Pointfree Funcoids*

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Abstract

It is a part of my Algebraic General Topology research.

I generalize the point-set notion of funcoids to pointfree topology notion of *pointfree funcoids*.

It seems that the theory of pointfree funcoids has use in the theory of n-ary funcoids.

Keywords: algebraic general topology, funcoids, reloids

A.M.S. subject classification: 54J05, 54A05, 54D99, 54E05, 54E15, 54E17, 54E99

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1 Draft status

This is a draft. No 100% warranty against errors.

Also the notation is inconsistent: sometimes used x sometimes $\mathcal{X}$.

[TODO: Some theorems about funcoids may generalize to pointfree funcoids having one side a complete atomic boolean lattice.]

2 Preface

This article generalizes the notions of funcoids introduced in [3], I call this generalization *pointfree funcoids*.

∗. This document has been written using the GNU $\text{T}_\text{E}\text{X}_{\text{MACS}}$ text editor (see `www.texmacs.org`).

It seems that pointfree funcoids are necessary for research of an other generalization of funcoids, n-ary funcoids. However in this article I do not attempt to research n-ary funcoids.

3 Notation

First we use the notation introduced in [2] and [3].

In addition to $\operatorname{atoms}^{\mathfrak{A}} x$, the set of atoms under an element x of a poset $\mathfrak{A}$ we will write just $\operatorname{atoms}^{\mathfrak{A}}$ for all atoms of a poset $\mathfrak{A}$.

4 Pointfree funcoids

4.1 Definition

Definition 1. *Pointfree funcoid* is a quadruple $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$ where $\mathfrak{A}$ and $\mathfrak{B}$ are posets, $\alpha \in \mathfrak{B}^{\mathfrak{A}}$ and $\beta \in \mathfrak{A}^{\mathfrak{B}}$ such that

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B}: (y \not\asymp^{\mathfrak{B}} \alpha x \Leftrightarrow x \not\asymp^{\mathfrak{A}} \beta y).$$

Definition 2. The *source* $\operatorname{Src}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{A}$ and *destination* $\operatorname{Dst}(\mathfrak{A}; \mathfrak{B}; \alpha; \beta) = \mathfrak{B}$ for every pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Definition 3. I will denote $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ the set of pointfree funcoids from $\mathfrak{A}$ to $\mathfrak{B}$ (that is with source $\mathfrak{A}$ and destination $\mathfrak{B}$), for every posets $\mathfrak{A}$ and $\mathfrak{B}$.

Proposition 4. If $\mathfrak{A}$ and $\mathfrak{B}$ have least elements, then $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ has least element.

Proof. It is $(\mathfrak{A}; \mathfrak{B}; 0^{\mathfrak{A}}; 0^{\mathfrak{B}})$. □

Definition 5. $\langle(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)\rangle \overset{\text{def}}{=} \alpha$ for a pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Definition 6. $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)^{-1} = (\mathfrak{B}; \mathfrak{A}; \beta; \alpha)$ for a pointfree funcoid $(\mathfrak{A}; \mathfrak{B}; \alpha; \beta)$.

Proposition 7. If f is a pointfree funcoid then f^{-1} is also a pointfree funcoid.

Proof. Follows from symmetry in the definition of pointfree funcoid. □

Obvious 8. $(f^{-1})^{-1} = f$ for a pointfree funcoid f.

Definition 9. The relation $[f] \in \mathscr{P}(\operatorname{Src} f \times \operatorname{Dst} f)$ is defined by the formula (for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$)

$$x [f] y \overset{\text{def}}{=} y \not\asymp^{\operatorname{Dst} f} \langle f\rangle x.$$

Obvious 10. $x [f] y \Leftrightarrow y \not\asymp^{\operatorname{Dst} f} \langle f\rangle x \Leftrightarrow x \not\asymp^{\operatorname{Src} f} \langle f^{-1}\rangle y$ for every pointfree funcoid f and $x \in \operatorname{Src} f$, $y \in \operatorname{Dst} f$.

Obvious 11. $[f^{-1}] = [f]^{-1}$ for a pointfree funcoid f.

Theorem 12. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets. Then:

1. If $\mathfrak{A}$ is separable, for given value of $\langle f\rangle$ exists no more than one $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$;
2. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable, for given value of $[f]$ exists no more than one $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

Proof. Let $f, g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

1. Let $\langle f \rangle = \langle g \rangle$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have $x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle y \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle f \rangle x \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle g \rangle x \Leftrightarrow x \not\asymp^{\mathfrak{A}} \langle g^{-1} \rangle y$ and thus by separability of $\mathfrak{A}$ we have $\langle f^{-1} \rangle y = \langle g^{-1} \rangle y$ that is $\langle f^{-1} \rangle = \langle g^{-1} \rangle$ and so $f = g$.
2. Let $[f] = [g]$. Then for every $x \in \mathfrak{A}$, $y \in \mathfrak{B}$ we have $y \not\asymp^{\mathfrak{B}} \langle f \rangle x \Leftrightarrow x [f] y \Leftrightarrow x [g] y \Leftrightarrow y \not\asymp^{\mathfrak{B}} \langle g \rangle x$ and thus by separability of $\mathfrak{B}$ we have $\langle f \rangle x = \langle g \rangle x$ that is $\langle f \rangle = \langle g \rangle$. Similarly we have $\langle f^{-1} \rangle = \langle g^{-1} \rangle$. Thus $f = g$. □

Theorem 13. Let f is a pointfree funcoid with distributive lattice $\operatorname{Src} f$ with least element and separable distributive lattice $\operatorname{Dst} f$ with least element. Then $\langle f \rangle (i \cup^{\operatorname{Src} f} j) = \langle f \rangle i \cup^{\operatorname{Dst} f} \langle f \rangle j$ for every $i, j \in \operatorname{Src} f$.

Proof.

$$\begin{aligned}
\star \langle f \rangle (i \cup^{\operatorname{Src} f} j) &= \\
\{ y \in \mathfrak{A} \mid y \cap^{\operatorname{Dst} f} \langle f \rangle (i \cup^{\operatorname{Src} f} j) \neq 0 \} &= \\
\{ y \in \mathfrak{A} \mid (i \cup^{\operatorname{Src} f} j) \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0 \} &= \\
\{ y \in \mathfrak{A} \mid (i \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y) \cup^{\operatorname{Src} f} (j \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y) \neq 0 \} &= \\
\{ y \in \mathfrak{A} \mid i \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq \emptyset \vee j \cap^{\operatorname{Src} f} \langle f^{-1} \rangle y \neq 0 \} &= \\
\{ y \in \mathfrak{A} \mid y \cap^{\operatorname{Dst} f} \langle f \rangle i \neq \emptyset \vee y \cap^{\operatorname{Dst} f} \langle f \rangle j \neq 0 \} &= \\
\{ y \in \mathfrak{A} \mid (y \cap^{\operatorname{Dst} f} \langle f \rangle i) \cup^{\operatorname{Dst} f} (y \cap^{\operatorname{Dst} f} \langle f \rangle j) \neq 0 \} &= \\
\{ y \in \mathfrak{A} \mid y \cap^{\operatorname{Dst} f} (\langle f \rangle i \cup^{\operatorname{Dst} f} \langle f \rangle j) \neq 0 \} &= \\
\star (\langle f \rangle i \cup^{\operatorname{Dst} f} \langle f \rangle j).
\end{aligned}$$

Thus $\langle f \rangle (i \cup^{\operatorname{Src} f} j) = \langle f \rangle i \cup^{\operatorname{Src} f} \langle f \rangle j$ by separability. □

Proposition 14. Let f is a pointfree funcoid with separable distributive lattice $\operatorname{Src} f$ with least element and separable distributive lattice $\operatorname{Dst} f$ with least element. Then:

1. $\mathcal{K} [f] \mathcal{I} \cup \mathcal{J} \Leftrightarrow \mathcal{K} [f] \mathcal{I} \vee \mathcal{K} [f] \mathcal{J}$ for every $\mathcal{I}, \mathcal{J} \in \mathfrak{F}(\operatorname{Dst} f)$, $\mathcal{K} \in \mathfrak{F}(\operatorname{Src} f)$.
2. $\mathcal{I} \cup \mathcal{J} [f] \mathcal{K} \Leftrightarrow \mathcal{I} [f] \mathcal{K} \vee \mathcal{J} [f] \mathcal{K}$ for every $\mathcal{I}, \mathcal{J} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{K} \in \mathfrak{F}(\operatorname{Dst} f)$.

Proof. 1. $\mathcal{K} [f] \mathcal{I} \cup \mathcal{J} \Leftrightarrow (\mathcal{I} \cup \mathcal{J}) \cap \langle f \rangle \mathcal{K} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow (\mathcal{I} \cap \langle f \rangle \mathcal{K}) \cup (\mathcal{J} \cap \langle f \rangle \mathcal{K}) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{I} \cap \langle f \rangle \mathcal{K} \neq 0^{\mathfrak{F}(B)} \vee \mathcal{J} \cap \langle f \rangle \mathcal{K} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{K} [f] \mathcal{I} \vee \mathcal{K} [f] \mathcal{J}$.

2. Similar. □

?? [TODO: $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}$ theorem.]

4.2 Composition of pointfree funcoids

Definition 15. *Composition* of pointfree funcoids is defined by the formula

$$(\mathfrak{B}; \mathfrak{C}; \alpha_2; \beta_2) \circ (\mathfrak{A}; \mathfrak{B}; \alpha_1; \beta_1) = (\mathfrak{A}; \mathfrak{C}; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

Definition 16. I will call funcoids f and g *composable* when $\operatorname{Dst} f = \operatorname{Src} g$.

Proposition 17. If f, g are pointfree funcoids and $\operatorname{Dst} f = \operatorname{Src} g$ then $g \circ f$ is pointfree funcoid.

Proof. Let $f = (\mathfrak{A}; \mathfrak{B}; \alpha_1; \beta_1)$, $g = (\mathfrak{B}; \mathfrak{C}; \alpha_2; \beta_2)$. For every $x, y \in \mathfrak{A}$ we have

$$y \not\asymp^{\mathfrak{C}} (\alpha_2 \circ \alpha_1) x \Leftrightarrow y \not\asymp^{\mathfrak{C}} \alpha_2 \alpha_1 x \Leftrightarrow \alpha_1 x \not\asymp^{\mathfrak{B}} \beta_2 y \Leftrightarrow x \not\asymp^{\mathfrak{A}} \beta_1 \beta_2 y \Leftrightarrow x \not\asymp^{\mathfrak{A}} (\beta_1 \circ \beta_2) y.$$

So $(\mathfrak{A}; \mathfrak{C}; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)$ is a pointfree funcoid. □

Obvious 18. $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$ for every composable pointfree funcoids f and g.

Theorem 19. $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ for every composable pointfree funcoids f and g.

Proof.

$$\langle (g\circ f)^{-1}\rangle = \langle f^{-1}\rangle \circ \langle g^{-1}\rangle = \langle f^{-1}\circ g^{-1}\rangle;$$
$$\langle ((g\circ f)^{-1})^{-1}\rangle = \langle g\circ f\rangle = \langle (f^{-1}\circ g^{-1})^{-1}\rangle.$$

□

Proposition 20. $(h\circ g)\circ f = h\circ (g\circ f)$ for every composable pointfree funcoids f, g, h.

Proof.
$\langle (h\circ g)\circ f\rangle = \langle h\circ g\rangle \circ \langle f\rangle = \langle h\rangle \circ \langle g\rangle \circ \langle f\rangle = \langle h\rangle \circ \langle g\circ f\rangle = \langle h\circ (g\circ f)\rangle$.
$\langle ((h\circ g)\circ f)^{-1}\rangle = \langle f^{-1}\circ (h\circ g)^{-1}\rangle = \langle f^{-1}\circ g^{-1}\circ h^{-1}\rangle = \langle (g\circ f)^{-1}\circ h^{-1}\rangle = \langle (h\circ (g\circ f))^{-1}\rangle$. □

4.3 Pointfree funcoid as continuation

Theorem 21. Let f is a pointfree funcoid. Then for every $x\in \operatorname{Src} f$, $y\in \operatorname{Dst} f$ we have

1. If $(\operatorname{Src} f;\mathfrak{Z})$ is a filtrator with separable core then $x[f]y\Leftrightarrow \forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x: X[f]y$.
2. If $(\operatorname{Dst} f;\mathfrak{Z})$ is a filtrator with separable core then $x[f]y\Leftrightarrow \forall Y\in \operatorname{up}^{(\operatorname{Dst} f;\mathfrak{Z})} x: x[f]Y$.

