This is a draft which I will never publish. It is because this theory is unnecessarity comlex but has no applications.

Quasi-cartesian functions

Above we have defined several different kinds of product. These products resemble cartesian product. Saying this formally, these functions are quasi-cartesian as defined below.

First (before formal definitions) I will give an example of a quasi-cartesian function. The first and the most prominent example is certain quasi-cartesian situation $\mathfrak{S}$ together with the well known quasi-cartesian function *cartesian product of an indexed family of sets*. Below is a quasi-cartesian situation $\mathfrak{S}$:

1. *Forms* are small sets.
2. *Arguments* are pairs $(\mho; r)$ where $\mho$ is a form and $r \in \mathscr{P}\mho$.
3. The *form corresponding to an argument* $(\mho; r)$ is $\mho$.
4. *Zero* for the form $\mho$ is defined by the formula $Z\,\mho = (\mho; \emptyset)$.

The *quasi-cartesian function* f from $\mathfrak{S}$ to $\mathfrak{S}$ is defined by the formula

$$f\,(\lambda i \in D \colon (\mho_i; r_i)) = \Big(\prod \mho_i;\ \prod r_i \Big).$$

Now proceed to the formal definitions:

Definition 1. *A quasi-cartesian situation $\mathfrak{S}$ is:*

1. *a set F (forms);*
2. *a set X (arguments);*
3. *a function $\rho \in F^X$ (forms of arguments);*
4. *a function $Z \in X^F$ (zeros)*

such that $\rho \circ Z \circ \rho = \rho$.

Definition 2. *The set* ZC *is the set of such small indexed families of arguments that for every small indexed family x of arguments*

$$x \in \mathrm{ZC} \Leftrightarrow \exists i \in \operatorname{dom} x \colon x_i = Z(\rho(x_i)). \tag{1}$$

Remark 3. For theorems below we will need only that ZC is a set of indexed families of arguments. The formula (1) is not required, but there are no need to generalize here.

Let fix two quasi-cartesian situations $\mathfrak{S}_0$ (*source*) and $\mathfrak{S}_1$ (*destination*).

Definition 4. *A pre-quasi-cartesian function is a function f such that the image of f is a subset of X_1 and every element of the domain of f is an indexed family of elements of X_0 such that:*

1. *$x \in \mathrm{ZC}_0 \Leftrightarrow f x = Z_1(\rho_1\, f x)$ for every $x \in \operatorname{dom} f$;*
2. *$\rho_0 \circ x = \rho_0 \circ y \Rightarrow \rho_1\, f x = \rho_1\, f y$ for every $x, y \in \operatorname{dom} f$.*

There exists a function Υ (*aggregation*) conforming to the formula $\Upsilon(\rho_0 \circ x) = \rho_1 fx$.

A pre-quasi-cartesian function can be described first defining a function Υ from small indexed families of forms into forms such that $\rho_1 fx = \Upsilon(\rho_0 \circ x)$ and $x \in \mathrm{ZC}_0 \Leftrightarrow fx = Z_1 \Upsilon(\rho_0 \circ x)$.

Definition 5. *A quasi-cartesian function is such pre-quasi-cartesian function f that $f|_{\{x \in X_0^{\operatorname{dom} \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_0}$ is an injection for every indexed family $\mathfrak{A}$ of forms.*

Definition 6. *A pre-quasi-cartesian function with injective aggregation is a pre-quasi-cartesian function for which the Υ function is injective.*

Exercise 1. Prove that the above defined "cartesian product of an indexed family of sets" is a quasi-cartesian function for two quasi-cartesian systems with injective aggregation.

Definition 7. *Restriction of a quasi-cartesian situation is this quasi-cartesian situation with the set of arguments X replaced by a smaller set X' such that* $\operatorname{im} Z \subseteq X'$ *and forms of arguments ρ replaced with $\rho' = \rho|_{X'}$.*

Proposition 8. *Every restriction of a quasi-cartesian situation is a quasi-cartesian situation.*

Proof. We need to prove $\rho' \circ Z \circ \rho' = \rho'$. This formula follows from $\rho' = \rho|_X$ and $\operatorname{dom} \rho' \supseteq \operatorname{im} Z$. □

Definition 9. *Restriction of a pre-quasi-cartesian function is the restriction of the source quasi-cartesian situation, the destination quasi-cartesian situation, together with a restriction of the quasi-cartesian function to indexed families of the new set of (source) arguments.*

Obvious 10. *Restriction of a pre-quasi-cartesian function is a pre-quasi-cartesian function.*

Obvious 11. *Restriction of a quasi-cartesian function is a quasi-cartesian function.*

Obvious 12. *Restriction of a (pre-)quasi-cartesian situation with injective aggregation is a (pre-)quasi-cartesian situation with injective aggregation.*

When $\operatorname{card} \langle f \rangle \{x\} = 1$ for a binary relation f, we will denote $f(x)$ or fx the element of the singleton $\langle f \rangle \{x\}$.

