Rewrite plan for my research monograph

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This is my plan to rewrite my draft research monograph “Algebraic General Topology. Volume 1”.

Say that the book intended mainly to represent the current status of my research.

Add thanks to Todd Trimble and Andreas Blass, Robert Martin Solovay.

Shall I denote sets like $\frac{x \in A}{P(x)}$ instead of $\{x \in A | P(x)\}$?

Remove multiple labels of the same theorem.

Notations for meets and joins for partially ordered categories.

1 Immediate actions

What I am going to do next:

1. Replace hardcoded item references with LaTeX cross-references.
2. Rewrite with arbitrary primary filtrators rather than filters. (Instead $\mathfrak{F}$ say “$\mathfrak{A}$ is base of a primary filtrator”.)
3. Join theorems into implication tuples. (Move theorems back not forward! Otherwise a dependent theorem may occur before it.)
4. Generalize theorems using the formula of meet of filters for distrubutive lattices for meet-semilattices.

2 Implications tuples

In the old version of my book there are multiple situations where there are both theorems of the form $B \Rightarrow C$ and $A \Rightarrow C$ where the $A \Rightarrow C$ is proved using that $A \Rightarrow B$ and $B \Rightarrow C$. It is accompanied with verbal explanation, that in these theorems C is the most important component of the theorems and $B \Rightarrow C$ is a boring generalization of the “main” theorem $A \Rightarrow C$.

I decided to rewrite every occurence of this case as single theorem $A \Rightarrow B \Rightarrow C$ with multiple implications. (There may be more than two implications, in general it is a $P_1 \Rightarrow \ldots \Rightarrow P_n$ implication.)

To describe such implications verbally, I define implications tuples.

Definition 1. An *implications tuple* is a tuple $(P_1, \ldots, P_n)$ such that $P_1 \Rightarrow \ldots \Rightarrow P_n$.

Obvious 2. $(P_1, \ldots, P_n)$ is an implications tuple iff $P_i \Rightarrow P_j$ for every $i < j$ (where $i, j \in \{1, \ldots, n\}$).

The following is an example of a theorem using an implication tuple:

Example 3. The following is an implications tuple:

1. A;
2. B;
3. C.

This example means just that $A \Rightarrow B \Rightarrow C$.

I prefer here a verbal description instead of symbolic implications $A \Rightarrow B \Rightarrow C$, because A, B, C may be long English phrases and they may not fit into the formula layout.

The main (intuitive) idea of the theorem is expressed by the implication $P_1 \Rightarrow P_n$, the rest implications ($P_2 \Rightarrow P_n$, $P_3 \Rightarrow P_n$, ...) are purely technical, as they express generalizations of the main idea.

For uniformity theorems in the section about filters and filtrators start with the same P_1: "$(\mathfrak{A}; \mathfrak{Z})$ is a powerset filtrator."

3 Theory of filters

Explicitly define order and lattice operations for ideals, free stars, and mixers.

For filtrators, filters, funcoids, and reloids write lattice operations without explicit posets.

(About free stars and similar things) say that presence of least element is the same as presence of the entire poset are equivalent if there is the least element.

I have "discovered" that there are four sets (including the set of filters itself) isomorphic to the set of filters on any poset.

See http://www.mathematics21.org/binaries/dual-filters.pdf

As there are several isomorphic sets, it makes sense to describe it more generally than the special case of the set of filters.

We shall differentiate between $\operatorname{st} \mathcal{A} = \langle \operatorname{dual} \rangle^\ast \neg \operatorname{up} \mathcal{A} = \neg \langle \operatorname{dual} \rangle^\ast \operatorname{up} \mathcal{A}$ and $\partial \mathcal{A}$. There is also the isomorphism with boolean lattices; how to denote it?

To describe this I redefine *primary filtrator* (earlier defined as a filtrator $(\mathfrak{F}; \mathfrak{Z})$ where $\mathfrak{F}$ is the set of filters on a poset $\mathfrak{Z}$) in an other (non-equivalent) way.

Definition 4. A *primary filtrator* is such a filtrator $(\mathfrak{A}; \mathfrak{Z})$ that $\mathfrak{A}$ is isomorphic to the set of filters on the poset $\mathfrak{Z}$.

Definition 5. A primary filtrator *over* a poset $\mathfrak{Z}$ is a primary filtrator of the form $(\mathfrak{A}; \mathfrak{Z})$.

Theorem 6. For every poset $\mathfrak{Z}$ there exists a primary filtrator over $\mathfrak{Z}$.

Proof. See [1]. □

What I called "complete free stars" could be better called *principal free stars*. The same is true for ideals and mixers. I should write explicit characterizations of principality for all four kinds of "filter objects".

Proposition 7. All filters on a finite poset are principal.

Get an elementary proof from http://math.stackexchange.com/questions/1206777/an-elementary-proof-about-filters (also elementary.tm)

http://math.stackexchange.com/questions/462270/are-all-atoms-of-the-lattice-of-filters-principal-filters

I've conjectured that filters on every meet-semilattice are co-brouwerian. It is not so! (take a non-distribuitive finite lattice) So distributivity is necessary.

Proposition 8. $\partial \Omega(U)$ is the set of infinite subsets of U.

Proposition 9. $\operatorname{Cor} a = \operatorname{Cor}' a = a$ for every element a of the core of a filtrator.

4 2-staroids

Definition 10. 2-staroid is a binary relation δ between two posets such that $\{Y \in \mathfrak{B} \mid \exists X \in \mathfrak{A}: X \mathrel{\delta} Y\}$ and $\{X \in \mathfrak{A} \mid \exists Y \in \mathfrak{B}: X \mathrel{\delta} Y\}$ are free stars. [TODO: Say that 2-staroids are a special case of staroids.]

