

Analogously \( \mathcal{X} \cap \beta' \mathcal{Y} \neq 0^{\mathfrak{A}}(A) \) \( \iff \mathcal{X} (\mathcal{Y} \mathcal{X} \delta) \).

1. Let define the relation \( \delta \in \mathfrak{F}(\mathfrak{A} \times \mathfrak{B}) \) by the formula \( \mathcal{X} \mathcal{Y} \delta \mathcal{X} \mathcal{Y} \iff \uparrow^{\mathcal{Y}} \cap \alpha \mathcal{X} \neq 0^{\mathfrak{B}}(B) \).

That \( (\uparrow^{\mathcal{Y}} \cap \alpha \mathcal{X} \neq 0^{\mathfrak{B}}(B)) \) is obvious (if \( 0^{\mathfrak{B}} \) and \( 0^{\mathfrak{A}} \) are defined). We have \( I \cup J \delta K \iff \uparrow^{B} \cap \alpha(I \cup J) \neq 0^{\mathfrak{B}}(B) \iff \uparrow^{B} \cap \alpha I \neq 0^{\mathfrak{B}}(B) \lor \uparrow^{B} \cap \alpha J \neq 0^{\mathfrak{B}}(B) \iff I \delta K \lor J \delta K \) and

\[
K \delta I \cup J \iff \uparrow^{B} \cap \alpha K \neq 0^{\mathfrak{B}}(B) \iff (\uparrow^{B} I \cup \uparrow^{B} J) \cap \alpha K \neq 0^{\mathfrak{B}}(B) \iff \uparrow^{B} I \cap \alpha K \neq 0^{\mathfrak{B}}(B) \lor \uparrow^{B} J \cap \alpha K \neq 0^{\mathfrak{B}}(B).
\]

That is the formulas (‘) are true.

Accordingly to the above there exists a funcoid \( f \) such that

\[
\mathcal{X} [f] \mathcal{Y} \mathcal{X} (\mathcal{Y} \mathcal{X} \delta).
\]

Need to generalize all theorems dependent on it.

### 4 Staroids and multifuncoids

Change notation \([f] \rightarrow [f]^*; \quad [\uparrow^{\mathcal{Y}} f] \rightarrow [f] \) (for staroids).

I assumed that upgrading a staroid is a staroid without proof. Fill this hole. (Definition 17.69 for an example.) This is addressed in theorem 17.83.

\([\uparrow^{\mathcal{Y}} f] \rightarrow \bigcap_{X \in f} \mathcal{X} \) for a free staroid \( f \) on powersets. Can we use this formula instead of upgrading in the book?

Because the set of free stars is identified with the set of filters, the set of staroids (of a given form) can be identified with the set of multifuncoids on primary filtrators \((\mathfrak{A}, 5)\).
This allows to thoroughly revise the theory of staroids and multifuncoids.

The last chapter of my book ("Identity staroids") contains errors. I am going to rewrite it after switching to this new notation.

[TODO: Generalize "Funcoids are filters" for staroids (call it hyperfuncoids.)]

**Theorem 12.** \( \langle \| \| f \rangle_k A = \prod_{A \in \text{up} a} \langle f \rangle_k A \) for every multifuncoid \( f \) of the form \( \mathfrak{A} \) where \( k \in \text{arity} f \) and \( \mathfrak{A} \) is a poset of filter objects on a boolean lattice and \( a \) is an \( (\text{arity} f) \setminus \{k\} \) family of filters.

[TODO: Say exactly which family of filters is meant.]

