Products in dagger categories with complete ordered Mor-sets

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[TODO: This is a rough draft. It is not yet checked for errors.]

1 Prerequisites

I have made this article, except of its last section (about funcoids and reloids) self-contained.

To understand the section on funcoids and reloids you need first read my book [1].

Note 1. What I previously denoted $\prod F$ is now denoted $\prod^{(L)} F$ (and likewise for $\coprod$). The other draft articles referring to this article may be not yet updated.

2 Some terminology

Some terminology from my book [1]:

Let f be a morphism f in a dagger category whose Mor-sets are complete lattices (with an order $\sqsubseteq$ and lattice operations $\bigsqcup$ and $\bigsqcap$), having the order agreed with the dagger.

Definition 2. The morphism f is:

- *monovalued* iff $f \circ f^{\dagger} \sqsubseteq 1_{\mathrm{Dst}\, f}$;
- *injective* iff $f^{\dagger} \circ f \sqsubseteq 1_{\mathrm{Src}\, f}$;
- *entirely defined* iff $f^{\dagger} \circ f \sqsupseteq 1_{\mathrm{Src}\, f}$;
- *surjective* iff $f \circ f^{\dagger} \sqsupseteq 1_{\mathrm{Dst}\, f}$.

Definition 3. A morphism f of a partially ordered category is *metamonovalued* when $(\bigsqcap G) \circ f = \bigsqcap_{g \in G} (g \circ f)$ whenever G is a set of funcoids with a suitable domain and image.

Definition 4. A morphism f of a partially ordered category is *metainjective* when $f \circ (\bigsqcap G) = \bigsqcap_{g \in G} (f \circ g)$ whenever G is a set of funcoids with a suitable domain and image.

Definition 5. A morphism f of a partially ordered category is *metacomplete* when $f \circ (\bigsqcup G) = \bigsqcup_{g \in G} (f \circ g)$ whenever G is a set of funcoids with a suitable domain and image.

Definition 6. A morphism f of a partially ordered category is *co-metacomplete* when $(\bigsqcup G) \circ f = \bigsqcup_{g \in G} (g \circ f)$ whenever G is a set of funcoids with a suitable domain and image.

Obvious 7. Every function $f \in \mathrm{Mor}_{\mathbf{Set}}(A; B)$ considered as a morphism $f \in \mathrm{Mor}_{\mathbf{Rel}}(A; B)$ is monovalued, entirely defined, metamonovalued, metacomplete, and co-metacomplete.

Obvious 8. Every morphism $f \in \mathrm{Mor}_{\mathbf{Rel}}(A; B)$ is metacomplete and co-metacomplete.

Definition 9. Let μ and ν are endomorphisms. A monovalued entirely defined morphism $f \in \mathrm{Mor}(\mathrm{Ob}\, \mu; \mathrm{Ob}\, \nu)$ is *continuous* (denoted $f \in \mathrm{C}(\mu; \nu)$) iff $f \circ \mu \sqsubseteq \nu \circ f$.

In the book [1] is proved:

Proposition 10. $f \in \mathrm{C}(\mu;\nu) \Leftrightarrow \mu \sqsubseteq f^{\dagger} \circ \nu \circ f \Leftrightarrow f \circ \mu \circ f^{\dagger} \sqsubseteq \nu$ for every monovalued entirely defined morphism f.

Proposition 11. [TODO: Check correct usage of these implications below!]

1. Every entirely defined monovalued morphism is metamonovalued and metacomplete.
2. Every surjective injective morphism is metainjective and co-metacomplete.

Proof. 1. Let f be an entirely defined monovalued morphism.
$(\bigsqcap G) \circ f \sqsubseteq \bigsqcap_{g \in G} (g \circ f)$ by monotonicity of composition.
Using the fact that f is monovalued and entirely defined:
$\left(\bigsqcap_{g \in G} (g \circ f)\right) \circ f^{\dagger} \sqsubseteq \bigsqcap_{g \in G} (g \circ f \circ f^{\dagger}) \sqsubseteq \bigsqcap G$;
$\bigsqcap_{g \in G} (g \circ f) \sqsubseteq \left(\bigsqcap_{g \in G} (g \circ f)\right) \circ f^{\dagger} \circ f \sqsubseteq (\bigsqcap G) \circ f$.
So $(\bigsqcap G) \circ f = \bigsqcap_{g \in G} (g \circ f)$.
Let f be a entirely defined monovalued morphism.
$f \circ (\bigsqcup G) \sqsupseteq \bigsqcup_{g \in G} (f \circ g)$ by monotonicity of composition.
Using the fact that f is entirely defined and monovalued:
$f^{\dagger} \circ \left(\bigsqcup_{g \in G} (f \circ g)\right) \sqsupseteq \bigsqcup_{g \in G} (f^{\dagger} \circ f \circ g) \sqsupseteq \bigsqcap G$;
$\bigsqcup_{g \in G} (f \circ g) \sqsupseteq f \circ f^{\dagger} \circ \bigsqcup_{g \in G} (f \circ g) \sqsupseteq f \circ (\bigsqcup G)$.
So $f \circ (\bigsqcup G) = \bigsqcup_{g \in G} (f \circ g)$.
2. By duality. □

3 General product in partially ordered dagger category

To understand the below better, you can restrict your imagination to the case when $\mathcal{C}$ is the category Rel.

