About myself

I'm not a professional mathematician, I work as a programmer.
I have been studying in a university in Russia but have not finished my study.
So, I know little beyond my specialization.
Nevertheless in my free time I discovered a new theory which would completely overturn general topology.

About this lesson

In this lesson I present my discovery, the theory of funcoids and reloids.

I will not give here proofs of my results, as you can read my actual articles if you get interested in knowing the details.

The motivation for study of funcoids and reloids is that they are an elegant generalization of “spaces” (topological, pretopological, proximity, uniform spaces) and of binary relations between elements of spaces.

For brevity I will be sometimes a little informal in this lesson, for instance considering composition of funcoids (see below) without explicitly formulating that they are composable.

What is Algebraic General Topology?

I have introduced and researched objects called *funcoids*, *reloids*, and their generalizations.

I have named the theory of these objects *Algebraic General Topology*.

See

http://www.mathematics21.org

My generalizations

Funcoids are a generalization of:

- proximity spaces
- pretopological spaces
- preclosures
- digraphs (that is binary relations)

Reloids are a generalization of:

- uniform spaces
- digraphs (that is binary relations)

Usage of funcoids and reloids

That funcoids and reloids are a common generalization of spaces and functions (functions are a special case of binary relations), it makes them a smart tool for expressing properties of functions in regard of spaces.

For example, the statement "f is a continuous function from a space μ to a space ν" can be expressed by the formula:

$$f \circ \mu \sqsubseteq \nu \circ f.$$

Algebraic General Topology is a generalization of customary general topology but is much more elegant than the customary general topology.

Filters

The theory of funcoids and reloids is based on the theory of filters.

I've written an article on the theory of filters and their generalizations:

$$\mathrm{http://www.mathematics21.org}$$

In that article I consider filters on arbitrary posets and generalizations thereof. But in this lecture we will consider only filters on the lattice of subsets of some set.

Lattices and Filters

In order not to confuse poset/lattice operators with set-theoretic operators, I will denote partial order as $\sqsubseteq$ and lattice operators as $\sqcup$, $\sqcap$, $\bigsqcup$, $\bigsqcap$.
For my notation to be consistent, I need to order filters *reverse* to set theoretic inclusion of filters. I will denote $\mathfrak{F}$ the lattice of filters (on some set) including the improper filter ordered reverse to set-theoretic inclusion of filters:

$$\mathcal{A} \sqsubseteq \mathcal{B} \Leftrightarrow \mathcal{A} \supseteq \mathcal{B}.$$

More about filters

I will denote $\operatorname{Base}(\mathcal{A})$ the set on which the filter $\mathcal{A}$ is defined. I will denote the principal filter on a set A corresponding to a set X as

$$\uparrow^A X.$$

Ultrafilters are atoms of the lattice of filter objects (atomic filters).

Lattice of filters

The lattice $\mathfrak{F}(A)$ of reverse ordered filters (on some set A) is:

- having minimum and maximum $0^{\mathfrak{F}(A)}$ and $1^{\mathfrak{F}(A)}$
- atomistic
- complete
- distributive
- co-Brouwerian ($\mathcal{A} \sqcup \prod S = \prod \{ \mathcal{A} \sqcup \mathcal{X} \mid \mathcal{X} \in S \}$)

Read more about such lattices and more general posets in my article:

“Filters on Posets and Generalizations”

Generalized proximities

The most natural way to introduce funcoids is generalizing proximity spaces.

Let δ be a proximity on a set $\mho$. It can be extended from subsets of $\mho$ to filters on $\mho$ by the formula

$$\mathcal{X}\,\delta'\,\mathcal{Y} \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y} : X\,\delta\,Y.$$

I've proved that there exist two functions $\alpha : \mathfrak{F}(\mho) \to \mathfrak{F}(\mho)$ and $\beta : \mathfrak{F}(\mho) \to \mathfrak{F}(\mho)$ such that

$$\mathcal{X}\,\delta'\,\mathcal{Y} \Leftrightarrow \mathcal{Y} \sqcap \alpha\,\mathcal{X} \neq 0^{\mathfrak{F}(\mho)} \Leftrightarrow \mathcal{X} \sqcap \beta\,\mathcal{Y} \neq 0^{\mathfrak{F}(\mho)}.$$

