Multidimensional Funcoids

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Abstract

First I define a product of two funcoids. Then I define multifuncoids and staroids as generalizations of funcoids. Using staroids I define a product of an arbitrary (possibly infinite) family of funcoids and some other products.

1 Draft status

It is a rough draft.

This article is outdated. Read the book instead.

2 Notation

This article presents a generalization of concepts from [1] and [3].

In this article I will use $\sqsupseteq$ to denote order in a poset and $\sqcap$, $\sqcup$ to denote meets and joins on a semilattice. I reserve $\supseteq$, $\cap$, and $\cup$ for set-theoretic supset-relation, intersection, and union.

For a poset $\mathfrak{A}$ I will denote $\operatorname{Least}(\mathfrak{A})$ the set of least elements of $\mathfrak{A}$. (This set always has either one or zero elements.)

With this notation we do not need the concept of *filter objects* ([4]), we will use the standard set of filters, but the order $\sqsubseteq$ on the lattice of filters will be opposite the set theoretic inclusion $\subseteq$ of filters.

3 Product of two funcoids

3.1 Lemmas

Lemma 1. Let A, B, C are sets, $f \in \mathsf{FCD}(A; B)$, $g \in \mathsf{FCD}(B; C)$, $h \in \mathsf{FCD}(A; C)$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

Proof. See [1]. □

Lemma 2. Let A, B, C are sets, $f \in \mathsf{RLD}(A; B)$, $g \in \mathsf{RLD}(B; C)$, $h \in \mathsf{RLD}(A; C)$. Then

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

Proof. See [1]. □

Lemma 3. $f \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \langle f \rangle \mathcal{B}$ for elements $\mathcal{A} \in \mathfrak{A}$ and $\mathcal{B} \in \mathfrak{B}$ of some posets $\mathfrak{A}$, $\mathfrak{B}$ with least elements and $f \in \mathsf{FCD}(\mathfrak{A}; \mathfrak{B})$.

Proof. $\langle f \circ (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \rangle \mathcal{X} = \left(\begin{cases} \langle f \rangle \mathcal{B} & \text{if } \mathcal{X} \not\asymp \mathcal{A} \\ 0 & \text{if } \mathcal{X} \asymp \mathcal{A} \end{cases} \right) = \langle \mathcal{A} \times^{\mathsf{FCD}} \langle f \rangle \mathcal{B} \rangle \mathcal{X}.$ □

3.2 Definition

Definition 4. I will call a *quasi-invertible category* a partially ordered dagger category such that it holds

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{\dagger} \tag{1}$$

for every morphisms $f \in \operatorname{Hom}(A; B)$, $g \in \operatorname{Hom}(B; C)$, $h \in \operatorname{Hom}(A; C)$, where A, B, C are objects of this category.

Inverting this formula, we get $f^{\dagger} \circ g^{\dagger} \not\asymp h^{\dagger} \Leftrightarrow g^{\dagger} \not\asymp f \circ h^{\dagger}$. After replacement of variables, this gives: $f^{\dagger} \circ g \not\asymp h \Leftrightarrow g \not\asymp f \circ h$

As it follows from [1], the category of funcoids and the category of reloids are quasi-invertible (taking $f^{\dagger} = f^{-1}$). Moreover by [3] the category of pointfree funcoids between lattices of filters on boolean lattices are quasi-invertible.

Definition 5. The *cross-composition product* of morphisms f and g of a quasi-invertible category is the pointfree funcoid $\operatorname{Hom}(\operatorname{Src} f; \operatorname{Src} g) \to \operatorname{Hom}(\operatorname{Dst} f; \operatorname{Dst} g)$ defined by the formulas (for every $a \in \operatorname{Hom}(\operatorname{Src} f; \operatorname{Src} g)$ and $b \in \operatorname{Hom}(\operatorname{Dst} f; \operatorname{Dst} g)$):

$$\left\langle f \times^{(C)} g \right\rangle a = g \circ a \circ f^{\dagger} \quad \text{and} \quad \left\langle \left(f \times^{(C)} g \right)^{-1} \right\rangle b = g^{\dagger} \circ b \circ f.$$

The cross-composition product is a pointfree funcoid from $\operatorname{Hom}(\operatorname{Src} f; \operatorname{Src} g)$ to $\operatorname{Hom}(\operatorname{Dst} f; \operatorname{Dst} g)$.

We need to prove that it is really a pointfree funcoid that is that

$$b \not\asymp \left\langle f \times^{(C)} g \right\rangle a \Leftrightarrow a \not\asymp \left\langle \left(f \times^{(C)} g \right)^{-1} \right\rangle b.$$

This formula means $b \not\asymp g \circ a \circ f^{\dagger} \Leftrightarrow a \not\asymp g^{\dagger} \circ b \circ f$ and can be easily proved applying the formula (1) two times.

Proposition 6. $a \left[f \times^{(C)} g \right] b \Leftrightarrow a \circ f^{\dagger} \not\asymp g^{\dagger} \circ b$.