Proof. We will prove only the second because the first is similar.
$x[f]y\Leftrightarrow y\not\asymp^{\operatorname{Dst} f}\langle f\rangle x\Leftrightarrow \forall Y\in \operatorname{up} y: Y\not\asymp^{\operatorname{Dst} f}\langle f\rangle x\Leftrightarrow \forall Y\in \operatorname{up} y: x[f]Y$. □

Corollary 22. Let f is a pointfree funcoid and $(\operatorname{Src} f;\mathfrak{Z}_0)$, $(\operatorname{Dst} f;\mathfrak{Z}_1)$ are filtrators with separable core. Then

$$x[f]y\Leftrightarrow \forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z}_0)} x, Y\in \operatorname{up}^{(\operatorname{Dst} f;\mathfrak{Z}_1)} y: X[f]Y.$$

Proof. Apply the theorem twice. □

Theorem 23. Let f is a pointfree funcoid. Let $(\operatorname{Src} f;\mathfrak{Z})$ is a primary filtrator over a distributive lattice and $\operatorname{Dst} f$ is a separable meet-semilattice with least element. Then for every $x\in \operatorname{Src} f$

$$\langle f\rangle x = \bigcap^{\operatorname{Dst} f}\langle\langle f\rangle\rangle \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x.$$

Proof. $(\operatorname{Src} f;\mathfrak{Z})$ is a filtrator with separable core by the theorem 37 in [2]. Thus we can apply the previous theorem for every $y\in \operatorname{Dst} f$:

$$y\not\asymp^{\operatorname{Dst} f}\langle f\rangle x\Leftrightarrow x[f]y\Leftrightarrow \forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x: X[f]y\Leftrightarrow \forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x: y\not\asymp^{\operatorname{Dst} f}\langle f\rangle X.$$

Let's denote $W=\left\{ y\cap^{\operatorname{Dst} f}\langle f\rangle X \mid X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x\right\}$. We will prove that W is a generalized filter base. To prove this enough to show that $V=\left\{ \langle f\rangle X \mid X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x\right\}$ is a generalized filter base.

Let $\mathcal{P},\mathcal{Q}\in V$. Then $\mathcal{P}=\langle f\rangle A$, $\mathcal{Q}=\langle f\rangle B$ where $A,B\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x$; $A\cap^{\mathfrak{Z}} B\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x$ (used the fact that it is a primary filtrator) and $\mathcal{R}\subseteq \mathcal{P}\cap^{\mathfrak{F}}\mathcal{Q}$ for $\mathcal{R}=\langle f\rangle (A\cap^{\mathfrak{Z}} B)\in V$. So V is a generalized filter base and thus W is a generalized filter base.

$0\notin W\Leftrightarrow \bigcap^{\operatorname{Dst} f} W\not\asymp 0$ by the properties of generalized filter bases. That is

$$\forall X\in \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x: y\cap^{\operatorname{Dst} f}\langle f\rangle X\neq \emptyset \Leftrightarrow y\cap^{\operatorname{Dst} f}\bigcap^{\operatorname{Dst} f}\langle\langle f\rangle\rangle \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x\neq\emptyset.$$

Comparing with the above, $y\cap^{\operatorname{Dst} f}\langle f\rangle x\neq \emptyset\Leftrightarrow y\cap^{\operatorname{Dst} f}\bigcap^{\operatorname{Dst} f}\langle\langle f\rangle\rangle \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x\neq\emptyset$. So $\langle f\rangle x = \bigcap^{\operatorname{Dst} f}\langle\langle f\rangle\rangle \operatorname{up}^{(\operatorname{Src} f;\mathfrak{Z})} x$ because $\operatorname{Dst} f$ is separable. □

Theorem 24. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices.

1. A function $\alpha\in \mathfrak{B}^{\mathfrak{Z}_0}$ conforming to the formulas (for every $I,J\in \mathfrak{Z}_0$)

$$\alpha 0 = 0,\quad \alpha (I\cup^{\mathfrak{Z}_0} J) = \alpha I\cup^{\mathfrak{B}}\alpha J$$

can be continued to the function $\langle f\rangle$ for a unique $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$;

$$\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{B}}\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X} \tag{1}$$

for every $\mathcal{X}\in\mathfrak{A}$.

2. A relation $\delta\in\mathscr{P}(\mathfrak{Z}_0\times\mathfrak{Z}_1)$ conforming to the formulas (for every I, J, $K\in\mathfrak{Z}_0$ and I', J', $K'\in\mathfrak{Z}_1$)

$$\begin{array}{l}\neg(0\,\delta\,I),\ \ I\cup^{\mathfrak{Z}_0}J\,\delta\,K'\Leftrightarrow I\,\delta\,K'\vee J\,\delta\,K',\\ \neg(I\,\delta\,0),\ \ K\,\delta\,I\cup^{\mathfrak{Z}_1}J'\Leftrightarrow K\,\delta\,I'\vee K\,\delta\,J'\end{array} \tag{2}$$

can be continued to the relation $[f]$ for a unique $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$;

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X},Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}\colon X\,\delta\,Y \tag{3}$$

for every $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{B}$.

Proof. Existence of no more than one such pointfree funcoids and formulas (1) and (3) follow from two previous theorems.

2. $\{Y\in\mathfrak{Z}_1\mid X\,\delta\,Y\}$ is obviously a free star for every $X\in\mathfrak{Z}_0$. By properties of filters on boolean lattices, there exist a unique filter object αX such that $\partial(\alpha X)=\{Y\in\mathfrak{Z}_1\mid X\,\delta\,Y\}$ for every $X\in\mathfrak{Z}_0$. Thus $\alpha\in\mathfrak{B}^{\mathfrak{Z}_0}$. Similarly can be defined $\beta\in\mathfrak{A}^{\mathfrak{Z}_1}$ by the formula $\partial(\beta X)=\{X\in\mathfrak{Z}_0\mid X\,\delta\,Y\}$. Let's continue the functions α and β to $\alpha'\in\mathfrak{B}^{\mathfrak{A}}$ and $\beta'\in\mathfrak{A}^{\mathfrak{B}}$ by the formulas

$$\alpha'\mathcal{X}=\bigcap^{\mathfrak{B}}\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\quad\text{and}\quad\beta'\mathcal{X}=\bigcap^{\mathfrak{A}}\langle\beta\rangle\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{X}$$

and δ to $\delta'\in\mathscr{P}(\mathfrak{A}\times\mathfrak{B})$ by the formula

$$\mathcal{X}\,\delta'\,\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X},Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}\colon X\,\delta\,Y.$$

$\mathcal{Y}\cap^{\mathfrak{B}}\alpha'\mathcal{X}\neq0\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{B}}\bigcap^{\mathfrak{B}}\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\neq\emptyset\Leftrightarrow\bigcap^{\mathfrak{B}}\langle\mathcal{Y}\cap^{\mathfrak{B}}\rangle\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\neq0$. Let's prove that

$$W=\langle\mathcal{Y}\cap^{\mathfrak{B}}\rangle\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$ is a generalized filter base.

If $\mathcal{A},\mathcal{B}\in\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$ then exist $X_1,X_2\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$ such that $\mathcal{A}=\alpha X_1$ and $\mathcal{B}=\alpha X_2$.

Then $\alpha(X_1\cap^{\mathfrak{Z}_0}X_2)\in\langle\alpha\rangle\mathrm{up}\,\mathcal{X}$. So $\langle\alpha\rangle\mathrm{up}\,\mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

By properties of generalized filter bases, $\bigcap^{\mathfrak{F}}\langle\mathcal{Y}\cap^{\mathfrak{B}}\rangle\langle\alpha\rangle\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\neq0$ is equivalent to

$$\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}\colon\mathcal{Y}\cap^{\mathfrak{B}}\alpha X\neq\emptyset,$$

what is equivalent to $\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$, $Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}$: $Y\cap^{\mathfrak{B}}\alpha X\neq\emptyset\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$, $Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}$: $Y\in\partial(\alpha X)\Leftrightarrow\forall X\in\mathrm{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$, $Y\in\mathrm{up}^{(\mathfrak{B};\mathfrak{Z}_1)}\mathcal{Y}$: $X\,\delta\,Y$. Combining the equivalencies we get $\mathcal{Y}\cap^{\mathfrak{B}}\alpha'\mathcal{X}\neq0\Leftrightarrow X\,\delta'\,Y$. Analogously $\mathcal{X}\cap^{\mathfrak{A}}\beta'\mathcal{Y}\neq\emptyset\Leftrightarrow X\,\delta'\,Y$. So $\mathcal{Y}\cap^{\mathfrak{B}}\alpha'\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{X}\cap^{\mathfrak{A}}\beta'\mathcal{Y}\neq0$, that is $(\mathfrak{A};\mathfrak{B};\alpha';\beta')$ is a pointfree funcoid. From the formula $\mathcal{Y}\cap^{\mathfrak{B}}\alpha'\mathcal{X}\neq\emptyset\Leftrightarrow X\,\delta'\,Y$ follows that $[(\mathfrak{A};\mathfrak{B};\alpha';\beta')]$ is a continuation of δ.

1. Let define the relation $\delta\in\mathscr{P}(\mathfrak{Z}_0\times\mathfrak{Z}_1)$ by the formula $X\,\delta\,Y\Leftrightarrow Y\cap^{\mathfrak{B}}\alpha X\neq\emptyset$.

That $\neg(0\;\delta\;I')$ and $\neg(I\;\delta\;0)$ is obvious. We have $K\;\delta\;I'\cup^{\mathfrak{Z}_1}J'\Leftrightarrow(I'\cup^{\mathfrak{Z}_1}J')\cap^{\mathfrak{B}}\alpha K\neq0\Leftrightarrow(I'\cup^{\mathfrak{B}}J')\cap^{\mathfrak{B}}\alpha K\neq0\Leftrightarrow(I'\cap^{\mathfrak{B}}\alpha K)\cup^{\mathfrak{B}}(J'\cup^{\mathfrak{B}}\alpha K)\neq0\Leftrightarrow I'\cap^{\mathfrak{B}}\alpha K\neq\emptyset\vee J'\cup^{\mathfrak{B}}\alpha K\neq\emptyset\Leftrightarrow K\;\delta\;I'\vee K\;\delta\;J'$ and $I\cup^{\mathfrak{Z}_0}J\;\delta\;K'\Leftrightarrow K'\cap^{\mathfrak{B}}\alpha(I\cup^{\mathfrak{Z}_0}J)\neq0\Leftrightarrow K'\cap^{\mathfrak{B}}(\alpha I\cup^{\mathfrak{B}}\alpha J)\neq0\Leftrightarrow(K'\cap^{\mathfrak{B}}\alpha I)\cup^{\mathfrak{B}}(K'\cap^{\mathfrak{B}}\alpha J)\neq0\Leftrightarrow K'\cap^{\mathfrak{B}}\alpha I\neq0\vee K'\cap^{\mathfrak{B}}\alpha J\neq\emptyset\Leftrightarrow I\,\delta\,K'\vee J\,\delta\,K'$.

That is the formulas (2) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

$\forall X\in\mathfrak{Z}_0,Y\in\mathfrak{Z}_1\colon(Y\cap^{\mathfrak{F}}\langle f\rangle X\neq0\Leftrightarrow X[f]Y\Leftrightarrow Y\cap^{\mathfrak{B}}\alpha X\neq0)$, consequently $\forall X\in\mathfrak{Z}_0\colon\alpha X=\langle f\rangle X$ because our filtrator is with separable core. So $\langle f\rangle$ is a continuation of α. □

4.4 The preorder of funcoids

The *preorder of pointfree funcoids* is defined by the formula $f\subseteq g\Leftrightarrow[f]\subseteq[g]$ for every pointfree funcoids f and g.