3.1 Infimum product

Let $\mathcal{C}$ be a dagger category, each Mor-set of which is a complete lattice (having order agreed with the dagger).

We will designate some morphisms as *principal* and require that principal morphisms are both metacomplete and co-metacomplete. (For a particular example of the category Rel, all morphisms are considered principal.)

Let $\prod^{(Q)} X$ be an object for each indexed family X of objects.

Let π be a partial function mapping elements $X \in \operatorname{dom} \pi$ (which consists of small indexed families of objects of $\mathcal{C}$) to indexed families $\prod^{(Q)} X \to X_i$ of principal morphisms (called *projections*) for every $i \in \operatorname{dom} X$.

We will denote particular morphisms as π_i^X.

Remark 12. In some important examples the function π is entire, that is $\operatorname{dom} \pi$ is the set of all small indexed families of objects of $\mathcal{C}$. However there are also some important examples where it is partial.

Definition 13. *Infimum product* $\prod F$ (such that π is defined at $\lambda j \in n: \operatorname{Src} F_j$ and $\lambda j \in n: \operatorname{Dst} F_j$) is defined by the formula

$$\prod^{(L)} F = \bigsqcap_{i \in \operatorname{dom} F} \left(\left(\pi_i^{\lambda j \in n: \operatorname{Dst} F_j}\right)^{\dagger} \circ F_i \circ \pi_i^{\lambda j \in n: \operatorname{Src} F_j} \right).$$

This formula can be (over)simplified to:

$$\prod^{(L)} F = \bigsqcap_{i \in \operatorname{dom} F} \left(\left(\pi_i^{\operatorname{Dst} \circ F}\right)^{\dagger} \circ F_i \circ \pi_i^{\operatorname{Src} \circ F} \right).$$

Remark 14. $\left(\pi_i^{\lambda j\in n:\operatorname{Dst} F_j}\right)^\dagger \circ F_i \circ \pi_i^{\lambda j\in n:\operatorname{Src} F_j} \in \operatorname{Mor}\left(\prod^{(Q)}_{j\in n} \operatorname{Src} F_j; \prod^{(Q)}_{j\in n} \operatorname{Dst} F_j\right)$ are properly defined and have the same sources and destination (whenever $i\in\operatorname{dom} F$ is), thus the meet in the formulas is properly defined.

Remark 15. Thus

$$F_0 \times^{(L)} F_1 = \left(\left(\pi_0^{(\operatorname{Dst} F_0;\operatorname{Dst} F_1)}\right)^\dagger \circ F_0 \circ \pi_0^{(\operatorname{Src} F_0;\operatorname{Src} F_1)}\right) \sqcap \left(\left(\pi_1^{(\operatorname{Dst} F_0;\operatorname{Dst} F_1)}\right)^\dagger \circ F_1 \circ \pi_1^{(\operatorname{Src} F_0;\operatorname{Src} F_1)}\right)$$

that is product is defined by a pure algebraic formula.

Proposition 16. $\prod^{(L)} F = \max\left\{\Phi \in \operatorname{Mor}\left(\prod^{(Q)}_{j\in n} \operatorname{Src} F_j; \prod^{(Q)}_{j\in n} \operatorname{Dst} F_j\right) \mid \forall i\in n\colon \Phi \sqsubseteq \left(\pi_i^{\lambda j\in n:\operatorname{Dst} F_j}\right)^\dagger \circ F_i \circ \pi_i^{\lambda j\in n:\operatorname{Src} F_j}\right\}$.

Proof. By definition of meet on a complete lattice. □

Corollary 17. $\prod^{(L)} F = \bigsqcup\left\{\Phi \in \operatorname{Mor}\left(\prod^{(Q)}_{j\in n} \operatorname{Src} F_j; \prod^{(Q)}_{j\in n} \operatorname{Dst} F_j\right) \mid \forall i\in n\colon \Phi \sqsubseteq \left(\pi_i^{\lambda j\in n:\operatorname{Dst} F_j}\right)^\dagger \circ F_i \circ \pi_i^{\lambda j\in n:\operatorname{Src} F_j}\right\}$.

