

Product funcoids and reloids*

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Abstract

It's introduced product of two funcoids and of two reloids. This is a preliminary partial draft.

Table of contents

1 Introduction	1
2 Lemmas	1
3 Definition	1
4 Displacement	2
5 Conjectures	3
Bibliography	3

1 Introduction

Read [3], [1], and [2] first to understand this article.

2 Lemmas

Lemma 1. Let A, B, C are sets, $f \in \text{FCD}(A; B)$, $g \in \text{FCD}(B; C)$, $h \in \text{FCD}(A; C)$. Then

$$g \circ f \not\approx h \Leftrightarrow g \not\approx h \circ f^{-1}.$$

Proof. See [1]. □

Lemma 2. Let A, B, C are sets, $f \in \text{RLD}(A; B)$, $g \in \text{RLD}(B; C)$, $h \in \text{RLD}(A; C)$. Then

$$g \circ f \not\approx h \Leftrightarrow g \not\approx h \circ f^{-1}.$$

Proof. See [1]. □

Lemma 3. $f \circ (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) = \mathcal{A} \times^{\text{FCD}} \langle f \rangle \mathcal{B}$ for ??.

Proof. $\langle f \circ (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \rangle \mathcal{X} = \left(\begin{cases} \langle f \rangle \mathcal{B} & \text{if } \mathcal{X} \not\approx \mathcal{A} \\ 0 & \text{if } \mathcal{X} \approx \mathcal{A} \end{cases} \right) = \mathcal{A} \times^{\text{FCD}} \langle f \rangle \mathcal{B}$. □

3 Definition

Definition 4. I will call a *quasi-invertible category* a partially ordered dagger category such that holds

$$g \circ f \not\approx h \Leftrightarrow g \not\approx h \circ f^\dagger \tag{1}$$

*. This document has been written using the GNU \TeX MACS text editor (see www.texmacs.org).

for every morphisms $f \in \text{Hom}(A; B)$, $g \in \text{Hom}(B; C)$, $h \in \text{Hom}(A; C)$, where A, B, C are objects of this category.

As it follows from [1], the category of functors and the category of reoids are quasi-invertible (taking $f^\dagger = f^{-1}$). Moreover by [2] the category of pointfree functors between lattices of filter objects on boolean lattices are quasi-invertible.

Definition 5. The *cross-composition product* of morphisms f and g of a quasi-invertible category is the pointfree functor defined by the formulas (for every $a \in \text{Hom}(\text{Src } f; \text{Src } g)$ and $b \in \text{Hom}(\text{Dst } f; \text{Dst } g)$):

$$\langle f \times_1 g \rangle a = g \circ a \circ f^\dagger \quad \text{and} \quad \langle (f \times_1 g)^{-1} \rangle b = g^\dagger \circ b \circ f.$$

The cross-composition product is a pointfree functor from $\text{Hom}(\text{Src } f; \text{Src } g)$ to $\text{Hom}(\text{Dst } f; \text{Dst } g)$.

We need to prove that it is really a pointfree functor that is that

$$b \not\prec \langle f \times_1 g \rangle a \Leftrightarrow a \not\prec \langle (f \times_1 g)^{-1} \rangle b.$$

This formula means $b \not\prec g \circ a \circ f^\dagger \Leftrightarrow a \not\prec g^\dagger \circ b \circ f$ and can be easily proved applying the formula (1) two times.

Proposition 6. $a \prec \langle f \times_1 g \rangle b \Leftrightarrow a \circ f^\dagger \not\prec g^\dagger \circ b$.

Proof. From the lemma. □

Proposition 7. $a \prec \langle f \times_1 g \rangle b \Leftrightarrow f[a \times_1 b]g$.

Proof. $f[a \times_1 b]g \Leftrightarrow f \circ a^\dagger \not\prec b^\dagger \circ g \Leftrightarrow a \circ f^\dagger \not\prec g^\dagger \circ b \Leftrightarrow a \prec \langle f \times_1 g \rangle b$. □

Theorem 8. $(f \times_1 g)^\dagger = f^\dagger \times_1 g^\dagger$.