Remark 25. It is enough to define preorder of pointfree funcoids on every set $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ where $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements. We do not need to compare funcoids with different sources or destinations.

Theorem 26. If $\mathfrak{A}$ and $\mathfrak{B}$ are separable posets then $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a poset.

Proof. From the theorem 12. □

Theorem 27. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices. Then for $R\in\mathscr{P}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $X\in\mathfrak{Z}_0$, $Y\in\mathfrak{Z}_1$ we have:

1. $X\left[\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\right]Y\Leftrightarrow\exists f\in R: X[f]Y$;
2. $\left\langle\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\right\rangle X=\bigcup^{\mathfrak{B}}\{\langle f\rangle X\mid f\in R\}$.

Proof.

2. $\alpha X\overset{\mathrm{def}}{=}\bigcup^{\mathfrak{B}}\{\langle f\rangle X\mid f\in R\}$ (by corollary 8 in [2] all joins on $\mathfrak{B}$ exist). We have $\alpha 0=0$;

$$\begin{aligned}\alpha(I\cup^{\mathfrak{Z}_0}J) &= \bigcup\nolimits^{\mathfrak{F}}\{\langle f\rangle(I\cup^{\mathfrak{Z}_0}J)\mid f\in R\}\\ &\ \bigcup\nolimits^{\mathfrak{F}}\{\langle f\rangle(I\cup^{\mathfrak{A}}J)\mid f\in R\}\\ &= \bigcup\nolimits^{\mathfrak{F}}\{\langle f\rangle I\cup^{\mathfrak{B}}\langle f\rangle J\mid f\in R\}\\ &= \bigcup\nolimits^{\mathfrak{B}}\{\langle f\rangle I\mid f\in R\}\cup^{\mathfrak{B}}\bigcup\nolimits^{\mathfrak{B}}\{\langle f\rangle J\mid f\in R\}\\ &= \alpha I\cup^{\mathfrak{B}}\alpha J\end{aligned}$$

(used the theorem 13). By the theorem 24 the function α can be continued to $\langle h\rangle$ for a $h\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$. Obviously

$$\forall f\in R: h\supseteq f. \tag{4}$$

And h is the least element of $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ for which holds the condition (4). So $h=\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R$.

1. $X[\bigcup^{\mathsf{FCD}}R]Y\Leftrightarrow Y\cap^{\mathfrak{B}}\langle\bigcup^{\mathsf{FCD}}R\rangle X\neq 0\Leftrightarrow Y\cap^{\mathfrak{B}}\bigcup^{\mathfrak{B}}\{\langle f\rangle X\mid f\in R\}\neq 0\Leftrightarrow\exists f\in R: Y\cap^{\mathfrak{B}}\langle f\rangle X\neq 0\Leftrightarrow\exists f\in R: X[f]Y$ (used the theorem 40 in [2]). □

Corollary 28. If $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices then $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a complete lattice.

Proof. $\mathfrak{A}$ and $\mathfrak{B}$ are separable accordingly obvious 20 in [2].

Then apply [1] taking in account the theorem 12. □

Theorem 29. Let $\mathfrak{A}$ and $\mathfrak{B}$ are distributive lattices with least elements. Then:

1. $\left\langle f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g\right\rangle\mathcal{X}=\langle f\rangle\mathcal{X}\cup^{\mathfrak{B}}\langle g\rangle\mathcal{X}$ for every $\mathcal{X}\in\mathfrak{A}$;
2. $\left[f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g\right]=[f]\cup[g]$.

Proof.

1. Let $\alpha\mathcal{X}\overset{\mathrm{def}}{=}\langle f\rangle\mathcal{X}\cup^{\mathfrak{B}}\langle g\rangle\mathcal{X}$; $\beta\mathcal{Y}\overset{\mathrm{def}}{=}\langle f^{-1}\rangle\mathcal{Y}\cup^{\mathfrak{A}}\langle g^{-1}\rangle\mathcal{Y}$ for every $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{B}$. Then

$$\begin{aligned}\mathcal{Y}\cap^{\mathfrak{B}}\alpha\mathcal{X}\neq\emptyset &\Leftrightarrow (\mathcal{Y}\cap^{\mathfrak{B}}\langle f\rangle\mathcal{X})\cup^{\mathfrak{B}}(\mathcal{Y}\cap^{\mathfrak{B}}\langle g\rangle\mathcal{X})\neq 0\\ &\Leftrightarrow \mathcal{Y}\cap^{\mathfrak{B}}\langle f\rangle\mathcal{X}\neq 0\vee\mathcal{Y}\cap^{\mathfrak{B}}\langle g\rangle\mathcal{X}\neq 0\\ &\Leftrightarrow \mathcal{X}\cap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{Y}\neq 0\vee\mathcal{X}\cap^{\mathfrak{A}}\langle g^{-1}\rangle\mathcal{Y}\neq 0\\ &\Leftrightarrow (\mathcal{X}\cap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{Y})\cup^{\mathfrak{A}}(\mathcal{X}\cap^{\mathfrak{A}}\langle g^{-1}\rangle\mathcal{Y})\neq 0\\ &\Leftrightarrow \mathcal{X}\cap^{\mathfrak{A}}\beta\mathcal{Y}\neq 0.\end{aligned}$$

So $h=(\mathfrak{A};\mathfrak{B};\alpha;\beta)$ is a pointfree funcoid. Obviously $h\supseteq f$ and $h\supseteq g$. If $p\supseteq f$ and $p\supseteq g$ for some $p\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ then $\langle p\rangle\mathcal{X}\supseteq\langle f\rangle\mathcal{X}\cup^{\mathfrak{F}}\langle g\rangle\mathcal{X}=\langle h\rangle\mathcal{X}$ that is $p\supseteq h$. So $f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g=h$.

2. $\mathcal{X}\left[f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g\right]\mathcal{Y}\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{B}}\left\langle f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g\right\rangle\mathcal{X}\neq 0\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{B}}(\langle f\rangle\mathcal{X}\cup^{\mathfrak{B}}\langle g\rangle\mathcal{X})\neq 0\Leftrightarrow(\mathcal{Y}\cap^{\mathfrak{B}}\langle f\rangle\mathcal{X})\cup^{\mathfrak{B}}(\mathcal{Y}\cap^{\mathfrak{B}}\langle f\rangle\mathcal{Y})\neq 0\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{B}}\langle f\rangle\mathcal{X}\neq 0\vee\mathcal{Y}\cap^{\mathfrak{B}}\langle g\rangle\mathcal{X}\neq 0\Leftrightarrow\mathcal{X}[f]\mathcal{Y}\vee\mathcal{X}[g]\mathcal{Y}$ for every $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{B}$. □

4.5 More on composition of pointfree funcoids

Proposition 30. $[g\circ f]=[g]\circ\langle f\rangle=\langle g^{-1}\rangle^{-1}\circ[f]$ for every composable pointfree funcoids f and g.

Proof. $\mathcal{X}[g\circ f]\mathcal{Y}\Leftrightarrow\mathcal{Y}\not\asymp^{\mathrm{Dst}\,g}\langle g\circ f\rangle\mathcal{X}\Leftrightarrow\mathcal{Y}\not\asymp^{\mathrm{Dst}\,g}\langle g\rangle\langle f\rangle\mathcal{X}\Leftrightarrow\langle f\rangle\mathcal{X}[g]\mathcal{Y}\Leftrightarrow\mathcal{X}([g]\circ\langle f\rangle)\mathcal{Y}$ for every $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{B}$. Thus $[g\circ f]=[g]\circ\langle f\rangle$. $[g\circ f]=[(f^{-1}\circ g^{-1})^{-1}]=[f^{-1}\circ g^{-1}]^{-1}=([f^{-1}]\circ\langle g^{-1}\rangle)^{-1}=\langle g^{-1}\rangle^{-1}\circ[f]$. □

Theorem 31. Let f and g are pointfree funcoids and $\mathfrak{A}=\mathrm{Dst}\,f=\mathrm{Src}\,g$ is an atomic poset. Then for every $\mathcal{X}\in\mathrm{Src}\,f$ and $\mathcal{Z}\in\mathrm{Dst}\,g$

$$\mathcal{X}[g\circ f]\mathcal{Z}\Leftrightarrow\exists y\in\mathrm{atoms}^{\mathfrak{A}}:(\mathcal{X}[f]y\wedge y[g]\mathcal{Z}).$$

Proof.

$$\begin{aligned}\exists y\in\mathrm{atoms}^{\mathfrak{A}}:(\mathcal{X}[f]y\wedge y[g]\mathcal{Z}) &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{A}}:(\mathcal{Z}\not\asymp^{\mathrm{Dst}\,g}\langle g\rangle y\wedge y\not\asymp^{\mathfrak{A}}\langle f\rangle\mathcal{X})\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{A}}:(\mathcal{Z}\not\asymp^{\mathrm{Dst}\,g}\langle g\rangle y\wedge y\subseteq\langle f\rangle\mathcal{X})\\ &\Rightarrow \mathcal{Z}\not\asymp^{\mathrm{Dst}\,g}\langle g\rangle\langle f\rangle\mathcal{X}\\ &\Leftrightarrow \mathcal{X}[g\circ f]\mathcal{Z}.\end{aligned}$$

Reversely, if $\mathcal{X}[g\circ f]\mathcal{Z}$ then $\langle f\rangle\mathcal{X}[g]\mathcal{Z}$, consequently exists $y\in\mathrm{atoms}^{\mathfrak{A}}\langle f\rangle\mathcal{X}$ such that $y[g]\mathcal{Z}$; we have $\mathcal{X}[f]y$. □

Theorem 32. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are posets and $\mathfrak{B}$ is atomic. Then:

1. $f\circ(g\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}h)=f\circ g\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})}f\circ h$ for $g,h\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $f\in\mathsf{FCD}(\mathfrak{B};\mathfrak{C})$.
2. $(g\cup^{\mathsf{FCD}(\mathfrak{B};\mathfrak{C})}h)\circ f=g\circ f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})}h\circ f$ for $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $g,h\in\mathsf{FCD}(\mathfrak{B};\mathfrak{C})$.

Proof. I will prove only the first equality because the other is analogous.

For every $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{C}$

$$\begin{aligned}\mathcal{X}\left[f\circ(g\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}h)\right]\mathcal{Z} &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{B}}:(\mathcal{X}\left[g\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}h\right]y\wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{B}}:((\mathcal{X}[g]y\vee\mathcal{X}[h]y)\wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{B}}:((\mathcal{X}[g]y\wedge y[f]\mathcal{Z})\vee(\mathcal{X}[h]y\wedge y[f]\mathcal{Z}))\\ &\Leftrightarrow \exists y\in\mathrm{atoms}^{\mathfrak{B}}:(\mathcal{X}[g]y\wedge y[f]\mathcal{Z})\vee\exists y\in\mathrm{atoms}^{\mathfrak{B}}:(\mathcal{X}[h]y\wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \mathcal{X}[f\circ g]\mathcal{Z}\vee\mathcal{X}[f\circ h]\mathcal{Z}\\ &\Leftrightarrow \mathcal{X}\left[f\circ g\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{C})}f\circ h\right]\mathcal{Z}.\end{aligned}$$

□

4.6 Domain and range of a pointfree funcoid

Let $\mathfrak{A}$ is a meet-semilattice.

[TODO: Simple (not restricted) identity pointfree funcoids.]

Definition 33. Let $\mathcal{A}\in\mathfrak{A}$. The *diagonal pointfree funcoid* $I^{\mathsf{FCD}(\mathfrak{A})}_{\mathcal{A}}=(\mathfrak{A};\mathfrak{A};\mathcal{A}\cap^{\mathfrak{A}};\mathcal{A}\cap^{\mathfrak{A}})$.

Proposition 34. The diagonal pointfree funcoid is a pointfree funcoid.

Proof. We need to prove that $(\mathcal{A}\cap^{\mathfrak{A}}\mathcal{X})\not\asymp^{\mathfrak{A}}\mathcal{Y}\Leftrightarrow(\mathcal{A}\cap^{\mathfrak{A}}\mathcal{Y})\not\asymp^{\mathfrak{A}}\mathcal{X}$ what is obvious. □

Obvious 35. $\left(I^{\mathsf{FCD}(\mathfrak{A})}_{\mathcal{A}}\right)^{-1}=I^{\mathsf{FCD}(\mathfrak{A})}_{\mathcal{A}}$.