Theorem 18. Let π_i^X be metamonovalued morphisms. If $S \in \mathscr{P}(\operatorname{Mor}(A_0;B_0) \times \operatorname{Mor}(A_1;B_1))$ for some sets A_0, B_0, A_1, B_1 then

$$\bigsqcap \left\{a \times^{(L)} b \mid (a;b) \in S\right\} = \bigsqcap \operatorname{dom} S \times^{(L)} \bigsqcap \operatorname{im} S.$$

Proof. $\bigsqcap \{a \times b \mid (a;b) \in S\} = \bigsqcap \left\{\left(\left(\pi_0^{(\operatorname{Dst} a;\operatorname{Dst} b)}\right)^\dagger \circ a \circ \pi_0^{(\operatorname{Src} a;\operatorname{Src} b)}\right) \sqcap \left(\left(\pi_1^{(\operatorname{Dst} a;\operatorname{Dst} b)}\right)^\dagger \circ b \circ \pi_1^{(\operatorname{Src} a;\operatorname{Src} b)}\right) \mid (a;b)\in S\right\} = \bigsqcap \left\{\left(\pi_0^{(\operatorname{Dst} a;\operatorname{Dst} b)}\right)^\dagger \circ a \circ \pi_0^{(\operatorname{Src} a;\operatorname{Src} b)} \mid a \in \operatorname{dom} S\right\} \sqcap \bigsqcap \left\{\left(\pi_1^{(\operatorname{Dst} a;\operatorname{Dst} b)}\right)^\dagger \circ b \circ \pi_1^{(\operatorname{Src} a;\operatorname{Src} b)} \mid b \in \operatorname{im} S\right\} = \left(\left(\pi_0^{(\operatorname{Dst} a;\operatorname{Dst} b)}\right)^\dagger \circ \bigsqcap \{a \mid a\in\operatorname{dom} S\} \circ \pi_0^{(\operatorname{Src} a;\operatorname{Src} b)}\right) \sqcap \left(\left(\pi_1^{(\operatorname{Dst} a;\operatorname{Dst} b)}\right)^\dagger \circ \bigsqcap \{b \mid b\in\operatorname{im} S\} \circ \pi_1^{(\operatorname{Src} a;\operatorname{Src} b)}\right) = \left(\left(\pi_0^{(\operatorname{Dst} a;\operatorname{Dst} b)}\right)^\dagger \circ \left(\bigsqcap \operatorname{dom} S\right) \circ \pi_0^{(\operatorname{Src} a;\operatorname{Src} b)}\right) \sqcap \left(\left(\pi_1^{(\operatorname{Dst} a;\operatorname{Dst} b)}\right)^\dagger \circ \left(\bigsqcap \operatorname{im} S\right) \circ \pi_1^{(\operatorname{Src} a;\operatorname{Src} b)}\right) = \bigsqcap \operatorname{dom} S \times \bigsqcap \operatorname{im} S$. □

Corollary 19. $\left(a_0 \times^{(L)} b_0\right) \sqcap \left(a_1 \times^{(L)} b_1\right) = (a_0 \sqcap a_1) \times^{(L)} (b_0 \sqcap b_1)$.

Corollary 20. $a_0 \times^{(L)} b_0 \not\asymp a_1 \times^{(L)} b_1 \Leftrightarrow a_0 \not\asymp a_1 \wedge b_0 \not\asymp b_1$.

3.2 Infimum product for endomorphisms

Let F is an indexed family of endomorphisms of $\mathcal{C}$.

I will denote $\operatorname{Ob} f$ the object (source and destination) of an endomorphism f.

Let also π_i^X be a monovalued entirely defined morphism (for each $i\in\operatorname{dom} F$).

Then $\prod^{(L)} F = \bigsqcap_{i\in\operatorname{dom} F}\left(\left(\pi_i^{\lambda j\in n:\operatorname{Ob} F_j}\right)^\dagger \circ F_i \circ \pi_i^{\lambda j\in n:\operatorname{Ob} F_j}\right)$ (if π is defined at $\lambda j\in n\colon \operatorname{Ob} F_j$).

Abbreviate $\pi_i = \pi_i^{\lambda j\in n:\operatorname{Ob} F_j}$.

So $\prod^{(L)} F = \bigsqcap_{i\in\operatorname{dom} F}\left((\pi_i)^\dagger \circ F_i \circ \pi_i\right)$.

$\prod^{(L)} F = \max\left\{\Phi \in \operatorname{End}\left(\prod^{(Q)}_{j\in n} \operatorname{Ob} F_j\right) \mid \forall i\in n\colon \Phi \sqsubseteq (\pi_i)^\dagger \circ F_i \circ \pi_i\right\}$.