Proposition 13. *For pre-quasi-cartesian function f we have*

$$(\langle f \rangle \{x \in X_0^{\operatorname{dom} \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\}) \setminus \{Z_1(\Upsilon_f \mathfrak{A})\} = \langle f \rangle (\{x \in X_0^{\operatorname{dom} \mathfrak{A}} \mid \rho \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_1).$$

Proof.

$$\begin{aligned} \langle f \rangle (\{x \in X_0^{\operatorname{dom} \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_0) &= \\ \langle f \rangle \{x \in X_0^{\operatorname{dom} \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A} \wedge x \notin \mathrm{ZC}_0\} &= \\ \langle f \rangle \{x \in X_0^{\operatorname{dom} \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A} \wedge fx \neq Z_1 \Upsilon_f (\rho_0 \circ x)\} &= \\ \langle f \rangle \{x \in X_0^{\operatorname{dom} \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A} \wedge fx \neq Z_1 \Upsilon_f \mathfrak{A}\} &= \\ (\langle f \rangle \{x \in X_0^{\operatorname{dom} \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\}) \setminus \{Z_1(\Upsilon_f \mathfrak{A})\}. & \end{aligned}$$

□

Proposition 14. $\operatorname{card} \langle f^{-1} \rangle \{y\} = 1$ *if* $y \in (\langle f \rangle \{x \in X_0^{\operatorname{dom} \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\}) \setminus \{Z_1(\Upsilon_f \mathfrak{A})\}$.

Proof. card $\langle f^{-1}\rangle\{y\}\geqslant 1$ is obvious. It remains to show that $y\,[f^{-1}]\,a\wedge y\,[f^{-1}]\,b\Rightarrow a=b$ for every a and b. Really, let $y\,[f^{-1}]\,a\wedge y\,[f^{-1}]\,b$. Then $y=fa$ and $y=fb$ and thus $a=b$ because

$y\in\langle f\rangle(\{x\in X_0^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}_0)$ and $f|_{\{x\in X_0^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}_0}$ is an injection. □

Fix quasi-cartesian situations $\mathfrak{S}_A$, $\mathfrak{S}_B$, $\mathfrak{S}_C$ and quasi-cartesian functions f: $\mathfrak{S}_A\rightarrow\mathfrak{S}_B$ and g: $\mathfrak{S}_A\rightarrow\mathfrak{S}_C$ such that $\operatorname{dom} f=\operatorname{dom} g$. Let $\mathfrak{A}$ is a small indexed family of forms. [TODO: Check below formulations (it is possible that I've done little errors confusing A, B, and C).]

For a small indexed family $\mathfrak{A}$ of forms let:

$\varphi_{\mathfrak{A}}=g\circ\mathrm{id}_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\circ f^{-1}$.

Obvious 15. $\varphi_{\mathfrak{A}}=g|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\circ f^{-1}=g\circ\left(f|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\right)^{-1}$.

Proposition 16. *$\varphi_{\mathfrak{A}}$ is a function and* $\operatorname{dom}\varphi_{\mathfrak{A}}=\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}$ *and for every* $y\in\operatorname{dom}\varphi_{\mathfrak{A}}$ *we have*

$$\varphi_{\mathfrak{A}}\,y=\begin{cases}Z_C(\Upsilon_g\,\mathfrak{A}) & \text{if } y=Z_B(\Upsilon_f\,\mathfrak{A});\\ g\,f^{-1}\,y & \text{if } y\neq Z_B(\Upsilon_f\,\mathfrak{A}).\end{cases}$$

Proof. It follows from the previous proposition. □

Theorem 17. $\varphi_{\mathfrak{A}}=g\circ f^{-1}|_{\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}$.

Proof. If $y\in(\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\})\setminus\{Z_B(\Upsilon_f\,\mathfrak{A})\}$ then card $\langle f^{-1}\rangle\{y\}=1$ and thus $\langle f\rangle\{y\}\in\{x\in X_B^{\operatorname{dom}\mathfrak{A}}\mid\rho_B\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}_B$. Consequently

$$\langle\varphi_{\mathfrak{A}}\rangle\{y\}=\langle g\circ f^{-1}\rangle\{y\}=\left\langle g\circ f^{-1}|_{\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\right\rangle\{y\}.$$

$\langle\varphi_{\mathfrak{A}}\rangle\{Z_B(\Upsilon_f\,\mathfrak{A})\}=Z_C(\Upsilon_g\,\mathfrak{A})$ and

$$\begin{aligned}\left\langle g\circ f^{-1}|_{\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\right\rangle\{Z_C(\Upsilon_g\,\mathfrak{A})\} &=\\ \langle g\circ f^{-1}\rangle\{Z_C(\Upsilon_g\,\mathfrak{A})\} &=\\ \langle g\rangle(\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}\cap\mathrm{ZC}_A) &=\\ Z_C(\Upsilon_g\,\mathfrak{A}).\end{aligned}$$

Thus $\langle\varphi_{\mathfrak{A}}\rangle\{y\}=\left\langle g\circ f^{-1}|_{\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\right\rangle\{y\}$ for every $y\in\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}$. □

Theorem 18. *$\varphi_{\mathfrak{A}}$ is a bijection*

$$\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}\rightarrow\langle g\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}.$$

Proof. That $\varphi_{\mathfrak{A}}$ is a surjection

$$\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}\}\rightarrow\langle g\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}\}$$

follows from a proposition above and symmetry. To prove that it is an injection is enough to show that:

1. $g\,f^{-1}\,y\neq Z_C(\Upsilon_g\,\mathfrak{A})$ if $y\neq Z_B(\Upsilon_f\,\mathfrak{A})$ for every $y\in\operatorname{dom}\varphi_{\mathfrak{A}}$.

2. $g\circ f^{-1}|_{(\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\})\setminus\{Z_B(\Upsilon_f\mathfrak{A})\}}$ is injective.