Obvious 36. $\mathcal{X}\left[I^{\mathsf{FCD}(\mathfrak{A})}_{\mathcal{A}}\right]\mathcal{Y}\Leftrightarrow\mathcal{A}\not\asymp^{\mathfrak{A}}\mathcal{X}\cap^{\mathfrak{A}}\mathcal{Y}$ for any $\mathcal{X},\mathcal{Y}\in\mathfrak{A}$.

Definition 37. I will define *restricting* of a pointfree funcoid f to an element $\mathcal{A}\in\mathrm{Src}\,f$ by the formula $f|_{\mathcal{A}}\stackrel{\mathrm{def}}{=}f\circ I^{\mathsf{FCD}(\mathrm{Src}\,f)}_{\mathcal{A}}$.

Definition 38. Let f is a pointfree funcoid whose source has greatest element.

Image of f will be defined by the formula $\operatorname{im} f = \langle f \rangle 1$.

Definition 39. *Domain* of a pointfree funcoid f is defined by the formula $\operatorname{dom} f = \operatorname{im} f^{-1}$ (when f has a poset with greatest element as its destination).

Proposition 40. $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \sqcap^{\operatorname{Src} f} \operatorname{dom} f)$ for every pointfree funcoid f whose destination is a separable poset with greatest element and source is a meet-semilattice and $\mathcal{X} \in \operatorname{Src} f$.

Proof. For every $\mathcal{Y} \in \operatorname{Dst} f$ we have $\mathcal{Y} \not\asymp^{\operatorname{Dst} f} \langle f \rangle (\mathcal{X} \sqcap^{\operatorname{Src} f} \operatorname{dom} f) \neq 0 \Leftrightarrow \mathcal{X} \sqcap^{\operatorname{Src} f} \operatorname{dom} f \sqcap^{\operatorname{Src} f} \langle f^{-1} \rangle \mathcal{Y} \neq 0 \Leftrightarrow \mathcal{X} \sqcap^{\operatorname{Src} f} \operatorname{im} f^{-1} \sqcap^{\operatorname{Src} f} \langle f^{-1} \rangle \mathcal{Y} \neq 0 \Leftrightarrow \mathcal{X} \sqcap^{\operatorname{Src} f} \langle f^{-1} \rangle \mathcal{Y} \neq 0 \Leftrightarrow \mathcal{Y} \not\asymp^{\operatorname{Dst} f} \langle f \rangle \mathcal{X}$. Thus $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \sqcap^{\operatorname{Src} f} \operatorname{dom} f)$ by separability of $\operatorname{Dst} f$. □

Proposition 41. $\mathcal{X} \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \Leftrightarrow (\langle f \rangle \mathcal{X}$ is not least) for every pointfree funcoid f whose destination is a poset with greatest element and source is a meet-semilattice and $\mathcal{X} \in \operatorname{Src} f$.

Proof. $\mathcal{X} \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \Leftrightarrow \mathcal{X} \not\asymp^{\operatorname{Src} f} \langle f^{-1} \rangle 1 \Leftrightarrow 1 \not\asymp^{\operatorname{Dst} f} \langle f \rangle \mathcal{X} \Leftrightarrow (\langle f \rangle \mathcal{X}$ is not least). □

Corollary 42. $\operatorname{dom} f = \bigsqcup^{\operatorname{Src} f} \{ a \in \operatorname{atoms}^{\operatorname{Src} f} \mid \langle f \rangle a \neq \emptyset \}$ for every pointfree funcoid f whose destination is a bounded poset and source is an atomistic meet-semilattice.

Proof. For every $a \in \operatorname{atoms}^{\operatorname{Src} f}$ we have $a \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \Leftrightarrow a \not\asymp^{\operatorname{Src} f} \operatorname{im} f^{-1} \Leftrightarrow a \not\asymp^{\operatorname{Src} f} \langle f^{-1} \rangle 1 \Leftrightarrow 1 \not\asymp \langle f \rangle a \Leftrightarrow \langle f \rangle a \neq 0$. So

$\operatorname{dom} f = \bigsqcup^{\operatorname{Src} f} \{ a \in \operatorname{atoms}^{\operatorname{Src} f} \mid a \not\asymp^{\operatorname{Src} f} \operatorname{dom} f \} = \bigsqcup^{\operatorname{Src} f} \{ a \in \operatorname{atoms}^{\operatorname{Src} f} \mid \langle f \rangle a \neq 0 \}$. □

Proposition 43. $\operatorname{dom} f|_{\mathcal{A}}^{\mathsf{FCD}(\operatorname{Src} f)} = \mathcal{A} \sqcap^{\operatorname{Src} f} \operatorname{dom} f$ for every pointfree funcoid f and $\mathcal{A} \in \operatorname{Src} f$ where $\operatorname{Src} f$ is a meet-semilattice and $\operatorname{Dst} f$ has greatest element.

Proof. $\operatorname{dom} f|_{\mathcal{A}}^{\mathsf{FCD}(\operatorname{Src} f)} = \operatorname{im} \left(I_{\mathcal{A}}^{\mathsf{FCD}(\operatorname{Src} f)} \circ f^{-1} \right) = \left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\operatorname{Src} f)} \right\rangle \langle f^{-1} \rangle 1 = \mathcal{A} \sqcap^{\operatorname{Src} f} \langle f^{-1} \rangle 1 = \mathcal{A} \sqcap^{\operatorname{Src} f} \operatorname{dom} f$. □

4.7 Category of funcoids

[TODO: Move this up in the text.]

I will define the category $\mathsf{FCD}^{\mathfrak{A}}$ of pointfree funcoids:

- The class of objects are posets.
- The set of morphisms from $\mathfrak{A}$ to $\mathfrak{B}$ is $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.
- The composition is the composition of pointfree funcoids.
- Identity morphism for an object $\mathfrak{A}$ is $(\mathfrak{A}; \mathfrak{A}; (=)|_{\mathfrak{A}}; (=)|_{\mathfrak{A}})$.

To prove that it is really a category is trivial.

4.8 Specifying funcoids by functions or relations on atomic filter objects

Theorem 44. Let $\mathfrak{A}$ is an atomic poset and $(\mathfrak{B}; \mathfrak{Z}_1)$ is a primary filtrator over a boolean lattice. Then for every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{X} \in \mathfrak{A}$ we have

$$\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}.$$

Proof. For every $Y \in \mathfrak{Z}_1$ we have

$$\begin{aligned} Y \not\asymp^{\mathfrak{B}} \langle f \rangle \mathcal{X} &\Leftrightarrow \mathcal{X} \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \\ &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : x \not\asymp^{\mathfrak{A}} \langle f^{-1} \rangle Y \\ &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} : Y \not\asymp^{\mathfrak{B}} \langle f \rangle x. \end{aligned}$$

Thus $\partial \langle f \rangle \mathcal{X} = \bigcup \langle \partial \rangle \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X} = \partial \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ (used the theorem 46 in [2]). Consequently $\langle f \rangle \mathcal{X} = \bigsqcup^{\mathfrak{B}} \langle \langle f \rangle \rangle \operatorname{atoms}^{\mathfrak{A}} \mathcal{X}$ by the corollary 15 in [2]. □

Theorem 45. Let f is a pointfree funcoid. Then for every $\mathcal{X}\in\operatorname{Src} f$ and $\mathcal{Y}\in\operatorname{Dst} f$

1. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\operatorname{atoms}^{\operatorname{Src} f}\mathcal{X}: x[f]\mathcal{Y}$ if $\operatorname{Src} f$ is an atomic poset.
2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists y\in\operatorname{atoms}^{\operatorname{Dst} f}\mathcal{Y}: \mathcal{X}[f]y$ if $\operatorname{Dst} f$ is an atomic poset.

Proof. I will prove only the second as the first is similar.

If $\mathcal{X}[f]\mathcal{Y}$, then $\mathcal{Y}\not\asymp^{\mathfrak{F}}\langle f\rangle\mathcal{X}$, consequently exists $y\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{Y}$ such that $y\not\asymp^{\mathfrak{F}}\langle f\rangle\mathcal{X}$, $\mathcal{X}[f]y$. The reverse is obvious. □

Corollary 46. If f is a pointfree funcoid with both source and destination being atomic posets, then for every $\mathcal{X}\in\operatorname{Src} f$ and $\mathcal{Y}\in\operatorname{Dst} f$

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\operatorname{atoms}^{\operatorname{Src} f}\mathcal{X}, y\in\operatorname{atoms}^{\operatorname{Dst} f}\mathcal{Y}: x[f]y.$$

Proof. Apply the theorem twice. □

Theorem 47. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices.

1. A function $\alpha\in\mathfrak{B}^{\operatorname{atoms}^{\mathfrak{A}}}$ such that (for every $a\in\operatorname{atoms}^{\mathfrak{A}}$)
$$\alpha a\subseteq\bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\circ\operatorname{atoms}^{\mathfrak{A}}\right\rangle\operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a \tag{5}$$
can be continued to the function $\langle f\rangle$ for a unique $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$;
$$\langle f\rangle\mathcal{X}=\bigcup^{\mathfrak{B}}\langle\alpha\rangle\operatorname{atoms}^{\mathfrak{F}}\mathcal{X} \tag{6}$$
for every $\mathcal{X}\in\mathfrak{A}$.
2. A relation $\delta\in\mathscr{P}(\operatorname{atoms}^{\mathfrak{A}}\times\operatorname{atoms}^{\mathfrak{B}})$ such that (for every $a\in\operatorname{atoms}^{\mathfrak{A}}$, $b\in\operatorname{atoms}^{\mathfrak{B}}$)
$$\forall X\in\operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a, Y\in\operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)}b\,\exists x\in\operatorname{atoms}^{\mathfrak{A}}X, y\in\operatorname{atoms}^{\mathfrak{B}}Y: x\,\delta\,y\Rightarrow a\,\delta\,b \tag{7}$$
can be continued to the relation $[f]$ for a unique $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$;
$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\operatorname{atoms}^{\mathfrak{A}}\mathcal{X}, y\in\operatorname{atoms}^{\mathfrak{B}}\mathcal{Y}: x\,\delta\,y \tag{8}$$
for every $\mathcal{X}\in\mathfrak{A}$, $\mathcal{Y}\in\mathfrak{B}$.

Proof. Existence of no more than one such funcoids and formulas (6) and (8) follow from the theorem 44 and corollary 46 and the fact that our filtrators are with separable core.

1. Consider the function $\alpha'\in\mathfrak{B}^{\mathfrak{Z}_0}$ defined by the formula (for every $X\in\mathfrak{Z}_0$)

$$\alpha' X=\bigcup^{\mathfrak{B}}\langle\alpha\rangle\operatorname{atoms}^{\mathfrak{A}}X.$$

Obviously $\alpha' 0=0$. For every $I,J\in\mathfrak{Z}_0$

$$\begin{aligned}
\alpha'(I\cup J) &= \bigcup^{\mathfrak{B}}\langle\alpha'\rangle\operatorname{atoms}^{\mathfrak{B}}(I\cup J)\\
&= \bigcup^{\mathfrak{B}}\langle\alpha'\rangle(\operatorname{atoms}^{\mathfrak{B}}I\cup\operatorname{atoms}^{\mathfrak{B}}J)\\
&= \bigcup^{\mathfrak{B}}(\langle\alpha'\rangle\operatorname{atoms}^{\mathfrak{B}}I\cup\langle\alpha'\rangle\operatorname{atoms}^{\mathfrak{B}}J)\\
&= \bigcup^{\mathfrak{B}}\langle\alpha'\rangle\operatorname{atoms}^{\mathfrak{B}}I\cup^{\mathfrak{B}}\bigcup^{\mathfrak{F}}\langle\alpha'\rangle\operatorname{atoms}^{\mathfrak{B}}J.\\
&= \alpha' I\cup^{\mathfrak{B}}\alpha' J.
\end{aligned}$$

Let continue α' till a funcoid f (by the theorem 24): $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{B}}\langle\alpha'\rangle\operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)}\mathcal{X}$.