Proof. From the lemma. □

Proposition 7. $a \left[f \times^{(C)} g \right] b \Leftrightarrow f \left[a \times^{(C)} b \right] g$.

Proof. $f \left[a \times^{(C)} b \right] g \Leftrightarrow f \circ a^{\dagger} \not\asymp b^{\dagger} \circ g \Leftrightarrow a \circ f^{\dagger} \not\asymp g^{\dagger} \circ b \Leftrightarrow a \left[f \times^{(C)} g \right] b$. □

Theorem 8. $\left(f \times^{(C)} g \right)^{\dagger} = f^{\dagger} \times^{(C)} g^{\dagger}$.

Proof. For every funcoids $a \in \operatorname{Hom}(\operatorname{Src} f; \operatorname{Src} g)$ and $b \in \operatorname{Hom}(\operatorname{Dst} f; \operatorname{Dst} g)$ we have:

$\left\langle \left(f \times^{(C)} g \right)^{\dagger} \right\rangle b = g^{\dagger} \circ b \circ f = g^{\dagger} \circ b \circ f = \left\langle f^{\dagger} \times^{(C)} g^{\dagger} \right\rangle b$.

$\left\langle \left(\left(f \times^{(C)} g \right)^{\dagger} \right)^{\dagger} \right\rangle a = \left\langle f \times^{(C)} g \right\rangle a = g \circ a \circ f^{\dagger} = \left\langle \left(f^{\dagger} \times^{(C)} g^{\dagger} \right)^{\dagger} \right\rangle a$. □

Theorem 9. Let f, g are morphisms of a quasi-invertible category where $\operatorname{Dst} f$ and $\operatorname{Dst} g$ are f.o. on boolean lattices. Then for every f.o. $\mathcal{A}_0 \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B}_0 \in \mathfrak{F}(\operatorname{Src} g)$

$$\left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) = \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0.$$

Proof. For every atom $a_1 \times^{\mathsf{FCD}} b_1$ ($a_1 \in \operatorname{atoms}^{\operatorname{Dst} f}$, $b_1 \in \operatorname{atoms}^{\operatorname{Dst} g}$) of the lattice of funcoids we have:

$a_1 \times^{\mathsf{FCD}} b_1 \not\asymp \left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \Leftrightarrow \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g \right] a_1 \times^{\mathsf{FCD}} b_1 \Leftrightarrow (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \circ f^{\dagger} \not\asymp g^{\dagger} \circ (a_1 \times^{\mathsf{FCD}} b_1) \Leftrightarrow \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \not\asymp a_1 \times^{\mathsf{FCD}} \langle g^{\dagger} \rangle b_1 \Leftrightarrow \langle f \rangle \mathcal{A}_0 \not\asymp a_1 \wedge \langle g^{\dagger} \rangle b_1 \not\asymp \mathcal{B}_0 \Leftrightarrow \langle f \rangle \mathcal{A}_0 \not\asymp a_1 \wedge \langle g \rangle \mathcal{B}_0 \not\asymp b_1 \Leftrightarrow \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0 \not\asymp a_1 \times^{\mathsf{FCD}} b_1$. Thus $\left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) = \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0$ because the lattice $\mathsf{FCD}(\mathfrak{F}(\operatorname{Dst} f); \mathfrak{F}(\operatorname{Dst} g))$ is atomically separable (corollary 64 in [3]). □

Proposition 10. $\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g \right] \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \Leftrightarrow \mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1$ for every $\mathcal{A}_0 \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{A}_1 \in \mathfrak{F}(\operatorname{Dst} f)$, $\mathcal{B}_0 \in \mathfrak{F}(\operatorname{Src} g)$, $\mathcal{B}_1 \in \mathfrak{F}(\operatorname{Dst} g)$.

Proof. $\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0 \left[f \times^{(C)} g \right] \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \Leftrightarrow \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \not\asymp \left\langle f \times^{(C)} g \right\rangle (\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \Leftrightarrow \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \not\asymp \langle f \rangle \mathcal{A}_0 \times^{\mathsf{FCD}} \langle g \rangle \mathcal{B}_0 \Leftrightarrow \mathcal{A}_1 \not\asymp \langle f \rangle \mathcal{A}_0 \wedge \mathcal{B}_1 \not\asymp \langle g \rangle \mathcal{B}_0 \Leftrightarrow \mathcal{A}_0 [f] \mathcal{A}_1 \wedge \mathcal{B}_0 [g] \mathcal{B}_1$. □

4 Function spaces of posets

Definition 11. Let $\mathfrak{A}_i$ is a family of posets indexed by some set $\operatorname{dom}\mathfrak{A}$. We will define order of families of posets by the formula

$$a \sqsubseteq b \Leftrightarrow \forall i \in \operatorname{dom}\mathfrak{A}: a_i \sqsubseteq b_i.$$

I will call this new poset $\mathfrak{A} = \prod \mathfrak{A}$ *the function space* of posets and the above order *product order*.