Really,

1. Let $y\neq Z_B(\Upsilon_f\mathfrak{A})$ for some $y\in\operatorname{dom}\varphi_{\mathfrak{A}}$. Then $f^{-1}y\notin\mathrm{ZC}_A$ because otherwise $fx\neq Z_B(\Upsilon_f\mathfrak{A})$ for some $x\in\mathrm{ZC}_A$. Consequently $g\,f^{-1}y\neq Z_C(\Upsilon_g\mathfrak{A})$.

2. $f^{-1}|_{\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}\setminus\{Z_B(\Upsilon_f\mathfrak{A})\}}$ is obviously injective.

   $g|_{\langle f^{-1}\rangle((\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\})\setminus\{Z_B(\Upsilon_f\mathfrak{A})\})}$ is injective because $f^{-1}\;y\notin\mathrm{ZC}_A$ for $y\neq Z_B(\Upsilon_f\mathfrak{A})$.

   Thus $g\circ f^{-1}|_{(\langle f\rangle\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\})\setminus\{Z_B(\Upsilon_f\mathfrak{A})\}}$ is injective. □

As shown by the below theorems, every two quasi-cartesian functions are equivalent up to a bijection:

Theorem 19. $\varphi_{\mathfrak{A}}\circ f|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}=g|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}$.

Proof. If $x\in\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}_A$ then [TODO: Make clear that multivalued functions are not applied below. Rewrite the proof for clarity.]

$$\begin{aligned}(\varphi_{\mathfrak{A}}\circ f|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}})x &=\\ g\,\mathrm{id}_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\,f^{-1}\,f|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\,x &=\\ g\,\mathrm{id}_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\,x &=\\ gx &=\\ (g|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}})x.\end{aligned}$$

If $x\in\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}\cap\mathrm{ZC}_A$ then

$(f|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}})x=Z_B\,\Upsilon_f\,\mathfrak{A}$ and $(g|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}})x=Z_C\,\Upsilon_g\,\mathfrak{A}$;

$\langle f^{-1}\circ(f|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}})\rangle\{x\}=\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}\cap\mathrm{ZC}_A$. Thus it is easy to show that

$\langle g\circ\mathrm{id}_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}}\circ f^{-1}\circ(f|_{\{x\in X_A^{\operatorname{dom}\mathfrak{A}}\mid\rho_A\circ x=\mathfrak{A}\}})\rangle\{x\}=\{Z_C\,\Upsilon_g\,\mathfrak{A}\}$. □

Now let also f and g be with injective aggregation.

Let $\Phi=g\circ f^{-1}$.

Lemma 20. *The set of all* $\langle f\rangle\{x\in X_0^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}\}$, *for* $\mathfrak{A}$ *being small indexed families of forms, is a partition of the set* $\operatorname{im} f$ *where* f *is a quasi-cartesian function with injective aggregation* $\mathfrak{S}_1\to\mathfrak{S}_2$.

Proof. Let denote this set S. That $\bigcup S=\operatorname{im} f$ is obvious.

Suppose $A=\langle f\rangle\{x\in X_0^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}_0\}$ and $B=\langle f\rangle\{x\in X_0^{\operatorname{dom}\mathfrak{A}}\mid\rho_0\circ x=\mathfrak{A}_1\}$ for families $\mathfrak{A}_0\neq\mathfrak{A}_1$ of forms. Then for every $a\in A$ we have $a=fx$ where $\rho_0\circ x=\mathfrak{A}_0$. Thus $\rho_1\,a=\Upsilon\mathfrak{A}_0$ and $\rho_1\,b=\Upsilon\mathfrak{A}_1$; $\rho_1\,a\neq\rho_1\,b$; $a\neq b$. So S is a disjoint set. □

Theorem 21. Φ *is a bijection* $\operatorname{im} f\to\operatorname{im} g$.

Proof. From the lemma. □

Theorem 22. $\Phi \circ f = g$.

Proof. Because $\Phi \circ f|_{\langle f \rangle \{x \in X_A^{\operatorname{dom} \mathfrak{A}} \mid \rho_A \circ x = \mathfrak{A}\}} = g|_{\langle f \rangle \{x \in X_A^{\operatorname{dom} \mathfrak{A}} \mid \rho_A \circ x = \mathfrak{A}\}}$ and the lemma. □

Theorem 23.

1. *$\varphi_{\mathfrak{A}} = \Phi|_{\langle f \rangle \{x \in X_A^{\operatorname{dom} \mathfrak{A}} \mid \rho_A \circ x = \mathfrak{A}\}}$ for every small indexed family $\mathfrak{A}$ of forms.*

2. *Φ is the union of all functions $\varphi_{\mathfrak{A}}$ where $\mathfrak{A}$ is a small indexed family of forms.*

Proof. Both are trivial from the above. □

Definition 24. *A product-projection system is a quasi-cartesian function together with a function* Pr *whose values are indexed families, such that for every $x \in \operatorname{dom} f$:*

$$x \notin \mathrm{ZC}_0 \Rightarrow \operatorname{Pr} f x = x.$$

[TODO: Also: $f(\operatorname{Pr} y) = y$ if $y \in \operatorname{im} f$.]

[TODO: Particular product-projection systems.]