Let's prove the reverse of (5):

$$\begin{aligned}
\bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\circ\operatorname{atoms}^{\mathfrak{A}}\right\rangle\operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a &= \bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\right\rangle\langle\operatorname{atoms}^{\mathfrak{A}}\rangle\operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)}a\\
&\subseteq \bigcap^{\mathfrak{B}}\left\langle\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\right\rangle\{\{a\}\}\\
&= \bigcap^{\mathfrak{B}}\left\{\left(\bigcup^{\mathfrak{B}}\circ\langle\alpha\rangle\right)\{a\}\right\}\\
&= \bigcap^{\mathfrak{B}}\left\{\bigcup^{\mathfrak{B}}\langle\alpha\rangle\{a\}\right\}\\
&= \bigcap^{\mathfrak{B}}\left\{\bigcup^{\mathfrak{B}}\{\alpha a\}\right\}=\bigcap^{\mathfrak{B}}\{\alpha a\}=\alpha a.
\end{aligned}$$

Finally,

$$\alpha a = \bigcap^{\mathfrak{B}} \left\langle \bigcup^{\mathfrak{B}} \circ \langle \alpha \rangle \circ \operatorname{atoms}^{\mathfrak{A}} \right\rangle \operatorname{up} a = \bigcap^{\mathfrak{B}} \langle \alpha' \rangle \operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)} a = \langle f \rangle a,$$

so $\langle f \rangle$ is a continuation of α.

2. Consider the relation $\delta' \in \mathscr{P}(\mathfrak{Z}_0 \times \mathfrak{Z}_1)$ defined by the formula (for every $X \in \mathfrak{Z}_0$, $Y \in \mathfrak{Z}_1$)

$$X \, \delta' \, Y \Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y.$$

Obviously $\neg(X \, \delta' \, 0)$ and $\neg(0 \, \delta' \, Y)$.

$$\begin{aligned} (I \cup J) \, \delta' \, Y &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}}(I \cup J), y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y \\ &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} I \cup \operatorname{atoms}^{\mathfrak{A}} J, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y \\ &\Leftrightarrow \exists x \in \operatorname{atoms}^{\mathfrak{A}} I, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y \vee \exists x \in \operatorname{atoms}^{\mathfrak{A}} J, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y \\ &\Leftrightarrow I \, \delta' \, Y \vee J \, \delta' \, Y; \end{aligned}$$

analogously $X \, \delta' \, (I \cup J) \Leftrightarrow X \, \delta' \, I \vee X \, \delta' \, J$. Let's continue δ' till a funcoid f (by the theorem 23):

$$\mathcal{X} [f] \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)} \mathcal{X}, Y \in \operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)} \mathcal{Y} : X \, \delta' \, Y$$

The reverse of (7) implication is trivial, so

$$\forall X \in \operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)} a, Y \in \operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)} b \, \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y \Leftrightarrow a \, \delta \, b.$$

$\forall X \in \operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)} a, Y \in \operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)} b \, \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y \Leftrightarrow \forall X \in \operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)} a, Y \in \operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)} b$: $X \, \delta' \, Y \Leftrightarrow a [f] b$.

So $a \, \delta \, b \Leftrightarrow a [f] b$, that is $[f]$ is a continuation of δ. □

One of uses of the previous theorem is the proof of the following theorem:

Theorem 48. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. If $R \in \mathscr{P} \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $x \in \operatorname{atoms}^{\mathfrak{A}}$, $y \in \operatorname{atoms}^{\mathfrak{B}}$, then

1. $\left\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R \right\rangle x = \bigcap^{\mathfrak{B}} \{ \langle f \rangle x \mid f \in R \}$;
2. $x \left[\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R \right] y \Leftrightarrow \forall f \in R : x [f] y$.

Proof.

2. Let denote $x \, \delta \, y \Leftrightarrow \forall f \in R : x [f] y$.

$$\begin{gathered} \forall X \in \operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)} a, Y \in \operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)} b \, \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x \, \delta \, y \Leftrightarrow \\ \forall f \in R, X \in \operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)} a, Y \in \operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)} b \, \exists x \in \operatorname{atoms}^{\mathfrak{A}} X, y \in \operatorname{atoms}^{\mathfrak{B}} Y : x [f] y \Rightarrow \\ \forall f \in R, X \in \operatorname{up}^{(\mathfrak{A};\mathfrak{Z}_0)} a, Y \in \operatorname{up}^{(\mathfrak{B};\mathfrak{Z}_1)} b : X [f] Y \Rightarrow \\ \forall f \in R : a [f] b \Leftrightarrow \\ a \, \delta \, b. \end{gathered}$$

So, by the theorem 47, δ can be continued till $[p]$ for some $p \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

For every $q \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ such that $\forall f \in R : q \subseteq f$ we have $x [q] y \Rightarrow \forall f \in R : x [f] y \Leftrightarrow x \, \delta \, y \Leftrightarrow x [p] y$, so $q \subseteq p$. Consequently $p = \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R$.

From this $x \left[\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R \right] y \Leftrightarrow \forall f \in R : x [f] y$.

1. From the former $y \in \operatorname{atoms}^{\mathfrak{B}} \left\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R \right\rangle x \Leftrightarrow y \cap^{\mathfrak{B}} \left\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R \right\rangle x \neq \emptyset \Leftrightarrow \forall f \in R$: $y \cap^{\mathfrak{B}} \langle f \rangle x \neq \emptyset \Leftrightarrow y \in \bigcap \langle \operatorname{atoms}^{\mathfrak{B}} \rangle \{ \langle f \rangle x \mid f \in R \} \Leftrightarrow y \in \operatorname{atoms}^{\mathfrak{B}} \bigcap^{\mathfrak{B}} \{ \langle f \rangle x \mid f \in R \}$ for every $y \in \operatorname{atoms}^{\mathfrak{B}}$.

$\mathfrak{B}$ is atomically separable by the corollary 17 in [2]. Thus

$$\left\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} R \right\rangle x = \bigcap^{\mathfrak{B}} \{ \langle f \rangle x \mid f \in R \}.$$ □

Theorem 49. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are posets of filter object over some boolean lattices, $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$, $g \in \mathsf{FCD}(\mathfrak{B}; \mathfrak{C})$, $h \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{C})$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

Proof.

$$\begin{aligned} g\circ f\not\asymp h &\Leftrightarrow\\ \exists a\in\operatorname{atoms}1^{\mathfrak{F}(A)}, c\in\operatorname{atoms}1^{\mathfrak{F}(C)}: a[(g\circ f)\cap h]c &\Leftrightarrow\\ \exists a\in\operatorname{atoms}1^{\mathfrak{F}(A)}, c\in\operatorname{atoms}1^{\mathfrak{F}(C)}: (a[g\circ f]c\wedge a[h]c) &\Leftrightarrow\\ \exists a\in\operatorname{atoms}1^{\mathfrak{F}(A)}, b\in\operatorname{atoms}1^{\mathfrak{F}(B)}, c\in\operatorname{atoms}1^{\mathfrak{F}(C)}: (a[f]b\wedge b[g]c\wedge a[h]c) &\Leftrightarrow\\ \exists b\in\operatorname{atoms}1^{\mathfrak{F}(B)}, c\in\operatorname{atoms}1^{\mathfrak{F}(C)}: (b[g]c\wedge b[h\circ f^{-1}]c) &\Leftrightarrow\\ \exists b\in\operatorname{atoms}1^{\mathfrak{F}(B)}, c\in\operatorname{atoms}1^{\mathfrak{F}(C)}: b[g\cap(h\circ f^{-1})]c &\Leftrightarrow\\ g\not\asymp h\circ f^{-1}. & \end{aligned}$$

□

4.9 Direct product of elements

Definition 50. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets and $\mathcal{A}\in\mathfrak{A}$, $\mathcal{B}\in\mathfrak{B}$. *Direct product* of $\mathcal{A},\mathcal{B}\in\mathfrak{A}$ is such a pointfree funcoid $\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ that

$$\mathcal{X}\left[\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}\right]\mathcal{Y}\Leftrightarrow\mathcal{X}\not\asymp^{\mathfrak{A}}\mathcal{A}\wedge\mathcal{Y}\not\asymp^{\mathfrak{B}}\mathcal{B}.$$

Proposition 51. $\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}$ is really a pointfree funcoid and for every $\mathcal{X}\in\mathfrak{A}$

$$\left\langle\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}\right\rangle\mathcal{X}=\begin{cases}\mathcal{B} & \text{if } \mathcal{X}\not\asymp^{\mathfrak{A}}\mathcal{A};\\ 0 & \text{if } \mathcal{X}\asymp^{\mathfrak{A}}\mathcal{A}.\end{cases}$$

Proof. Obvious. □

Proposition 52. Let $\mathfrak{A}$ and $\mathfrak{B}$ are bounded posets, $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $\mathcal{A}\in\mathfrak{A}$, $\mathcal{B}\in\mathfrak{B}$. Then

$$f\subseteq\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}\Leftrightarrow\operatorname{dom}f\subseteq\mathcal{A}\wedge\operatorname{im}f\subseteq\mathcal{B}.$$

Proof. If $f\subseteq\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}$ then $\operatorname{dom}f\subseteq\operatorname{dom}(\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B})\subseteq\mathcal{A}$, $\operatorname{im}f\subseteq\operatorname{im}(\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B})\subseteq\mathcal{B}$.
If $\operatorname{dom}f\subseteq\mathcal{A}\wedge\operatorname{im}f\subseteq\mathcal{B}$ then

$$\forall\mathcal{X}\in\mathfrak{A},\mathcal{Y}\in\mathfrak{B}: (\mathcal{X}[f]\mathcal{Y}\Rightarrow\mathcal{X}\not\asymp^{\mathfrak{F}}\mathcal{A}\wedge\mathcal{Y}\not\asymp^{\mathfrak{F}}\mathcal{B});$$

consequently $f\subseteq\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}$. □

Theorem 53. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. For every $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $\mathcal{A}\in\mathfrak{A}$, $\mathcal{B}\in\mathfrak{B}$

$$f\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}(\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B})=I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})}\circ f\circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}.$$

Proof. From above $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a (complete) lattice.
$h\overset{\text{def}}{=}I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})}\circ f\circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}$. For every $\mathcal{X}\in\mathfrak{A}$

$$\langle h\rangle\mathcal{X}=\left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})}\right\rangle\langle f\rangle\left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}\right\rangle\mathcal{X}=\mathcal{B}\cap^{\mathfrak{B}}\langle f\rangle(\mathcal{A}\cap^{\mathfrak{A}}\mathcal{X}).$$

From this, as easy to show, $h\subseteq f$ and $h\subseteq\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}$. If $g\subseteq f\wedge g\subseteq\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}$ for a $g\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ then $\operatorname{dom}g\subseteq\mathcal{A}$, $\operatorname{im}g\subseteq\mathcal{B}$,

$$\langle g\rangle\mathcal{X}=\mathcal{B}\cap^{\mathfrak{B}}\langle g\rangle(\mathcal{A}\cap^{\mathfrak{A}}\mathcal{X})\subseteq\mathcal{B}\cap^{\mathfrak{B}}\langle f\rangle(\mathcal{A}\cap^{\mathfrak{A}}\mathcal{X})=\left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})}\right\rangle\langle f\rangle\left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}\right\rangle\mathcal{X}=\langle h\rangle\mathcal{X},$$

$g\subseteq h$. So $h=f\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}(\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B})$. □

Corollary 54. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. For every $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $\mathcal{A}\in\mathfrak{A}$ we have $f|_{\mathcal{A}}=f\cap(\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}1)$.