Definition of funcoids

Let's fix two sets A and B.
The pair of two functions $\alpha\colon \mathfrak{F}(A) \to \mathfrak{F}(B)$ and $\beta\colon \mathfrak{F}(B) \to \mathfrak{F}(A)$ such that

$$\mathcal{Y} \sqcap \alpha\, \mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X} \sqcap \beta\, \mathcal{Y} \neq 0^{\mathfrak{F}(A)}$$

denotes a *funcoid*. Strictly speaking, a funcoid is a quadruple $(A; B; \alpha; \beta)$ conforming to the above formula.
Thus funcoids are a generalization of proximity spaces.
I call funcoids $(A; B; \alpha; \beta)$ funcoids from A to B and denote the set of funcoids from A to B as $\mathsf{FCD}(A; B)$.

Source and destination of a funcoid

The source and the destination of a funcoid $f = (A; B; F)$ are

$$\operatorname{Src} f = A; \quad \operatorname{Dst} f = B.$$

Components of a funcoid: part 1

Let $f = (A; B; \alpha; \beta)$ be a funcoid. Then by definition:

$$\langle f \rangle = \alpha.$$

A funcoid can be inverted (the inverse is also a funcoid):

$$f^{-1} = (B; A; \beta; \alpha).$$

Components of a funcoid: part 2

We have

$$\langle f^{-1} \rangle = \beta.$$

Thus a funcoid f has two components:

$$\langle f \rangle = \alpha \quad \text{and} \quad \langle f^{-1} \rangle = \beta.$$

An important property of funcoids: a funcoid f is completely characterized by just one of its components, say $\langle f \rangle$. Moreover f is determined by values of $\langle f \rangle$ on principal filters.

Funcoids and relations between filters

By definition

$$\mathcal{X} \left[f \right] \mathcal{Y} \Leftrightarrow \mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)}.$$

We have

$$\mathcal{X} \left[f \right] \mathcal{Y} \Leftrightarrow \mathcal{Y} \sqcap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X} \sqcap \langle f \rangle \mathcal{Y} \neq 0^{\mathfrak{F}(A)}.$$

For brevity I will also define:

$$\langle f \rangle^* X = \langle f \rangle \uparrow^A X \quad \text{and} \quad X \left[f \right]^* Y \Leftrightarrow \uparrow^A X \left[f \right] \uparrow^B Y.$$

A funcoid f is completely characterized by the relation $[f]$ and even by $\langle f \rangle^*$ or $[f]^*$.

Principal funcoids

Let A and B be sets.
For every binary relation $F \in \mathscr{P}(A \times B)$ there exists a funcoid $\uparrow^{\mathsf{FCD}(A;B)} F \in \mathsf{FCD}(A;B)$ defined by the formula (for every $X \in \mathscr{P}A$)

$$\langle \uparrow^{\mathsf{FCD}(A;B)} F \rangle^* X = \uparrow^B F[X].$$

This funcoid is unique because a funcoid is determined by the values of its first component on principal filters.
I call funcoids corresponding to a binary relation by the formula above as *principal funcoids*.

Funcoids & pretopologies

Let α be a pretopology, so α is a function $\mho \to \mathfrak{F}(\mho)$. Then there exists a funcoid f such that

$$\langle f \rangle^* X = \bigsqcup \{\alpha(x) \mid x \in X\}$$

(the join is taken on the lattice of filters).
So funcoids are a generalization of pretopologies.

Funcoids & preclosures

Let F be a preclosure (for example, F may be a topological space in Kuratowski sense). Then there exists a funcoid f such that

$$\langle f\rangle^* X = \uparrow^B F(X).$$

Thus funcoids are a generalization of preclosures.

