Rewrite plan for my research monograph

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This is my plan to rewrite my draft research monograph “Algebraic General Topology. Volume 1”.

See http://www.mathematics21.org/binaries/volume-1.pdf for the old draft. Say that the book intended mainly to represent the current status of my research. Add thanks to Todd Trimble and Andreas Blass, Robert Martin Solovay. Shall I denote sets like \[ \mathcal{A} \] instead of \( \{ x \in A \mid P(x) \} \)?

Continuity of functions between metric spaces as generalized continuity. Draw a big diagram of conjunctions and implications of different properties of filtrators (and (p.f.) funcoids also).

Unify terminology: Say powerset filter instead filter on a set.

1 Immediate actions

What I am going to do next:

1. Replace hardcoded item references with \LaTeX cross-references.
2. Rewrite with arbitrary primary filtrators rather than filters. (Instead \( \mathcal{A} \) say “\( \mathcal{A} \) is base of a primary filtrator”.)
3. Join theorems into implication tuples.
4. Generalize theorems using the formula of meet of filters for distributive lattices for meet-semilattices.

2 Implications tuples

In the old version of my book there are multiple situations where there are both theorems of the form \( B \Rightarrow C \) and \( A \Rightarrow C \) where the \( A \Rightarrow C \) is proved using that \( A \Rightarrow B \) and \( B \Rightarrow C \). It is accompanied with verbal explanation, that in these theorems \( C \) is the most important component of the theorems and \( B \Rightarrow C \) is a boring generalization of the “main” theorem \( A \Rightarrow C \).

I decided to rewrite every occurrence of this case as single theorem \( A \Rightarrow B \Rightarrow C \) with multiple implications. (There may be more than two implications, in general it is \( P_1 \Rightarrow ... \Rightarrow P_n \) implication.)

To describe such implications verbally, I define implications tuples.

Definition 1. An implications tuple is a tuple \((P_1, ..., P_n)\) such that \( P_1 \Rightarrow ... \Rightarrow P_n \).

Obvious 2. \((P_1, ..., P_n)\) is an implications tuple iff \( P_i \Rightarrow P_j \) for every \( i < j \) (where \( i, j \in \{1, ..., n\} \)).

The following is an example of a theorem using an implication tuple:

Example 3. The following is an implications tuple:

1. \( A \);
2. \( B \);
3. \( C \).

This example means just that \( A \Rightarrow B \Rightarrow C \).

I prefer here a verbal description instead of symbolic implications \( A \Rightarrow B \Rightarrow C \), because \( A, B, C \) may be long English phrases and they may not fit into the formula layout.
The main (intuitive) idea of the theorem is expressed by the implication $P_1 \Rightarrow P_n$, the rest implications ($P_2 \Rightarrow P_n$, $P_3 \Rightarrow P_n$, ...) are purely technical, as they express generalizations of the main idea.

For uniformity theorems in the section about filters and filtrators start with the same $P_1$: “$(\mathfrak{A}; 3)$ is a powerset filtrator.”

### 3 Theory of filters

I have “discovered” that there are four sets (including the set of filters itself) isomorphic to the set of filters on any poset.


As there are several isomorphic sets, it makes sense to describe it more generally than the special case of the set of filters.

We shall differentiate between $s \mathcal{A} = \langle \text{dual} \rangle^* \setminus \text{up} \mathcal{A} = \neg \langle \text{dual} \rangle^* \setminus \text{up} \mathcal{A}$ and $\partial \mathcal{A}$. There is also the isomorphism with boolean lattices; how to denote it? Probably, I should not merge chapters on funcoids and pointfree funcoids, not to disturb a reader with this difference, when he only starts to learn funcoids. In pointfree funcoids chapter, make two variants of the theorem: with general posets and with boolean lattices.

To describe this I redefine primary filtrator (earlier defined as a filtrator $(\mathfrak{F}; 3)$ where $\mathfrak{F}$ is the set of filters on a poset $3$) in an other (non-equivalent) way. [TODO: Are filtered filtrators the same as primary filtrators?]

**Definition 4.** A primary filtrator is such a filtrator $(\mathfrak{F}; 3)$ that $\mathfrak{F}$ is isomorphic to the set of filters on the poset $3$.

**Definition 5.** A primary filtrator over a poset $3$ is a primary filtrator of the form $(\mathfrak{F}; 3)$. Define $\mathfrak{F}$ as the set of elements of an arbitrary primary filtrator. Call $\mathfrak{F}$ as filter objects.

Probably we don’t need the term filter objects in the chapter about filtrators? Then we can use this term instead for denoting one fixed set isomorphic to filters? Can we denote it $\mathfrak{F}$?

**Theorem 6.** For every poset $3$ there exists a primary filtrator over $3$.

**Proof.** See [1].

**Remark 7.** The concept of a primary filtrator over a poset $3$ is a new incarnation of my old concept of filter objects (see [1]). I refused this idea (in the old version of the book) but now I returned it back. The main idea of filter objects is that the set of principal filters is equated with the base set. And now again $3 \subseteq \mathfrak{F}$ what essentially means that principal filters are element of the base set $3$.

However, this has a drawback. If $S \in \mathcal{P} 3$ we need explicitly differentiate between $\bigsqcup^A S$ and $\prod^A S$.

What I called “complete free stars” could be better called principal free stars. The same is true for ideals and mixers. I should write explicit characterizations of principality for all four kinds of “filter objects”.

Define two isomorphic filtrators $(\mathfrak{F}; 3')$ and $(\mathfrak{F}; 3)$ for a shifted filtrator $(\mathfrak{F}; 3; \uparrow)$. (BTW, define isomorphic filtrators.)