Proposition 12. The function space for posets is also a poset.

Proof.

Reflexivity. Obvious.

Antisymmetry. Obvious.

Transitivity. Obvious. □

Obvious 13. $\mathfrak{A}$ has least element iff each $\mathfrak{A}_i$ has a least element. In this case

$$\operatorname{Least}(\mathfrak{A}) = \prod_{i \in \operatorname{dom}\mathfrak{A}} \operatorname{Least}(\mathfrak{A}_i).$$

Proposition 14. $a \not\asymp b \Leftrightarrow \exists i \in \operatorname{dom}\mathfrak{A}: a_i \not\asymp b_i$ for every $a, b \in \prod \mathfrak{A}$.

Proof. $a \not\asymp b \Leftrightarrow \exists c \in \prod \mathfrak{A}: (c \sqsubseteq a \wedge c \sqsubseteq b) \Leftrightarrow \exists c \in \prod \mathfrak{A} \forall i \in \operatorname{dom}\mathfrak{A}: (c_i \sqsubseteq a_i \wedge c_i \sqsubseteq b_i) \Leftrightarrow \forall i \in \operatorname{dom}\mathfrak{A} \exists x \in \prod \mathfrak{A}: (x \sqsubseteq a_i \wedge x \sqsubseteq b_i) \Leftrightarrow \forall i \in \operatorname{dom}\mathfrak{A}: a_i \not\asymp b_i$. □

Proposition 15.

1. If $\mathfrak{A}_i$ are join-semilattices then $\mathfrak{A}$ is a join-semilattice and
$$A \sqcup B = \lambda i \in \operatorname{dom}\mathfrak{A}: A i \sqcup B i. \tag{2}$$
2. If $\mathfrak{A}_i$ are meet-semilattices then $\mathfrak{A}$ is a meet-semilattice and
$$A \sqcap B = \lambda i \in \operatorname{dom}\mathfrak{A}: A i \sqcap B i. \tag{3}$$

Proof. It is enough to prove the formula (2).

It's obvious that $\lambda i \in \operatorname{dom}\mathfrak{A}: A i \sqcup B i \sqsupseteq A, B$.

Let $C \sqsupseteq A, B$. Then (for every $i \in \operatorname{dom}\mathfrak{A}$) $C i \sqsupseteq A i$ and $C i \sqsupseteq B i$. Thus $C i \sqsupseteq A i \sqcup B i$ that is $C \sqsupseteq \lambda i \in \operatorname{dom}\mathfrak{A}: A i \sqcup B i$. □

Corollary 16. If $\mathfrak{A}_i$ are lattices then $\mathfrak{A}$ is a lattice.

Obvious 17. If $\mathfrak{A}_i$ are distributive lattices then $\mathfrak{A}$ is a distributive lattice.

Obvious 18. If $\mathfrak{A}_i$ are (co-)brouwerian lattices then $\mathfrak{A}$ is a (co-)brouwerian lattice.

Proposition 19. If $\mathfrak{A}_i$ are boolean lattices then $\prod \mathfrak{A}$ is a boolean lattice.

Proof. We need to prove only that every element $a \in \prod \mathfrak{A}$ has a complement. But this complement is evidently $\lambda i \in \operatorname{dom} a: \overline{a_i}$. □

Proposition 20. If $\mathfrak{A}_i$ are lattices then for every $S \in \mathscr{P} \prod \mathfrak{A}$

1. $\bigsqcup S = \lambda i \in \operatorname{dom}\mathfrak{A}: \bigsqcup \{ x_i \mid x \in S \}$ whenever $\bigsqcup \{ x_i \mid x \in S \}$ exists;
2. $\bigsqcap S = \lambda i \in \operatorname{dom}\mathfrak{A}: \bigsqcap \{ x_i \mid x \in S \}$ whenever $\bigsqcap \{ x_i \mid x \in S \}$ exists.

Proof. It's enough to prove the first formula.

$(\lambda i \in \operatorname{dom}\mathfrak{A}: \bigsqcup \{ x_i \mid x \in S \})_i = \bigsqcup \{ x_i \mid x \in S \} \sqsupseteq x_i$ for every $x \in S$ and $i \in \operatorname{dom}\mathfrak{A}$.

Let $y \sqsupseteq x$ for every $x \in S$. Then $y_i \sqsupseteq x_i$ for every $i \in \operatorname{dom}\mathfrak{A}$ and thus $y_i \sqsupseteq \bigsqcup \{ x_i \mid x \in S \} = (\lambda i \in \operatorname{dom}\mathfrak{A}: \bigsqcup \{ x_i \mid x \in S \})_i$ that is $y \sqsupseteq \lambda i \in \operatorname{dom}\mathfrak{A}: \bigsqcup \{ x_i \mid x \in S \}$.