Taking into account that π_i is a monovalued entirely defined morphism, we get:

Obvious 21. $\prod^{(L)} F = \max\left\{\Phi \in \operatorname{End}\left(\prod^{(Q)}_{j\in n} \operatorname{Ob} F_j\right) \mid \forall i\in n\colon \pi_i \in \mathrm{C}(\Phi; F_i)\right\}$.

Remark 22. The above formula may allow to define the product for non-dagger categories (but only for endomorphisms). In this writing I don't introduce a notation for this, however.

Corollary 23. $\pi_i \in \mathrm{C}\left(\prod^{(L)} F; F_i\right)$ for every $i\in\operatorname{dom} F$.

3.3 Category of continuous morphisms

Let $\pi_i = \pi_i^X$ (for $i \in \operatorname{dom} F$) be entirely defined monovalued morphisms (we suppose it is defined at X).

Let $\bigotimes$ of an indexed family of morphisms is a morphism; $\pi_i \circ \bigotimes f = f_i$; $\bigotimes_{i \in n} (\pi_i \circ f) = f$.

Definition 24. The category $\operatorname{cont}(\mathcal{C})$ is defined as follows:

- Objects are endomorphisms of the category $\mathcal{C}$.
- Morphisms are triples $(f; a; b)$ where a and b are objects and $f\colon \operatorname{Ob} a \to \operatorname{Ob} b$ is an entirely defined monovalue principal morphism of the category $\mathcal{C}$ such that $f \in \mathrm{C}(a; b)$ (in other words, $f \circ a \sqsubseteq b \circ f$).
- Composition of morphisms is defined by the formula $(g; b; c) \circ (f; a; b) = (g \circ f; a; c)$.
- Identity morphisms are $(a; a; \mathrm{id}_a^{\mathcal{C}})$.

It is really a category:

Proof. We need to prove that: composition of morphisms is a morphism, composition is associative, and identity morphisms can be canceled on the left and on the right.

That composition of morphisms is a morphism by properties of generalized continuity.

That composition is associative is obvious.

That identity morphisms can be canceled on the left and on the right is obvious. □

Remark 25. The "physical" meaning of this category is:

- Objects (endomorphisms of $\mathcal{C}$) are spaces.
- Morphisms are continuous functions between spaces.
- $f \circ a \sqsubseteq b \circ f$ intuitively means that f combined with an infinitely small is less than infinitely small combined with f (that is f is continuous).

Definition 26. $\pi_i^{\operatorname{cont}(\mathcal{C})} = \left(\prod^{(L)} F; F_i; \pi_i \right)$.

Proposition 27. π_i are continuous, that is $\pi_i^{\operatorname{cont}(\mathcal{C})}$ are morphisms.

Proof. We need to prove $\pi_i \in \mathrm{C}\left(\prod^{(L)} F; F_i \right)$ but that was proved above. □

Lemma 28. $f \in \operatorname{Mor}_{\operatorname{cont}(\mathcal{C})}\left(Y; \prod^{(L)} F \right)$ is continuous iff all $\pi_i \circ f$ are continuous.

Proof.

⇒. Let $f \in \operatorname{Mor}_{\operatorname{cont}(\mathcal{C})}\left(Y; \prod^{(L)} F \right)$. Then $f \circ Y \sqsubseteq \left(\prod^{(L)} F \right) \circ f$; $\pi_i \circ f \circ Y \sqsubseteq \pi_i \circ \left(\prod^{(L)} F \right) \circ f$; $\pi_i \circ f \circ Y \sqsubseteq \left(\prod^{(L)} F \right) \circ \pi_i \circ f$. Thus $\pi_i \circ f$ is continuous.

⇐. Let all $\pi_i \circ f$ be continuous. Then $\pi_i^{\operatorname{cont}(\mathcal{C})} \circ f \in \operatorname{Mor}_{\operatorname{cont}(\mathcal{C})}(Y; F_i)$; $\pi_i^{\operatorname{cont}(\mathcal{C})} \circ f \circ Y \sqsubseteq F_i \circ \pi_i^{\operatorname{cont}(\mathcal{C})} \circ f$. We need to prove $Y \sqsubseteq f^{\dagger} \circ \left(\prod^{(L)} F \right) \circ f$ that is

$$Y \sqsubseteq f^{\dagger} \circ \prod_{i \in n} ((\pi_i)^{\dagger} \circ F_i \circ \pi_i) \circ f$$

for what is enough (because f is metamonovalued)

$$Y \sqsubseteq \prod_{i \in n} (f^{\dagger} \circ (\pi_i)^{\dagger} \circ F_i \circ \pi_i \circ f)$$

what follows from $Y \sqsubseteq \prod_{i \in n} (f^{\dagger} \circ (\pi_i)^{\dagger} \circ \pi_i \circ f \circ Y)$ what is obvious. □

Theorem 29. $\prod^{(L)}$ together with $\bigotimes$ is a (partial) product in the category $\mathrm{cont}(\mathcal{C})$.