**Proof.** \( X \neq \langle \| \| f \rangle_k A \iff a \cup \{ (k; X) \} \in \text{GR} \] \( \langle \| \| f \rangle \) \( \text{up}(a \cup \{ (k; X) \}) \subseteq \text{GR} \[ \iff \forall A \in \text{up} a, \ X \in \text{up} X: A \cup \{ (k; X) \} \in \text{GR} \] \[ \iff \forall A \in \text{up} a, \ X \in \text{up} X: X \neq \langle f \rangle_k A \iff (\text{because it is separable}) \] \( \forall A \in \text{up} a: X \neq \langle f \rangle_k A \iff (\text{by properties of generalized filter bases and the fact that } [f] \) \[ \text{is an upper set}) \iff X \neq \langle f \rangle_k A. \]

So \( \langle \| \| f \rangle_k A = \prod_{A \in \text{up} a} \langle f \rangle_k A \) because filters on boolean lattice are separable, \( \Box \)

**Example 13.** There is such anchored relation \( f \) that \( \| \| f \) is not a completary staroid. [TODO: Remove the conjecture about this.] [TODO: This also proves existence of non completary staroids (but not for powersets).]

**Proof.** (based on an Andreas Blass’s proof)

Take \( f \) the set of functions \( x: \mathbb{N} \rightarrow \mathbb{N} \) where \( x_0 \) an arbitrary natural number and \( x_i = \begin{cases} 0 & \text{if } n \leq x_0 \\ 1 & \text{if } n > x_0 \end{cases} \) for \( i = 1, 2, 3, \ldots \).

Let \( L_0(0) = L_1(0) = \Omega(N), \ L_0(i) = \uparrow \{0\} \) and \( L_1(i) = \uparrow \{1\} \) for \( i > 0 \).

Let \( X \in \text{up}(L_0 \cup L_1) \) that is \( X \in \text{up} L_0 \cap \text{up} L_1 \).

\( X \) contains all but finitely many elements of \( N \).

For \( i > 0 \) we have \( \{0, 1\} \subseteq X \).

Evidently, \( \prod X \) contains an element of \( f \).

Now consider any fixed \( c \in \{0, 1\}^N \). There is at most one \( k \in \mathbb{N} \) such that the sequence \( x = [k; c(1); c(2); \ldots] \) (i.e. \( c \) with \( c(0) \) replaced by \( k \)) is in \( f \). Let \( Q = N \setminus \{k\} \) if there is such a \( k \) and \( Q = N \) otherwise.

Take \( Y_i = \begin{cases} Q & \text{if } i = 0 \\ \{c(i)\} & \text{if } i > 0 \end{cases} \) for \( i = 1, 2, 3, \ldots \). We have \( Y \in \text{up}(\lambda i \in N: \mathcal{L}_c(i)) \).

But evidently \( \prod Y \) does not contain an element of \( f \). \( \Box \)

**Example 14.** There exists such an (infinite) set \( N \) and \( N \)-ary relation \( f \) that \( \mathcal{P} \in \| \| f \) but there are no indexed family \( a \in \prod_{i \in N} \text{atoms} \mathcal{P}_i \) of atomic filter such that \( a \in \| \| f \) that is \( \forall A \in \text{up} a: f \neq \| \| \).

**Proof.** Take \( L_0, L_1 \) and \( f \) from the proof of example 13. Take \( \mathcal{P} = L_0 \cup L_1 \). If \( a \in \prod_{i \in N} \text{atoms} \mathcal{P}_i \) then there exists \( c \in \{0, 1\}^N \) such that \( a_1 \subseteq \mathcal{L}_c(i) \) (because \( \mathcal{L}_c(i) \) \( \neq 0 \)). Then from that example it follows that \( (\lambda i \in N: \mathcal{L}_c(i)) \notin \text{GR} \| \| f \) and thus \( a \notin \text{GR} \| \| f \). \( \Box \)

**Example 15.** There is such an anchored relation \( F \) that for some \( k \in \text{dom} F \)

\[ \langle \| \| F \rangle_k \mathcal{L} \neq \prod_{a \in \prod_{i \in \text{dom}(F) \setminus \{k\}} \text{atoms} \mathcal{L}_i} \langle \| \| F \rangle_k a. \]

**Proof.** Take \( \mathcal{P} \in \text{GR} \mathcal{F} \) from the previous counter-example. We have \( \forall a \in \prod_{i \in \text{dom} F} \text{atoms} \mathcal{P}_i : a \notin \text{GR} \mathcal{P} \).