Thus $\bigsqcup S = \lambda i \in \operatorname{dom}\mathfrak{A}: \bigsqcup \{ x_i \mid x \in S \}$ by the definition of join. □

Corollary 21. If $\mathfrak{A}_i$ are complete lattices then $\mathfrak{A}$ is a complete lattice.

Proposition 22. If each $\mathfrak{A}_i$ is a separable poset with least element (for some index set n) then $\prod \mathfrak{A}$ is a separable poset.

Proof. Let $a \neq b$. Then $\exists i \in \operatorname{dom} \mathfrak{A}: a_i \neq b_i$. So $\exists x \in \mathfrak{A}_i: (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = (((\operatorname{dom} \mathfrak{A}) \setminus \{i\}) \times \{0\}) \cup \{(i; x)\}$. Then $y \not\asymp a$ and $y \asymp b$. □

Obvious 23. If every $\mathfrak{A}_i$ is a poset with least element 0_i, then the set of atoms of $\prod \mathfrak{A}$ is

$$\{(\{k\} \times \operatorname{atoms}^{\mathfrak{A}_k}) \cup (\lambda i \in (\operatorname{dom} \mathfrak{A}) \setminus \{k\}: 0_i) \mid k \in \operatorname{dom} \mathfrak{A}\}.$$

Proposition 24. If every $\mathfrak{A}_i$ is an atomistic poset with least element 0_i, then $\prod \mathfrak{A}$ is an atomistic poset.

Proof. $x_i = \bigsqcup \operatorname{atoms} x_i$ for every $x_i \in \mathfrak{A}_i$. Thus

$$x = \lambda i \in \operatorname{dom} x: x_i = \bigsqcup_{i \in \operatorname{dom} x} \operatorname{atoms} x_i = \bigsqcup_{i \in \operatorname{dom} x} \lambda j \in \operatorname{dom} x: \begin{cases} x_i & \text{if } j = i \\ 0_i & \text{if } j \neq i. \end{cases}$$

Take join two times. □

Corollary 25. If $\mathfrak{A}_i$ are atomistic complete lattices, then $\prod \mathfrak{A}$ is atomically separable.

Proof. Proposition 14 in [4]. □

Proposition 26. Let $(\mathfrak{A}_{i \in n}; \mathfrak{Z}_{i \in n})$ is a family of filtrators. Then $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a filtrator.

Proof. We need to prove that $\prod \mathfrak{Z}$ is a sub-poset of $\prod \mathfrak{A}$. First $\prod \mathfrak{Z} \subseteq \prod \mathfrak{A}$ because $\mathfrak{Z}_i \subseteq \mathfrak{A}_i$ for each $i \in n$.

Let $A, B \in \prod \mathfrak{Z}$ and $A \subseteq^{\prod \mathfrak{Z}} B$. Then $\forall i \in n: A_i \subseteq^{\mathfrak{Z}_i} B_i$; consequently $\forall i \in n: A_i \subseteq^{\mathfrak{A}_i} B_i$ that is $A \subseteq^{\prod \mathfrak{A}} B$. □

Proposition 27. Let $(\mathfrak{A}_{i \in n}; \mathfrak{Z}_{i \in n})$ is a family of filtrators.

1. The filtrator $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is (finitely) join-closed if every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is (finitely) join-closed.
2. The filtrator $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is (finitely) meet-closed if every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is (finitely) meet-closed.

Proof. Let every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is finitely join-closed. Let $A, B \in \prod \mathfrak{Z}$. Then $A \sqcup^{\prod \mathfrak{Z}} B = \lambda \in n: A_i \sqcap^{\mathfrak{Z}_i} B_i = \lambda \in n: A_i \sqcup^{\mathfrak{A}_i} B_i = A \sqcup^{\prod \mathfrak{A}} B$.

Let now every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is finitely join-closed. Let $S \in \mathscr{P} \prod \mathfrak{Z}$. Then $\bigsqcup^{\prod \mathfrak{Z}} S = \lambda i \in \operatorname{dom} \mathfrak{A}: \bigsqcup^{\mathfrak{Z}_i} \{x_i \mid x \in S\} = \lambda i \in \operatorname{dom} \mathfrak{A}: \bigsqcup^{\mathfrak{A}_i} \{x_i \mid x \in S\} = \bigsqcup^{\prod \mathfrak{A}} S$.

The rest follows from symmetry. □

Proposition 28. If each $(\mathfrak{A}_i; \mathfrak{Z}_i)$ where $i \in n$ (for some index set n) is a down-aligned filtrator with separable core (for some index set n) then $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is with separable core.

Proof. Let $a \neq b$. Then $\exists i \in n: a_i \neq b_i$. So $\exists x \in \mathfrak{Z}_i: (x \not\asymp a_i \wedge x \asymp b_i)$ (or vice versa).