Composition of funcoids

The composition of binary relations induces for principal funcoids composition which complies with the formulas:

$$\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle \quad \text{and} \quad \langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle.$$

We can define *composition* for funcoids by the same formulas. Strictly speaking the composition of funcoids is defined by the formula:

$$(B; C; \alpha_2; \beta_2) \circ (A; B; \alpha_1; \beta_1) = (A; C; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

Composition of funcoids is associative:

$$h \circ (g \circ f) = (h \circ g) \circ f.$$

Alternate representations of funcoids

Above I defined funcoids as quadruples. But a funcoid can be represented in two other ways:

- as a binary relation $\delta \in \mathscr{P}(\mathscr{P}A \times \mathscr{P}B)$ between sets
- as a function $\alpha: \mathscr{P}A \rightarrow \mathfrak{F}(B)$ from sets to filters

Below I will show the exact conditions required for δ and α in order to represent a funcoid. Funcoids from A to B bijectively correspond to such δ and α.

Funcoids as binary relations

A binary relation $\delta \in \mathscr{P}(\mathscr{P}A \times \mathscr{P}B)$ corresponds to a funcoid if and only if it complies to the formulas (for all suitable sets I, J, K):

$$\neg(\emptyset \,\delta\, I);\;\; I \cup J \,\delta\, K \Leftrightarrow I \,\delta\, K \vee J \,\delta\, K;$$
$$\neg(I \,\delta\, \emptyset);\;\; K \,\delta\, I \cup J \Leftrightarrow K \,\delta\, I \vee K \,\delta\, J.$$

The funcoid f and relation δ are related by the formulas:

$$\mathcal{X} \,[f]\, \mathcal{Y} \Leftrightarrow \forall X \in \mathcal{X}, Y \in \mathcal{Y} : X \,\delta\, Y;$$
$$X \,\delta\, Y \Leftrightarrow X \,[f]^{*}\, Y.$$

Funcoids as functions

A function $\alpha \in \mathscr{P} A \to \mathfrak{F}(B)$ corresponds to a funcoid if and only if it complies to the formulas (for all sets $I, J \in \mathscr{P} A$):

$$\alpha \emptyset = 0^{\mathfrak{F}(B)}; \qquad \alpha(I \cup J) = \alpha I \sqcup \alpha J.$$

The funcoid f and function α are related by the formulas:

$$\langle f \rangle \mathcal{X} = \bigsqcap \{\alpha X \mid X \in \mathcal{X}\};$$
$$\alpha X = \langle f \rangle^* X.$$

Order of funcoids

The set $\mathsf{FCD}(A; B)$ of funcoids from A to B is a poset with order defined by the formula:

$$f \sqsubseteq g \Leftrightarrow [f] \subseteq [g].$$

Moreover it is a complete, distributive, co-Brouwerian, atomistic lattice.

Values of a join or meet of funcoids

For every $R \in \mathscr{P}\mathsf{FCD}(A; B)$ and $X \in \mathscr{P}A$, $Y \in \mathscr{P}B$

1. $X \left[\bigsqcup R\right]^* Y \Leftrightarrow \exists f \in R: X \left[f\right]^* Y$;
2. $\langle \bigsqcup R \rangle^* X = \bigsqcup \{ \langle f \rangle^* X \mid f \in R \}$.

For every $R \in \mathscr{P}\mathsf{FCD}(A; B)$ and x, y being ultrafilters on A and B correspondingly we have:

1. $x \left[\bigsqcap R\right] y \Leftrightarrow \forall f \in R: x \left[f\right] y$;
2. $\langle \bigsqcap R \rangle x = \bigcap \{ \langle f \rangle x \mid f \in R \}$.

Funcoidal product

The funcoidal product of filters is a generalization of the Cartesian product of sets.

Let $\mathcal{A}$ and $\mathcal{B}$ be filters. Then there exists a unique funcoid (the *funcoidal product* of $\mathcal{A}$ and $\mathcal{B}$) $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ such that

$$\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \sqcap \mathcal{A} \neq 0^{\mathrm{Base}(\mathcal{A})}; \\ 0^{\mathrm{Base}(\mathcal{B})} & \text{if } \mathcal{X} \sqcap \mathcal{A} = 0^{\mathrm{Base}(\mathcal{A})}; \end{cases}$$

$$\mathcal{X} \left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right] \mathcal{Y} \Leftrightarrow \mathcal{X} \sqcap \mathcal{A} \neq 0^{\mathrm{Base}(\mathcal{A})} \wedge \mathcal{Y} \sqcap \mathcal{B} \neq 0^{\mathrm{Base}(\mathcal{B})}.$$

Restricted identity funcoid

Restricted identity funcoids are a generalization of identity relations on a set.