**Proposition 8.** All filters on a finite poset are principal.

$\mathfrak{F}(A) \rightarrow \mathfrak{F}(\mathcal{P} A)$ to eliminate ambiguities. Filter in a set vs filter on a poset, not to make terminology confusion. (Say that it is not a standard terminology.)

Define $\text{Based} \mathcal{F} = \{(\text{Base}(\mathcal{F}); \mathcal{F}) \mid \mathcal{F} \in \text{up} \mathcal{F}\}$ for every element $\mathcal{F}$ of a primary filtrator. The reverse function is named GR. The filtrator whose elements are like such is also a primary filtrator. For them define order, identity, and cartesian product. Use this to eliminate double $\text{GR/xyGR}$ notation, use $\text{up}$ instead of both.
\textbf{Theorem 9.} \[ \prod S = [\cup S]_\cap \] (for every meet-semilattice \( \mathfrak{F} \)) Thus \( \mathfrak{F} \) is a complete lattice. [\text{FIXME:} [\cup S]_\cap \) is a filter base, not filter.]

\textbf{Proof.} For every \( A \in S \) we have \( \cup S \supseteq A \) thus \([\cup S]_\cap \supseteq A \) that is \([\cup S]_\cap \subseteq A \).

If filter \( \mathcal{A} \subseteq \mathcal{X} \) for every \( \mathcal{X} \in S \) then \( \mathcal{A} \supseteq \cup S \) then \( \mathcal{A} \supseteq [\cup S]_\cap \) that is \( \mathcal{A} \subseteq [\cup S]_\cap \).

So \( \prod S = [\cup S]_\cap \).

Get an elementary proof from http://math.stackexchange.com/questions/1206777/an-elementary-proof-about-lters (also elementary-proof-about-filters (also elementary.tm)

http://math.stackexchange.com/questions/462270/are-all-atoms-of-the-lattice-of-filters-principal-filters

I've conjectured that filters on every meet-semilattice are co-brouwerian. It is not so! (take a non-distributive finite lattice) So distributivity is necessary.

\textbf{Proposition 10.} \( \partial \Omega(U) \) is the set of infinite subsets of \( U \).

\section{2-staroids}

Describe pointfree funcoids between filters also through \((-)^* \) instead \((-) \).

\textbf{Definition 11.} 2-staroid is a binary relation \( \delta \) between two posets such that \( \{Y \in \mathfrak{B} \mid \exists X \in \mathfrak{A}: X \delta Y \} \) and \( \{X \in \mathfrak{A} \mid \exists Y \in \mathfrak{B}: X \delta Y \} \) are free stars. [TODO: Say that 2-staroids are a special case of staroids.]

\textbf{Definition 12.} \( \| \) \( \delta \) is the binary relation defined by the formula
\[ \mathcal{X} (\| \delta) \mathcal{Y} \Leftrightarrow \forall X \in \text{up} \mathcal{X}, Y \in \text{up} \mathcal{Y}: X \delta Y. \]

\textbf{Obvious 13.} Provided that \( \mathfrak{A} \) and \( \mathfrak{B} \) are join-semilattice with least elements, \( \delta \in \mathcal{P}(\mathfrak{A} \times \mathfrak{B}) \) is a 2-staroid iff
\[ \neg (I \delta 0), I \cup J \delta K \Leftrightarrow I \delta K \vee J \delta K \quad (\text{for every} \ I, J \in \mathfrak{A}, K \in \mathfrak{B}), \]
\[ \neg (0 \delta I), K \delta I \cup J \Leftrightarrow K \delta I \vee K \delta J \quad (\text{for every} \ I, J \in \mathfrak{B}, K \in \mathfrak{A}). \]

The following theorem was generalized below (for arbitrary distributive lattices rather than as boolean lattices): [TODO: It seems to be generalized for all meet-semilattices!]

\textbf{Theorem 14.} Fix distributive lattices \( \mathfrak{A} \) and \( \mathfrak{B} \). Let \( L_F = \lambda f \in \text{FCD}(\mathfrak{G}(\mathfrak{A}); \mathfrak{G}(\mathfrak{B})): (f)^* \) and \( L_R = \lambda f \in \text{FCD}(\mathfrak{G}(\mathfrak{B}); \mathfrak{G}(\mathfrak{A})): [f]^* \).

1. \( L_F \) is a bijection from the set \( \text{FCD}(\mathfrak{A}; \mathfrak{B}) \) to the set of functions \( \alpha \in \mathfrak{G}(\mathfrak{B})^\mathfrak{A} \) that obey the conditions (for every \( I, J \in \mathfrak{A} \))
\[ \alpha 0 = 0 \mathfrak{G}(\mathfrak{B}), \quad \alpha (I \cup J) = \alpha I \cup \alpha J. \] (1)

For such \( \alpha \) it holds (for every \( \mathcal{X} \in \mathfrak{G}(\mathfrak{A}) \))
\[ \langle L_F^{-1} \alpha \rangle \mathcal{X} = \prod \langle \alpha \rangle \mathcal{X}. \] (2)

2. \( L_R \) is a bijection from the set \( \text{FCD}(\mathfrak{A}; \mathfrak{B}) \) to the set of 2-staroids (denoted \( \delta \)).

For such \( \delta \) it holds (for every \( \mathcal{X} \in \mathfrak{G}(\mathfrak{A}), \mathcal{Y} \in \mathfrak{G}(\mathfrak{B}) \))
\[ \mathcal{X} \ [L_R^{-1} \delta] \mathcal{Y} \Leftrightarrow \mathcal{X} (\| \delta) \mathcal{Y}. \] (3)

\textbf{Proof.} Note that \( \mathfrak{G}(\mathfrak{A}) \) and \( \mathfrak{G}(\mathfrak{B}) \) are also distributive lattices.