5 Staroids and multifuncoids

$(\operatorname{val} f)_i L$ and $\langle f\rangle_i L$ coincide.

Change notation $[f] \to [f]^*$, $[\upuparrows f]^* \to [f]$ (for staroids).

I assumed that upgrading a staroid is a staroid without proof. Fill this hole. (Definition 17.69 for an example.) This is addressed in theorem 17.83.

Because the set of free stars is identified with the set of filters, the set of staroids (of a given form) can be identified with the set of multifuncoids on primary filtrators $(\mathfrak{A}_i; \mathfrak{Z}_i)$. Can all staroids and all multifuncoids be identified?

This allows to thoroughly revise the theory of staroids and multifuncoids.

The last chapter of my book ("Identity staroids") contains errors. I am going to rewrite it after switching to this new notation.

[TODO: Generalize "Funcoids are filters" for staroids (call it *hyperfuncoids*).]

Theorem 11. $\langle\upuparrows f\rangle_k a = \bigsqcap^{\mathfrak{F}}_{A\in\operatorname{up} a} \langle f\rangle_k A$ for every multifuncoid f of the form $\mathfrak{A}$ where $k\in\operatorname{arity} f$ and $\mathfrak{A}_k$ is a poset of filter objects on a boolean lattice and a is an $(\operatorname{arity} f)\setminus\{k\}$ family of filters. [TODO: Say exactly which family of filters is meant.]

Proof. $\mathcal{X} \not\asymp \langle\upuparrows f\rangle_k\ a \Leftrightarrow a\cup\{(k;\mathcal{X})\}\in \mathrm{GR}\ [\upuparrows f] \Leftrightarrow \operatorname{up}(a\cup\{(k;\mathcal{X})\})\subseteq \mathrm{GR}\ [f] \Leftrightarrow \forall A\in\operatorname{up} a, X\in\operatorname{up}\mathcal{X}: A\cup\{(k;X)\}\in\mathrm{GR}\ [f] \Leftrightarrow \forall A\in\operatorname{up} a, X\in\operatorname{up}\mathcal{X}: X\not\asymp\langle f\rangle_k\ A \Leftrightarrow$ (because it is separable)$\Leftrightarrow \forall A\in\operatorname{up} a: \mathcal{X}\not\asymp\langle f\rangle_k A \Leftrightarrow$ (by properties of generalized filter bases and the fact that $[f]$ is an upper set)$\Leftrightarrow \mathcal{X}\not\asymp \bigsqcap^{\mathfrak{F}}_{A\in\operatorname{up} a} \langle f\rangle_k A$.

So $\langle\upuparrows f\rangle_k a = \bigsqcap^{\mathfrak{F}}_{A\in\operatorname{up} a} \langle f\rangle_k A$ because filters on boolean lattice are separable, □

Example 12. There is such anchored relation f that $\upuparrows\uparrow f$ is not a completary staroid. [TODO: Remove the conjecture about this.] [TODO: This also proves existence of non completary staroids (but not for powersets).]

Proof. (based on an Andreas Blass's proof)

Take f the set of functions $x: \mathbb{N}\to\mathbb{N}$ where x_0 an arbitrary natural number and $x_i = \begin{cases} 0 & \text{if } n\leqslant x_0 \\ 1 & \text{if } n > x_0\end{cases}$ for $i = 1, 2, 3, \ldots$.

Let $\mathcal{L}_0(0) = \mathcal{L}_1(0) = \Omega(\mathbb{N})$, $\mathcal{L}_0(i) = \uparrow\{0\}$ and $\mathcal{L}_1(i) = \uparrow\{1\}$ for $i > 0$.

Let $X\in\operatorname{up}(\mathcal{L}_0\sqcup\mathcal{L}_1)$ that is $X\in\operatorname{up}\mathcal{L}_0\cap\operatorname{up}\mathcal{L}_1$.

X_0 contains all but finitely many elements of $\mathbb{N}$.

For $i > 0$ we have $\{0, 1\}\subseteq X$.

Evidently, $\prod X$ contains an element of f.

Now consider any fixed $c\in\{0,1\}^{\mathbb{N}}$. There is at most one $k\in\mathbb{N}$ such that the sequence $x = [\![k; c(1); c(2); \ldots]\!]$ (i.e. c with $c(0)$ replaced by k) is in f. Let $Q = \mathbb{N}\setminus\{k\}$ if there is such a k and $Q = \mathbb{N}$ otherwise.

Take $Y_i = \begin{cases} Q & \text{if } i = 0 \\ \{c(i)\} & \text{if } i > 0\end{cases}$ for $i = 1, 2, 3, \ldots$. We have $Y\in\operatorname{up}(\lambda i\in\mathbb{N}: \mathcal{L}_{c(i)}(i))$.

But evidently $\prod Y$ does not contain an element of f. □

Example 13. There exists such an (infinite) set N and N-ary relation f that $\mathcal{P}\in\upuparrows f$ but there are no indexed family $a\in\prod_{i\in N}\operatorname{atoms}\mathcal{P}_i$ of atomic filters such that $a\in\mathrm{GR}\upuparrows f$ that is $\forall A\in\operatorname{up} a$: $f\not\asymp\prod A$.

