# Pointfree binary relations

Victor Porton

November 24, 2015

**Abstract**

I define **pointfree binary relations**, a way to describe binary relations and more general structures without referring to particular "points" (elements).

In this short article I define **pointfree binary relations**, that is a way to describe binary relations and more general structures without referring to particular "points" (elements).

**Definition.** Remind that **binary relations** (also called morphisms of the category **Rel**) are triples $(A, B, f)$ where $A$, $B$ are sets and $f \in \mathscr{P}(A \times B)$.

**Definition.** **Endo-relations** are relations of the form $(A, A, f)$ for some set $A$ and $f \in \mathscr{P}(A \times A)$.

**Definition.** **Isomorphism** between relations $\mu \in \mathscr{P}(A_0 \times A_0)$ and $\nu \in \mathscr{P}(A_1 \times A_1)$ is a bijection $f : A_1 \to A_0$ such that $\nu = f^{-1} \circ \mu \circ f$.

I remind that **precategory** is defined as category without requirement of existence of identity morphisms.

**Definition.** **Ordered precategory** is a precategory, each Hom-set of which is a poset, subject to the inequalities

$$f_0 \leq f_1 \wedge g_0 \leq g_1 \Rightarrow g_0 \circ f_0 \leq g_1 \circ f_1.$$

**Definition.** **Isomorphism** of ordered precategories is a map which is both precategory isomorphism and order isomorphism.

**Definition.** **System of pointfree relations** is:

- an ordered precategory (I call morphisms of which **pointfree (binary) relations**; the composition is denoted $\circ$);
- for each Hom-sets two posets: **possible domains** and **possible images**;
- **domain** dom and **image** im (maps from pointfree relations to possible domains and possible images of its Hom-set), subject to the inequalities

$$f \leq g \Rightarrow \operatorname{dom} f \leq \operatorname{dom} g \wedge \operatorname{im} f \leq \operatorname{im} g.$$

**Definition.** System of pointfree relations **induced** by binary relations is the system of pointfree relations whose precategory is the category **Rel** of binary relations, possible domains and images for $\mathbf{Rel}(A,B)$ are $\mathscr{P}A$ and $\mathscr{P}B$ correspondingly and domain and image are domain and image as usually defined for binary relations.

The pointfree relation induced by a binary relation is this binary relation considered as a pointfree binary relation.

**Definition.** **Pointfree endo-relations** are endo-relations whose source and destination (in our precategory) is the same.

**Obvious 1.** *System of pointfree relations induced by binary relations is really a system of pointfree relations.*

**Remark.** The category of pointfree relations induced by binary relations is a large category (its set of objects is a proper class).

**Definition.** The **isomorphism** between pointfree endo-relations $\mu$ and $\nu$ of a system of pointfree relations is an isomorphism $f$ of the precategory of this system such that $\nu = f^{-1} \circ \mu \circ f$.

**Obvious 2.** *The precategory for the system of pointfree relations induced by binary relations is a category.*

Let describe reversal $f \mapsto f^{-1}$ of a binary relation $f$ in pointfree terms:

$f$ is a join (i.e., supremum) of atomic binary relation. Reversal of an atomic relation $t$ is defined as the unique relation $t^{-1}$ conforming to the formulas $\operatorname{dom} t^{-1} = \operatorname{im} t$, $\operatorname{im} t^{-1} = \operatorname{dom} t$. So $f^{-1}$ is the supremum of such $t^{-1}$. Trivially combining these steps we get reversal of a binary relation described in pointfree terms.

Thus we have proved:

**Proposition 1.** *Reversal of binary relations can be restored, knowing only induced pointfree relations (up to isomorphism of systems of pointfree relations).*

**Proposition 2.** *Identity relation for binary relations can be restored, knowing only induced pointfree relations (up to isomorphism of systems of pointfree relations).*

*Proof.* Identity relation id is the identity element of our semigroup (and thus does not depend on the isomorphism). □

**Proposition 3.** *The set of bijections for relations* $\mathbf{Rel}(A,B)$ *can be restored, knowing only induced pointfree relations (up to isomorphism of systems of pointfree relations).*

*Proof.* Bijections $f : A \to B$ are characterized by the formulas $f \circ f^{-1} = \mathrm{id}_B$ and $f^{-1} \circ f = \mathrm{id}_A$. □

The following theorem states that pointfree endo-relations induced by binary endo-relations are essentially the same as these binary endo-relations themselves. Thus pointfree binary endo-relations are a generalization of binary endo-relations.