2. Define \( \alpha \in \mathfrak{G}(\mathfrak{B}) \) by the formula
\[ \partial (\alpha \mathcal{X}) = \{Y \in \mathfrak{B} \mid \exists X \in \mathfrak{A}: X \delta Y \} \]
for every $X \in \mathfrak{A}$. (It is properly defined because $\{Y \in \mathfrak{B} \mid \exists X \in \mathfrak{A} : X \not\delta Y\}$ is a free star.) Similarly define
\[
\partial(\beta X) = \{X \in \mathfrak{A} \mid \exists Y \in \mathfrak{B} : X \not\delta Y\}.
\]

Let’s continue $\alpha$ and $\beta$ to $\alpha' \in \mathfrak{B}(\mathfrak{B})$ and $\beta' \in \mathfrak{B}(\mathfrak{A})$ by the formulas:
\[
\alpha' X = \bigcap \{\alpha\}^* \uparrow X \quad \text{and} \quad \beta' Y = \bigcap \{\beta\}^* \uparrow Y.
\]

$\mathcal{Y} \cap \alpha' X \neq 0^{\delta(B)} \iff \mathcal{Y} \cap \bigcap \{\alpha\}^* \uparrow X \neq 0^{\delta(B)} \iff \bigcap \{\mathcal{Y}\} \cap \{\alpha\}^* \uparrow X \neq 0^{\delta(B)}$. Let’s prove that
\[
W = \{\mathcal{Y}\} \cap \{\alpha\}^* \uparrow X
\]
is a generalized filter base: To prove it is enough to show that $\{\alpha\}^* \uparrow X$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \{\alpha\}^* \uparrow X$ then exist $X_1, X_2 \in \uparrow X$ such that $\mathcal{A} = \alpha X_1, \mathcal{B} = \alpha X_2$.

Then $\alpha(X_1 \cap X_2) \in \{\alpha\}^* \uparrow X$. So $\{\alpha\}^* \uparrow X$ is a generalized filter base and thus $W$ is a generalized filter base.

By properties of generalized filter bases, $\bigcap \{\mathcal{Y}\} \cap \{\alpha\}^* \uparrow X \neq 0^{\delta(B)}$ is equivalent to
\[
\forall X \in \uparrow X : \mathcal{Y} \cap \alpha X \neq 0^{\delta(B)},
\]
what is equivalent to $\forall X \in \mathcal{X}, \mathcal{Y} \in \mathcal{Y} : \uparrow^B Y \cap \alpha X \neq 0^{\delta(B)} \iff \forall X \in \mathcal{X}, \mathcal{Y} \in \mathcal{Y} : Y \in \partial(\alpha X) \iff \forall X \in \mathcal{X}, Y \in \mathcal{Y} : X \not\delta Y \iff \mathcal{X} \not\delta \mathcal{Y}$. Combining the equivalencies we get $\mathcal{Y} \cap \alpha' X \neq 0^{\delta(B)} \iff \mathcal{X} \not\delta \mathcal{Y}$.

Analogously $\mathcal{X} \cap \beta' \mathcal{Y} \neq 0^{\delta(A)} \iff \mathcal{X} \not\delta \mathcal{Y}$. So $\mathcal{Y} \cap \alpha' X \neq 0^{\delta(A)} \iff \mathcal{X} \cap \beta' \mathcal{Y} \neq 0^{\delta(A)}$, that is $(\mathcal{A}; \mathcal{B}; \alpha'; \beta')$ is a funcoid. From the formula $\mathcal{Y} \cap \alpha' X \neq 0^{\delta(B)} \iff \mathcal{X} \not\delta \mathcal{Y}$ it follows that
\[
X [(\mathcal{A}; \mathcal{B}; \alpha'; \beta')] \mathcal{Y} \iff \uparrow^B Y \cap \alpha' \uparrow^A X \neq 0^{\delta(B)} \iff \uparrow^A X \not\delta \uparrow^B Y \iff X \not\delta Y.
\]

1. Let define the relation $\delta \in \mathfrak{P}(\mathfrak{A} \times \mathfrak{B})$ by the formula $X \not\delta Y \iff \uparrow^B Y \cap \alpha X \neq 0^{\delta(B)}$.

That $\neg (I \not\delta 0^B)$ and $\neg (0^B \not\delta I)$ is obvious (if $0^A$ and $0^B$ are defined). We have $I \cup J \not\delta K \iff \uparrow^B K \cap (I \cup J) \neq 0^{\delta(B)} \iff \uparrow^B K \cap (\alpha I \cup \alpha J) \neq 0^{\delta(B)} \iff \uparrow^B K \cap \alpha I \neq 0^{\delta(B)} \vee \uparrow^B K \cap \alpha J \neq 0^{\delta(B)} \iff I \not\delta K \vee J \not\delta K$ and
\[
\uparrow^B I \cap \uparrow^B J \cap \alpha K \neq 0^{\delta(B)} \iff (\uparrow^B I \cup \uparrow^B J) \cap \alpha K \neq 0^{\delta(B)} \iff \uparrow^B I \cap \alpha K \not\delta \uparrow^B J \cap \alpha K
\]
and
\[
\uparrow^B J \cap \alpha K \not\delta \uparrow^B I \cap \alpha K
\]
That is the formulas (i) are true.

Accordingly to the above there exists a funcoid $f$ such that
\[
\mathcal{X} X[f] \mathcal{Y} \iff \mathcal{X} (\not\delta \mathcal{Y}).
\]

Need to generalize all theorems dependent on it.

5 Staroids and multifuncoids

(val $f)_L$ and $(f)_L$ coincide.

Change notation $[f] \to [f]^*$, $[\not\delta f]^* \to [f]$ (for staroids).