Some examples of quasi-cartesian situations and functions

Definition 25. *Let C is a category with zero morphisms. The corresponding quasi-cartesian situation is:*

- *Forms are pairs of objects.*
- *Arguments are morphisms.*
- *Form of an argument x is $(\operatorname{Src} x; \operatorname{Dst} x)$.*
- *Zero for form $(A; B)$ is the zero morphism 0_{AB}.*

Let us prove it is really a quasi-cartesian situation.

Proof. We need to prove $\rho \circ Z \circ \rho = \rho$. Really, let f is an argument. Then $\rho Z \rho f = \rho Z (\operatorname{Src} f; \operatorname{Dst} f) = \rho 0_{\operatorname{Src} f, \operatorname{Dst} f} = (\operatorname{Src} f; \operatorname{Dst} f) = \rho f$. □

The above definition immediately gives rise of quasi-cartesian situations for binary relations (the category **Rel**), pointfree funcoids (the category of small pointfree funcoids), reloids (the category of small reloids).

Definition 26. *The quasi-cartesian situation of anchored relations:*

- *Forms F are small indexed families of sets.*
- *Arguments are small anchored relations.*
- *Form of an argument is the arity of anchored relation.*

- *Zero Z for a form is the empty relation of that form.*

Proposition 27. *The quasi-cartesian situation of anchored relations is really a quasi-cartesian situation.*

Proof. We need to prove $\rho\circ Z\circ\rho=\rho$. Really let f is an anchored relation of the form $\mathfrak{A}$. Then $Z\rho f$ is the zero relation of the same form ρf. Consequently $\rho Z\rho f=\rho f$. □

Proposition 28. *Reindexation product (for small indexed families of relation) is a quasi-cartesian function with injective aggregation from the quasi-cartesian situation of anchored relations to the same quasi-cartesian situation.*

Proof. First prove that it is a pre-quasi-cartesian function. We need to show that for every small indexed families x, y of anchored relations:

1. $x\in\mathrm{ZC}\Leftrightarrow\prod^{(D)}x=Z\left(\rho\prod^{(D)}x\right)$;

2. $\rho\circ x=\rho\circ y\Rightarrow\rho\prod^{(D)}x=\rho\prod^{(D)}y$;

that is

1. $x\in\mathrm{ZC}\Leftrightarrow\prod^{(D)}x=Z\left(\mathrm{arity}\prod^{(D)}x\right)$;

2. $\mathrm{arity}\circ x=\mathrm{arity}\circ y\Rightarrow\mathrm{arity}\prod^{(D)}x=\mathrm{arity}\prod^{(D)}y$;

that is

1. $x\in\mathrm{ZC}\Leftrightarrow\prod^{(D)}x=Z\left(\mathrm{arity}\prod^{(D)}x\right)$;

2. $\mathrm{arity}\circ x=\mathrm{arity}\circ y\Rightarrow\mathrm{uncurry}(\mathrm{arity}\circ x)=\mathrm{uncurry}(\mathrm{arity}\circ y)$;

but these formulas are obvious.

Next prove that it is a quasi-cartesian function. We need to show that for every indexed family of sets

$$\left(\prod^{(D)}x\right)|_{\{x\in X^{\mathrm{dom}\,\mathfrak{A}}\mid\rho\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}}$$

is injection. This follows from the known fact that $(\prod x)|_{\{x\in X^{\mathrm{dom}\,\mathfrak{A}}\mid\rho\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}}$ is an injection.

Last, we need to prove that it is with injective aggregation. Define $\Upsilon(\rho\circ x)=\rho\prod^{(D)}x$ that is $\Upsilon(\mathrm{arity}\circ x)=\mathrm{uncurry}(\mathrm{arity}\circ x)$ that is $\Upsilon p=\mathrm{uncurry}\,p$. Obviously this Υ is injective. □

Proposition 29. *Ordinated product (for small indexed families of relation) is a quasi-cartesian function from the quasi-cartesian situation of anchored relations to the same quasi-cartesian situation.*

Proof. First prove that it is a pre-quasi-cartesian function. We need to show that for every small indexed families x, y of anchored relations:

1. $x\in\mathrm{ZC}\Leftrightarrow\prod^{(\mathrm{ord})}x=Z\left(\rho\prod^{(\mathrm{ord})}x\right)$;

2. $\rho\circ x=\rho\circ y\Rightarrow\rho\prod^{(\mathrm{ord})}x=\rho\prod^{(\mathrm{ord})}y$;

that is

1. $x\in\mathrm{ZC}\Leftrightarrow\prod^{(\mathrm{ord})}x=Z\left(\mathrm{arity}\prod^{(\mathrm{ord})}x\right)$;

2. $\mathrm{arity}\circ x=\mathrm{arity}\circ y\Rightarrow\mathrm{arity}\prod^{(\mathrm{ord})}x=\mathrm{arity}\prod^{(\mathrm{ord})}y$;

that is

1. $x\in\mathrm{ZC}\Leftrightarrow\prod^{(\mathrm{ord})}x=Z\left(\mathrm{arity}\prod^{(\mathrm{ord})}x\right)$;

2. $\mathrm{arity}\circ x=\mathrm{arity}\circ y\Rightarrow\sum(\mathrm{arity}\circ x)=\sum(\mathrm{arity}\circ y)$;

but these formulas are obvious.