Proof. Take $\mathcal{L}_0$, $\mathcal{L}_1$ and f from the proof of example 12. Take $\mathcal{P} = \mathcal{L}_0\sqcup\mathcal{L}_1$. If $a\in\prod_{i\in N}\operatorname{atoms}\mathcal{P}_i$ then there exists $c\in\{0,1\}^N$ such that $a_i\sqsubseteq\mathcal{L}_{c(i)}(i)$ (because $\mathcal{L}_{c(i)}(i)\neq 0$). Then from that example it follows that $(\lambda i\in N: \mathcal{L}_{c(i)}(i))\notin\mathrm{GR}\upuparrows f$ and thus $a\notin\mathrm{GR}\upuparrows f$. □

Example 14. There is such an anchored relation F that for some $k\in\operatorname{dom} F$

$$\langle\upuparrows\uparrow F\rangle^*{}_k\mathcal{L} \neq \bigsqcup^{\mathfrak{F}}_{a\in\prod_{i\in(\operatorname{dom} F)\setminus\{k\}}\operatorname{atoms}\mathcal{L}_i} \langle\upuparrows\uparrow F\rangle^*{}_k a.$$

Proof. Take $\mathcal{P} \in \mathrm{GR}\, F$ from the previous counter-example. We have

$$\forall a \in \prod_{i \in \mathrm{dom}\, F} \mathrm{atoms}\, \mathcal{P}_i : a \notin \mathrm{GR}\, \mathcal{P}.$$

Take $k = 1$.

Let $\mathcal{L} = \mathcal{P}|_{(\mathrm{dom}\, F) \setminus \{k\}}$. Then $a \notin \mathrm{GR} \upuparrows \uparrow F$ and thus $a_k \asymp \langle \upuparrows \uparrow F \rangle^\ast{}_k\, a|_{(\mathrm{dom}\, F) \setminus \{k\}}$.

Consequently $\mathcal{P}_k \asymp \langle \upuparrows \uparrow F \rangle^\ast{}_k\, a|_{(\mathrm{dom}\, F) \setminus \{k\}}$ and thus $\mathcal{P}_k \asymp \bigsqcup^{\mathfrak{F}}_{a \in \prod_{i \in (\mathrm{dom}\, F) \setminus \{k\}} \mathrm{atoms}\, \mathcal{L}_i} \langle \upuparrows \uparrow F \rangle^\ast{}_k\, a$ because $\mathcal{P}_k$ is principal.

But $\mathcal{P}_k \not\asymp \langle \upuparrows \uparrow F \rangle^\ast{}_k\, \mathcal{L}$. Thus follows $\langle \upuparrows \uparrow F \rangle^\ast{}_k\, \mathcal{L} \neq \bigsqcup^{\mathfrak{F}}_{a \in \prod_{i \in (\mathrm{dom}\, F) \setminus \{k\}} \mathrm{atoms}\, \mathcal{L}_i} \langle \upuparrows \uparrow F \rangle^\ast{}_k\, a$. □

cross-composition-funcoids.tm

Multifuncoid is a function α distributive over join of every single argument? (It seem that no).

6 Typed sets

Use typed sets instead of sets.

Definition 15. *Typed set* is a pair $(U; A)$ of a set U and its subset A. But we should go more general: We can define *typed element* as a pair $(\mathfrak{A}; a)$ where $\mathfrak{A}$ is a poset and $a \in \mathfrak{A}$. Note that cartesian product can be defined for the special case if it is a powerset.

Remark 16. *Typed sets* is an awkward formalization of type theory sets in ZFC (U is meant to express the *type* of the set). This book could be better written using type theory instead of ZFC, but I want my book to be understandable for everyone knowing ZFC.

Definition 17. $\mathscr{P}(U; A) = \{(U; X) \mid X \in \mathscr{P} A\}$.

For typed sets define order, binary cartesian product (into the category Rel).

Define typed elements?

$\mathscr{T} A = \{(A; X) \mid X \in \mathscr{P} A\} = \{A\} \times \mathscr{P} A$

$\mathfrak{T} \mathfrak{A} = \{(\mathfrak{A}; a) \mid a \in \mathfrak{A}\} = \{\mathfrak{A}\} \times \mathfrak{A}$

Consider typed sets denoted $\bot^{\mathscr{T} A}$ and $\top^{\mathscr{T} A}$. (It is consistent with $\bot^{\mathfrak{F} A}$ and $\top^{\mathfrak{F} A}$.)

$\langle f \rangle^\ast$ and $[f]^\ast$ should be defined for typed sets, not sets.

We shall consider a primary filtrator $(\mathfrak{F} A; \mathscr{T} A)$ for every set A to define funcoids and reloids.

Notwithstanding the above, funcoids and reloids are defined between sets, not typed sets.

7 Other

Generalization of down-aligned (and up-aligned): A filtrator $(\mathfrak{A}; \mathfrak{Z})$ is down-closed if $\forall a \in \mathfrak{A} \exists b \in \mathfrak{Z}: b \sqsubseteq a$.

Funcoids can be alternatively defined as: $Y \not\asymp \langle f \rangle^\ast X \Leftrightarrow X \not\asymp \langle f^{-1} \rangle^\ast Y$ where $\langle f \rangle^\ast : A \to \mathfrak{F}(B)$ and $\langle f^{-1} \rangle^\ast : B \to \mathfrak{F}(A)$.

For a binary relation f replace $\langle f \rangle$ with $\langle f \rangle^\ast$ for clarity of notation.

$0 \to \bot$, $1 \to \top$.

$\mathrm{up}^{(\mathfrak{A}; \mathfrak{Z})} \to \mathrm{up}^{\mathfrak{A}}$.

Define $\bigsqcup_{X \in S} F(X) = \bigsqcup \{F(X) \mid X \in S\}$ (with index of the operator symbol).