I assumed that upgrading a staroid is a staroid without proof. Fill this hole. (Definition 17.69 for an example.) This is addressed in theorem 17.83.

\[
\uparrow^*_{\text{base}} \mathcal{F} g = \bigcap X \in g X \quad \text{for a free star} g \text{ on powersets}. \quad \text{Can we use this formula instead of upgrading in the book?}
\]

Because the set of free stars is identified with the set of filters, the set of staroids (of a given form) can be identified with the set of multifuncoids on primary filtrators ($\mathfrak{A}_i; \mathfrak{Z}_i$). Can all staroids and all multifuncoids be identified?

This allows to thoroughly revise the theory of staroids and multifuncoids.

The last chapter of my book (“Identity staroids”) contains errors. I am going to rewrite it after switching to this new notation.

[TODO: Generalize “Funcoids are filters” for staroids (call it hyperfuncoids).]
Theorem 15. $(\| \| \uparrow f) \ k a = \bigcap_{A \in \up a} \langle f \rangle \ k A$ for every multifuncoid of the form $\mathfrak{A}$ where $k \in \text{arity } f$ and $\mathfrak{A} \ k a$ is a poset of filter objects on a boolean lattice and $a$ is an (arity $f$) \ k family of filters. [TODO: Say exactly which family of filters is meant.]

Proof. $\mathcal{X} \neq (\| \| \uparrow f) \ k a \iff a \cup \{(k; \mathcal{X})\} \in \mathsf{GR}$ $(\| \| \uparrow f) \up a (a \cup \{(k; \mathcal{X})\}) \subseteq \mathsf{GR} [f] \forall A \in \up a$, $X \in \up \mathcal{X}: A \cup \{(k; X)\} \in \mathsf{GR}$ $(\| \| \uparrow A) \up a X \subseteq (\| \| \uparrow f) \ k A \iff (\text{because it is separable}) \forall A \in \up a; \mathcal{X} \neq (\| \| \uparrow f) \ k A \Rightarrow (\text{by properties of generalized filter bases and the fact that } [f]$ is an upper set) $\forall X \neq \mathcal{X}$ $\exists A \in \up a (\| \| \uparrow f) \ k A$.

So $(\| \| \uparrow f) \ k a = \bigcap_{A \in \up a} \langle f \rangle \ k A$ because filters on boolean lattice are separable, \Box

Example 16. There is such an anchored relation $f$ that $\| \| \uparrow f$ is not a completary staroid. [TODO: Remove the conjecture about this.] [TODO: This also proves existence of non completary staroids (but not for powersets).]

Proof. (based on an Andreas Blass’s proof)

Take $f$ the set of functions $x: N \to N$ where $x_0$ an arbitrary natural number and $x_i = \{ 0 \ \text{if } n \leq x_0 \}$ for $i = 1, 2, 3, \ldots$.

Let $L_0(0) = \Omega(N)$, $L_0(0) = \| \{0\} \|$ and $L_0(1) = \| \{1\} \|$ for $i > 0$.

Let $X \in (L_0 \cup L_1)$ that is $X \in \up L_0 \cap \up L_1$.

$X$ contains all but finitely many elements of $N$.

For $i > 0$, we have $\{0, 1\} \subseteq X$.

Evidently, $\prod X$ contains an element of $f$.

Now consider any fixed $c \in (0, 1)^N$. There is at most one $k \in N$ such that the sequence $x = [k; c(1); c(2); \ldots]$ (i.e. $c$ with $c(0)$ replaced by $k$) is in $f$. Let $Q = N \backslash \{k\}$ if there is such a $k$ and $Q = N$ otherwise.

Take $Y_i = \{\{c(i)\} \cup \{i\} \}$ for $i = 1, 2, 3, \ldots$. We have $Y \in \up \lambda \in N; L_{c(i)}(i)$.

But evidently $\prod Y$ does not contain an element of $f$. \Box

Example 17. There exists such an (infinite) set $N$ and $N$-ary relation $f$ that $P \in \| \| \uparrow f$ but there are no indexed family $a \in \prod_{i \in N} \text{atoms } P_i$ of atomic filter such that $a \in \| \| \uparrow f$ that is $\forall A \in \up a; f \neq \prod A$.

Proof. Take $L_0$, $L_1$ and $f$ from the proof of example 16. Take $P = L_0 \cup L_1$. If $a \in \prod_{i \in N} \text{atoms } P_i$ then there exists $c \in (0, 1)^N$ such that $a_i \subseteq L_{c(i)}(i)$ (because $L_{c(i)}(i) \neq 0$). Then from that example it follows that $(\lambda \in N; L_{c(i)}(i)) \neq \mathsf{GR} [f]$ and thus $a \notin \mathsf{GR} [f]$.

Example 18. There is such an anchored relation $F$ that for some $k \in \text{dom } F$

$\langle \| \| \uparrow F \rangle \ k A \neq \bigcup_{a \in \prod_{i \in \text{dom } F} \{a\}} \text{atoms } L_i \langle \| \| \uparrow F \rangle \ k a$.

Proof. Take $P \in \mathsf{GR} F$ from the previous counter-example. We have

\begin{align*}
\forall a \in \prod_{i \in \text{dom } F} \text{atoms } P_i; a \notin \mathsf{GR} P.
\end{align*}

Take $k = 1$.

Let $L = \mathcal{P}\{(\text{dom } F) \backslash \{k\}\}$. Then $a \notin \| \| \uparrow F \|$ and thus $a_k \succ \langle \| \| \uparrow F \rangle \ k a |_{\text{dom } F} \backslash \{k\}$.