Next prove that it is a quasi-cartesian function. We need to show that for every indexed family of sets

$$\left(\prod^{(D)}x\right)|_{\{x\in X^{\mathrm{dom}\,\mathfrak{A}}\mid\rho\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}}$$

is injection. This follows from the known fact that $(\prod x)|_{\{x\in X^{\mathrm{dom}\,\mathfrak{A}}\mid\rho\circ x=\mathfrak{A}\}\setminus\mathrm{ZC}}$ is an injection. [TODO: More detailed proof.] □

Definition 30. *The quasi-cartesian situation of pointfree funcoids over posets with least elements is:*

1. *Forms are pairs* $(\mathfrak{A};\mathfrak{B})$ *of posets with least elements.*

2. *Arguments are pointfree funcoids.*

3. *The form of an argument* f *is* $(\mathrm{Src}\,f;\mathrm{Dst}\,f)$.

4. *Zero of the form* $(\mathfrak{A};\mathfrak{B})$ *is* $0^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}=(\mathfrak{A}\times\{0^{\mathfrak{B}}\};\mathfrak{B}\times\{0^{\mathfrak{A}}\})$. *(It exists because* $\mathfrak{A}$ *and* $\mathfrak{B}$ *have least elements.)*

Proposition 31. *It is really a quasi-cartesian situation.*

Proof. We need to prove $\rho\circ Z\circ\rho=\rho$. Really,

$$\rho Z\rho f=\rho Z(\mathrm{Src}\,f;\mathrm{Dst}\,f)=\rho 0^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}=(\mathrm{Src}\,f;\mathrm{Dst}\,f)=\rho f.$$ □

Definition 32. *The quasi-cartesian situation of binary relations is:*

1. *Forms are pairs* $(A;B)$ *of sets.*

2. *Arguments are* **Rel**-*morphisms;*

3. *The form of an argument* f *is* $(\mathrm{Src}\,f;\mathrm{Dst}\,f)$.

4. *Zero of the form* $(A;B)$ *is the* **Rel**-*morphism* $(\emptyset;A;B)$.

Proposition 33. *It is really a quasi-cartesian situation.*

Proof. We need to prove $\rho \circ Z \circ \rho = \rho$. Really,

$$\rho Z \rho f = \rho Z(\mathrm{Src}\, f; \mathrm{Dst}\, f) = \rho\, (\emptyset; \mathrm{Src}\, f; \mathrm{Dst}\, f) = (\mathrm{Src}\, f; \mathrm{Dst}\, f) = \rho f. \qquad \square$$

Definition 34. *The quasi-cartesian situation of reloids is:*

1. *Forms are pairs* $(A; B)$ *of sets.*
2. *Arguments are reloids.*
3. *The form of an argument* f *is* $(\mathrm{Src}\, f; \mathrm{Dst}\, f)$.
4. *Zero of the form* $(A; B)$ *is* $0^{\mathsf{RLD}(A; B)}$.

Proposition 35. *It is really a quasi-cartesian situation.*

Proof. We need to prove $\rho \circ Z \circ \rho = \rho$. Really,

$$\rho Z \rho f = \rho Z(\mathrm{Src}\, f; \mathrm{Dst}\, f) = \rho 0^{\mathsf{RLD}(\mathrm{Src}\, f; \mathrm{Dst}\, f)} = (\mathrm{Src}\, f; \mathrm{Dst}\, f) = \rho f. \qquad \square$$

Next we need to prove that cross-composition product of some particular categories with star-morphisms are quasi-cartesian functions with injective aggregation.

Theorem 36. *Cross-composition product (for small indexed families of relations) is a quasi-cartesian function with injective aggregation from the quasi-cartesian situation* $\mathfrak{S}_0$ *of binary relations to the quasi-cartesian situation* $\mathfrak{S}_1$ *of pointfree funcoids over posets with least elements.*

Proof. First prove that it is a pre-quasi-cartesian function. We need to show that for every small indexed families x, y of **Rel**-morphisms:

1. $x \in \mathrm{ZC}_0 \Leftrightarrow \prod^{(C)} x = Z_1\left(\rho_1 \prod^{(C)} x \right)$;
2. $\rho_0 \circ x = \rho_0 \circ y \Rightarrow \rho_1 \prod^{(C)} x = \rho_1 \prod^{(C)} y$;

$\prod^{(C)} x = Z_1\left(\rho_1 \prod^{(C)} x \right) \Leftrightarrow \prod^{(C)} x = Z_1(\mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Src}\, x_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Dst}\, x_i))) \Leftrightarrow \prod^{(C)} x = 0^{\mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Src}\, x_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Dst}\, x_i))} \Leftrightarrow \forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Src}\, x_i) : \left\langle \prod^{(C)} x \right\rangle a = 0^{\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Dst}\, x_i)} \Leftrightarrow \forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Src}\, x_i) : \mathrm{StarComp}(a; x) = 0^{\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Dst}\, x_i)} \Leftrightarrow \forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Src}\, x_i) : \mathrm{GR}\, \mathrm{StarComp}(a; x) = \emptyset$;

$\forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Src}\, x_i) : \mathrm{GR}\, \mathrm{StarComp}(a; x) = \emptyset \Leftarrow x \in \mathrm{ZC}_0$.