Proof. For every functors $a \in \text{Hom}(\text{Src } f; \text{Src } g)$ and $b \in \text{Hom}(\text{Dst } f; \text{Dst } g)$ we have:

$$\begin{aligned} \langle (f \times_1 g)^\dagger \rangle b &= g^\dagger \circ b \circ f = g^\dagger \circ b \circ f = \langle f^\dagger \times_1 g^\dagger \rangle b. \\ \langle ((f \times_1 g)^\dagger)^\dagger \rangle a &= \langle f \times_1 g \rangle a = g \circ a \circ f^\dagger = \langle (f^\dagger \times_1 g^\dagger)^\dagger \rangle a. \end{aligned} \quad \square$$

Theorem 9. Let f, g are morphisms of a quasi-invertible category where $\text{Dst } f$ and $\text{Dst } g$ are f.o. on boolean lattices. Then for every f.o. $a_0 \in \mathfrak{F}(\text{Src } f)$, $b_0 \in \mathfrak{F}(\text{Src } g)$ [TODO: change notation $a_0 \rightarrow \mathcal{A}_0$, $b_0 \rightarrow \mathcal{B}_0$]

$$\langle f \times_1 g \rangle (a_0 \times^{\text{FCD}} b_0) = \langle f \rangle a_0 \times^{\text{FCD}} \langle g \rangle b_0.$$

Proof. For every atom $a_1 \times^{\text{FCD}} b_1$ ($a_1 \in \text{atoms}^{\text{Dst } f}$, $b_1 \in \text{atoms}^{\text{Dst } g}$) of the lattice of functors we have:

$a_1 \times^{\text{FCD}} b_1 \not\prec \langle f \times_1 g \rangle (a_0 \times^{\text{FCD}} b_0) \Leftrightarrow a_0 \times^{\text{FCD}} b_0 \prec \langle f \times_1 g \rangle a_1 \times^{\text{FCD}} b_1 \Leftrightarrow (a_0 \times^{\text{FCD}} b_0) \circ f^\dagger \not\prec g^\dagger \circ (a_1 \times^{\text{FCD}} b_1) \Leftrightarrow \langle f \rangle a_0 \times^{\text{FCD}} b_0 \not\prec a_1 \times^{\text{FCD}} \langle g^\dagger \rangle b_1 \Leftrightarrow \langle f \rangle a_0 \not\prec a_1 \wedge \langle g^\dagger \rangle b_1 \not\prec b_0 \Leftrightarrow \langle f \rangle a_0 \not\prec a_1 \wedge \langle g \rangle b_0 \not\prec b_1 \Leftrightarrow \langle f \rangle a_0 \times^{\text{FCD}} \langle g \rangle b_0 \not\prec a_1 \times^{\text{FCD}} b_1$. Thus $\langle f \times_1 g \rangle (a_0 \times^{\text{FCD}} b_0) = \langle f \rangle a_0 \times^{\text{FCD}} \langle g \rangle b_0$ because the lattice $\text{FCD}(\mathfrak{F}(\text{Dst } f); \mathfrak{F}(\text{Dst } g))$ is atomically separable (corollary 64 in [2]). □

Proposition 10. $a_0 \times^{\text{FCD}} b_0 \prec \langle f \times_1 g \rangle a_1 \times^{\text{FCD}} b_1 \Leftrightarrow a_0 \prec \langle f \rangle a_1 \wedge b_0 \prec \langle g \rangle b_1$ for every $a_0 \in \mathfrak{F}(\text{Src } f)$, $a_1 \in \mathfrak{F}(\text{Dst } f)$, $b_0 \in \mathfrak{F}(\text{Src } g)$, $b_1 \in \mathfrak{F}(\text{Dst } g)$.