Take $y = ((n \setminus \{i\}) \times \{0\}) \cup \{(i; x)\}$. Then we have $y \not\asymp a$ and $y \asymp b$ and $y \in \mathfrak{Z}$. □

Proposition 29. Let every $\mathfrak{A}_i$ is a bounded lattice. Every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is a central filtrator iff $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a central filtrator.

Proof. $x \in Z(\prod \mathfrak{A}) \Leftrightarrow \exists y \in \prod \mathfrak{A}: \left(x \sqcap y = 0^{\prod \mathfrak{A}} \wedge x \sqcup y = 1^{\prod \mathfrak{A}}\right) \Leftrightarrow \exists y \in \prod \mathfrak{A} \forall i \in \operatorname{dom} \mathfrak{A}: (x_i \sqcap y_i = 0^{\mathfrak{A}_i} \wedge x_i \sqcup y_i = 1^{\mathfrak{A}_i}) \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A} \exists y \in \mathfrak{A}_i: (x_i \sqcap y_i = 0^{\mathfrak{A}_i} \wedge x_i \sqcup y_i = 1^{\mathfrak{A}_i}) \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: x_i \in Z(\mathfrak{A}_i)$. □

Proposition 30. For every element a of a product filtrator $(\prod \mathfrak{A}; \prod \mathfrak{Z})$:

1. $\operatorname{up} a = \prod_{i \in \operatorname{dom} a} \operatorname{up} a_i$;

2. $\operatorname{down} a = \prod_{i \in \operatorname{dom} a} \operatorname{down} a_i$.

Proof. We will prove only the first as the second is dual.

$\operatorname{up} a = \{c \in \prod \mathfrak{Z} \mid c \sqsupseteq a\} = \{c \in \prod \mathfrak{Z} \mid \forall i \in \operatorname{dom} a: c_i \sqsupseteq a_i\} = \{c \in \prod \mathfrak{Z} \mid \forall i \in \operatorname{dom} a: c_i \in \operatorname{up} a_i\} = \prod_{i \in \operatorname{dom} a} \operatorname{up} a_i$. □

Proposition 31. If every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is a filtered complete lattice filtrator, then $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a filtered complete lattice filtrator.

Proof. That $\prod \mathfrak{A}$ is a complete lattice is already proved above. We have for every $a \in \prod \mathfrak{A}$

$\bigsqcap^{\prod \mathfrak{A}} \operatorname{up} a = \lambda i \in \operatorname{dom} \mathfrak{A}: \bigsqcap \{x_i \mid x \in \operatorname{up} a\} = \lambda i \in \operatorname{dom} \mathfrak{A}: \bigsqcap \{x \mid x \in \operatorname{up} a_i\} = \lambda i \in \operatorname{dom} \mathfrak{A}: \bigsqcap \operatorname{up} a_i = \lambda i \in \operatorname{dom} \mathfrak{A}: a_i = a$. □

Obvious 32. If every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is a prefiltered complete lattice filtrator, then $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a prefiltered complete lattice filtrator.

Proposition 33. Let $\mathfrak{A}_i$ is a non-empty poset. Every $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is a semifiltered complete lattice filtrator iff $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a semifiltered complete lattice filtrator.

Proof. $\operatorname{up} a \supseteq \operatorname{up} b \Leftrightarrow \lambda i \in \operatorname{dom} \mathfrak{A}: \operatorname{up} a_i \supseteq \operatorname{up} b_i \Rightarrow \lambda i \in \operatorname{dom} \mathfrak{A}: a_i \sqsubseteq b_i \Leftrightarrow a \sqsubseteq b$ for every $a, b \in \prod \mathfrak{A}$ (used the fact that $\operatorname{up} a_i \neq 0$ because up is injective). □

Proposition 34. Let $(\mathfrak{A}_i; \mathfrak{Z}_i)$ are filtrators and each $\mathfrak{Z}_i$ is a complete lattice. For $a \in \prod \mathfrak{A}$:

1. $\operatorname{Cor} a = \lambda i \in \operatorname{dom} a: \operatorname{Cor} a_i$;
2. $\operatorname{Cor}' a = \lambda i \in \operatorname{dom} a: \operatorname{Cor}' a_i$.

Proof. We will prove only the first, because the second is dual.

$\operatorname{Cor} a = \bigsqcap^{\prod \mathfrak{Z}} \operatorname{up} a = \lambda i \in \operatorname{dom} a: \bigsqcap^{\mathfrak{Z}_i} \{x_i \mid x \in \operatorname{up} a\} = \lambda i \in \operatorname{dom} a: \bigsqcap^{\mathfrak{Z}_i} \{x \mid x \in \operatorname{up} a_i\} = \lambda i \in \operatorname{dom} a: \bigsqcap^{\mathfrak{Z}_i} \operatorname{up} a_i = \lambda i \in \operatorname{dom} a: \operatorname{Cor} a_i$. □

Proposition 35. If each $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is a filtrator with (co-)separable core, then $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a filtrator with (co-)separable core.

Proof. We will prove only for separable core, as co-separable core is dual.