Take \( k = 1 \).

Let \( \mathcal{L} = \mathcal{P}|_{\text{dom}(F) \setminus \{k\}} \). Then \( a \notin \| \| F \) and thus \( a_k \in \langle \| \| F \rangle_k a|_{\text{dom}(F) \setminus \{k\}} \).

Consequently \( \mathcal{P}_k \simeq \langle \| \| F \rangle_k a|_{\text{dom}(F) \setminus \{k\}} \) and thus \( \mathcal{P}_k \simeq \bigcup_{a \in \prod_{i \in \text{dom}(F) \setminus \{k\}} \text{atoms} \mathcal{L}_i} \langle \| \| F \rangle_k a \) because \( \mathcal{P}_k \) is principal.

But \( \mathcal{P}_k \notin \| \| F \rangle_k \mathcal{L} \). Thus follows \( \langle \| \| F \rangle_k \mathcal{L} \neq \bigcup_{a \in \prod_{i \in \text{dom}(F) \setminus \{k\}} \text{atoms} \mathcal{L}_i} \langle \| \| F \rangle_k a. \) \( \Box \)

cross-composition-funcoids.tm
5 Other

Generalization of down-aligned (and up-aligned): A ltrator \((A; Z)\) is down-closed if:

\[ b \leq a \iff a \not\in A \implies b \not\in Z. \]

What is \(a \not\in A \implies b \not\in Z\) as a weaker axiom than finite meet closedness.

Pointfree funcoids on lters are equivalent to 2-staroids. Can we use it to prove something?

What is \(a \not\in A \implies b \not\in Z\) in terms of 2-multifuncoids?

Funcoids can be alternatively defined as:

\[ (f)^* X \iff X \implies f^{-1} Y \text{ where } \langle f \rangle^* : A \rightarrow \mathcal{F}(B) \text{ and } \langle f^{-1} \rangle^* : B \rightarrow \mathcal{F}(A). \]

For a binary relation \(f\) replace \((f)\) with \((f)^*\) for clarity of notation.

0 \rightarrow \bot, 1 \rightarrow \top.

Proposition 16. \( \bigcap \{ \up{X} \mid X \in \up{\text{base } F} \} \subseteq \up{X} \subseteq \bigcup \{ \text{base } F \mid X \in \up{\text{base } F} \} \)

Proof. ??

Define \( \bigcup_{X \in S} F(X) = \bigcup \{ F(X) \mid X \in S \} \) (with index of the operator symbol).

My commentary on Todd Trimble’s notes. Also his constructive proof that the poset of funcoids is a frame.

Denote \(W(A; B) = \{ (A; B; F) \mid F \in W[A; B] \}\) for a set \(W\) (here \(W\) can be FCD, RLD, \Gamma, etc.) and use this notation \(W[A; B]\) where appropriate.

Change notation \(A \rightarrow U\) in the “Multireloids” section.

Proposition 18.34 - define what is \(X\).

6 Other new theorems

funcoids-are-filters.tm

funcoids-are-frame.pdf

Theorem 17. The set of funcoids is with separable core.

Proof. Because filters on distributive lattices are with separable core.

Theorem 18. The set of funcoids is with co-separable core.

Proof. Let \(f, g \in \text{FCD}(A; B)\) and \(f \sqcup g = 1\). Then for every \(X \in \mathcal{P}A\) we have

\[ (f)^* X \sqcup (g)^* X = 1 \iff \text{Cor } (f)^* X \sqcup \text{Cor } (g)^* X = 1 \iff (\text{CoCompl } f)^* X \sqcup (\text{CoCompl } g)^* X = 1. \]

Thus \( (\text{CoCompl } f)^* X = 1 \); \( f \sqcup g = 1 \iff \text{CoCompl } f \sqcup \text{CoCompl } g = 1 \).

