Partially Ordered Dagger Categories*

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Abstract

Defined partially ordered dagger categories. For such categories defined monovalued morphisms and entirely defined morphisms.

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1 Partially ordered categories

Definition 1. I will call a *partially ordered (pre)category* a (pre)category together with partial order $\subseteq$ on each of its Hom-sets with the additional requirement that

$$f_1 \subseteq f_2 \wedge g_1 \subseteq g_2 \Rightarrow g_1 \circ f_1 \subseteq g_2 \circ f_2$$

for any morphisms f_1, g_1, f_2, g_2 such that $\operatorname{Src} f_1 = \operatorname{Src} f_2 \wedge \operatorname{Dst} f_1 = \operatorname{Dst} f_2 = \operatorname{Src} g_1 = \operatorname{Src} g_2 \wedge \operatorname{Dst} g_1 = \operatorname{Dst} g_2$.

2 Dagger categories

Definition 2. I will call a *dagger precategory* a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^\dagger$.

In other words, a *dagger precategory* is a precategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

Definition 3. I will call a *dagger category* a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^\dagger$.

In other words, a *dagger category* is a category equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A:

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
3. $(1_A)^\dagger = 1_A$.

Theorem 4. If a category is a dagger precategory then it is a dagger category.

Proof. We need to prove only that $(1_A)^\dagger = 1_A$. Really

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A.$$ □

*. This document has been written using the GNU TeXmacs text editor (see `www.texmacs.org`).

For a partially ordered dagger (pre)category I will additionally require (for any morphisms f and g)

$$f^{\dagger} \subseteq g^{\dagger} \Leftrightarrow f \subseteq g.$$

Definition 5. A morphism f of a dagger category is called *unitary* when it is an isomorphism and $f^{\dagger} = f^{-1}$.

Definition 6. *Symmetric* (endo)morphism of a dagger precategory is such a morphism f that $f = f^{\dagger}$.

Definition 7. *Transitive* (endo)morphism of a precategory is such a morphism f that $f = f \circ f$.

Theorem 8. The following conditions are equivalent for a morphism f of a dagger precategory:

1. f is symmetric and transitive.
2. $f = f^{\dagger} \circ f$.

Proof.

(1) ⇒ (2). If f is symmetric and transitive then $f^{\dagger} \circ f = f \circ f = f$.

(2) ⇒ (1). $f^{\dagger} = (f^{\dagger} \circ f)^{\dagger} = f^{\dagger} \circ f^{\dagger\dagger} = f^{\dagger} \circ f = f$, so f is symmetric. $f = f^{\dagger} \circ f = f \circ f$, so f is transitive. □

2.1 Monovalued and entirely defined morphisms

Definition 9. For a partially ordered dagger category I will call *monovalued* morphism such a morphism f that $f \circ f^{\dagger} \subseteq 1_{\operatorname{Dst} f}$.

Definition 10. For a partially ordered dagger category I will call *entirely defined* morphism such a morphism f that $f^{\dagger} \circ f \supseteq 1_{\operatorname{Src} f}$.

Definition 11. For a given partially ordered dagger category C the *category of monovalued (entirely defined) morphisms* of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined) morphisms of C with the composition of morphisms the same as in C.

We need to prove that these are really categories, that is that composition of monovalued (entirely defined) morphisms is monovalued (entirely defined) and that identity morphisms are monovalued and entirely defined.

Proof.

Monovalued. Let f and g are monovalued morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. $(g \circ f) \circ (g \circ f)^{\dagger} = g \circ f \circ f^{\dagger} \circ g^{\dagger} \subseteq g \circ 1_{\operatorname{Dst} f} \circ g^{\dagger} = g \circ 1_{\operatorname{Src} g} \circ g^{\dagger} = g \circ g^{\dagger} \subseteq 1_{\operatorname{Dst} g} = 1_{\operatorname{Dst}(g \circ f)}$. So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following: $1_A \circ (1_A)^{\dagger} = 1_A \circ 1_A = 1_A = 1_{\operatorname{Dst} 1_A} \subseteq 1_{\operatorname{Dst} 1_A}$.

Entirely defined. Let f and g are entirely defined morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. $(g \circ f)^{\dagger} \circ (g \circ f) = f^{\dagger} \circ g^{\dagger} \circ g \circ f \supseteq f^{\dagger} \circ 1_{\operatorname{Src} g} \circ f = f^{\dagger} \circ 1_{\operatorname{Dst} f} \circ f = f^{\dagger} \circ f \supseteq 1_{\operatorname{Src} f} = 1_{\operatorname{Src}(g \circ f)}$. So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following: $(1_A)^{\dagger} \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\operatorname{Src} 1_A} \subseteq 1_{\operatorname{Src} 1_A}$. □