Proof. $f\cap(\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}1)=I_1^{\mathsf{FCD}(\mathfrak{B})}\circ f\circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}=f\circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}=f|_{\mathcal{A}}$. □

Corollary 55. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. For every $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathcal{A} \in \mathfrak{A}$, $\mathcal{B} \in \mathfrak{B}$ we have

$$f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}) \neq 0 \Leftrightarrow \mathcal{A}[f]\mathcal{B}.$$

Proof. $f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}) \neq 0 \Leftrightarrow \langle f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B})\rangle 1 \neq 0 \Leftrightarrow \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})} \circ f \circ I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}\right\rangle 1 \neq 0 \Leftrightarrow \left\langle I_{\mathcal{B}}^{\mathsf{FCD}(\mathfrak{B})}\right\rangle \langle f\rangle \left\langle I_{\mathcal{A}}^{\mathsf{FCD}(\mathfrak{A})}\right\rangle 1 \neq 0 \Leftrightarrow \mathcal{B} \cap^{\mathfrak{B}} \langle f\rangle (\mathcal{A} \cap^{\mathfrak{A}} 1) \neq 0 \Leftrightarrow \mathcal{B} \cap^{\mathfrak{B}} \langle f\rangle \mathcal{A} \neq 0 \Leftrightarrow \mathcal{A}[f]\mathcal{B}$. □

Theorem 56. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. Then the poset $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is separable.

Proof. Let $f, g \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $f \neq g$. By the theorem 12 $[f] \neq [g]$. That is there exist $x, y \in \mathfrak{A}$ such that $x[f]y \not\Leftrightarrow x[g]y$ that is $f \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (x \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} y) \neq \emptyset \not\Leftrightarrow g \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (x \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} y) \neq \emptyset$. Thus $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is separable. □

Theorem 57. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. If $S \in \mathscr{P}(\mathfrak{A} \times \mathfrak{B})$ then

$$\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \{\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\} = \bigcap^{\mathfrak{A}} \operatorname{dom} S \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \bigcap^{\mathfrak{B}} \operatorname{im} S.$$

Proof. If $x \in \operatorname{atoms}^{\mathfrak{A}}$ then by the theorem 48

$$\left\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \{\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\}\right\rangle x = \bigcap^{\mathfrak{B}} \{\langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}\rangle x \mid (\mathcal{A}; \mathcal{B}) \in S\}.$$

If $x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S \neq 0$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{A}} \mathcal{A} \neq \emptyset \wedge \langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}\rangle x = \mathcal{B});$$
$$\{\langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}\rangle x \mid (\mathcal{A}; \mathcal{B}) \in S\} = \operatorname{im} S;$$

if $x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S = 0$ then

$$\exists (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{A}} \mathcal{A} = 0 \wedge \langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}\rangle x = 0);$$
$$\{\langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}\rangle x \mid (\mathcal{A}; \mathcal{B}) \in S\} \ni 0.$$

So

$$\left\langle \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \{\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\}\right\rangle x = \begin{cases} \bigcap^{\mathfrak{B}} \operatorname{im} S & \text{if } x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S \neq 0; \\ 0 & \text{if } x \cap^{\mathfrak{A}} \bigcap^{\mathfrak{A}} \operatorname{dom} S = 0. \end{cases}$$

From this by corollary 46 (taking in account 47 in [2]) follows the statement of the theorem. □

Corollary 58. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices.

For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{A}$ and $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{B}$

$$(\mathcal{A}_0 \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_0) \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A}_1 \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_1) = (\mathcal{A}_0 \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{A}_1) \times^{\mathsf{FCD}^{\mathfrak{A}}} (\mathcal{B}_0 \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_1).$$

Proof. $(\mathcal{A}_0 \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_0) \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{A}_1 \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_1) = \bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \{\mathcal{A}_0 \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_1\}$ what is by the last theorem equal to $(\mathcal{A}_0 \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{A}_1) \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} (\mathcal{B}_0 \cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}_1)$. □

Theorem 59. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. If $\mathcal{A} \in \mathfrak{A}$ then $\mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ is a complete homomorphism of the lattice $\mathfrak{A}$ to a complete sublattice of the lattice $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$, if also $\mathcal{A} \neq 0$ then it is an isomorphism.

Proof. Let $S \in \mathscr{P}\mathfrak{A}$, $X \in \mathfrak{Z}_0$, $x \in \operatorname{atoms}^{\mathfrak{A}}$.

$$\begin{aligned} \left\langle \bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \rangle S\right\rangle X &= \bigcup^{\mathfrak{B}} \{\langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \mathcal{B}\rangle X \mid \mathcal{B} \in S\} \\ &= \begin{cases} \bigcup^{\mathfrak{B}} S & \text{if } X \cap^{\mathfrak{A}} \mathcal{A} \neq 0 \\ 0 & \text{if } X \cap^{\mathfrak{A}} \mathcal{A} = 0 \end{cases} \\ &= \left\langle \mathcal{A} \times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})} \bigcup^{\mathfrak{B}} S\right\rangle X. \end{aligned}$$

Thus $\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\langle\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\rangle=\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\bigcup^{\mathfrak{B}}S$ by the theorem 23 (taking in account obvious 20 in [2]).

$$\begin{aligned}\langle\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\langle\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\rangle S\rangle x &= \bigcap^{\mathfrak{B}}\{\langle\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\mathcal{B}\rangle x \mid \mathcal{B}\in S\}\\ &= \begin{cases}\bigcap^{\mathfrak{B}}S & \text{if } x\cap^{\mathfrak{A}}\mathcal{A}\neq 0\\ 0 & \text{if } x\cap^{\mathfrak{A}}\mathcal{A}=0\end{cases}\\ &= \langle\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\bigcap^{\mathfrak{B}}S\rangle x.\end{aligned}$$

Thus $\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\langle\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\rangle S=\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\bigcap^{\mathfrak{B}}S$ by the theorem 48.

If $\mathcal{A}\neq 0$ then obviously the function $\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ is injective. □

Proposition 60. Let $\mathfrak{A}$ is a meet-semilattice and $\mathfrak{B}$ is a poset with least element. If a is an atom of $\mathfrak{A}$, $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ then $f|_a=a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\langle f\rangle a$.

Proof. Let $\mathcal{X}\in\mathfrak{A}$.

$$\mathcal{X}\cap^{\mathfrak{A}}a\neq 0\Rightarrow\langle f|_a\rangle\mathcal{X}=\langle f\rangle a,\quad \mathcal{X}\cap^{\mathfrak{A}}a=0\Rightarrow\langle f|_a\rangle\mathcal{X}=0.$$ □

4.10 Atomic pointfree funcoids

Theorem 61. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. A $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is an atom of the poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ iff there exist $a\in\operatorname{atoms}^{\mathfrak{A}}$ and $b\in\operatorname{atoms}^{\mathfrak{B}}$ such that $f=a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b$.

Proof.

- **⇒.** Let f is an atom of the poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$. Let's get elements $a\in\operatorname{atoms}^{\mathfrak{A}}\operatorname{dom}f$ and $b\in\operatorname{atoms}^{\mathfrak{B}}\langle f\rangle a$. Then for every $\mathcal{X}\in\mathfrak{A}$

  $$\mathcal{X}\asymp^{\mathfrak{A}}a\Rightarrow\langle a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\rangle\mathcal{X}=0\subseteq\langle f\rangle\mathcal{X},\quad \mathcal{X}\not\asymp^{\mathfrak{A}}a\Rightarrow\langle a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\rangle\mathcal{X}=b\subseteq\langle f\rangle\mathcal{X}.$$

  So $a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\subseteq f$; because f is atomic we have $f=a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b$.

- **⇐.** Let $a\in\operatorname{atoms}^{\mathfrak{A}}$, $b\in\operatorname{atoms}^{\mathfrak{B}}$, $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$. If $b\asymp^{\mathfrak{F}}\langle f\rangle a$ then $\neg(a[f]b)$, $f\cap^{\mathsf{FCD}}(a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b)=0$ (because $\mathfrak{A}$ and $\mathfrak{B}$ are bounded meet-semilattices); if $b\subseteq\langle f\rangle a$ then $\forall\mathcal{X}\in\mathfrak{A}:(\mathcal{X}\not\asymp^{\mathfrak{F}}a\Rightarrow\langle f\rangle\mathcal{X}\supseteq b)$, $f\supseteq a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b$. Consequently $f\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}(a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b)=0\vee f\supseteq a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b$; that is $a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b$ is an atomic filter object. □

Theorem 62. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. Then $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is atomic.

Proof. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ and $f\neq 0$. Then $\operatorname{dom}f\neq 0$, thus exists $a\in\operatorname{atoms}^{\mathfrak{A}}\operatorname{dom}f$. So $\langle f\rangle a\neq 0$ thus exists $b\in\operatorname{atoms}^{\mathfrak{B}}\langle f\rangle a$. Finally the atomic pointfree funcoid $a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\subseteq f$. □

Theorem 63. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. Then the poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is separable.

Proof. Let $f,g\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $f\subset g$. Then taking in account the theorem 44 exists $a\in\operatorname{atoms}^{\mathfrak{A}}$ such that $\langle f\rangle a\subset\langle g\rangle a$. By corollary 17 in [2] $\mathfrak{B}$ is atomically separable. So exists $b\in\operatorname{atoms}^{\mathfrak{B}}$ such that $\langle f\rangle a\cap^{\mathfrak{B}}b=\emptyset$ and $b\subseteq\langle g\rangle a$. For every $x\in\operatorname{atoms}^{\mathfrak{A}}$

$$\langle f\rangle a\cap^{\mathfrak{B}}\langle a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\rangle a=\langle f\rangle a\cap^{\mathfrak{B}}b=0,$$
$$x\neq a\Rightarrow\langle f\rangle x\cap^{\mathfrak{B}}\langle a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\rangle x=\langle f\rangle x\cap^{\mathfrak{B}}0=0.$$

Thus $\langle f\rangle x\cap^{\mathfrak{B}}\langle a\times b\rangle x=0$ and consequently $f\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}(a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b)=0$.

$$\langle a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\rangle a=b\subseteq\langle g\rangle a,$$
$$x\neq a\Rightarrow\langle a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\rangle x=0\subseteq\langle g\rangle a.$$

Thus $\langle a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\rangle x\subseteq\langle g\rangle x$ and consequently $a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b\subseteq g$.

So the lattice of funcoids is separable by the theorem 19 in [2]. □

Corollary 64. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. The poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is:

1. separable;
2. atomically separable;
3. conforming to Wallman's disjunction property.

Proof. By the theorem 22 in [2]. □

Remark 65. For more ways to characterize (atomic) separability of the lattice of pointfree funcoids see [2], subsections "Separation subsets and full stars" and "Atomically separable lattices".

Corollary 66. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices. The poset $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is an atomistic lattice.

Proof. By the theorem 27 $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a complete lattice. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$. Suppose contrary to the statement to be proved that $\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}f\subset f$. Then exists $a\in\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}f$ such that $a\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\bigcup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}f=0$ what is impossible. □

Proposition 67. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices.
$\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}(f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g)=\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}f\cup\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g$ for every $f,g\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

Proof. $(a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b)\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}(f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g)\neq\emptyset\Leftrightarrow a\left[f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g\right]b\Leftrightarrow a[f]b\vee a[g]b\Leftrightarrow (a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b)\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}f\neq\emptyset\vee(a\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}b)\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g\neq\emptyset$ for every $a\in\operatorname{atoms}^{\mathfrak{A}}$ and $b\in\operatorname{atoms}^{\mathfrak{B}}$ (used the corollary 55 and theorem 29). □

Corollary 68. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices. For every f, $g,h\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$, $R\in\mathscr{P}\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$:

1. $f\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}(g\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}h)=(f\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}g)\cup^{\mathsf{FCD}^{\mathfrak{F}}}(f\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}h)$;
2. $f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R=\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\left\langle f\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\right\rangle R$.

Proof. We will take in account that the lattice of funcoids is an atomistic lattice (corollary 66). To be concise I will write atoms instead of $\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ and $\cap$ and $\cup$ instead of $\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$ and $\cup^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}$.