Let $\mathcal{A}$ be a filter. Then there exists a unique funcoid (the *restricted identity funcoid* on $\mathcal{A}$) $\mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}}$ such that:

$$\langle \mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}} \rangle \mathcal{X} = \mathcal{X} \sqcap \mathcal{A};$$

$$\mathcal{X} \left[\mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}} \right] \mathcal{Y} \Leftrightarrow \mathcal{X} \sqcap \mathcal{Y} \sqcap \mathcal{A} \neq 0^{\mathrm{Base}(\mathcal{A})}.$$

Restriction of funcoids

Restriction of a funcoid f to a filter $\mathcal{A}$ is defined by the formula

$$f|_{\mathcal{A}} = f \circ \mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}}.$$

It follows

$$f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{F}(\mathrm{Dst}\, f)}).$$

T_0-, T_1- and T_2-separable funcoids

A funcoid f is T_1-*separable* when

$$\forall \alpha \in \operatorname{Src} f, \beta \in \operatorname{Dst} f : (\alpha \neq \beta \Rightarrow \neg(\{\alpha\} [f]^* \{\beta\})).$$

An endofuncoid (a funcoid with the same source and destination) is:

1. T_0-separable when $f \sqcap f^{-1}$ is T_1-separable.
2. T_2-separable when $f^{-1} \circ f$ is T_1-separable.

Some properties of funcoids

Let f, g be funcoids, $\mathcal{X}$, $\mathcal{Z}$ be filters. Then $\mathcal{X} \left[g \circ f \right] \mathcal{Z}$ iff there exists an ultrafilter y such that $\mathcal{X} \left[f \right] y$ and $y \left[g \right] \mathcal{Z}$.

Let f, g, h be funcoids. Then

$$f \circ (g \sqcup h) = f \circ g \sqcup f \circ h.$$

Also

$$f \sqcap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathrm{id}^{\mathsf{FCD}}_{\mathcal{B}} \circ f \circ \mathrm{id}^{\mathsf{FCD}}_{\mathcal{A}}.$$

Reloids

Reloids are a trivial generalization of uniform spaces.
Roughly speaking, a reloid is a filter on a Cartesian product of two sets.
To be precise, I define a reloid as a triple $f = (A; B; F)$ where A and B are sets and F is a filter on $A \times B$.
Note that reloids are also a generalization of binary relations.
The reverse reloid f^{-1} is defined as follows:

$$f^{-1} = (A; B; F)^{-1} = (B; A; F^{-1}).$$

I will also denote $\mathrm{GR}(A; B; F) = F$.

Principal reloids

Let F be a binary relation between sets A and B. Then

$$\uparrow^{\mathsf{RLD}(A;B)} F = (A; B; \uparrow^{A \times B} F)$$

is so called the *principal reloid* corresponding to the relation F.

Composition of reloids

Let $f = (A;\ B;\ F)$ and $g = (B;\ C;\ G)$ be reloids. The composition $g \circ f$ is defined by the formula

$$g \circ f = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} (Y \circ X) \mid X \in \mathrm{GR}\, F, Y \in \mathrm{GR}\, G \right\}.$$

In other words, the composition corresponds to the filter (on $A \times C$) defined by the base

$$\{ Y \circ X \mid X \in \mathrm{GR}\, F, Y \in \mathrm{GR}\, G \}.$$

Composition of reloids is associative.

Reloidal product

The reloidal product of filters is a generalization of the Cartesian product of sets.