Consequently $\mathcal{P}_k \succ \langle \| \| \uparrow F \rangle \ k a |_{\text{dom } F} \backslash \{k\}$ and thus $\mathcal{P}_k \approx \mathcal{P}_k \bigcup_{a \in \prod_{i \in \text{dom } F} \{a\}} \text{atoms } L_i \langle \| \| \uparrow F \rangle \ k a$ because $P_k$ is principal.

But $\mathcal{P}_k \neq \langle \| \| \uparrow F \rangle \ k L$. Thus follows $\langle \| \| \uparrow F \rangle \ k L \neq \bigcup_{a \in \prod_{i \in \text{dom } F} \{a\}} \text{atoms } L_i \langle \| \| \uparrow F \rangle \ k a$. \Box

cross-composition-funcoids.tn

Multifuncoid is a function $\alpha$ distributive over join of every single argument?

Identity pointfree funcoid correspond to a $2$-staroid not for every poset. Consider a non-starrish finite poset. For an argument being a free star, its enough for the poset to be starrish; when we need it to be a primary filtrator? (If it is less that the primary filtrator then to $2$-staroid it may correspond no funcoid.)
6 Typed sets

Use typed sets instead of sets.

**Definition 19.** Typed set is a pair \((U; A)\) of a set \(U\) and its subset \(A\). But we should go more general: We can define typed element as a pair \((\mathfrak{A}; a)\) where \(\mathfrak{A}\) is a poset and \(a \in \mathfrak{A}\). Note that cartesian product can be defined for the special case if it is a powerset.

**Remark 20.** Typed sets is an awkward formalization of type theory sets in ZFC \((U\) is meant to express the type of the set). This book could be better written using type theory instead of ZFC, but I want my book to be understandable for everyone knowing ZFC.

**Definition 21.** \(\mathcal{P}(U; A) = \{ (U; X) \mid X \in \mathcal{P}A \}\). For typed sets define order, binary cartesian product (into the category Rel).

Define typed elements?
\(\mathcal{F}A = \{ (A; X) \mid X \in \mathcal{P}A \} = \{ A \} \times \mathcal{P}A\)

Consider typed sets denoted \(\bot \mathcal{F}A\) and \(\top \mathcal{F}A\). (It is consistent with \(\bot \mathfrak{A}\) and \(\top \mathfrak{A}\).)

\((f)^*\) and \([f]^*\) should be defined for typed sets, not sets.

We shall consider a primary filtrator \((\mathfrak{F}; \mathcal{F}A)\) for every set \(A\) to define funcoids and reloids.

Notwithstanding the above, funcoids and reloids are defined between sets, not typed sets.

7 Other

Generalization of down-aligned (and up-aligned): A filtrator \((\mathfrak{A}; \mathfrak{B})\) is down-closed if \(\forall a \in \mathfrak{A}\exists b \in \mathfrak{B}: b \sqsubseteq a\). a \(\not\in \mathfrak{B}\) \(\iff\) a \(\not\in \mathfrak{A}\) b as a weaker axiom than finite meet closedness.

Pointfree funcoids on filters are equivalent to 2-staroids. Can we use it to prove something?

What is \(a \times^{\text{FCD}} b\) in terms of 2-multifuncoids?

Funcoids can be alternatively defined as: \(Y \not\in (f)^* X \Leftrightarrow X \not\in (f^{-1})^* Y\) where \((f)^*: A \rightarrow \mathfrak{F}(B)\) and \((f^{-1})^*: B \rightarrow \mathfrak{F}(A)\).

For a binary relation \(f\) replace \((f)\) with \((f)^*\) for clarity of notation.

0 \(\rightarrow \bot\), 1 \(\rightarrow \top\).
\(\text{up}^{\mathfrak{A}; \mathfrak{B}} \rightarrow \text{up}^{\mathfrak{A}}\).

**Proposition 22.** \(\prod^{(\text{base F})} \text{up}_{\text{base F}} f = \bigcup_{X \in \text{up}_{\text{base F}} f} \text{up}_{\text{base F}} X\)

**Proof.** [1]

Define \(\bigsqcup_{X \in S} F(X) = \bigsqcup \{ F(X) \mid X \in S \}\) (with index of the operator symbol).

My commentary on Todd Trimble’s notes. Also his constructive proof that the poset of funcoids is a frame.

Denote \(W(A; B) = \{(A; B; F) \mid F \in W[A ; B]\}\) for a set \(W\) (here \(W\) can be FCD, RLD, \(\Gamma\), etc.) and use this notation \(W[A; B]\) where appropriate.

Change notation \(A \rightarrow U\) in the “Multireloids” section.

Proposition 18.34 - define what is \(X\).

Use explicit \(p\text{FCD}\) to denote both pointfree funcoids and multifuncoids.

I overcomplicated the definition of \textit{image} for pointfree funcoids. It should be just \((f)\top\) (because it is used exclusively this way). Is there any single reason to define it in this general complicated way? it seems there is none. Also prove \(\text{im} f = \max \langle (f)\rangle\text{Src} f\).

8 Other new theorems

funcoids-are-filters.tm
The set of funcoids is with co-separable core.

**Proof.** Let \( f, g \in \text{FCD}(A; B) \) and \( f \sqcup g = 1 \). Then for every \( X \in \mathcal{P}A \) we have
\[
\langle f \rangle^* X \sqcup \langle g \rangle^* X = 1 \iff \text{Cor} \langle f \rangle^* X \sqcup \text{Cor} \langle g \rangle^* X = 1 \iff \langle \text{CoCompl} f \rangle^* X \sqcup \langle \text{CoCompl} g \rangle^* X = 1.
\]
Thus \( \langle \text{CoCompl} f \sqcup \text{CoCompl} g \rangle^* X = 1 \);
\[
f \sqcup g = 1 \implies \text{CoCompl} f \sqcup \text{CoCompl} g = 1. \tag{4}
\]
Applying the dual of the formulas (4) to the formula (4) we get:
\[
f \sqcup g = 1 \implies \text{Compl} \text{CoCompl} f \sqcup \text{Compl} \text{CoCompl} g = 1
\]
that is \( f \sqcup g = 1 \implies \text{Cor} f \sqcup \text{Cor} g = 1 \). So \( \text{FCD}(A; B) \) is with co-separable core. [TODO: Say that the filtrator of complete funcoids is also with co-separable core.]