Applying the dual of the formulas (4) to the formula (4) we get:

\[ f \sqcup g = 1 \implies \text{Compl } \text{CoCompl } f \sqcup \text{Compl } \text{CoCompl } g = 1 \]

that is \( f \sqcup g = 1 \implies \text{Cor } f \sqcup \text{Cor } g = 1 \). So \(\text{FCD}(A; B)\) is with co-separable core. [TODO: Say that the filtrator of complete funcoids is also with co-separable core.]

Proposition 19. ComplFCD(A; B) and ComplRLD(A; B) are co-brouwerian lattices. [TODO: remove the question.]

Proof. It follows from the fact that these lattices are isomorphic to families of filters (which are complete co-brouwerian lattices) and obvious 17.21.

Proposition 20. Every semifiltered filtrator is filtered. [TODO: The reverse implication is already proved.]
Proof. $a = \bigcap \varGamma$ up a is equivalent to a is a greatest lower bound of up a. That is the implication that b is lower bound of up a implies $a \supseteq b$.

b is lower bound of up a implies up $b \supseteq up a$. So as it is semifiltered $a \supseteq b$. □

6.1 Hyperfuncoids

Let $\mathcal{A}$ is an indexed family of sets.

Products are $\prod A$ for $A \in \prod \mathcal{A}$.

Hyperfuncoids are filters $\mathfrak{F} \Gamma$ on the lattice $\Gamma$ of all finite unions of products.

Problem 21. Is $\bigcap^{\text{FCD}}$ a bijection from hyperfuncoids $\mathfrak{F} \Gamma$ to:

1. prestaroids on $\mathcal{A}$;
2. staroids on $\mathcal{A}$;
3. completary staroids on $\mathcal{A}$?

If yes, is up $\text{f}$ defining the inverse bijection?

If not, characterize the image of the function $\bigcap^{\text{FCD}}$ defined on $\mathfrak{F} \Gamma$.

6.2 Relationships between funcoids and reloids

Lemma 22. If $a$, $b$, $c$ are filters on powersets and $b \neq 0$, then

$$\bigcup^{\text{RLD}} \{ G \circ F \mid F \in \text{atoms}^{\text{RLD}}(a \times^{\text{RLD}} b), G \in \text{atoms}^{\text{RLD}}(b \times^{\text{RLD}} c) \} = a \times^{\text{RLD}} c.$$  

Proof. $a \times^{\text{RLD}} c = (b \times^{\text{RLD}} c) \circ (a \times^{\text{RLD}} b) = \text{(corollary 7.18)} = \bigcup^{\text{RLD}} \{ G \circ F \mid F \in \text{atoms}^{\text{RLD}}(a \times^{\text{RLD}} b), G \in \text{atoms}^{\text{RLD}}(b \times^{\text{RLD}} c) \}$. □

Theorem 23. $(\text{RLD})_{\text{in}}(g \circ f) = (\text{RLD})_{\text{in}} g \circ (\text{RLD})_{\text{in}} f$ for every composable funcoids $f$ and $g$.

[TODO: remove the conjecture as it is now proved.]

Proof. $(\text{RLD})_{\text{in}} g \circ (\text{RLD})_{\text{in}} f = \text{(corollary 7.18)} = \bigcup^{\text{RLD}} \{ G \circ F \mid F \in \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} f, G \in \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} g \}$

Let $F$ be an atom of the poset $\text{RLD}(\text{Src} f; \text{Dst} f)$.

$F \in \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} f \Rightarrow \text{dom } F \times^{\text{RLD}} \text{im } F \subseteq \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} f \Rightarrow \text{(because (RLD)$_{\text{in}}$ f is a funcoidal reloid)} \Rightarrow \text{dom } F \times^{\text{RLD}} \text{im } F \subseteq \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} f$ but $\text{dom } F \times^{\text{RLD}} \text{im } F \subseteq \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} f$ is obvious.