Let $\mathcal{A}$ and $\mathcal{B}$ be filters. Then the *reloidal product* of $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula:

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(A;B)} (A \times B) \mid A \in \mathcal{A}, B \in \mathcal{B} \right\}.$$

In other words, the reloidal product is the reloid defined by the base

$$\{ A \times B \mid A \in \mathcal{A}, B \in \mathcal{B} \}.$$

Restricted identity reloid

The *identity reloid* on a set A is defined as $\mathrm{id}^{\mathsf{RLD}(A)} = \uparrow^{\mathsf{RLD}(A;A)}\mathrm{id}_A$.

Similarly to the above defined restricted identity funcoid, we can also define the *restricted identity reloid*

$$\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = \mathrm{id}^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A}))} \sqcap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))}\right).$$

We have

$$\mathrm{id}^{\mathsf{RLD}}_{\mathcal{A}} = \bigsqcap \left\{\uparrow^{\mathsf{RLD}(\mathrm{Base}(A);\mathrm{Base}(A))}\mathrm{id}_A \mid A \in \mathrm{up}\,\mathcal{A}\right\}.$$

Restriction of a reloid

Restriction of a reloid f to a filter $\mathcal{A}$ is defined by the formula

$$f|_{\mathcal{A}} = f \circ \mathrm{id}_{\mathcal{A}}$$

or

$$f|_{\mathcal{A}} = f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Base}(\mathcal{A}))}).$$

Some properties of reloids

Let f, g, h be reloids. Then

$$f \circ (g \sqcup h) = f \circ g \sqcup f \circ h.$$

Also

$$f \sqcap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathrm{id}_{\mathcal{B}}^{\mathsf{RLD}} \circ f \circ \mathrm{id}_{\mathcal{A}}^{\mathsf{RLD}}.$$

Ordered and dagger categories

I call a *partially ordered category* a category together with a partial order on each of its Hom-sets.

A *dagger category* is a category together with a function $f \mapsto f^\dagger$ on the set of morphisms which inverses the source and the destination of the morphism and is subject to the following conditions:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
3. $(1_A)^\dagger = 1_A$.

Categories of funcoids and reloids

Funcoids with objects being sets and composition of funcoids form a category which I call *the category of funcoids*.
The same holds for reloids.
Categories of funcoids and reloids are both partially ordered dagger categories with “the dagger” defined as

$$f \mapsto f^{-1}.$$

The order and the dagger agree:

$$f^{\dagger} \sqsubseteq g^{\dagger} \Leftrightarrow f \sqsubseteq g.$$

Special morphisms

Let f be a morphism of a partially ordered dagger category.

f is *monovalued* when $f \circ f^{\dagger} \sqsubseteq 1_{\mathrm{Dst}\, f}$.

f is *entirely defined* when $f^{\dagger} \circ f \sqsupseteq 1_{\mathrm{Src}\, f}$.

f is *injective* when $f^{\dagger} \circ f \sqsubseteq 1_{\mathrm{Src}\, f}$.

f is *surjective* when $f \circ f^{\dagger} \sqsupseteq 1_{\mathrm{Dst}\, f}$.

It's easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective binary relations as morphisms of the category **Rel**.

Monovalued funcoids

I will denote "atoms a" the set of atoms under a for an element a of a poset.

The following statements are equivalent for a funcoid f:

1. f is monovalued.
2. $\forall a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)} : \langle f \rangle a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} \cup \left\{ 0^{\mathfrak{F}(\operatorname{Dst} f)} \right\}$.
3. $\forall \mathcal{I}, \mathcal{J} \in \mathfrak{F}(\operatorname{Dst} f) : \langle f^{-1} \rangle (\mathcal{I} \sqcap \mathcal{J}) = \langle f^{-1} \rangle \mathcal{I} \sqcap \langle f^{-1} \rangle \mathcal{J}$.
4. $\forall I, J \in \mathscr{P}(\operatorname{Dst} f) : \langle f^{-1} \rangle^{*} (I \cap J) = \langle f^{-1} \rangle^{*} I \sqcap \langle f^{-1} \rangle^{*} J$.

Consequently a principal funcoid is monovalued iff its corresponding binary relation is monovalued (a function).