$\forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x : \mathrm{Src}\, x_i) : \mathrm{GR}\, \mathrm{StarComp}(a; x) = \emptyset \Rightarrow \mathrm{GR}\, \mathrm{StarComp}\left(\left(\mathrm{dom}\, x; \prod_{i \in \mathrm{dom}\, x} \mathrm{Src}\, x_i \right); x \right) = \emptyset \Rightarrow \nexists L \in \mho^{\mathrm{arity}\, a} \exists y \in \prod_{i \in \mathrm{dom}\, x} \mathrm{Src}\, x_i \forall i \in \mathrm{arity}\, a : y_i\ x_i\ L_i \Leftrightarrow \neg \forall i \in \mathrm{arity}\, a \exists L \in \mho, y \in \mathrm{Src}\, x_i : y x_i L \Leftrightarrow \neg \forall i \in \mathrm{arity}\, a : x_i \neq 0 \Leftrightarrow x \in \mathrm{ZC}_0$.

Thus $x \in \mathrm{ZC}_0 \Leftrightarrow \prod^{(C)} x = Z_1\left(\rho_1 \prod^{(C)} x \right)$.

If $\rho_0 \circ x = \rho_0 \circ y$ then $\mathrm{arity}\, x = \mathrm{arity}\, y = n$ for some index set n.

$\rho_0 \circ x = \rho_0 \circ y \Rightarrow \lambda i \in n$: $(\mathrm{Src}\, x = \mathrm{Src}\, y \wedge \mathrm{Dst}\, x = \mathrm{Dst}\, y) \Rightarrow \rho_1 \prod^{(C)} x = \mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Dst}\, x_i)) = \mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, y\colon \mathrm{Src}\, y_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, y\colon \mathrm{Dst}\, y_i)) = \rho_1 \prod^{(C)} y$.

We have proved that it is a pre-quasi-cartesian function.

Next prove that it is a quasi-cartesian function, that is

$$\left(\prod^{(C)}\right)|_{\{x \in X_0^{\mathrm{dom}\,\mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_0}$$

is an injection for every indexed family $\mathfrak{A}$ of forms. Let $x \in \{x \in X_0^{\mathrm{dom}\,\mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_0$. To prove that it is an injection we will restore the value of x from $\prod^{(C)} x$.

$\left\langle \prod^{(C)} x \right\rangle \{p\} = \mathrm{StarComp}(\{p\}; x)$ for every $p \in \mho^n$.

$L \in \mathrm{GR}\,\mathrm{StarComp}(\{p\}; x) \Leftrightarrow \forall i \in n\colon p_i\, x_i\, L_i \Leftrightarrow \forall i \in n\colon L_i \in \langle x_i \rangle \{p_i\}$ for every $L \in \mho^n$.

Thus $\mathrm{GR}\,\mathrm{StarComp}(\{p\}; x) = \prod_{i \in n} \langle x_i \rangle \{p_i\}$.

Since $x_i \neq 0$ there exist p such that $\langle x_i \rangle \{p_i\} \neq 0$. Take $k \in n$, $p'_i = p_i$ for $i \neq k$ and $p_k = q$ for an arbitrary value q; then

$$\langle x_k \rangle \{q\} = \mathrm{Pr}_k \prod_{i \in n} \langle x_i \rangle \{p'_i\} = \mathrm{Pr}_k\, \mathrm{GR}\, \mathrm{StarComp}(\{p'\}; x) = \mathrm{Pr}_k\, \mathrm{GR} \left\langle \prod^{(C)} x \right\rangle \{p'\}.$$

So the value of x can be restored from $\prod^{(C)} x$ by this formula.

It remained to prove that it is with injective aggregation.

We have $\Upsilon F = (\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon F_{i,0}); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon F_{i,1}))$ for every form F.

It is really an injection because $\mathrm{StarHom}(-)$ are disjoint. □

Theorem 37. *Cross-composition product (for small indexed families of pointfree funcoids between separable atomic posets with least elements and atomistic posets) is a quasi-cartesian function (with injective aggregation) from the quasi-cartesian situation $\mathfrak{S}_0$ of pointfree funcoids over posets with least elements to the quasi-cartesian situation $\mathfrak{S}_1$ of pointfree funcoids over posets with least elements.*

Proof. First prove that it is a pre-quasi-cartesian function. We need to show that for every small indexed families x, y of pointfree funcoids:

1. $x \in \mathrm{ZC}_0 \Leftrightarrow \prod^{(C)} x = Z_1\left(\rho_1 \prod^{(C)} x\right)$;

2. $\rho_0 \circ x = \rho_0 \circ y \Rightarrow \rho_1 \prod^{(C)} x = \rho_1 \prod^{(C)} y$;

$\prod^{(C)} x = Z_1\left(\rho_1 \prod^{(C)} x\right) \Leftrightarrow \prod^{(C)} x = Z_1(\mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Dst}\, x_i))) \Leftrightarrow \prod^{(C)} x = 0^{\mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x: \mathrm{Src}\, x_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x: \mathrm{Dst}\, x_i))} \Leftrightarrow \forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i)\colon \left\langle \prod^{(C)} x \right\rangle a = 0^{\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x: \mathrm{Dst}\, x_i)} \Leftrightarrow \forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i)\colon \mathrm{StarComp}(a; x) = 0^{\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x: \mathrm{Dst}\, x_i)} \Leftrightarrow \forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i)\colon \mathrm{GR}\,\mathrm{StarComp}(a; x) = \emptyset$;

$\forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i)\colon \mathrm{GR}\,\mathrm{StarComp}(a; x) = \emptyset \Leftarrow x \in \mathrm{ZC}_0$.