Proof. Obvious.

Check http://math.stackexchange.com/questions/102632/how-to-check-whether-it-is-a-direct-product/102677#102677 □

4 On duality

We will consider duality where both the category $\mathcal{C}$ and orders on Mor-sets are replaced with their dual. I will denote $A \overset{\text{dual}}{\longleftrightarrow} B$ when two formulas A and B are dual with this duality.

Proposition 30. $f \in \mathrm{C}(\mu; \nu) \overset{\text{dual}}{\longleftrightarrow} f^\dagger \in \mathrm{C}(\nu^\dagger; \mu^\dagger)$.

Proof. $f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \circ \mu \sqsubseteq \nu \circ f \overset{\text{dual}}{\longleftrightarrow} \mu^\dagger \circ f^\dagger \sqsupseteq f^\dagger \circ^\dagger \nu^{-1} \Leftrightarrow f^\dagger \in \mathrm{C}(\nu^\dagger; \mu^\dagger)$. □

f is entirely defined$\Leftrightarrow f^\dagger \circ f \sqsupseteq 1_{\mathrm{Src}\, f} \overset{\text{dual}}{\longleftrightarrow} f^\dagger \circ f \sqsubseteq 1_{\mathrm{Src}\, f} \Leftrightarrow f$ is injective$\Leftrightarrow f^\dagger$ is monovalued.

f is monovalued$\Leftrightarrow f \circ f^\dagger \sqsubseteq 1_{\mathrm{Dst}\, f} \overset{\text{dual}}{\longleftrightarrow} f \circ f^\dagger \sqsupseteq 1_{\mathrm{Dst}\, f} \Leftrightarrow f$ is surjective$\Leftrightarrow f^\dagger$ is entirely defined.

5 General coproduct in partially ordered dagger category

The below is the dual of the above, proofs are omitted as they are dual.

Let ι_i [TODO: What is i?] are entirely defined monovalued morphisms to an object Z.

Let $\iota_i \overset{\text{dual}}{\longleftrightarrow} \pi_i$ that is $\iota_i = (\pi_i)^\dagger$. We have the above equivalent to π_i being monovalued and entirely defined.

5.1 Supremum coproduct

Let $\mathcal{C}$ be a dagger category, each Mor-set of which is a complete lattice (having order agreed with the dagger).

We will designate some morphisms as *principal* and require that principal morphisms are both metacomplete and co-metacomplete. (For a particular example of the category Rel, all morphisms are considered principal.)

Let $\coprod^{(Q)} X$ be an object for each indexed family X of objects.

Let ι be a partial function mapping elements $X \in \operatorname{dom} \iota$ (which consists of small indexed families of objects of $\mathcal{C}$) to indexed families $X_i \to \coprod^{(Q)} X$ of principal morphisms (called *injections*) for every $i \in \operatorname{dom} X$.

Definition 31. *Supremum coproduct* $\coprod^{(L)} F$ (such that ι is defined at $\lambda j \in n\colon \mathrm{Dst}\, F_j$ and $\lambda j \in n\colon \mathrm{Src}\, F_j$) is defined by the formula

$$\coprod^{(L)} F = \bigsqcup_{i \in \operatorname{dom} F} \left(\iota_i^{\lambda j \in n : \mathrm{Src}\, F_j} \circ F_i^\dagger \circ \left(\iota_i^{\lambda j \in n : \mathrm{Dst}\, F_j} \right)^\dagger \right).$$

This formula can be (over)simplified to:

$$\coprod^{(L)} F = \bigsqcup_{i \in \operatorname{dom} F} \left(\iota_i^{\mathrm{Src} \circ F} \circ F_i^\dagger \circ \left(\iota_i^{\mathrm{Dst} \circ F} \right)^\dagger \right).$$

Remark 32. $\iota_i^{\lambda j \in n : \mathrm{Src}\, F_j} \circ F_i \circ \left(\iota_i^{\lambda j \in n : \mathrm{Dst}\, F_j} \right)^\dagger \in \mathrm{Mor}\left(\coprod^{(Q)}_{j \in n} \mathrm{Src}\, F_j; \coprod^{(Q)}_{j \in n} \mathrm{Dst}\, F_j \right)$ are properly defined and have the same sources and destination (whenever $i \in \operatorname{dom} F$ is), thus the meet in the formulas is properly defined.