$x \asymp^{\prod \mathfrak{A}} y \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: x_i \asymp^{\mathfrak{A}_i} y_i \Rightarrow \forall i \in \operatorname{dom} \mathfrak{A} \exists X \in \operatorname{up} x_i: X \asymp^{\mathfrak{A}_i} y_i \Leftrightarrow \exists X \in \operatorname{up} x \forall i \in \operatorname{dom} \mathfrak{A}: X_i \asymp^{\mathfrak{A}_i} y_i \Leftrightarrow \exists X \in \operatorname{up} x: X \asymp^{\prod \mathfrak{A}} y$ for every $x, y \in \prod \mathfrak{A}$. □

Obvious 36.

1. If each $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is a down-aligned filtrator, then $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a down-aligned filtrator.
2. If each $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is an up-aligned filtrator, then $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is an up-aligned filtrator.

Proposition 37. If every b_i is substractive from a_i where a and b are n-indexed families of distributive lattices with least elements (where n is an index set), then $a \setminus b = \lambda i \in n: a_i \setminus b_i$.

Proof. We need to prove $(\lambda i \in n: a_i \setminus b_i) \sqcap b = 0$ and $a \sqcup b = b \sqcup (\lambda i \in n: a_i \setminus b_i)$.

Really, $(\lambda i \in n: a_i \setminus b_i) \sqcap b = \lambda i \in n: (a_i \setminus b_i) \sqcap b_i = 0$ and $b \sqcup (\lambda i \in n: a_i \setminus b_i) = \lambda i \in n: b_i \sqcup (a_i \setminus b_i) = \lambda i \in n: b_i \sqcup a_i = a \sqcup b$. □

Proposition 38. If every $\mathfrak{A}_i$ is a distributive lattice, then $a \setminus^* b = \lambda i \in \operatorname{dom} \mathfrak{A}: a_i \setminus^* b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \setminus^* b_i$ is defined.

Proof. We need to prove that $\lambda i \in \operatorname{dom} \mathfrak{A}: a_i \setminus^* b_i = \bigsqcap \{z \in \prod \mathfrak{A} \mid a \sqsubseteq b \sqcup z\}$.

To prove it is enough to show $a_i \setminus^* b_i = \bigsqcap \{z_i \mid z \in \prod \mathfrak{A}, a \sqsubseteq b \sqcup z\}$ that is $a_i \setminus^* b_i = \bigsqcap \{z \in \mathfrak{A}_i \mid a_i \sqsubseteq b_i \sqcup z\}$ what is true by definition. □

Proposition 39. If every $\mathfrak{A}_i$ is a distributive lattice with least element, then $a \# b = \lambda i \in \operatorname{dom} \mathfrak{A}: a_i \# b_i$ for every $a, b \in \prod \mathfrak{A}$ whenever every $a_i \# b_i$ is defined.

Proof. We need to prove that $\lambda i\in\operatorname{dom}\mathfrak{A}: a_i \# b_i = \bigsqcap \{z \in \prod \mathfrak{A} \mid z \sqsubseteq a \wedge z \asymp b\}$.

To prove it is enough to show $a_i \# b_i = \bigsqcap \{z_i \mid z \in \prod \mathfrak{A}, z \sqsubseteq a \wedge z \asymp b\}$ that is $a_i \# b_i = \bigsqcap \{z \in \mathfrak{A}_i \mid z \sqsubseteq a_i \wedge \forall j \in \operatorname{dom}\mathfrak{A}: z_j \asymp b_j\}$ that is $a_i \# b_i = \bigsqcap \{z \in \mathfrak{A}_i \mid z \sqsubseteq a_i \wedge z \asymp b_i\}$ (take $z_i = 0$ for $j \neq i$) what is true by definition. □

Proposition 40. Let every $\mathfrak{A}_i$ is a poset with least element and a_i^* is defined. Then $a^* = \lambda i \in n: a_i^*$.

Proof. We need to prove that $\lambda i \in \operatorname{dom}\mathfrak{A}: a_i^* = \bigsqcup \{c \in \mathfrak{A} \mid c \asymp a\}$. To prove this it is enough to show that $a_i^* = \bigsqcup \{c_i \mid c \in \mathfrak{A}, c \asymp a\}$ that is $a_i^* = \bigsqcup \{c_i \mid c \in \mathfrak{A}, \forall j \in n: c_j \asymp a_j\}$ that is $a_i^* = \bigsqcup \{c_i \mid c \in \mathfrak{A}, c_i \asymp a_i\}$ (take $c_i = 0$ for $j \neq i$) that is $a_i^* = \bigsqcup \{c \in \mathfrak{A} \mid c \asymp a_i\}$ what is true by definition. □

Corollary 41. Let every $\mathfrak{A}_i$ is a poset with least element and a_i^+ is defined. Then $a^+ = \lambda i \in n: a_i^+$.