Proof. $a_0 \times^{\text{FCD}} b_0 \prec \langle f \times_1 g \rangle a_1 \times^{\text{FCD}} b_1 \Leftrightarrow a_1 \times^{\text{FCD}} b_1 \not\prec \langle f \times_1 g \rangle (a_0 \times^{\text{FCD}} b_0) \Leftrightarrow a_1 \times^{\text{FCD}} b_1 \not\prec \langle f \rangle a_0 \times^{\text{FCD}} \langle g \rangle b_0 \Leftrightarrow a_1 \not\prec \langle f \rangle a_0 \wedge b_1 \not\prec \langle g \rangle b_0 \Leftrightarrow a_0 \prec \langle f \rangle a_1 \wedge b_0 \prec \langle g \rangle b_1$. □

4 Displacement

Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices.

$(\mathfrak{A}'; \mathfrak{Z}_0)$ and $(\mathfrak{B}'; \mathfrak{Z}_1)$ are finitely join-closed filtrators with \mathfrak{A}' and \mathfrak{B}' being separable distributive lattices.

Let $f \in \text{FCD}(\mathfrak{A}'; \mathfrak{B}')$.

Then $\text{DP}(f)$ (*displacement* of f) is the pointfree funcoïd such that $X[\text{DP}(f)]Y \Leftrightarrow X[f]Y$ for every $X \in \mathfrak{Z}_0$, $Y \in \mathfrak{Z}_1$.

The pointfree funcoïd $\text{DP}(f)$ exists.

Proof. We need to prove that $\neg(0^{\mathfrak{Z}_0}[f]Y)$, $\neg(X[f]0^{\mathfrak{Z}_1})$, and $I \cup^{\mathfrak{Z}_0} J[f]Y \Leftrightarrow I[f]Y \vee J[f]Y$; $X[f]I \cup^{\mathfrak{Z}_0} J \Leftrightarrow X[f]I \vee X[f]J$ for every suitable I, J . $\neg(0^{\mathfrak{Z}_0}[f]Y)$ and $\neg(X[f]0^{\mathfrak{Z}_1})$ are obvious. $I \cup^{\mathfrak{Z}_0} J[f]Y \Leftrightarrow I \cup^{\mathfrak{A}'} J[f]Y \Leftrightarrow I[f]Y \vee J[f]Y$; $X[f]I \cup^{\mathfrak{Z}_0} J \Leftrightarrow X[f]I \cup^{\mathfrak{B}'} J \Leftrightarrow X[f]I \vee X[f]J$. \square

Remark 11. In the case if \mathfrak{A}' and \mathfrak{B}' are lattices of filter objects, displacement of a pointfree funcoïd may produce a (non-pointfree) funcoïd.

We will define *displaced product* of funcoïds f and g by the formula: $f \times_2 g = \text{DP}(f \times_1 g)$ where $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are some primary filtrators on sets, $(\mathfrak{A}'; \mathfrak{Z}_0)$ and $(\mathfrak{B}'; \mathfrak{Z}_1)$ are filtrators of funcoïds.

Displaced product of two reloïds is defined similarly.

Remark 12. Displaced product of funcoïds is a funcoïd.

Remark 13. Displaced product of reloïds is a funcoïd.

5 Conjectures

Conjecture 14. (both for funcoïds and reloïds)

1. $(\text{RLD})_{\text{in}} a[f \times_2 g](\text{RLD})_{\text{in}} b \Leftrightarrow a[f \times g]b$;
2. $(\text{RLD})_{\text{out}} a[f \times_2 g](\text{RLD})_{\text{out}} b \Leftrightarrow a[f \times g]b$;
3. $(\text{FCD}) a[f \times g](\text{FCD}) b \Leftrightarrow a[f \times_2 g]b$.

[TODO: Associativity of products. Continuity of products. Need to check that it coincides with Tychonov's topology.]

Bibliography

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