Remark 33. Thus

$$F_0 \amalg^{(L)} F_1 = \left(\iota_0^{(\mathrm{Src}\, F_0;\mathrm{Src}\, F_1)} \circ F_0^\dagger \circ \left(\iota_0^{(\mathrm{Dst}\, F_0;\mathrm{Dst}\, F_1)}\right)^\dagger\right) \sqcup \left(\iota_1^{(\mathrm{Src}\, F_0;\mathrm{Src}\, F_1)} \circ F_1^\dagger \circ \left(\iota_1^{(\mathrm{Dst}\, F_0;\mathrm{Dst}\, F_1)}\right)^\dagger\right)$$

that is coproduct is defined by a pure algebraic formula.

Proposition 34. $\coprod^{(L)} F = \min\left\{\Phi \in \mathrm{End}\left(\coprod_{j\in n}^{(Q)} \mathrm{Ob}\, F_j\right) \mid \forall i \in n: \Phi \sqsupseteq \iota_i^{\lambda j\in n:\mathrm{Src}\, F_j} \circ F_i^\dagger \circ \left(\iota_i^{\lambda j\in n:\mathrm{Dst}\, F_j}\right)^\dagger\right\}$.

Proof. By definition of meet on a complete lattice. □

Corollary 35. $\coprod^{(L)} F = \bigsqcap\left\{\Phi \in \mathrm{End}\left(\coprod_{j\in n}^{(Q)} \mathrm{Ob}\, F_j\right) \mid \forall i \in n: \Phi \sqsupseteq \iota_i^{\lambda j\in n:\mathrm{Src}\, F_j} \circ F_i^\dagger \circ \left(\iota_i^{\lambda j\in n:\mathrm{Dst}\, F_j}\right)^\dagger\right\}$.

Theorem 36. Let π_i^X be metainjective morphisms. If $S \in \mathscr{P}(\mathrm{Mor}(A_0; B_0) \times \mathrm{Mor}(A_1; B_1))$ for some sets A_0, B_0, A_1, B_1 then

$$\bigsqcup \left\{a \times^{(L)} b \mid (a; b) \in S\right\} = \bigsqcup \mathrm{dom}\, S \times^{(L)} \bigsqcup \mathrm{im}\, S.$$

Corollary 37. $\left(a_0 \amalg^{(L)} b_0\right) \sqcup \left(a_1 \amalg^{(L)} b_1\right) = (a_0 \sqcap a_1) \amalg^{(L)} (b_0 \sqcap b_1)$.

Corollary 38. $a_0 \amalg^{(L)} b_0 \equiv a_1 \amalg^{(L)} b_1 \Leftrightarrow a_0 \equiv a_1 \wedge b_0 \equiv b_1$.

5.2 Supremum coproduct for endomorphisms

Let F be an indexed family of endomorphisms of $\mathcal{C}$.

I will denote $\mathrm{Ob}\, f$ the object (source and destination) of an endomorphism f.

Let also ι_i be a monovalued entirely defined morphism (for each $i \in \mathrm{dom}\, F$).

Definition 39. $\coprod^{(L)} F = \bigsqcup_{i\in\mathrm{dom}\, F}\left(\iota_i^{\lambda j\in n:\mathrm{Ob}\, F_j} \circ F_i^\dagger \circ \left(\iota_i^{\lambda j\in n:\mathrm{Ob}\, F_j}\right)^\dagger\right)$ (if ι is defined at $\lambda j \in n: \mathrm{Ob}\, F_j$). (I call it *supremum coproduct*).

Abbreviate $\iota_i = \iota_i^{\lambda j\in n:\mathrm{Ob}\, F_j}$.

So $\coprod F = \bigsqcup_{i\in\mathrm{dom}\, F}\left(\iota_i \circ F_i^\dagger \circ (\iota_i)^\dagger\right)$.

$\coprod F = \min\left\{\Phi \in \mathrm{End}\left(\coprod_{j\in n}^{(Q)} \mathrm{Ob}\, F_j\right) \mid \forall i \in n: \Phi \sqsupseteq \iota_i \circ F_i^\dagger \circ (\iota_i)^\dagger\right\}$.

Taking into account that ι_i is a monovalued entirely defined morphism, we get:

Obvious 40. $\coprod^{(L)} = \min\left\{\Phi \in \mathrm{End}\left(\coprod_{j\in n}^{(Q)} \mathrm{Ob}\, F_j\right) \mid \forall i \in n: \iota_i \in \mathrm{C}\left(F_i^\dagger; \Phi\right)\right\}$.

Corollary 41. $\iota_i \in \mathrm{C}\left(F_i; \coprod^{(L)} F\right)$ for every $i \in \mathrm{dom}\, F$.

5.3 Category of continuous morphisms

[TODO: What is X?]

Let ι_i (for $i \in \mathrm{dom}\, F$) be entirely defined monovalued and metacomplete morphisms.

Let $\bigoplus$ of an indexed family of morphisms is a morphism; $\left(\bigoplus f\right) \circ \iota_i = f_i$; $\bigoplus_{i\in n} (f \circ \iota_i) = f$ (a dual of the above).