So $F \in \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} f \Leftarrow \text{dom } F \times^{\text{RLD}} \text{im } F \subseteq (\text{RLD})_{\text{in}} f \Rightarrow (\text{FCD})(\text{dom } F \times^{\text{RLD}} \text{im } F) \subseteq (\text{FCD})(\text{RLD})_{\text{in}} f \Leftarrow \text{dom } F \times^{\text{FCD}} \text{im } F \subseteq f$.

But $\text{dom } F \times^{\text{FCD}} \text{im } F \subseteq f \Rightarrow (\text{RLD})_{\text{in}} \text{dom } F \times^{\text{FCD}} \text{im } F \subseteq (\text{RLD})_{\text{in}} f \Rightarrow \text{dom } F \times^{\text{RLD}} \text{im } F \subseteq (\text{RLD})_{\text{in}} f$.

So $F \in \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} f \Rightarrow \text{dom } F \times^{\text{FCD}} \text{im } F \subseteq f$.

$$\text{dom } F \times^{\text{RLD}} \text{im } G = \bigcup^{\text{RLD}} \{ G' \circ F' \mid F' \in \text{atoms}^{\text{RLD}} (\text{dom } F \times^{\text{RLD}} \text{im } F), G' \in \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} g \}$$

But $\text{dom } F \times^{\text{RLD}} \text{im } G \subseteq \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} f \Rightarrow G \in \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} g$.

Thus $(\text{RLD})_{\text{in}} g \circ (\text{RLD})_{\text{in}} f = \bigcup^{\text{RLD}} \{ \text{dom } F \times^{\text{RLD}} \text{im } G \mid F \in \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} f, G \in \text{atoms}^{\text{RLD}} (\text{RLD})_{\text{in}} g \}$.
(RLD)_{in}(g ∘ f) = ∪_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{RLD}} (f)\{x\} \} = \bigcup_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{FCD}} (f)\{x\} \}.

Now it becomes obvious that \((RLD)_{in} g ∘ (RLD)_{in} f = (RLD)_{in}(g ∘ f)\).

\section{Complete reloids}

\textbf{Theorem 24.} \((\text{FCD})\) and \((\text{RLD})_{out}\) form mutually inverse bijections between complete reloids and complete funcoids.

\textbf{Proof.} Consider the bijection \(\bigcup_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{RLD}} (f)\{x\} \} \rightarrow \bigcup_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{FCD}} (f)\{x\} \}\) from complete reloids into complete funcoids, where \(f\) ranges the set of complete funcoids. But this bijection is exactly \((\text{FCD})\): \(\text{ComplRLD}(A; B) \rightarrow \text{ComplFCD}(A; B)\) because \(\bigcup_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{RLD}} (f)\{x\} \} = \bigcup_{x ∈ \text{Src} f} \bigcup_{y ∈ \text{Dst} f} \{ \{ x \} ×_{\text{FCD}} \{ y \}\} = \bigcup_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{FCD}} (f)\{x\} \} \). Thus we have proved that \((\text{FCD})\): \(\text{ComplRLD}(A; B) \rightarrow \text{ComplFCD}(A; B)\) is a bijection.

It remains to prove that \((\text{RLD})_{out} g = \bigcup_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{RLD}} (g)\{x\} \}\) for every complete funcoid \(g\).

Really, \((\text{RLD})_{out} g \supseteq \bigcup_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{RLD}} (g)\{x\} \}\).

It remains to prove that \(\bigcup_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{RLD}} (g)\{x\} \} \supseteq (\text{RLD})_{out} g\).

Let \(L ∈ \text{up} \bigcup_{x ∈ \text{Src} f} \{ \{ x \} ×_{\text{RLD}} (g)\{x\} \}\). We will prove \(L ∈ \text{up} (\text{RLD})_{out} g\).

We can limit to the case when \(L\) is a reloidal product. Then

\(L ∈ \bigcap \{ \text{up} \{ \{ x \} ×_{\text{RLD}} (g)\{x\} \} \mid x ∈ \text{Src} f \} = \bigcap \{ \{ x \} × Y \mid Y ∈ \text{up} \{ g\{x\} \} \mid x ∈ \text{Src} f \} \).