**Theorem 1.** *Having specified up to isomorphism (of systems of pointfree relations) the system of pointfree relations induced by binary relations, we can restore the endo-relation to which corresponds a given pointfree endo-relation up to isomorphism (of binary relations).*

*Proof.* Fix a set $A$.

Let $\mu \in \mathbf{Rel}(A, A)$ be a binary relation (which we are going to restore up to isomorphism from the corresponding pointfree relation).

We will consider possible domains and possible images $D = \mathscr{P}A$.

Let $A'$ be the set of atoms of $D$, that is one-element sets.

Take (using axiom of choice) an arbitrary bijection $f : A' \to A$. (Note that it can be done using only the system of pointfree relations induced by the binary relations up to isomorphism.)

Let $y_0, y_1 \in A'$ be atoms. Take elements $p, q \in \mathbf{Rel}(A', A')$ such that $\operatorname{im} p = y_0$, $\operatorname{dom} q = y_1$ (they exist because our system of pointfree relations is induced by binary relations).

Define binary relation $\mu' \in \mathbf{Rel}(A', A')$ as: $y_0 \; \mu' \; y_1$ iff $q \circ f^{-1} \circ \mu \circ f \circ p \neq \bot$ (for all $y_0, y_1 \in A'$).

Thus we have restored relation $\mu'$ from the corresponding pointfree relation. $\mu'$ was defined using only properties of orders and the precategory. Thus it is invariant under isomorphism (of systems of pointfree relations).

Let's prove that $\mu' = f^{-1} \circ \mu \circ f$ (and thus $\mu'$ is isomorphic to $\mu$). Really,

$$y_0 \; \mu' \; y_1 \Leftrightarrow q \circ f^{-1} \circ \mu \circ f \circ p \neq \bot \Leftrightarrow \\ \exists x \in \operatorname{im} p, y \in \operatorname{dom} q : (x, y) \in f^{-1} \circ \mu \circ f \Leftrightarrow \\ (\operatorname{im} p \times \operatorname{dom} q) \cap f^{-1} \circ \mu \circ f \neq \emptyset \Leftrightarrow y_0 \; (f^{-1} \circ \mu \circ f) \; y_1.$$

If we take another isomorphism $f_2$ instead of $f$, then the induced binary relation

$$\mu'_2 = f_2^{-1} \circ \mu \circ f_2 = f_2^{-1} \circ f \circ \mu' \circ f^{-1} \circ f_2 = (f_2^{-1} \circ f) \circ \mu' \circ (f_2^{-1} \circ f)^{-1}.$$

Thus $\mu'_2$ is isomorphic to $\mu'$ (and therefore isomorphic to $\mu$), because the composition $f_2^{-1} \circ f$ is a bijection. We have determined $\mu'$ up to isomorphism of binary relations.

If $\mu_2$ is a binary relation isomorphic to $\mu'$, then $\mu_2$ is isomorphic to $\mu$ and thus $\mu_2 = f^{-1} \circ \mu \circ f$ for some bijection $f$, so obviously $\mu_2$ induces the same (up to isomorphism) pointfree binary relation as $\mu$. So we have determined the sought-for class of pairwise isomorphic binary relations which induce (up to isomorphism) the same pointfree binary relation as induced by $\mu$ as these binary endo-relations which are isomorphic to $\mu'$. □

Now to the topic of both rationale and applications of this idea: I seriously research ([1]) what I call **reloids**, that is filters on binary cartesian products of sets. This seems promising to generalize for filters on pointfree relations instead of that special case of filters on binary relations. This research is both the source of the idea and an application of systems of pointfree relations.

**Remark.** Funcoids and reloids (see [1]) also are systems of pointfree relations.

# References

[1] Victor Porton. *Algebraic General Topology. Volume 1.* 2015.