Let $F_i \in \mathrm{End}\left(\coprod_{j\in n}^{(Q)} \mathrm{Ob}\, F_j\right)$ for all $i \in n$ (where n is some index set) (a self-dual of the above).

Definition 42. $\iota_i^{\mathrm{cont}(\mathcal{C})} = \left(\coprod^{(L)} F; F_i^\dagger; \iota_i\right)$.

Proposition 43. ι_i are continuous, that is $\iota_i^{\mathrm{cont}(\mathcal{C})}$ are morphisms.

Lemma 44. $f \in \mathrm{Mor}_{\mathrm{cont}(\mathcal{C})}\left(\coprod^{(L)} F; Y\right)$ is continuous iff all $f \circ \iota^{\mathrm{cont}(\mathcal{C})}$ are continuous.

Theorem 45. $\coprod^{(L)}$ together with $\bigoplus$ is a (partial) coproduct in the category $\mathrm{cont}(\mathcal{C})$.

6 Applying this to the theory of funcoids and reloids

6.1 Funcoids

Definition 46. $\mathrm{Fcd} \overset{\mathrm{def}}{=} \mathrm{cont}(\mathsf{FCD})$.

Let F be a family of endofuncoids.

The cartesian product $\prod^{(Q)} X \overset{\mathrm{def}}{=} \prod X$.

I define $\pi_i = \pi_i^X \in \mathsf{FCD}(\prod X; X_i)$ as the principal funcoid corresponding to the i-th projection. (Here π is entirely defined.)

The disjoint union $\coprod^{(Q)} X \overset{\mathrm{def}}{=} \coprod X$.

I define $\iota_i = \iota_i^X \in \mathsf{FCD}(X_i; \coprod X)$ as the principal funcoid corresponding to the i-th canonical injection. (Here ι is entirely defined.)

Let $\bigotimes$ and $\bigoplus$ be defined in the same way as in category Set.

Obvious 47. $\pi_i \circ \bigotimes f = f_i$; $\bigotimes_{i \in n} (\pi_i \circ f) = f$.

Obvious 48. $(\bigoplus f) \circ \iota_i = f_i$; $\bigoplus_{i \in n} (f \circ \iota_i) = f$.

It is easy to show that π_i is entirely defined monovalued, and ι_i is metacomplete and co-metacomplete.

Thus we are under conditions for both canonical products and canonical coproducts and thus both $\prod^{(L)} F$ and $\coprod^{(L)} F$ are defined.

6.2 Reloids

Definition 49. $\mathrm{Rld} \overset{\mathrm{def}}{=} \mathrm{cont}(\mathsf{RLD})$.

Let F be a family of endoreloids.

The cartesian product $\prod^{(Q)} X \overset{\mathrm{def}}{=} \prod X$.

I define $\pi_i = \pi_i^X \in \mathsf{RLD}(\prod X; X_i)$ as the principal reloid corresponding to the i-th projection. (Here π is entirely defined.)

The disjoint union $\coprod^{(Q)} X \overset{\mathrm{def}}{=} \coprod X$.

I define $\iota_i = \iota_i^{\mathsf{FCD}(Z_1)} \in \mathsf{RLD}(X_i; \coprod X)$ as the principal reloid corresponding to the i-th canonical injection. (Here ι is entirely defined.)

Let $\bigotimes$ and $\bigoplus$ are defined in the same way as in category Set.

Obvious 50. $\pi_i \circ \bigotimes f = f_i$; $\bigotimes_{i \in n} (\pi_i \circ f) = f$.

Obvious 51. $(\bigoplus f) \circ \iota_i = f_i$; $\bigoplus_{i \in n} (f \circ \iota_i) = f$.

It is easy to show that π_i is entirely defined monovalued, and ι_i is metacomplete and co-metacomplete.

Thus we are under conditions for both canonical products and canonical coproducts and thus both $\prod^{(L)} F$ and $\coprod^{(L)} F$ are defined.

It is trivial that for uniform spaces infimum product of reloids coincides with product uniformilty.

6.3 Cross-composition of pointfree funcoids

[TODO: This section is partially written.]

Let now $\mathcal{C}$ be the category of all small pointfree funcoids. Let principal morphisms be principal funcoids.

Define $\pi_i^X = \Pr_i (\mathsf{RLD})_{\mathrm{in}} X$ whenever X is a funcoid. [TODO: We should generalize it for multifuncoids or staroids.]

7 Initial and terminal objects

Initial object of Fcd is the endofuncoid $\uparrow^{\mathsf{FCD}(\emptyset;\emptyset)}\emptyset$. It is initial because it has precisely one morphism o (the empty set considered as a function) to any object Y. o is a morphism because $o \circ \uparrow^{\mathsf{FCD}(\emptyset;\emptyset)}\emptyset \sqsubseteq Y \circ o$.