Functions between spaces

Let μ and ν be funcoids corresponding to a (pre)topological spaces, or proximity spaces, or μ and ν be uniform spaces (that is reloids).

Let f be the funcoid or reloid corresponding to a function from the first space to the second space.

Continuous morphisms

Continuity, proximal continuity, and uniform continuity of f is expressed by the same formula:

$$f \circ \mu \sqsubseteq \nu \circ f.$$

In the case if f is monovalued and entirely defined, we have

$$f \circ \mu \sqsubseteq \nu \circ f \Leftrightarrow \mu \sqsubseteq f^{-1} \circ \nu \circ f \Leftrightarrow f \circ \mu \circ f^{-1} \sqsubseteq \nu.$$

This can be generalized for any partially ordered dagger categories.

Relationships of funcoids and reloids

For every sets A, B we have $\mathsf{FCD}(A;B)$ and $\mathsf{RLD}(A;B)$ interrelated by the below defined functions:

$$(\mathsf{FCD}) : \mathsf{RLD}(A;B) \to \mathsf{FCD}(A;B);$$
$$(\mathsf{RLD})_{\mathrm{in}} : \mathsf{FCD}(A;B) \to \mathsf{RLD}(A;B);$$
$$(\mathsf{RLD})_{\mathrm{out}} : \mathsf{FCD}(A;B) \to \mathsf{RLD}(A;B).$$

The funcoid induced by a reloid

Every reloid $f \in \mathsf{RLD}(A; B)$ induces a funcoid $(\mathsf{FCD}) f \in \mathsf{FCD}(A; B)$ by the following formulas:

$$\mathcal{X} \left[(\mathsf{FCD}) f \right] \mathcal{Y} \Leftrightarrow \forall F \in \mathrm{GR}\, f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(A;B)} F \right] \mathcal{Y};$$
$$\langle (\mathsf{FCD}) f \rangle \mathcal{X} = \bigcap \left\{ \langle \uparrow^{\mathsf{FCD}(A;B)} F \rangle \mathcal{X} \mid F \in \mathrm{GR}\, f \right\}.$$

We have for every composable reloids f and g:

$$(\mathsf{FCD})(g \circ f) = ((\mathsf{FCD}) g) \circ ((\mathsf{FCD}) f).$$

I will skip some minor facts on this topic.

The reloids induced by a funcoid

Every funcoid $f \in \mathsf{FCD}(A; B)$ induces a reloid from A to B in two ways, namely intersection of *outward* relations and union of *inward* direct products of filters:

$$(\mathsf{RLD})_{\mathrm{out}} f \overset{\mathrm{def}}{=} \bigsqcap \uparrow^{\mathsf{RLD}(A;B)} [\mathrm{GR}\, f];$$

$$(\mathsf{RLD})_{\mathrm{in}} f \overset{\mathrm{def}}{=} \bigsqcup \{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{A} \in \mathfrak{F}(A), \mathcal{B} \in \mathfrak{F}(B), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \sqsubseteq f \}.$$

It's simple to show that

$(\mathsf{RLD})_{\mathrm{in}} f = \bigsqcup \{ a \times^{\mathsf{RLD}} b \mid a$ is an atom of $\mathfrak{F}(A), b$ is an atom of $\mathfrak{F}(A), a \times^{\mathsf{FCD}} b \sqsubseteq f \}$.

I will skip some minor results.

Some Galois connections

For every funcoid f we have

$$(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f.$$

(FCD): $\mathsf{RLD}(A; B) \to \mathsf{FCD}(A; B)$ is the lower adjoint of $(\mathsf{RLD})_{\mathrm{in}}$: $\mathsf{FCD}(A; B) \to \mathsf{RLD}(A; B)$. Thus

1. $(\mathsf{FCD}) \bigsqcup S = \bigsqcup \{(\mathsf{FCD}) f \mid f \in S\}$;
2. $(\mathsf{RLD})_{\mathrm{in}} \bigsqcap T = \bigsqcap \{(\mathsf{RLD})_{\mathrm{in}} f \mid f \in T\}$

for every set S of reloids or T of funcoids.