Proof. By duality. □

5 Definition of staroids

Let n be a set. As an example, n may be an ordinal, n may be a natural number, considered as a set by the formula $n = \{0, \ldots, n-1\}$. Let $\mathfrak{A} = \mathfrak{A}_{i\in n}$ is a family of posets indexed by the set n.

Definition 42. I will call an *anchored relation* a pair $f = (\operatorname{form} f; \operatorname{GR} f)$ of a family $\operatorname{form}(f)$ of sets indexed by the some index set and a relation $\operatorname{GR}(f) \in \mathscr{P} \prod \operatorname{form}(f)$. I call $\operatorname{GR}(f)$ the *graph* of the anchored relation f. I denote $\operatorname{Anch}(\mathfrak{A})$ the set of small anchored relations of the form $\mathfrak{A}$.

Definition 43. An anchored relation *on powersets* is an anchored relation f such that every $(\operatorname{form} f)_i$ is a powerset.

I will denote $\operatorname{arity} f = \operatorname{dom}\operatorname{form} f$.

Definition 44. Every set of anchored relations of the same form constitutes a poset by the formula $f \sqsubseteq g \Leftrightarrow \operatorname{GR} f \subseteq \operatorname{GR} g$.

Definition 45. An anchored relation is an *anchored relation between posets* when every $(\operatorname{form} f)_i$ is a poset.

Definition 46. Let f is an anchored relation. For every $i \in \operatorname{arity} f$ and $L \in \prod ((\operatorname{form} f)|_{(\operatorname{arity} f)\setminus\{i\}})$

$$(\operatorname{val} f)_i L = \{X \in (\operatorname{form} f)_i \mid L \cup \{(i; X)\} \in \operatorname{GR} f\}$$

("val" is an abbreviation of the word "value".)

Obvious 47. $X \in (\operatorname{val} f)_i L \Leftrightarrow L \cup \{(i; X)\} \in \operatorname{GR} f$.

Proposition 48. f can be restored knowing $\operatorname{form}(f)$ and $(\operatorname{val} f)_i$ for some $i \in n$.

Proof. $\operatorname{GR} f = \{K \in \prod \operatorname{form} f \mid K \in \operatorname{GR} f\} = \{L \cup \{(i; X)\} \mid L \in \prod (\operatorname{form} f)|_{(\operatorname{arity} f)\setminus\{i\}}, X \in (\operatorname{form} f)_i, L \cup \{(i; X)\} \in \operatorname{GR} f\} = \{L \cup \{(i; X)\} \mid L \in \prod (\operatorname{form} f)|_{(\operatorname{arity} f)\setminus\{i\}}, X \in (\operatorname{val} f)_i L\}$. □

Definition 49. A *pre-staroid* is an anchored relation f between poset such that $(\operatorname{val} f)_i L$ is a free star for every $i \in \operatorname{arity} f$, $L \in \prod (\operatorname{form} f)|_{(\operatorname{arity} f)\setminus\{i\}}$.

Definition 50. A *staroid* is a pre-staroid whose graph is an upper set (on the poset if anchored relations of the form of this pre-staroid).

Proposition 51. If $L \in \prod \operatorname{form} f$ and $L_i = 0^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin f$ if f is an pre-staroid.

Proof. Let $K = L|_{(\operatorname{arity} f)\setminus\{i\}}$. We have $0 \notin (\operatorname{val} f)_i K$; $K \cup \{(i; 0)\} \notin f$; $L \notin f$. □

Definition 52. *Infinitary pre-staroid* is such a staroid whose arity is infinite; *finitary pre-staroid* is such a staroid whose arity is finite.

Next we will define *completary staroids*. First goes the general case, next simpler case for the special case of join-semilattices instead of arbitrary posets.

Definition 53. A *completary staroid* is a poset relation conforming to the formulas:

1. $\forall K \in \prod \operatorname{form} f: (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \operatorname{GR} f) \Leftrightarrow \exists c \in \{0,1\}^n: (\lambda i \in n: L_{c(i)} i) \in \operatorname{GR} f$ for every $L_0, L_1 \in \prod \operatorname{form} f$.

2. If $L \in \prod \operatorname{form} f$ and $L_i = 0^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin f$.

Lemma 54. Every completary staroid is an upper set.

Proof. Let f is a completary staroid. Let $L_0 \sqsubseteq L_1$ for some L_0, $L_1 \in \prod \operatorname{form} f$ and $L_0 \in f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n: L_{c(i)} i = \lambda i \in n: L_0 i = L_0 \in f$ and thus $L_1 \in f$ because $L_1 \sqsupseteq L_0 \wedge L_1 \sqsupseteq L_1$. □

Proposition 55. A relation between posets whose form is a family of join-semilattices is a completary staroid iff both:

1. $L_0 \sqcup L_1 \in \operatorname{GR} f \Leftrightarrow \exists c \in \{0,1\}^n: (\lambda i \in n: L_{c(i)} i) \in \operatorname{GR} f$ for every $L_0, L_1 \in \prod \operatorname{form} f$.