$\forall a \in \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i)\colon \mathrm{GR}\,\mathrm{StarComp}(a; x) = \emptyset \Rightarrow \mathrm{GR}\,\mathrm{StarComp}\big((\mathrm{dom}\, x; \prod_{i \in \mathrm{dom}\, x} \mathrm{Src}\, x_i); x\big) = \emptyset \Rightarrow \nexists L \in \mho^{\mathrm{arity}\, a} \exists y \in \prod_{i \in \mathrm{dom}\, x} \mathrm{Src}\, x_i \cap \prod_{i \in \mathrm{dom}\, x} \mathrm{atoms}\, \mathrm{Src}\, x_i \forall i \in \mathrm{arity}\, a\colon y_i\ [x_i]\ L_i \Leftrightarrow \nexists L \in \mho^{\mathrm{arity}\, a} \exists y \in \prod_{i \in \mathrm{dom}\, x} \mathrm{atoms}\, \mathrm{Src}\, x_i \forall i \in \mathrm{arity}\, a\colon y_i\ [x_i]\ L_i \Leftrightarrow \neg \forall i \in \mathrm{arity}\, a \exists L \in \mho, y \in \mathrm{atoms}\, \mathrm{Src}\, x_i\colon y\, [x_i]\, L \Rightarrow \neg \forall i \in \mathrm{arity}\, a\colon x_i \neq 0 \Leftrightarrow x \in \mathrm{ZC}_0$.

Thus $x \in \mathrm{ZC}_0 \Leftrightarrow \prod^{(C)} x = Z_1\Big(\rho_1 \prod^{(C)} x\Big)$.

If $\rho_0 \circ x = \rho_0 \circ y$ then $\mathrm{arity}\, x = \mathrm{arity}\, y = n$ for some index set n.

$\rho_0 \circ x = \rho_0 \circ y \Rightarrow \lambda i \in n\colon (\mathrm{Src}\, x = \mathrm{Src}\, y \wedge \mathrm{Dst}\, x = \mathrm{Dst}\, y) \Rightarrow \rho_1 \prod^{(C)} x = \mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Src}\, x_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, x\colon \mathrm{Dst}\, x_i)) = \mathsf{FCD}(\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, y\colon \mathrm{Src}\, y_i); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, y\colon \mathrm{Dst}\, y_i)) = \rho_1 \prod^{(C)} y$.

We have proved that it is a pre-quasi-cartesian function.

Next prove that it is a quasi-cartesian function, that is

$$\left(\prod^{(C)}\right)|_{\{x \in X_0^{\mathrm{dom}\, \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_0}$$

is an injection for every indexed family $\mathfrak{A}$ of forms. Let $x \in \{x \in X_0^{\mathrm{dom}\, \mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_0$. To prove that it is an injection we will restore the value of x from $\prod^{(C)} x$.

$\Big\langle \prod^{(C)} x \Big\rangle p = \mathrm{StarComp}(p; x)$ for every $p \in \prod_{i \in n} \mathrm{atoms}\, \mathrm{Src}\, x_i$.

It is easy to see that $\mathrm{GR}\, p \cap \prod_{i \in n} \mathrm{atoms}\, \mathrm{Src}\, x_i = \{p\}$. Thus

$L \in \mathrm{GR}\, \mathrm{StarComp}(p; x) \Leftrightarrow \forall i \in n\colon p_i\, [x_i]\, L_i \Leftrightarrow \forall i \in n\colon L_i \in \langle x_i \rangle p_i$ for every $L \in \prod_{i \in n} \mathrm{Src}\, x_i$.

Thus $\mathrm{GR}\, \mathrm{StarComp}(p; x) = \prod_{i \in n} \langle x_i \rangle p_i$.

Since $x_i \neq 0$ there exist p such that $\langle x_i \rangle p_i \neq 0$. Take $k \in n$, $p'_i = p_i$ for $i \neq k$ and $p'_k = q$ for an arbitrary value q; then

$$\langle x_k \rangle q = \mathrm{Pr}_k \prod_{i \in n} \langle x_i \rangle p'_i = \mathrm{Pr}_k\, \mathrm{GR}\, \mathrm{StarComp}(p'; x) = \mathrm{Pr}_k\, \mathrm{GR} \left\langle \prod^{(C)} x \right\rangle p'. \tag{2}$$

Note that the theorem ?? in [?] applies to every x_i.

So the value of x can be restored from $\prod^{(C)} x$ by this formula.

It remained to prove that it is with injective aggregation.

We have $\Upsilon F = (\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon F_{i,0}); \mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon F_{i,1}))$ for every form F.

It is really an injection because $\mathrm{StarHom}(-)$ are disjoint. □

Conjecture 38. *Cross-composition product (for small indexed families of reloids) is a quasi-cartesian function (with injective aggregation) from the quasi-cartesian situation $\mathfrak{S}_0$ of reloids to the quasi-cartesian situation $\mathfrak{S}_1$ of pointfree funcoids over posets with least elements.*

Remark 39. The above conjecture is unsolved even for product of two multipliers.