1. $\operatorname{atoms}(f\cap(g\cup h))=\operatorname{atoms}f\cap\operatorname{atoms}(g\cup h)=\operatorname{atoms}f\cap(\operatorname{atoms}g\cup\operatorname{atoms}h)=(\operatorname{atoms}f\cap\operatorname{atoms}g)\cup(\operatorname{atoms}f\cap\operatorname{atoms}h)=\operatorname{atoms}(f\cap g)\cup\operatorname{atoms}(f\cap h)=\operatorname{atoms}((f\cap g)\cup(f\cap h))$.
2. $\operatorname{atoms}\left(f\cup\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R\right)=\operatorname{atoms}f\cup\operatorname{atoms}\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}R=\operatorname{atoms}f\cup\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\langle\operatorname{atoms}\rangle R=\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\langle(\operatorname{atoms}f)\cup\rangle\langle\operatorname{atoms}\rangle R=\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\langle\operatorname{atoms}\rangle\langle f\cup\rangle R=\operatorname{atoms}\bigcap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{B})}\langle f\cup\rangle R$. (Used the following equality.)

$$\begin{aligned}
\langle(\operatorname{atoms}f)\cup\rangle\langle\operatorname{atoms}\rangle R &= \\
\{(\operatorname{atoms}f)\cup A\mid A\in\langle\operatorname{atoms}\rangle R\} &= \\
\{(\operatorname{atoms}f)\cup A\mid\exists C\in R: A=\operatorname{atoms}C\} &= \\
\{(\operatorname{atoms}f)\cup(\operatorname{atoms}C)\mid C\in R\} &= \\
\{\operatorname{atoms}(f\cup C)\mid C\in R\} &= \\
\{\operatorname{atoms}B\mid\exists C\in R: B=f\cup C\} &= \\
\{\operatorname{atoms}B\mid B\in\langle f\cup\rangle R\} &= \\
\langle\operatorname{atoms}\rangle\langle f\cup\rangle.
\end{aligned}$$

□

Corollary 69. Let $(\mathfrak{A};\mathfrak{Z}_0)$ and $(\mathfrak{B};\mathfrak{Z}_1)$ are primary filtrators over boolean lattices. Then $\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is a co-Brouwerian lattice.

Proposition 70. Let $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ are sets of filters over some boolean lattices and $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$, $g \in \mathsf{FCD}(\mathfrak{B}; \mathfrak{C})$. Let $\mathfrak{B}$ is an atomic poset. Then

$\operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{C})}(g \circ f) = \{x \times^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{C})} z \mid x \in \operatorname{atoms}^{\mathfrak{A}}, z \in \operatorname{atoms}^{\mathfrak{C}}, \exists y \in \operatorname{atoms}^{\mathfrak{B}}: (x \times^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} y \in \operatorname{atoms}^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} f \wedge y \times^{\mathsf{FCD}(\mathfrak{B}; \mathfrak{C})} z \in \operatorname{atoms}^{\mathsf{FCD}(\mathfrak{B}; \mathfrak{C})} g)\}$.

Proof. $(x \times^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{C})} z) \cap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{C})} (g \circ f) \neq \emptyset \Leftrightarrow x[g \circ f]z \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: (x[f]y \wedge y[g]z) \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{B}}: ((x \times^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} y) \cap^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} f \neq \emptyset \wedge (y \times^{\mathsf{FCD}(\mathfrak{B}; \mathfrak{C})} z) \cap^{\mathsf{FCD}(\mathfrak{B}; \mathfrak{C})} g \neq \emptyset)$ (were used the corollary 55 and theorem 31). □

4.11 Complete pointfree funcoids

Definition 71. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is *complete*, when for every $S \in \mathscr{P}\mathfrak{A}$ whenever both $\bigcup^{\mathfrak{A}} S$ and $\bigcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S$ are defined we have

$$\langle f \rangle \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S.$$

Proposition 72. Let $\mathfrak{A}$, $\mathfrak{B}$ are sets of filters over boolean lattices. A pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is complete iff $\langle f \rangle a = \bigcup^{\mathfrak{B}} \langle\langle f \rangle\rangle \operatorname{atoms}^{\mathfrak{A}} a$.

Proof. Direct implication is obvious. The reverse implication follows from that $\mathfrak{A}$ is atomistic. □

Remark 73. Let $\mathfrak{Z}$ is a join-semilattice with least element. I will call *pointfree generalized closure* such a function $\alpha \in \mathfrak{Z}^{\mathfrak{Z}}$ that

1. $\alpha 0 = 0$;
2. $\forall I, J \in \mathfrak{Z}: \alpha(I \cup^{\mathfrak{Z}} J) = \alpha I \cup^{\mathfrak{Z}} \alpha J$.

Definition 74. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. I will call a *co-complete pointfree funcoid* a pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ such that $\langle f \rangle|_{\mathfrak{Z}_0}$ is a pointfree generalized closure.

Proposition 75. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices. Co-complete pointfree funcoids $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ bijectively correspond to pointfree generalized closures $\mathfrak{Z}_1^{\mathfrak{Z}_0}$, where the bijection is $f \mapsto \langle f \rangle|_{\mathfrak{Z}_0}$.

Proof. Follows from the theorem 24. □

Theorem 76. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ is semifiltered, star-separable, down-aligned filtrator with finitely meet closed, join-closed, and separable core, where $\mathfrak{Z}_0$ is a complete boolean lattice and both $\mathfrak{Z}_0$ and $\mathfrak{A}$ are atomistic lattices.

Let $(\mathfrak{B}; \mathfrak{Z}_1)$ is a star-separable filtrator.

The following conditions are equivalent for every pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$:

1. f^{-1} is co-complete;
2. $\forall S \in \mathscr{P}\mathfrak{A}, J \in \mathfrak{Z}_1: (\bigcup^{\mathfrak{A}} S[f]J \Rightarrow \exists \mathcal{I} \in S: \mathcal{I}[f]J)$;
3. $\forall S \in \mathscr{P}\mathfrak{Z}_0, J \in \mathfrak{Z}_1: (\bigcup^{\mathfrak{Z}_0} S[f]J \Rightarrow \exists I \in S: I[f]J)$;
4. f is complete;
5. $\forall S \in \mathscr{P}\mathfrak{Z}_0: \langle f \rangle \bigcup^{\mathfrak{Z}_0} S = \bigcup^{\mathfrak{B}} \langle\langle f \rangle\rangle S$.

Proof. First note that the theorem 53 in [2] applies to the filtrator $(\mathfrak{A}; \mathfrak{Z}_0)$.

(3)⇒(1). For every $S \in \mathscr{P}\mathfrak{Z}_0$, $J \in \mathfrak{Z}_1$

$$\bigcup^{\mathfrak{Z}_0} S \cap^{\mathfrak{A}} \langle f^{-1} \rangle J \neq 0 \Rightarrow \exists I \in S: I \cap^{\mathfrak{B}} \langle f^{-1} \rangle J \neq 0, \tag{9}$$

consequently by the theorem 53 in [2] we have $\langle f^{-1} \rangle J \in \mathfrak{Z}_0$.

(1)⇒(2). For every $S \in \mathscr{P}\mathfrak{A}$, $J \in \mathfrak{Z}_1$ we have $\langle f^{-1} \rangle J \in \mathfrak{Z}_0$, consequently the formula (9) is true. From this follows (2).

(2)⇒(4). Let $\langle f \rangle \bigcup^{\mathfrak{Z}_0} S$ and $\bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle S$ are defined. $J \cap^{\mathfrak{B}} \langle f \rangle \bigcup^{\mathfrak{A}} S \neq 0 \Leftrightarrow \bigcup^{\mathfrak{A}} S [f] J \Leftrightarrow \exists \mathcal{I} \in S \colon \mathcal{I} [f] J \Leftrightarrow \exists \mathcal{I} \in S \colon J \cap^{\mathfrak{B}} \langle f \rangle \mathcal{I} \neq 0 \Leftrightarrow J \cap^{\mathfrak{B}} \bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle S \neq 0$ (used the theorem 53 in [2]). Thus $\langle f \rangle \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle S$ by star-separability of $(\mathfrak{B}; \mathfrak{Z}_1)$.

(5)⇒(3). Let $\langle f \rangle \bigcup^{\mathfrak{Z}_0} S$ is defined. Then $\bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle S$ is also defined because $\langle f \rangle \bigcup^{\mathfrak{Z}_0} S = \bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle S$. Then $\bigcup^{\mathfrak{Z}_0} S [f] J \Leftrightarrow J \cap^{\mathfrak{B}} \langle f \rangle \bigcup^{\mathfrak{Z}_0} S \neq 0 \Leftrightarrow J \cap^{\mathfrak{B}} \bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle S \neq 0$ what by the theorem 53 in [2] equivalent to $\exists I \in S \colon J \cap^{\mathfrak{B}} \langle f \rangle I \neq 0$ that is $\exists I \in S \colon I [f] J$.

(2)⇒(3), (4)⇒(5). By join-closedness of the core of $(\mathfrak{A}; \mathfrak{Z}_0)$. □

Theorem 77. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices and $\mathfrak{Z}_0$ is a complete boolean lattice. If R is a set of co-complete pointfree funcoids in $\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ then $\bigcup^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} R$ is a co-complete pointfree funcoid.

Proof. First, conditions of the theorem 76 apply.

Let R is a set of co-complete pointfree funcoids. Then for every $X \in \mathfrak{Z}_0$

$$\left\langle \bigcup^{\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})} R \right\rangle X = \bigcup^{\mathfrak{Z}_1} \{ \langle f \rangle X \mid f \in R \} \in \mathfrak{Z}_1$$

(used the theorem 27). □

Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets with least elements. I will denote $\mathrm{Compl}\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $\mathrm{CoCompl}\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ the sets of complete and co-complete funcoids correspondingly from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ with least elements.

Proposition 78.

1. Let $f \in \mathrm{Compl}\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $g \in \mathrm{Compl}\mathsf{FCD}(\mathfrak{B}; \mathfrak{C})$ where $\mathfrak{A}$ and $\mathfrak{C}$ are posets with least elements and $\mathfrak{B}$ is a complete lattice. Then $g \circ f \in \mathrm{Compl}\mathsf{FCD}(\mathfrak{A}; \mathfrak{C})$.
2. Let $f \in \mathrm{CoCompl}\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ and $g \in \mathrm{CoCompl}\mathsf{FCD}(\mathfrak{B}; \mathfrak{C})$ where $\mathfrak{A}$, $\mathfrak{B}$ and $\mathfrak{C}$ are posets with least elements and $(\mathfrak{A}; \mathfrak{Z}_0)$, $(\mathfrak{B}; \mathfrak{Z}_1)$, $(\mathfrak{C}; \mathfrak{Z}_2)$ are filtrators. Then $g \circ f \in \mathrm{CoCompl}\mathsf{FCD}(\mathfrak{A}; \mathfrak{C})$.

Proof.

1. Let $\bigcup^{\mathfrak{A}} S$ and $\bigcup^{\mathfrak{C}} \langle \langle g \circ f \rangle \rangle S$ are defined. Then

$$\langle g \circ f \rangle \bigcup^{\mathfrak{A}} S = \langle g \rangle \langle f \rangle \bigcup^{\mathfrak{A}} S = \langle g \rangle \bigcup^{\mathfrak{B}} \langle \langle f \rangle \rangle S = \bigcup^{\mathfrak{C}} \langle \langle g \rangle \rangle \langle \langle f \rangle \rangle S = \bigcup^{\mathfrak{C}} \langle \langle g \circ f \rangle \rangle S.$$

2. $\langle g \circ f \rangle \mathfrak{Z}_0 = \langle g \rangle \langle f \rangle \mathfrak{Z}_0 \in \mathfrak{Z}_2$ because $\langle f \rangle \mathfrak{Z}_0 \in \mathfrak{Z}_1$. □

Proposition 79. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices and $\mathfrak{Z}_0$ is a complete boolean lattice. Then $\mathrm{CoCompl}\mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ (with induced order) is a complete lattice.

Proof. Follows from the theorem 77. □

4.12 Completion and co-completion

Definition 80. Let $(\mathfrak{A}; \mathfrak{Z}_0)$ and $(\mathfrak{B}; \mathfrak{Z}_1)$ are primary filtrators over boolean lattices and $\mathfrak{Z}_1$ is a complete lattice.

Co-completion of a pointfree funcoid $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$ is pointfree funcoid $\mathrm{CoCompl}\, f$ defined by the formula (for every $X \in \mathfrak{Z}_0$)

$$\langle \mathrm{CoCompl}\, f \rangle X = \mathrm{Cor}\, \langle f \rangle X.$$

Proposition 81. Above defined co-completion always exists.