Proposition 18.34 - define what is X.

Use explicit $p\mathsf{FCD}$ to denote pointfree funcoids.

I overcomplicated the definition of *image* for pointfree funcoids. It should be just $\langle f \rangle \top$ (because it is used exclusively this way). Is there any single reason to define it in this general complicated way? it seems there is none. Also prove $\mathrm{im}\, f = \max \langle \langle f \rangle \rangle^\ast \mathrm{Src}\, f$.

Theorem 18. For a diagram to be commutative, it's enough if each simple cycle commutes. See https://en.wikipedia.org/wiki/Cycle_%28graph_theory%29 for a definition of simple cycles.

8 Other new theorems

funcoids-are-filters.tm
funcoids-are-frame.tm

Theorem 19. The set of funcoids is with separable core.

Proof. Because filters on distributive lattices are with separable core. □

Theorem 20. The set of funcoids is with co-separable core. [TODO: For pointfree funcoids?]

Proof. Let $f, g \in \mathsf{FCD}(A; B)$ and $f \sqcup g = 1$. Then for every $X \in \mathscr{P}A$ we have

$$\langle f \rangle^{*} X \sqcup \langle g \rangle^{*} X = 1 \Leftrightarrow \operatorname{Cor} \langle f \rangle^{*} X \sqcup \operatorname{Cor} \langle g \rangle^{*} X = 1 \Leftrightarrow \langle \operatorname{CoCompl} f \rangle^{*} X \sqcup \langle \operatorname{CoCompl} g \rangle^{*} X = 1.$$

Thus $\langle \operatorname{CoCompl} f \sqcup \operatorname{CoCompl} g \rangle^{*} X = 1$;

$$f \sqcup g = 1 \Rightarrow \operatorname{CoCompl} f \sqcup \operatorname{CoCompl} g = 1. \tag{1}$$

Applying the dual of the formulas (1) to the formula (1) we get:

$$f \sqcup g = 1 \Rightarrow \operatorname{Compl} \operatorname{CoCompl} f \sqcup \operatorname{Compl} \operatorname{CoCompl} g = 1$$

that is $f \sqcup g = 1 \Rightarrow \operatorname{Cor} f \sqcup \operatorname{Cor} g = 1$. So $\mathsf{FCD}(A; B)$ is with co-separable core. [TODO: Say that the filtrator of complete funcoids is also with co-separable core.] □

Proposition 21. $\operatorname{Compl}\mathsf{FCD}(A; B)$ and $\operatorname{Compl}\mathsf{RLD}(A; B)$ are co-brouwerian lattices.

Draw a triangural diagram of correspondence of $\operatorname{Compl}\mathsf{FCD}(A; B)$ and $\operatorname{Compl}\mathsf{RLD}(A; B)$ and indexed families of filters.

Proposition 22. Every semifiltered filtrator is filtered. [TODO: The reverse implication is already proved.] [TODO: See also http://math.stackexchange.com/questions/1198368/a-question-on-order-theory-an-ordered-set-and-its-subset]

Proof. $a = \bigsqcap^{\mathfrak{A}} \operatorname{up} a$ is equivalent to a is a greatest lower bound of $\operatorname{up} a$. That is the implication that b is lower bound of $\operatorname{up} a$ implies $a \sqsupseteq b$.

b is lower bound of $\operatorname{up} a$ implies $\operatorname{up} b \supseteq \operatorname{up} a$. So as it is semifiltered $a \sqsupseteq b$. □

8.1 Hyperfuncoids

Let $\mathfrak{A}$ is an indexed family of sets.

Products are $\prod A$ for $A \in \prod \mathfrak{A}$.

Hyperfuncoids are filters $\mathfrak{F}\Gamma$ on the lattice Γ of all finite unions of products.

Problem 23. Is $\bigsqcap^{\mathsf{FCD}}$ a bijection from hyperfuncoids $\mathfrak{F}\Gamma$ to:

1. prestaroids on $\mathfrak{A}$;
2. staroids on $\mathfrak{A}$;
3. completary staroids on $\mathfrak{A}$?

If yes, is $\operatorname{up}^{\Gamma}$ defining the inverse bijection?

If not, characterize the image of the function $\bigsqcap^{\mathsf{FCD}}$ defined on $\mathfrak{F}\Gamma$.

8.2 Relationships between funcoids and reloids

Lemma 24. If a, b, c are filters on powersets and $b \neq 0$, then

$$\bigsqcup^{\mathsf{RLD}} \{G \circ F \mid F \in \operatorname{atoms}^{\mathsf{RLD}}(a \times^{\mathsf{RLD}} b), G \in \operatorname{atoms}^{\mathsf{RLD}}(b \times^{\mathsf{RLD}} c)\} = a \times^{\mathsf{RLD}} c.$$

Proof. $a\times^{\mathsf{RLD}}c=(b\times^{\mathsf{RLD}}c)\circ(a\times^{\mathsf{RLD}}b)=$ (corollary 7.18) $=\bigsqcup^{\mathsf{RLD}}\{G\circ F\mid F\in\operatorname{atoms}^{\mathsf{RLD}}(a\times^{\mathsf{RLD}}b), G\in\operatorname{atoms}^{\mathsf{RLD}}(b\times^{\mathsf{RLD}}c)\}$. □

Theorem 25. $(\mathsf{RLD})_{\mathrm{in}}(g\circ f)=(\mathsf{RLD})_{\mathrm{in}}\,g\circ(\mathsf{RLD})_{\mathrm{in}}\,f$ for every composable funcoids f and g. [TODO: remove the conjecture as it is now proved.]