Proposition 52. Terminal objects of Fcd are exactly $\uparrow^{\mathfrak{F}}\{*\} \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}}\{*\} = \uparrow^{\mathsf{FCD}}\{(*;*)\}$ where $*$ is an arbitrary point.

Proof. In order for a function $f: X \to \uparrow^{\mathsf{FCD}}\{(*;*)\}$ be a morphism, it is required exactly $f \circ X \sqsubseteq \uparrow^{\mathsf{FCD}}\{(*;*)\} \circ f$

$f \circ X \sqsubseteq (f^{-1} \circ \uparrow^{\mathsf{FCD}}\{(*;*)\})^{-1}$; $f \circ X \sqsubseteq (\{*\} \times^{\mathsf{FCD}} \langle f^{-1}\rangle\{*\})^{-1}$; $f \circ X \sqsubseteq \langle f^{-1}\rangle\{*\} \times^{\mathsf{FCD}} \{*\}$ what true exactly when f is a constant function with the value $*$. □

If $n = \emptyset$ then $Z = \{\emptyset\}$; $\prod^{(L)} \emptyset = \max \mathsf{FCD}(Z;Z) = \uparrow^{\mathfrak{F}}\{\emptyset\} \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}}\{\emptyset\} = \uparrow^{\mathsf{FCD}}\{(\emptyset;\emptyset)\}$.

[TODO: Initial and terminal objects of Rld.]

8 Canonical product and subatomic product

[TODO: Confusion between filters on products and multireloids.]

Proposition 53. $\Pr_i^{\mathsf{RLD}}|_{\mathfrak{F}(Z)} = \langle \pi_i \rangle$ for every index i of a cartesian product Z.

Proof. If $\mathcal{X} \in \mathfrak{F}(Z)$ then $(\Pr_i^{\mathsf{RLD}}|_{\mathfrak{F}(Z)})\mathcal{X} = \Pr_i^{\mathsf{RLD}} \mathcal{X} = \bigsqcap \langle\uparrow\rangle\langle \Pr_i \rangle \mathcal{X} = \bigsqcap \langle \pi_i \rangle \mathrm{up}\, \mathcal{X} = \langle \pi_i \rangle \mathcal{X}$. □

Proposition 54. $\prod^{(A)} F = \bigsqcap_{i \in n} \left(\left(\pi_i^{\mathsf{FCD}(\prod_{i\in n} \mathrm{Dst}\, F)} \right)^{-1} \circ F_i \circ \pi_i^{\mathsf{FCD}(\prod_{i\in n} \mathrm{Src}\, F)} \right)$.

Proof. $a \left[\prod^{(A)} F \right] b \Leftrightarrow \forall i \in \mathrm{dom}\, F: \Pr_i^{\mathsf{RLD}} a \, [F_i] \, \Pr_i^{\mathsf{RLD}} b \Leftrightarrow \forall i \in \mathrm{dom}\, F: \left\langle \left(\pi_i^{\mathsf{FCD}(\prod_{i\in n} \mathrm{Dst}\, F)} \right)^{-1} \right\rangle [F_i] \left\langle \pi_i^{\mathsf{FCD}(\prod_{i\in n} \mathrm{Src}\, F)} \right\rangle \Leftrightarrow \forall i \in \mathrm{dom}\, F: a \left[\left(\pi_i^{\mathsf{FCD}(\prod_{i\in n} \mathrm{Dst}\, F)} \right)^{-1} \circ F_i \circ \pi_i^{\mathsf{FCD}(\prod_{i\in n} \mathrm{Src}\, F)} \right] b \Leftrightarrow a \left[\bigsqcap_{i\in n} \left(\left(\pi_i^{\mathsf{FCD}(\prod_{i\in n} \mathrm{Dst}\, F)} \right)^{-1} \circ F_i \circ \pi_i^{\mathsf{FCD}(\prod_{i\in n} \mathrm{Src}\, F)} \right) \right] b$ for ultrafilters a and b. □

Corollary 55. $\prod^{(L)} F = \prod^{(A)} F$ is F is a small indexed family of funcoids.

9 Further plans

Does the formula $\prod^{(L)}_{i\in n} (g_i \circ f_i) = \prod^{(L)} g \circ \prod^{(L)} f$ hold?

Conjecture 56. The categories Fcd and Rld are cartesian closed (actually two conjectures).

http://mathoverflow.net/questions/141615/how-to-prove-that-there-are-no-exponential-object-in-a-category suggests to investigate colimits to prove that there are no exponential object.

Coordinate-wise continuity.

Bibliography

[1] Victor Porton. *Algebraic General Topology. Volume 1*. 2013.