Proof. Existence of $\mathrm{Cor}\, \langle f \rangle X$ follows from completeness of $\mathfrak{Z}_1$.

We may apply the theorem 24 because

$$\operatorname{Cor}\langle f\rangle(X\cup^{\mathfrak{Z}_0}Y)=\operatorname{Cor}(\langle f\rangle X\cup^{\mathfrak{B}}\langle f\rangle Y)=\operatorname{Cor}\langle f\rangle X\cup^{\mathfrak{B}}\operatorname{Cor}\langle f\rangle Y$$

by the theorem 65 in [2]. □

Proposition 82. $\langle\operatorname{CoCompl} f\rangle X=\operatorname{Cor}'\langle f\rangle X$.

Proof. From the theorem 26 in [2]. (Existence of $\operatorname{Cor}'\langle f\rangle X$ follows from completeness of $\mathfrak{Z}_1$.) □

Obvious 83. Co-completion is always co-complete.

Obvious 84. For above defined always $\operatorname{CoCompl} f\subseteq f$.

4.13 Monovalued pointfree funcoids

Definition 85. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

The pointfree funcoid f is:

- *monovalued* when $f\circ f^{-1}\subseteq 1_{\mathfrak{B}}$.
- *injective* when $f^{-1}\circ f\subseteq 1_{\mathfrak{A}}$.

Monovaluedness is dual of injectivity.

Proposition 86. Let $\mathfrak{A}$ and $\mathfrak{B}$ are posets. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$.

The pointfree funcoid f is:

- monovalued iff $f\circ f^{-1}\subseteq I^{\mathsf{FCD}(\mathfrak{B})}_{\operatorname{im} f}$ if $\mathfrak{B}$ is a meet-semilattice;
- injective iff $f^{-1}\circ f\subseteq I^{\mathsf{FCD}(\mathfrak{A})}_{\operatorname{dom} f}$ if $\mathfrak{A}$ is a meet-semilattice.

Proof. Enough to prove $f\circ f^{-1}\subseteq 1_{\mathfrak{B}}\Leftrightarrow f\circ f^{-1}\subseteq I^{\mathsf{FCD}(\mathfrak{B})}_{\operatorname{im} f}$.

⇐. Obvious.

⇒. Let $f\circ f^{-1}\subseteq 1_{\mathfrak{B}}$. Then $\langle f\circ f^{-1}\rangle x\subseteq x$ and $\langle f\circ f^{-1}\rangle x\subseteq\operatorname{im} f$. Thus $\langle f\circ f^{-1}\rangle x\subseteq x\cap^{\mathfrak{B}}\operatorname{im} f=\left\langle I^{\mathsf{FCD}(\mathfrak{B})}_{\operatorname{im} f}\right\rangle x$. Thus $f\circ f^{-1}\subseteq I^{\mathsf{FCD}(\mathfrak{B})}_{\operatorname{im} f}$. □

Theorem 87. Let $\mathfrak{A}$ is an atomistic poset, $\mathfrak{B}$ is a poset with least element. The following statements are equivalent for every $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$:

1. f is monovalued.
2. $\forall a\in\operatorname{atoms}^{\mathfrak{A}}$: $\langle f\rangle a\in\operatorname{atoms}^{\mathfrak{B}}\cup\{0\}$.
3. $\forall\mathcal{I},\mathcal{J}\in\mathfrak{A}$: $\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})=\langle f^{-1}\rangle\mathcal{I}\cap^{\mathfrak{B}}\langle f^{-1}\rangle\mathcal{J}$.

Proof.

(2)⇒(3). Let $a\in\operatorname{atoms}^{\mathfrak{A}}$, $\langle f\rangle a=b$. Then because $b\in\operatorname{atoms}^{\mathfrak{B}}\cup\{0\}$

$$(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})\cap^{\mathfrak{B}}b\neq 0\Leftrightarrow\mathcal{I}\cap^{\mathfrak{B}}b\neq\emptyset\wedge\mathcal{J}\cap^{\mathfrak{B}}b\neq 0;$$
$$a[f](\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})\Leftrightarrow a[f]\mathcal{I}\wedge a[f]\mathcal{J};$$
$$(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})[f^{-1}]a\Leftrightarrow\mathcal{I}[f^{-1}]a\wedge\mathcal{J}[f^{-1}]a;$$
$$a\cap^{\mathfrak{A}}\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})\neq\emptyset\Leftrightarrow a\cap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{I}\neq\emptyset\wedge a\cap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{J}\neq 0;$$
$$a\cap^{\mathfrak{A}}\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})\neq\emptyset\Leftrightarrow a\cap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{I}\cap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{J}\neq 0;$$
$$\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})=\langle f^{-1}\rangle\mathcal{I}\cap^{\mathfrak{B}}\langle f^{-1}\rangle\mathcal{J}.$$

(3)⇒(1). $\langle f^{-1}\rangle a\cap^{\mathfrak{A}}\langle f^{-1}\rangle b=\langle f^{-1}\rangle(a\cap^{\mathfrak{A}}b)=\langle f^{-1}\rangle 0=0$ for every two distinct $a,b\in\operatorname{atoms}^{\mathfrak{B}}$. This is equivalent to $\neg(\langle f^{-1}\rangle a[f]b)$; $b\cap^{\mathfrak{A}}\langle f\rangle\langle f^{-1}\rangle a=0$; $b\cap^{\mathfrak{A}}\langle f\circ f^{-1}\rangle a=0$; $\neg(a[f\circ f^{-1}]b)$. So $a[f\circ f^{-1}]b\Rightarrow a=b$ for every $a,b\in\operatorname{atoms}^{\mathfrak{B}}$. This is possible only (corollary 46) when $f\circ f^{-1}\subseteq 1_{\mathfrak{B}}$.

¬(2)⇒¬(1). Suppose $\langle f\rangle a\notin \operatorname{atoms}^{\mathfrak{B}}\cup\{0\}$ for some $a\in\operatorname{atoms}^{\mathfrak{A}}$. Then there exist two atoms $p\neq q$ such that $\langle f\rangle a\supseteq p\wedge\langle f\rangle a\supseteq q$. Consequently $p\cap^{\mathfrak{B}}\langle f\rangle a\neq 0$; $a\cap^{\mathfrak{A}}\langle f^{-1}\rangle p\neq 0$; $a\subseteq\langle f^{-1}\rangle p$; $\langle f\circ f^{-1}\rangle p=\langle f\rangle\langle f^{-1}\rangle p\supseteq\langle f\rangle a\supseteq q$; $\langle f\circ f^{-1}\rangle p\nsubseteq p$ and $\langle f\circ f^{-1}\rangle p\neq 0$. So it cannot be $f\circ f^{-1}\subseteq 1_{\mathfrak{B}}$. □

Theorem 88. Let $\mathfrak{A}$ is separable distributive lattice with least element, $(\mathfrak{B};\mathfrak{Z})$ is a primary filtrator over a boolean lattice. A pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{B})$ is monovalued iff

$$\forall I,J\in\mathfrak{Z}:\langle f^{-1}\rangle(I\cap J)=\langle f^{-1}\rangle I\cap^{\mathfrak{A}}\langle f^{-1}\rangle J.$$

Proof. $\mathfrak{A}$ is a complete lattice (corollary 8 in [2]).

⇒. Obvious.

⇐. $\langle f^{-1}\rangle(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})=\bigcap^{\mathfrak{A}}\langle\langle f^{-1}\rangle\rangle\operatorname{up}(\mathcal{I}\cap^{\mathfrak{B}}\mathcal{J})=\bigcap^{\mathfrak{A}}\langle\langle f^{-1}\rangle\rangle\{I\cap^{\mathfrak{Z}}J\mid I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}\}=\bigcap^{\mathfrak{A}}\{\langle f^{-1}\rangle(I\cap^{\mathfrak{Z}}J)\mid I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}\}=\bigcap^{\mathfrak{A}}\{\langle f^{-1}\rangle I\cap^{\mathfrak{A}}\langle f^{-1}\rangle J\mid I\in\operatorname{up}\mathcal{I},J\in\operatorname{up}\mathcal{J}\}=\bigcap^{\mathfrak{A}}\{\langle f^{-1}\rangle I\mid I\in\operatorname{up}\mathcal{I}\}\cap^{\mathfrak{A}}\bigcap^{\mathfrak{F}}\{\langle f^{-1}\rangle J\mid J\in\operatorname{up}\mathcal{J}\}=\langle f^{-1}\rangle\mathcal{I}\cap^{\mathfrak{A}}\langle f^{-1}\rangle\mathcal{J}$ (used theorem 23, theorem 34 in [2], theorem 13, theorem 48 in [2]). □

4.14 Elements closed regarding a pointfree funcoid

Let $\mathfrak{A}$ is a poset with least element. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{A})$.

Definition 89. Let's call *closed* regarding a pointfree funcoid f such element $\mathcal{A}\in\mathfrak{A}$ that $\langle f\rangle\mathcal{A}\subseteq\mathcal{A}$.

Proposition 90. If $\mathcal{I}$ and $\mathcal{J}$ are closed (regarding pointfree funcoid f), S is a set of closed elements (regarding pointfree funcoid $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{A})$), then

1. $\mathcal{I}\cup^{\mathfrak{A}}\mathcal{J}$ is a closed element, if $\operatorname{Dst} f$ is a separable distributive lattice with least element;
2. $\bigcap^{\mathfrak{A}}S$ is a closed element.

Proof. $\langle f\rangle(\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J})=\langle f\rangle\mathcal{I}\cup^{\mathfrak{F}}\langle f\rangle\mathcal{J}\subseteq\mathcal{I}\cup^{\mathfrak{F}}\mathcal{J}$ (theorem 13), $\langle f\rangle\bigcap^{\mathfrak{A}}S\subseteq\bigcap^{\mathfrak{A}}\langle\langle f\rangle\rangle S\subseteq\bigcap^{\mathfrak{A}}S$. Consequently the elements $\mathcal{I}\cup^{\mathfrak{A}}\mathcal{J}$ and $\bigcap^{\mathfrak{A}}S$ are closed. □

Proposition 91. If S is a set of elements closed regarding a complete pointfree funcoid f, then the element $\bigcup^{\operatorname{Src} f}S$ is also closed regarding our funcoid.

Proof. $\langle f\rangle\bigcup^{\operatorname{Src} f}S=\bigcup^{\operatorname{Dst} f}\langle\langle f\rangle\rangle S\subseteq\bigcup^{\operatorname{Dst} f}S$. □

4.15 Connectedness regarding a pointfree funcoid

Let $\mathfrak{A}$ is a poset with least element. Let $f\in\mathsf{FCD}(\mathfrak{A};\mathfrak{A})$.

Definition 92. An element $\mathcal{A}\in\mathfrak{A}$ is called *connected* regarding a pointfree funcoid μ over $\mathfrak{A}$ when

$$\forall\mathcal{X},\mathcal{Y}\in\mathfrak{A}\setminus\{0\}:(\mathcal{X}\cup^{\mathfrak{A}}\mathcal{Y}=\mathcal{A}\Rightarrow\mathcal{X}[\mu]\mathcal{Y}).$$

Proposition 93. Let $(\mathfrak{A};\mathfrak{Z})$ is a co-separable filtrator. An $A\in\mathfrak{Z}$ is connected regarding a funcoid μ iff

$$\forall\mathcal{X},\mathcal{Y}\in\mathscr{P}\mho\setminus\{0\}:(X\cup^{\mathfrak{Z}}Y=A\Rightarrow X[\mu]Y).$$

Proof.

⇒. Obvious.

⇐. Follows from co-separability. □

Obvious 94. For $\mathfrak{A}$ being a set of filters over a boolean lattice, an element $\mathcal{A}\in\mathfrak{A}$ is connected regarding a funcoid μ iff it is connected regarding the funcoid $\mu\cap^{\mathsf{FCD}(\mathfrak{A};\mathfrak{A})}(\mathcal{A}\times^{\mathsf{FCD}(\mathfrak{A};\mathfrak{A})}\mathcal{A})$.

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