Proof. $(\mathsf{RLD})_{\mathrm{in}}\,g\circ(\mathsf{RLD})_{\mathrm{in}}\,f=$ (corollary 7.18) $=\bigsqcup^{\mathsf{RLD}}\{G\circ F\mid F\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f, G\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,g\}$

Let F be an atom of the poset $\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)$.

$F\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f\Rightarrow\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}F\not\asymp\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f\Rightarrow$ (because $(\mathsf{RLD})_{\mathrm{in}}\,f$ is a funcoidal reloid) $\Rightarrow\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}F\sqsubseteq\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f$ but $\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}F\sqsubseteq\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f\Rightarrow F\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f$ is obvious.

So $F\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f\Leftrightarrow\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}F\sqsubseteq(\mathsf{RLD})_{\mathrm{in}}\,f\Rightarrow(\mathsf{FCD})(\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}F)\sqsubseteq(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}}\,f\Leftrightarrow\operatorname{dom}F\times^{\mathsf{FCD}}\operatorname{im}F\sqsubseteq f$.

But $\operatorname{dom}F\times^{\mathsf{FCD}}\operatorname{im}F\sqsubseteq f\Rightarrow(\mathsf{RLD})_{\mathrm{in}}(\operatorname{dom}F\times^{\mathsf{FCD}}\operatorname{im}F)\sqsubseteq(\mathsf{RLD})_{\mathrm{in}}\,f\Leftrightarrow\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}F\sqsubseteq(\mathsf{RLD})_{\mathrm{in}}\,f$.

So $F\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f\Leftrightarrow\operatorname{dom}F\times^{\mathsf{FCD}}\operatorname{im}F\sqsubseteq f$.

$\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}G=\bigsqcup^{\mathsf{RLD}}\{G'\circ F'\mid F'\in\operatorname{atoms}^{\mathsf{RLD}}(\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}F), G'\in\operatorname{atoms}^{\mathsf{RLD}}(\operatorname{im}F\times^{\mathsf{RLD}}\operatorname{im}G)\}\sqsubseteq\bigsqcup^{\mathsf{RLD}}\{G'\circ F'\mid F'\in\operatorname{atoms}^{\mathsf{RLD}(\operatorname{Src}F;\operatorname{Dst}F)}, G'\in\operatorname{atoms}^{\mathsf{RLD}(\operatorname{Src}G;\operatorname{Dst}G)}, F'\sqsubseteq(\mathsf{RLD})_{\mathrm{in}}\,f, G'\sqsubseteq(\mathsf{RLD})_{\mathrm{in}}\,g\}=\bigsqcup^{\mathsf{RLD}}\{G'\circ F'\mid F'\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f, G'\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,g\}=(\mathsf{RLD})_{\mathrm{in}}\,g\circ(\mathsf{RLD})_{\mathrm{in}}\,f$.

Thus $(\mathsf{RLD})_{\mathrm{in}}\,g\circ(\mathsf{RLD})_{\mathrm{in}}\,f\sqsupseteq\bigsqcup^{\mathsf{RLD}}\{\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}G\mid F\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f, G\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,g\}$

But $(\mathsf{RLD})_{\mathrm{in}}\,g\circ(\mathsf{RLD})_{\mathrm{in}}\,f\sqsubseteq\bigsqcup^{\mathsf{RLD}}\{(\operatorname{dom}G\times^{\mathsf{RLD}}\operatorname{im}G)\circ(\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}F)\mid F\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f, G\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,g\}$.

Thus $(\mathsf{RLD})_{\mathrm{in}}\,g\circ(\mathsf{RLD})_{\mathrm{in}}\,f=\bigsqcup^{\mathsf{RLD}}\{\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}G\mid F\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,f, G\in\operatorname{atoms}^{\mathsf{RLD}}(\mathsf{RLD})_{\mathrm{in}}\,g\}=\bigsqcup^{\mathsf{RLD}}\{\operatorname{dom}F\times^{\mathsf{RLD}}\operatorname{im}G\mid F\in\operatorname{atoms}^{\mathsf{RLD}(\operatorname{Src}f;\operatorname{Dst}f)}, G\in\operatorname{atoms}^{\mathsf{RLD}(\operatorname{Dst}f;\operatorname{Dst}g)}, \operatorname{dom}F\times^{\mathsf{FCD}}\operatorname{im}F\sqsubseteq f, \operatorname{dom}G\times^{\mathsf{FCD}}\operatorname{im}G\sqsubseteq g\}$.

But

$$(\mathsf{RLD})_{\mathrm{in}}(g\circ f)=\bigsqcup^{\mathsf{RLD}}\{a\times^{\mathsf{RLD}}c\mid a\in\mathfrak{F}(\operatorname{Src}f), b\in\mathfrak{F}(\operatorname{Dst}f), c\in\mathfrak{F}(\operatorname{Dst}g), a\times^{\mathsf{FCD}}b\in\operatorname{atoms}^{\mathsf{FCD}}f, b\times^{\mathsf{FCD}}c\in\operatorname{atoms}^{\mathsf{FCD}}g\}.$$

Now it becomes obvious that $(\mathsf{RLD})_{\mathrm{in}}\,g\circ(\mathsf{RLD})_{\mathrm{in}}\,f=(\mathsf{RLD})_{\mathrm{in}}(g\circ f)$. □

8.3 Complete reloids

Theorem 26. (FCD) and $(\mathsf{RLD})_{\mathrm{out}}$ form mutually inverse bijections between complete reloids and complete funcoids.