**Proposition 25.** \( \text{ComplFCD}(A; B) \) and \( \text{ComplRLD}(A; B) \) are co-brouwerian lattices. [TODO: remove the question.]

**Proof.** It follows from the fact that these lattices are isomorphic to families of filters (which are complete co-brouwerian lattices) and obvious 17.21.

**Proposition 26.** Every semifiltered filtrator is filtered. [TODO: The reverse implication is already proved.] [TODO: See also http://math.stackexchange.com/questions/1198368/a-question-on-order-theory-an-ordered-set-and-its-subset]

**Proof.** \( a = \prod \mathfrak{A} \) up \( a \) is equivalent to \( a \) is a greatest lower bound of up \( a \). That is the implication that \( b \) is lower bound of up \( a \) implies \( a \sqsupseteq b \).
\[
b \text{ is lower bound of up } a \text{ implies up } b \sqsupseteq \text{up } a. \text{ So as it is semifiltered } a \sqsupseteq b.
\]

### 8.1 Hyperfuncoids

Let \( \mathfrak{A} \) is an indexed family of sets.

*Products* are \( \prod A \) for \( A \in \prod \mathfrak{A} \).

*Hyperfuncoids* are filters \( \mathfrak{A}^\Gamma \) on the lattice \( \Gamma \) of all finite unions of products.

**Problem 27.** Is \( \prod \text{FCD} \) a bijection from hyperfuncoids \( \mathfrak{A}^\Gamma \) to:
1. prestaroids on \( \mathfrak{A} \);
2. staroids on \( \mathfrak{A} \);
3. completary staroids on \( \mathfrak{A} \)?

If yes, is \( \text{up}^\Gamma \) defining the inverse bijection?
If not, characterize the image of the function \( \prod \text{FCD} \) defined on \( \mathfrak{A}^\Gamma \).

### 8.2 Relationships between funcoids and reloids

**Lemma 28.** If \( a, b, c \) are filters on powersets and \( b \neq 0 \), then
\[
\{ G \circ F \mid F \in \text{atoms}^{\text{RLD}}(a \times^{\text{RLD}} b), G \in \text{atoms}^{\text{RLD}}(b \times^{\text{RLD}} c) \} = a \times^{\text{RLD}} c.
\]
\[ a \times_{\text{RLD}} c = (b \times_{\text{RLD}} c) \circ (a \times_{\text{RLD}} b) = (\text{corollary 7.18}) = \bigcup_{x \in X} \{ G \circ F \mid F \in \text{atoms}_{\text{RLD}}(a \times_{\text{RLD}} b), G \in \text{atoms}_{\text{RLD}}(b \times_{\text{RLD}} c) \}. \]

\[ (\text{RLD})_{\text{in}}(g \circ f) = (\text{RLD})_{\text{in}} g \circ (\text{RLD})_{\text{in}} f \text{ for every composable funcoids } f \text{ and } g. \]

\[ \text{Proof.} \]

Therefore, we have closed the conjecture as it is now proved.}

\[ (\text{RLD})_{\text{in}}(g \circ f) = (\text{RLD})_{\text{in}} g \circ (\text{RLD})_{\text{in}} f \text{ for every composable funcoids } f \text{ and } g. \]

\[ \text{Proof.} \]

\[ (\text{RLD})_{\text{in}} g \circ (\text{RLD})_{\text{in}} f = (\text{corollary 7.18}) = \bigcup_{x \in X} \{ G \circ F \mid F \in \text{atoms}_{\text{RLD}}(\text{RLD})_{\text{in}} f, G \in \text{atoms}_{\text{RLD}}(\text{RLD})_{\text{in}} g \}. \]

\[ (\text{FCD})_{\text{out}} g \circ (\text{FCD})_{\text{out}} f = (\text{RLD})_{\text{out}} g \circ (\text{RLD})_{\text{out}} f \text{ for every composable funcoids } f \text{ and } g. \]

\[ \text{Proof.} \]

\[ (\text{RLD})_{\text{out}} g \circ (\text{RLD})_{\text{out}} f = (\text{RLD})_{\text{out}} f \circ (\text{RLD})_{\text{out}} g \]

\[ \text{for every composable funcoids } f \text{ and } g. \]

\[ (\text{FCD})_{\text{out}} g \circ (\text{FCD})_{\text{out}} f = (\text{RLD})_{\text{out}} g \circ (\text{RLD})_{\text{out}} f \text{ for every composable funcoids } f \text{ and } g. \]

\[ \text{Proof.} \]

\[ (\text{RLD})_{\text{out}} g \circ (\text{RLD})_{\text{out}} f = (\text{RLD})_{\text{out}} f \circ (\text{RLD})_{\text{out}} g \]

\[ \text{for every composable funcoids } f \text{ and } g. \]
We can limit to the case when \( L \) is a reloidal product. Then
\[
L \in \bigcap \{ \text{up}(x) \times \text{RLD}(g) \{x\} \mid x \in \text{Src} f \} = \bigcap \{ \{ x \} \times Y \mid Y \in \text{up}(g) \{x\} \mid x \in \text{Src} f \}.
\]
It’s enough to prove that \( L \in \text{up} g \). Really, \( \forall x \in \text{Src} f : \langle L \rangle^* \{x\} \in \text{up}(g) \{x\} \) because
\[
(L)^* \{x\} \supseteq (T)^* \{x\}
\]
and thus
\[
T = \bigcap \{ \{ x \} \times Y \mid Y \in \text{up} G(x) \mid x \in \text{Src} f \}.
\]
and thus
\[
\langle L \rangle^* \{x\} \supseteq \bigcap \{ \{(x) \times Y \}^* \{x'\} \mid x' = x, Y \in \text{up} G(x) \mid x \in \text{Src} f \} = \{Y \mid Y \in \text{up} G(x') \} = \text{up} G(x').
\]
So \( \langle L \rangle^* \{x\} \in \text{up}(g) \{x\} \) and thus \( L \in \text{up} g \).