Convergence of funcoids

A filter $\mathcal{F}$ *converges* to a filter $\mathcal{A}$ regarding to a funcoid μ ($\mathcal{F}\overset{\mu}{\to}\mathcal{A}$) iff $\mathcal{F} \sqsubseteq \langle\mu\rangle\mathcal{A}$. (This generalizes the standard definition of filter convergent to a point or to a set.)

A funcoid f *converges* to a filter $\mathcal{A}$ regarding to a funcoid μ ($f\overset{\mu}{\to}\mathcal{A}$) iff $\operatorname{im} f \sqsubseteq \langle\mu\rangle\mathcal{A}$ that is iff $\operatorname{im} f\overset{\mu}{\to}\mathcal{A}$.

A funcoid f *converges* to a filter $\mathcal{A}$ on a filter $\mathcal{B}$ regarding to a funcoid μ iff $f|_{\mathcal{B}}\overset{\mu}{\to}\mathcal{A}$.

We can define also convergence for a reloid f: $f\overset{\mu}{\to}\mathcal{A} \Leftrightarrow \operatorname{im} f \sqsubseteq \langle\mu\rangle\mathcal{A}$ or what is the same $f\overset{\mu}{\to}\mathcal{A} \Leftrightarrow (\mathsf{FCD})f\overset{\mu}{\to}\mathcal{A}$.

Limit of a funcoid

$\lim^{\mu} f = a$ iff

$$f \xrightarrow{\mu} \uparrow^{\operatorname{Dst} f} \{a\}$$

for a T_2-separable funcoid μ and a non-empty funcoid f.
It is defined correctly, that is f has no more than one limit.

Generalized limit

We can define a (generalized) limit for an arbitrary (discontinuous) function, for example any function on the set of reals, or more generally from any topological vector space to any topological vector space, etc.
An idea is that the limit should not change when translating to an other point of the space. Thus we need to fix a group G of translations (or any other transformations) of our space.

Conditions for a generalized limit

Let μ and ν be funcoids on a set $\mho$, and G be a group of functions.
Let D be a set such that

$$\forall r \in G : \operatorname{im} r \subseteq D \wedge \forall x, y \in D \exists r \in G : r(x) = y.$$

We require that μ and every $r \in G$ commute, that is $\mu \circ r = r \circ \mu$ (for r considered as a principal funcoid).
We require for every $y \in \mho$

$$\nu \sqsupseteq \langle \nu \rangle^* \{ y \} \times^{\mathsf{FCD}} \langle \nu \rangle^* \{ y \}.$$

Definition of generalized limit

$$\operatorname{xlim} f \stackrel{\text{def}}{=} \{\nu \circ f \circ r \mid r \in G\}$$

for any funcoid f. (Here G is considered as a set of principal funcoids.)
The limit at a point x is defined as

$$\operatorname{xlim}_x f = \operatorname{xlim} f|_{\langle\mu\rangle^*\{x\}}.$$

Limits and generalized limits

$$\tau(y) \overset{\text{def}}{=} \{ \langle \mu \rangle \{ x \} \times^{\mathsf{FCD}} \langle \nu \rangle \{ y \} \mid x \in D \}.$$

Theorem Let μ be a T_2-separable funcoid and ν be a non-empty funcoid such that $\nu \sqsubseteq \nu \circ \nu$. If $\lim_x f = y$ then $\operatorname{xlim}_x f = \tau(y)$.

This theorem establishes a bijective correspondence (namely τ) between limits and a subset of generalized limits.

The skipped topics

In my speech I have skipped the following topics:

- specifying a funcoid by its values on ultrafilters
- atomic funcoids
- complete funcoids and reloids, completion of funcoids and reloids
- connectedness of sets and of filters regarding funcoids and reloids
- other

Further research directions

I also have researched *pointfree funcoids*, a generalization of funcoids relevant for pointfree topology, and *multifuncoids*, its further generalization.
The future research topics include:

- categories related to funcoids
- compactness of funcoids
- other

I have also formulated a quite large number of open problems related to filters, funcoids, and reloids. If you need a research field, I suggest you to solve my open problems.
I also have written a book.