Theorem 40. *Reloidal product (for small indexed families of filters on powersets) with multireloid projections is a product-projection system with injective aggregation from the quasi-cartesian situation $\mathfrak{S}_0$ of filters to the quasi-cartesian situation $\mathfrak{S}_1$ of multireloids.*

Ordered quasi-cartesian situations

Definition 41. *An ordered quasi-cartesian situation is a quasi-cartesian situation together with a partial order on each its set of its arguments of each given form.*

Definition 42. *An order-preserving quasi-cartesian function from a quasi-cartesian situation* $\mathfrak{S}_0$ *to a quasi-cartesian situation* $\mathfrak{S}_1$ *is a quasi-cartesian function* σ *such that* $\sigma x \sqsubseteq \sigma y \Rightarrow x \sqsubseteq y$ *for every indexed family* $\mathfrak{A}$ *of forms and* $x, y \in \{x \in X_0^{\operatorname{dom}\mathfrak{A}} \mid \rho_0 \circ x = \mathfrak{A}\} \setminus \mathrm{ZC}_0$.

Obvious 43. *Every order-preserving quasi-cartesian function is a quasi-cartesian function with injective aggregation.*

Remark 44. Using the obvious fact above, we can prove again that the considered quasi-cartesian functions are with injective aggregation using the below proved statements that they are order-preserving.

Proposition 45. *Cross-composition product (for small indexed families of relations) is an order-preserving quasi-cartesian function from the quasi-cartesian situation* $\mathfrak{S}_0$ *of binary relations to the quasi-cartesian situation* $\mathfrak{S}_1$ *of pointfree funcoids over posets with least elements equipped with the usual orderings of these sets.*

Proof. We need to prove $\forall i \in n: (f_i \neq \emptyset \wedge g_i \neq \emptyset) \wedge \prod^C f \sqsubseteq \prod^C g \Rightarrow f \sqsubseteq g$ for every n-indexed families f and g of binary relations.

$\left\langle \prod^{(C)} f \right\rangle \prod a = \prod_{i \in n} \langle f_i \rangle a_i$.

Fix $k \in n$, $x \in \mho$. Let $a = \mho^{n \setminus \{k\}} \cup \{(k; x)\}$. Then

$$\left\langle \prod^{(C)} f \right\rangle \prod a = \prod_{i \in n} \begin{cases} \langle f_i \rangle \mho & \text{if } i \neq k; \\ \langle f_k \rangle \{x\} & \text{if } i = k. \end{cases} \quad \text{and} \quad \left\langle \prod^{(C)} g \right\rangle \prod a = \prod_{i \in n} \begin{cases} \langle g_i \rangle \mho & \text{if } i \neq k; \\ \langle g_k \rangle \{x\} & \text{if } i = k. \end{cases}$$

Taking into account that $\langle f_i \rangle \mho \neq \emptyset$ and $\langle g_i \rangle \mho \neq \emptyset$ for every $i \in n$, by properties of Cartesian product, we get $\langle f_k \rangle \{x\} \sqsubseteq \langle g_k \rangle \{x\}$ for every $x \in \mho$ and thus $f_k \sqsubseteq g_k$. □

Corollary 46. *Cross-composition product (for small indexed families of* **Rel***-morphisms) is an order-preserving quasi-cartesian function from the quasi-cartesian situation* $\mathfrak{S}_0$ *of* **Rel***-morphisms to the quasi-cartesian situation* $\mathfrak{S}_1$ *of pointfree funcoids over posets with least elements.*

Theorem 47. *Let each* $\mathfrak{A}_i$ *(for* $i \in n$ *where* n *is some index set) is a separable poset with least element. Then*

$$\forall i \in n: a_i \neq 0 \wedge \prod^{\mathsf{FCD}} a \sqsubseteq \prod^{\mathsf{FCD}} b \Rightarrow a \sqsubseteq b.$$

Proof. Suppose $a \not\sqsubseteq b$.

$\prod \mathfrak{A}$ is a separable poset, Thus it exists $y \not\asymp a$ such that $y \asymp b$.

We have $\exists i \in n: y_i \not\asymp a_i$ and $\forall i \in n: y_i \asymp b_i$.

Take $k \in n$ such that $y_k \not\asymp a_k$. We have $y_k \asymp b_k$.

Take $z_i = \begin{cases} a_i & \text{if } i \neq k; \\ y_k & \text{if } i = k \end{cases}$ for $i \in n$.

$\forall i \in n: z_i \not\asymp a_i$ (taken in account that $a_i \neq 0$) and $\exists i \in n: z_i \asymp b_i$.

So there exists z such that $z \in \prod^{\mathsf{FCD}} a$ and $z \notin \prod^{\mathsf{FCD}} b$.

$\prod^{\mathsf{FCD}} a \nsubseteq \prod^{\mathsf{FCD}} b$. □

Corollary 48. $\prod^{\mathsf{FCD}}$ *is an order-preserving quasi-cartesian function from the (defined in an obvious way) quasi-cartesian situation of separable posets with least elements to the (defined in an obvious way) quasi-cartesian situation of multifuncoids. [TODO: Write the definitions explicitly.]*

Theorem 49. *Cross-composition product (for small indexed families of pointfree funcoids between separable atomic posets with least elements and atomistic posets) is an order-preserving quasi-cartesian function from the quasi-cartesian situation $\mathfrak{S}_0$ of pointfree funcoids over posets with least elements to the quasi-cartesian situation $\mathfrak{S}_1$ of pointfree funcoids over posets with least elements.*

Proof. It follows from the formula (2). [TODO: More detailed proof.] □

[TODO: Ordinated product is a quasi-cartesian function with injective aggregation.]

[TODO: Reloidal product is an order-preserving quasi-cartesian function.]

[TODO: Upgrading/downgrading quasi-cartesian functions? This is related with displaced product. First prove that upgrading is injective and that injection composed with a quasi-cartesian function is quasi-cartesian.]