Proof. Consider the bijection $\bigsqcup_{x\in\operatorname{Src}f}(\{x\}\times^{\mathsf{RLD}}\langle f\rangle\{x\})\mapsto\bigsqcup_{x\in\operatorname{Src}f}(\{x\}\times^{\mathsf{FCD}}\langle f\rangle\{x\})$ from complete reloids into complete funcoids, where f ranges the set of complete funcoids. But this bijection is exactly $(\mathsf{FCD})\colon\mathsf{ComplRLD}(A;B)\to\mathsf{ComplFCD}(A;B)$ because $(\mathsf{FCD})\bigsqcup_{x\in\operatorname{Src}f}(\{x\}\times^{\mathsf{RLD}}\langle f\rangle\{x\})=\bigsqcup_{x\in\operatorname{Src}f}(\mathsf{FCD})(\{x\}\times^{\mathsf{RLD}}\langle f\rangle\{x\})=\bigsqcup_{x\in\operatorname{Src}f}(\{x\}\times^{\mathsf{FCD}}\langle f\rangle\{x\})$. Thus we have proved that $(\mathsf{FCD})\colon\mathsf{ComplRLD}(A;B)\to\mathsf{ComplFCD}(A;B)$ is a bijection.

It remains to prove that $(\mathsf{RLD})_{\mathrm{out}}\,g=\bigsqcup_{x\in\operatorname{Src}f}(\{x\}\times^{\mathsf{RLD}}\langle g\rangle\{x\})$ for every complete funcoid g (because $g=\bigsqcup_{x\in\operatorname{Src}f}(\{x\}\times^{\mathsf{FCD}}\langle g\rangle\{x\})$).

Really, $(\mathsf{RLD})_{\mathrm{out}}\,g\sqsupseteq\bigsqcup_{x\in\operatorname{Src}f}(\{x\}\times^{\mathsf{RLD}}\langle g\rangle\{x\})$.

It remains to prove that $\bigsqcup_{x\in\operatorname{Src}f}(\{x\}\times^{\mathsf{RLD}}\langle g\rangle\{x\})\sqsupseteq(\mathsf{RLD})_{\mathrm{out}}\,g$.

Let $L\in\operatorname{up}\bigsqcup_{x\in\operatorname{Src}f}(\{x\}\times^{\mathsf{RLD}}\langle g\rangle\{x\})$. We will prove $L\in\operatorname{up}(\mathsf{RLD})_{\mathrm{out}}\,g$.

We can limit to the case when L is a reloidal product. Then

$$L \in \bigcap \{\mathrm{up}(\{x\} \times^{\mathsf{RLD}} \langle g \rangle \{x\}) \mid x \in \mathrm{Src}\, f\} = \bigcap \{\{\{x\} \times Y \mid Y \in \mathrm{up}\, \langle g \rangle \{x\}\} \mid x \in \mathrm{Src}\, f\}.$$

It's enough to prove that $L \in \mathrm{up}\, g$. Really, $\forall x \in \mathrm{Src}\, f \colon \langle L \rangle^* \{x\} \in \mathrm{up}\, \langle g \rangle \{x\}$ because $\langle L \rangle^* \{x\} \supseteq \langle T \rangle^* \{x\}$ for

$$T = \bigcap \{\{\{x\} \times Y \mid Y \in \mathrm{up}\, G(x)\} \mid x \in \mathrm{Src}\, f\}.$$

and thus

$$\begin{aligned} \langle L \rangle^* \{x\} &\supseteq \\ \bigcap \{\{\langle \{x\} \times Y \rangle^* \{x'\} \mid x' = x, Y \in \mathrm{up}\, G(x)\} \mid x \in \mathrm{Src}\, f\} &= \\ \{Y \mid Y \in \mathrm{up}\, G(x')\} &= \\ \mathrm{up}\, G(x'). & \end{aligned}$$

So $\langle L \rangle^* \{x\} \in \mathrm{up}\, \langle g \rangle \{x\}$ and thus $L \in \mathrm{up}\, g$. □

Corollary 27. $f \neq g \Rightarrow (\mathsf{RLD})_{\mathrm{out}} f \neq (\mathsf{RLD})_{\mathrm{out}} g$ for complete funcoids f and g.

Theorem 28. Composition of complete reloids is complete.

Proof. Let f, g be complete reloids. Then $(\mathsf{FCD})(g \circ f) = (\mathsf{FCD})g \circ (\mathsf{FCD})f$. Thus (because $(\mathsf{FCD})(g \circ f)$ is a complete funcoid) we have $g \circ f = (\mathsf{RLD})_{\mathrm{out}}((\mathsf{FCD})g \circ (\mathsf{FCD})f)$, but $(\mathsf{FCD})g \circ (\mathsf{FCD})f$ is a complete funcoid, thus $g \circ f$ is a complete reloid. □

Theorem 29.

1. $(\mathsf{RLD})_{\mathrm{out}}\, g \circ (\mathsf{RLD})_{\mathrm{out}}\, f = (\mathsf{RLD})_{\mathrm{out}}(g \circ f)$ for composable complete funcoids f and g.
2. $(\mathsf{RLD})_{\mathrm{out}}\, g \circ (\mathsf{RLD})_{\mathrm{out}}\, f = (\mathsf{RLD})_{\mathrm{out}}(g \circ f)$ for composable co-complete funcoids f and g.

Proof. Let f, g are composable complete funcoids.

$(\mathsf{FCD})((\mathsf{RLD})_{\mathrm{out}}\, g \circ (\mathsf{RLD})_{\mathrm{out}}\, f) = (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}}\, g \circ (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}}\, f = g \circ f$.