\[\square\]

**Corollary 31.** \( f \neq g \Rightarrow (\text{RLD})_{\text{out}} f \neq (\text{RLD})_{\text{out}} g \) for complete funcoids \( f \) and \( g \).

**Theorem 32.** Composition of complete reloids is complete.

**Proof.** Let \( f, g \) be complete reloids. Then \( \langle \text{FCD} \rangle(g \circ f) = \langle \text{FCD} \rangle g \circ \langle \text{FCD} \rangle f \). Thus (because \( \langle \text{FCD} \rangle(g \circ f) \) is a complete funcoid) we have \( g \circ f = (\text{RLD})_{\text{out}} \langle \text{FCD} \rangle g \circ \langle \text{FCD} \rangle f \), but \( \langle \text{FCD} \rangle g \circ \langle \text{FCD} \rangle f \) is a complete funcoid, thus \( g \circ f \) is a complete reloid.

\[\square\]

**Theorem 33.**

1. \( (\text{RLD})_{\text{out}} g \circ (\text{RLD})_{\text{out}} f = (\text{RLD})_{\text{out}} (g \circ f) \) for composable complete funcoids \( f \) and \( g \).
2. \( (\text{RLD})_{\text{out}} g \circ (\text{RLD})_{\text{out}} f = (\text{RLD})_{\text{out}} (g \circ f) \) for composable co-complete funcoids \( f \) and \( g \).

(If so, it is dually true for co-complete funcoids.)

**Proof.** Let \( f, g \) are composable complete funcoids.
\[
\langle \text{FCD} \rangle((\text{RLD})_{\text{out}} g \circ (\text{RLD})_{\text{out}} f) = (\text{FCD})(\text{RLD})_{\text{out}} g \circ \langle \text{FCD} \rangle (\text{RLD})_{\text{out}} f = g \circ f.
\]
Thus (taking into account that \( (\text{RLD})_{\text{out}} g \circ (\text{RLD})_{\text{out}} f \) is complete) we have \( (\text{RLD})_{\text{out}} g \circ (\text{RLD})_{\text{out}} f = (\text{RLD})_{\text{out}} (g \circ f) \).

For co-complete funcoids it’s dual.

\[\square\]

9 Not yet written

Equality \( \bigcap \{ G \circ f = \bigcap_{g \in G} (g \circ f) \} \) for every \( G \) implies that \( f \) is a function. Generalize for funcoids and reloids.

Can we prove without axiom of choice that \( \text{FCD} \) is a co-brouwerian lattice using the fact that it is isomorphic to filters on \( \Gamma \)?

I do some research in:

- backward.pdf
- multireloids-relationships.pdf

Question: Can we restore the set of binary relations, knowing only order of \( \text{FCD}(A ; B) \)? Note that it is not the center of the lattice, as not all funcoids are in the center. Yes, it can be characterized as joins of complemented funcoids or joins of complemented atomic funcoids. Is every complemented funcoid principal? This way principality can be generalized for pointfree funcoids. The set of principal p.f. funcoids is join-closed. When filtrator of pointfree funcoids is filtered?

Should we extend filtrators with finite join/meet closed core to nullary closed (having bottom/top)? The old concept shall be named binary join/meet closed filtrators. These are related with up/down aligned filtrators.

\[ a \uparrow b = \bigcup \{ z \in A \mid a \supseteq b \cap z \} = \overline{b \setminus a} = b \cup a \] for filters. Also dual of second quasidifference.
Define Fréchet element for a filterator by the formula \( \Omega = \max \{ \mathcal{X} \in \mathfrak{F} \mid \text{Cor } \mathcal{X} = 0^3 \} \). (It uses the formula \( \text{Cor} \left[ \bigcup^\delta S \right] = \bigcup^\delta \text{Cor}S \) which in turn uses properties of Fréchet filter, so this would probably a circular proof.)

What about pseudocomplement filter of infinite joins and meets of filters? Write an explicit formula for composition with a complete reloid (with the function \( F(\alpha) \) to which the complete reloid bijectively corresponds). Using this formula prove that complete reloids are meta-complete. Also for funcoids.

**Conjecture 34.** \( \delta = [f] \) for a funcoid \( f \) iff all of the following: [TODO: generalize for staroids]

1. \( \neg(0 \delta \mathcal{V}) \)
2. \( \neg(\mathcal{X} \delta 0) \)
3. \( (\mathcal{I} \sqcup \mathcal{J}) \delta \mathcal{K} \Leftrightarrow \mathcal{I} \delta \mathcal{K} \lor \mathcal{J} \delta \mathcal{K} \);  
4. \( F \delta (\mathcal{I} \sqcup \mathcal{J}) \Leftrightarrow F \delta \mathcal{I} \lor F \delta \mathcal{J} \);  
5. \( \mathcal{X} [f] \bigcap S \Leftrightarrow \forall \mathcal{Y} \in S: \mathcal{X} [f] \mathcal{Y} \) for filtered set \( S \) of filters;  
6. \( \mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall \mathcal{X} \in S: \mathcal{X} [f] \mathcal{Y} \) for filtered set \( S \) of filters.