It’s enough to prove that \(L ∈ \text{up} g\). Really, \(∀x ∈ \text{Src} f; \langle L\rangle^*\{x\} ∈ \text{up} \{ g\{x\} \}\) because

\(\langle L\rangle^*\{x\} ⊇ \langle T\rangle^*\{x\}\)

for

\(T = \bigcap \{ \{ x \} × Y \mid Y ∈ \text{up} G(x) \} \mid x ∈ \text{Src} f \} \).

and thus

\(\bigcap \{ \{ x \} × Y \mid x = x', Y ∈ \text{up} G(x) \} \mid x ∈ \text{Src} f \} = \{ Y \mid Y ∈ \text{up} G(x') \} = \up G(x') \).

So \(\langle L\rangle^*\{x\} ∈ \text{up} \{ g\{x\} \}\) and thus \(L ∈ \text{up} g\). \hspace{1cm} \Box

\textbf{Corollary 25.} \(f \neq g \Rightarrow (\text{RLD})_{out} f \neq (\text{RLD})_{out} g\) for complete funcoids \(f\) and \(g\).

\textbf{Theorem 26.} Composition of complete reloids is complete.

\textbf{Proof.} Let \(f, g\) be complete reloids. Then \((\text{FCD})(g ∘ f) = (\text{FCD})g ∘ (\text{FCD})f\). Thus \(\text{ComplFCD}(A; B) \rightarrow \text{ComplFCD}(A; B)\) is a complete funcoid, thus \(g ∘ f\) is a complete funcoid, thus \(g ∘ f\) is a complete reloid. \hspace{1cm} \Box

\textbf{Theorem 27.}

1. \((\text{RLD})_{out} g ∘ (\text{RLD})_{out} f = (\text{RLD})_{out}(g ∘ f)\) for composable complete funcoids \(f\) and \(g\).

2. \((\text{RLD})_{out} g ∘ (\text{RLD})_{out} f = (\text{RLD})_{out}(g ∘ f)\) for composable co-complete funcoids \(f\) and \(g\). (If so, it is dually true for co-complete funcoids.)

\textbf{Proof.} Let \(f, g\) be composable complete funcoids.
(FCD)((RLD)_{out}g ∘ (RLD)_{out}f) = (FCD)((RLD)_{out}g ∘ (FCD)(RLD)_{out}f = g ∘ f).

Thus (taking into account that (RLD)_{out}g ∘ (RLD)_{out}f is complete) we have (RLD)_{out}g ∘ (RLD)_{out}f = (RLD)_{out}(g ∘ f).

For co-complete funcoids it’s dual. □

7 Not yet written

Can we prove without axiom of choice that FCD is a co-brouwerian lattice using the fact that it is isomorphic to filters on \( \Gamma \)?

I do some research in:

- backward.pdf
- multireloids-relationships.pdf

Question: Can we restore the set of binary relations, knowing only order of FCD(\( A; B \))? Note that it is not the center of the lattice, as not all funcoids are in the center. Yes, it can be characterized as joins of complemented funcoids or joins of complemented atomic funcoids. Is every complemented funcoid principal? This way principality can be generalized for pointfree funcoids. The set of principal p.f. funcoids is join-closed. When filtrator of pointfree funcoids is filtered?

Should we extend filtrators with finite join/meet closed core to nullary closed (having bottom/top)? The old concept shall be named binary join/meet closed filtrators. These are related with up/down aligned filtrators.

8 Errors

“Theorem 17.150. Anchored relations with objects being atomic posets and above defined compositions form a quasi-invertible category with star-morphisms.”

It is wrong, because composition of a star-morphism \( m \) with identify morphisms may be not equal to \( m \). In the definition of general cross-composition product we can replace quasi-invertible category with quasi-invertible pre-category.

Bibliography