Thus (taking into account that $(\mathsf{RLD})_{\mathrm{out}}\, g \circ (\mathsf{RLD})_{\mathrm{out}}\, f$ is complete) we have $(\mathsf{RLD})_{\mathrm{out}}\, g \circ (\mathsf{RLD})_{\mathrm{out}}\, f = (\mathsf{RLD})_{\mathrm{out}}(g \circ f)$.

For co-complete funcoids it's dual. □

9 Not yet written

Continuity in metric spaces is continuity in topology spaces; uniform continuity in metric spaces is continuity in proximity and uniform spaces.

Pointfree analog of the lattice Γ. Also consider the lattice of finite unions of funcoidal products of fitlers (and generalizations).

Introduce core of a lattice $\mathsf{FCD}(\mathscr{F}(\mathfrak{A}); \mathscr{F}(\mathfrak{B}))$ as $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$. Generalize it for staroids. Also filter on $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ can be considered as pointfree reloids.

Uniform spaces (or proximities?) are equivalent to sets of filters? (Do tornings bijectively correspond to uniform spaces?)

Is Cor a functor for a. funcoids; b. reloids? Isn't it adjoint of $\uparrow^{\mathsf{FCD}}$ or $\uparrow^{\mathsf{RLD}}$?

Adjunction of prefunctors: http://www.sciencedirect.com/science/article/pii/0304397585900623 (free download, also Google for "pre-adjunction", also "semi" instead of "pre") Are (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ adjunct?

Definition 30. A morphism of a category each Mor-sets of which is a meet-semilattice, is *weakly metamonovalued* iff $(g \sqcap h) \circ f = (g \circ f) \sqcap (h \circ f)$. Similarly define *weakly metainjective*.

Prove that monovalued, metamonovalued, and weakly metamonovalued are the same for Rel, FCD, and RLD.

What are pointfree funcoids between $\mathscr{P}A$ and $\mathscr{P}B$?

Conjecture 31. For an injective funcoid f:

1. $\langle f\rangle \bigsqcap^{\mathfrak{F}} S = \bigsqcap_{X \in S} \langle f\rangle X$ and $S \in \mathscr{P}\mathscr{P}\operatorname{Src} f$.
2. $\langle f\rangle \bigsqcap^{\mathfrak{F}} S = \bigsqcap_{X \in S} \langle f\rangle X$ and $S \in \mathscr{P}\mathfrak{F}\operatorname{Src} f$.

Equality $(\bigcap G) \circ f = \bigcap_{g \in G} (g \circ f)$ for every G implies that f is a function. Generalize for funcoids and reloids.

I do some research in:

- backward.pdf
- multireloids-relationships.pdf

Question: Can we restore the set of binary relations, knowing only order of $\mathsf{FCD}(A; B)$? Note that it is not the center of the lattice, as not all funcoids are in the center. Yes, it can be characterized as joins of complemented funcoids or joins of complemented atomic funcoids. Is every complemented funcoid principal? This way principality can be generalized for pointfree funcoids. The set of principal p.f. funcoids is join-closed. When filtrator of pointfree funcoids is filtered?

Should we extend filtrators with finite join/meet closed core to nullary closed (having bottom/top)? The old concept shall be named binary join/meet closed filtrators. These are related with up/down aligned filtrators.

$a \mathbin{?} b = \bigsqcup \{z \in \mathfrak{A} \mid a \sqsupseteq b \sqcap z\} = \overline{b \setminus a} = \bar{b} \sqcup a$ for filters. Also dual of second quasidifference.

Define Fréchet element for a filtrator by the formula $\Omega = \max \{\mathcal{X} \in \mathfrak{F} \mid \operatorname{Cor} \mathcal{X} = 0^{\mathfrak{Z}}\}$. (It uses the formula $\operatorname{Cor} \bigsqcup^{\mathfrak{F}} S = \bigsqcup^{\mathfrak{F}} \langle \operatorname{Cor} \rangle S$ which in turn uses properties of Fréchet filter, so this would probably a circular proof.)

What about pseudocomplement filter of infinite joins and meets of filters?

Write an explicit formula for composition with a complete reloid (with the function $F(\alpha)$ to which the complete reloid bijectively corresponds). Using this formula prove that complete reloids are meta-complete. Also for funcoids.

Conjecture 32. $\delta = [f]$ for a funcoid f iff all of the following: [TODO: generalize for staroids]

1. $\neg(0 \,\delta\, \mathcal{Y})$
2. $\neg(\mathcal{X} \,\delta\, 0)$
3. $(\mathcal{I} \sqcup \mathcal{J}) \,\delta\, \mathcal{K} \Leftrightarrow \mathcal{I} \,\delta\, \mathcal{K} \vee \mathcal{J} \,\delta\, \mathcal{K}$;
4. $\mathcal{K} \,\delta\, (\mathcal{I} \sqcup \mathcal{J}) \Leftrightarrow \mathcal{K} \,\delta\, \mathcal{I} \vee \mathcal{K} \,\delta\, \mathcal{J}$;
5. $\mathcal{X} \,[f] \bigsqcap S \Leftrightarrow \forall \mathcal{Y} \in S: \mathcal{X} \,[f]\, \mathcal{Y}$ for filtered set S of filters;
6. $\bigsqcap S \,[f]\, \mathcal{Y} \Leftrightarrow \forall \mathcal{X} \in S: \mathcal{X} \,[f]\, \mathcal{Y}$ for filtered set S of filters.