**Conjecture 35.** Every funcoid is a composition of a co-complete funcoid and complete funcoid (or vice versa?) [TODO: Try to prove it using the fact that a funcoid is a join of products of ultrafilters. What’s about reloids?] [TODO: If this conjecture is false, what about representing every funcoid as composition of three funcoids: complete, principal, and co-complete?]

The following are equivalent for a funcoid \( f \) (call it **strictly monovalued funcoid**):

1. \( f \) is a function restricted to a filter;
2. \( f \) corresponds to a monovalued reloid;
3. Is it equivalent to monovaluedness of \( (\text{RLD})_{in}f \) or \( (\text{RLD})_{out}f \)? \((\text{RLD})_{in}f \) is not monovalued if \( f \) is an identity
4. other? (no ideas: open question)

Use the above result for ordering of filters.

Homeomorphisms between funcoids. They are also isomorphisms between filters.  
Check current.tn and other files.  
Preservation of properties (reflexivity, symmetry, etc.) of funcoids and reloids by lattice operations.

### 9.1 Last axiom of proximity

As the following propositions show (merge them into one theorem), the last axiom of proximity is equivalent to transitivity of funcoids:

**Proposition 36.** If \( f \) is a transitive, symmetric funcoid, then the last axiom of proximity holds.

**Proof.** \( \neg(A [f] B) \Leftrightarrow \neg(A [f^{-1} \circ f] B) \Leftrightarrow (f) B \simeq (f) A \Leftrightarrow \exists M \in \text{Ob } f: M \simeq (f) A \land \text{Ob } f \simeq (f) B. \) \( \square \)

**Proposition 37.** For a reflexive funcoid, the last axiom of proximity implies that it is transitive and symmetric.

**Proof.** Let \( \neg(A [f] B) \) implies \( \exists M: M \simeq (f) A \land \text{Ob } f \simeq (f) B. \) Then \( \neg(A [f] B) \) implies \( \neg(A [f^{-1} \circ f] B) \) that is \( f \supseteq f^{-1} \circ f \) and thus \( f = f^{-1} \circ f. \) By theorem ??(about transitive endomorphisms) \( f \) is transitive and symmetric. \( \square \)

So proximity spaces are the same as reflexive, symmetric, transitive funcoids.  
Make a short separate chapter about proximities and uniformities. Remove all other definitions of uniform spaces, to be defined exactly once.
10 Misc

\[ \bigcup \{ F(x) \mid x \in A \} \rightarrow \bigcup_{x \in A} F(x) \] (first define this notation).

Change superfluous notation: \( \uparrow^{\text{FCD}(A;B)} f \rightarrow \uparrow^{\text{FCD}(A;B)} f \) and likewise for \( \text{RLD} \). The old notation is sometimes useful as in the definition \( \Delta = \prod \{ \uparrow^{\mathfrak{A}(R)}(-\epsilon; \epsilon) \mid \epsilon \in \mathbb{R}, \epsilon > 0 \} \).

Proofs that \( \uparrow^{\text{FCD}(g \circ f)} = \uparrow^{\text{FCD} g \circ \text{FCD} f} \) and \( \uparrow^{\text{RLD}(g \circ f)} = \uparrow^{\text{RLD} g \circ \text{RLD} f} \).

Probably, \( \text{id}^{C(A)} \rightarrow 1_{A}^{C} \) and leave \( \text{id}^{\text{FCD}}_{A} \) for restricted identity funcoids and reloids.

**Theorem 38.** A complete lattice is atomistic if and only if it is atomically separable.

**Proof.**

\[ \Rightarrow. \] Let our poset be atomistic. Then obviously atoms \( a \neq b \) for elements \( a \neq b \).

\[ \Leftarrow. \] Let “atoms” be injective. Consider an element \( a \) of our poset. Let \( b = \bigcup \) atoms \( a \). Obviously \( b \subseteq a \) and thus atoms \( b \subseteq a \). But if \( x \in \) atoms \( a \) then \( x \not\subseteq b \) and thus \( x \in \) atoms \( b \). So atoms \( a = \) atoms \( b \). By injectivity \( a = b \) that is \( a = \bigcup \) atoms \( a \). \( \square \)

**Definition 39.** \( \bigcup_{x} X = \bigcup^{C(Src X; Dst X)} X \) for a morphism \( X \) of a directed multigraph \( C \) each Mor-set of which is a poset. Similarly for \( \bigcap, \cup, \cap \). [TODO: Define \( C(A;B) \) and say that funcoids and reloids are directed multigraphs.] [TODO: Similarly for \( up \) and \( Cor/Cor \).]

Say: Whilst I have (mostly) thoroughly studied basic properties of funcoids, staroids (defined below) are yet much a mystery. For example, we do not know whether the set of staroids on powersets is atomic.

**Conjecture 40.** Let \( f \) be a Rel-morphism and \( A \in \mathfrak{F}(\text{Src } f) \). If \( F \in \uparrow^{\text{FCD}(f|A)} \) then there exists a Set-morphism \( G \) and set \( A \in \uparrow^{A} \) such that \( F = G|A \). [TODO: Prove it using (FCD) and (RLD)\_in?]

Prove that preclosure continuity is implied by metric continuity. Also proximal continuity.

**11 Errors**

“Theorem 17.150. Anchored relations with objects being atomic posets and above defined compositions form a quasi-invertible category with star-morphisms.”

It is wrong, because composition of a star-morphism \( m \) with identify morphisms may be not equal to \( m \). In the definition of general cross-composition product we can replace quasi-invertible category with quasi-invertible pre-category.

**Bibliography**