Conjecture 33. Every funcoid is a composition of a co-complete funcoid and complete funcoid (or vice versa?) [TODO: Try to prove it using the fact that a funcoid is a join of products of ultrafilters. What's about reloids?] [TODO: If this conjecture is false, what about representing every funcoid as compositon of three funcoids: complete, principal, and co-complete?]

The following are equivalent for a funcoid f (call it *strictly monovalued funcoid*):

1. f is a function restricted to a filter;
2. f corresponds to a monovalued reloid;
3. Is it equivalent to monovaluedness of $(\mathsf{RLD})_{\mathrm{in}} f$ or $(\mathsf{RLD})_{\mathrm{out}} f$? ($(\mathsf{RLD})_{\mathrm{in}} f$ is not monovalued if f is an identity)
4. other? (no ideas: open question)

Use the above result for ordering of filters.

Homeomorphisms between funcoids. They are also isomorphisms between filters?

Check current.tm and other files.

Preservatoin of properties (reflexivity, summetry, etc.) of funcoids and reloids by lattice operations.

9.1 Last axiom of proximity

As the following propositions show (merge them into one theorem), the last axiom of proximity is equivalent to transitivity of funcoids:

Proposition 34. If f is a transitive, symmetric funcoid, then the last axiom of proximity holds.

Proof. $\neg(A\,[f]\,B)\Leftrightarrow\neg(A\,[f^{-1}\circ f]\,B)\Leftrightarrow\langle f\rangle B\asymp\langle f\rangle A\Leftrightarrow\exists M\in\operatorname{Ob} f: M\asymp\langle f\rangle A\wedge\overline{M}\asymp\langle f\rangle B$. □

Proposition 35. For a reflexive funcoid, the last axiom of proximity implies that it is transitive and symmetric.

Proof. Let $\neg(A\ [f]\ B)$ implies $\exists M: M\asymp\langle f\rangle A\wedge\overline{M}\asymp\langle f\rangle B$. Then $\neg(A\ [f]\ B)$ implies $\neg(A\ [f^{-1}\circ f]\ B)$ that is $f\sqsupseteq f^{-1}\circ f$ and thus $f=f^{-1}\circ f$. By theorem ??(about transitive endomorphisms) f is transitive and symmetric. □

So proximity spaces are the same as reflexive, symmetric, transitive funcoids.

Remove all other definitions of uniform spaces, to be defined exactly once.

10 Misc

Say that (FCD) and $\uparrow^{\mathsf{FCD}}$, $\uparrow^{\mathsf{RLD}}$ are functors.

$\bigsqcup\{F(x)\mid x\in A\}\rightarrow\bigsqcup_{x\in A}F(x)$ (first define this notation).

Define $C(A;B)=\operatorname{Mor}_C(A;B)$.

Change superflusous notation: $\uparrow^{\mathsf{FCD}(A;B)}f\rightarrow\uparrow^{\mathsf{FCD}}(A;B;f)$ and likewise for RLD. The old notation is sometimes useful as in the definition $\Delta=\bigsqcap\left\{\uparrow^{\mathfrak{F}(\mathbb{R})}(-\varepsilon;\varepsilon)\mid\varepsilon\in\mathbb{R},\varepsilon>0\right\}$.

Proofs that $\uparrow^{\mathsf{FCD}}(g\circ f)=\uparrow^{\mathsf{FCD}}g\circ\uparrow^{\mathsf{FCD}}f$ and $\uparrow^{\mathsf{RLD}}(g\circ f)=\uparrow^{\mathsf{RLD}}g\circ\uparrow^{\mathsf{RLD}}f$.

Probably, $\mathrm{id}^{C(A)}\rightarrow 1^C_A$ and leave $\mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}}$ for restricted identity funcoids and reloids.

Theorem 36. A complete lattice is atomistic iff it is atomically separable.

Proof.

- **⇒.** Let our poset is atomistic. Then obviously $\operatorname{atoms}a\neq\operatorname{atoms}b$ for elements $a\neq b$.
- **⇐.** Let "atoms" be injective. Consider an element a of our poset. Let $b=\bigsqcup\operatorname{atoms}a$. Obviously $b\sqsubseteq a$ and thus $\operatorname{atoms}b\subseteq\operatorname{atoms}a$. But if $x\in\operatorname{atoms}a$ then $x\sqsubseteq b$ and thus $x\in\operatorname{atoms}b$. So $\operatorname{atoms}a=\operatorname{atoms}b$. By injectivity $a=b$ that is $a=\bigsqcup\operatorname{atoms}a$. □

Definition 37. $\bigsqcup^C X\stackrel{\text{def}}{=}\bigsqcup^{C(\operatorname{Src}X;\operatorname{Dst}X)}X$ for a morphism X of a directed multigraph C each Mor-set of which is a poset. Similarly for $\bigsqcap$, $\sqcup$, $\sqcap$.

Say: Whilst I have (mostly) throughly studied basic properties of funcoids, *staroids* (defined below) are yet much a mystery. For example, we do not know whether the set of staroids on powersets is atomic.

star-comparison.tm

cross-composition-funcoids.tm

todd-notes.tm

11 Errors

"Theorem 17.150. Anchored relations with objects being atomic posets and above defined compositions form a quasi-invertible category with star-morphisms."

It is wrong, because composition of a star-morphism m with identify morphisms may be not equal to m. In the definition of general cross-composition product we can replace quasi-invertible category with quasi-invertible pre-category.

Bibliography

[1] Victor Porton. Filters on posets and generalizations. *International Journal of Pure and Applied Mathematics*, 74(1):55–119, 2012. http://www.mathematics21.org/binaries/filters.pdf.