2. If $L \in \prod \operatorname{form} f$ and $L_i = 0^{(\operatorname{form} f)_i}$ for some $i \in \operatorname{arity} f$ then $L \notin f$.

Proof. Let the formulas (1) and (2) hold. Then f is an upper set: Let $L_0 \sqsubseteq L_1$ for some L_0, $L_1 \in \prod \operatorname{form} f$ and $L_0 \in f$. Then taking $c = n \times \{0\}$ we get $\lambda i \in n: L_{c(i)} i = \lambda i \in n: L_0 i = L_0 \in f$ and thus $L_1 = L_0 \sqcup L_1 \in f$.

Thus to finish the proof it is enough to show that

$$L_0 \sqcup L_1 \in \operatorname{GR} f \Leftrightarrow \forall K \in \prod \operatorname{form} f: (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \operatorname{GR} f)$$

under condition that $\operatorname{GR} f$ is an upper set. But this is obvious. □

Proposition 56. A completary staroid is a staroid.

Proof. Let f is a completary staroid.

Let $K \in \prod_{i \in (\operatorname{arity} f) \setminus \{i\}} (\operatorname{form} f)_i$. Let $L_0 = K \cup \{(i; X_0)\}$, $L_1 = K \cup \{(i; X_1)\}$ for some X_0, $X_1 \in \mathfrak{A}_i$. Then $X_0 \sqcup X_1 \in (\operatorname{val} f)_i K \Leftrightarrow L_0 \sqcup L_1 \in \operatorname{GR} f \Leftrightarrow \exists k \in \{0,1\}: K \cup \{(i; X_k)\} \in \operatorname{GR} f \Leftrightarrow K \cup \{(i; X_0)\} \in f \vee K \cup \{(i; X_1)\} \in \operatorname{GR} f \Leftrightarrow X_0 \in (\operatorname{val} f)_i K \vee X_1 \in (\operatorname{val} f)_i K$.

So $(\operatorname{val} f)_i K$ is a free star (taken in account that $K_i = 0^{(\operatorname{form} f)_i} \Rightarrow f \notin K$).

f is an upper set by the lemma. □

Lemma 57. Every finitary pre-staroid is completary.

Proof. $\exists c \in \{0,1\}^n: (\lambda i \in n: L_{c(i)} i) \in \operatorname{GR} f \Leftrightarrow \exists c \in \{0,1\}^{n-1}: (\{(n-1; L_0(n-1))\} \cup (\lambda i \in n-1: L_{c(i)} i)) \in \operatorname{GR} f \vee (\{(n-1; L_1(n-1))\} \cup (\lambda i \in n-1: L_{c(i)} i)) \in \operatorname{GR} f \Leftrightarrow \exists c \in \{0,1\}^{n-1}: L_0(n-1) \in (\operatorname{val} f)_{n-1}(\lambda i \in n-1: L_{c(i)} i) \vee L_1(n-1) \in (\operatorname{val} f)_{n-1}(\lambda i \in n-1: L_{c(i)} i) \Leftrightarrow \exists c \in \{0,1\}^{n-1} \forall K \in \prod \operatorname{form} f: (K \sqsupseteq L_0(n-1) \vee K \sqsupseteq L_1(n-1) \Rightarrow K \in (\operatorname{val} f)_{n-1}(\lambda i \in n-1: L_{c(i)} i)) \Leftrightarrow \exists c \in \{0,1\}^{n-1} \forall K_{n-1} \in (\operatorname{form} f)_{n-1}: (K_{n-1} \sqsupseteq L_0(n-1) \vee K_{n-1} \sqsupseteq L_1(n-1) \Rightarrow \{(n-1; K)\} \cup (\lambda i \in n-1: L_{c(i)} i)) \in \operatorname{GR} f \Leftrightarrow \ldots \Leftrightarrow \forall K \in \prod \operatorname{form} f: (K \sqsupseteq L_0 \wedge K \sqsupseteq L_1 \Rightarrow K \in \operatorname{GR} f)$. □

Exercise 1. Prove the simpler special case of the above theorem when the form is a family of join-semilattices.

Theorem 58. For finite arity the following are the same:

1. pre-staroids;

2. staroids;

3. completary staroids.

Proof. f is a finitary pre-staroid $\Rightarrow$ f is a finitary completary staroid.

f is a finitary completary staroid $\Rightarrow$ f is a finitary staroid.

f is a finitary staroid $\Rightarrow$ f is a finitary pre-staroid. □

Definition 59. We will denote the set of staroids, pre-staroids, and completary staroids of a form $\mathfrak{A}$ correspondingly as $\mathsf{Strd}(\mathfrak{A})$, $\mathsf{pStrd}(\mathfrak{A})$, and $\mathsf{cStrd}(\mathfrak{A})$.

6 Upgrading and downgrading a set regarding a filtrator

Let fix a filtrator $(\mathfrak{A}; \mathfrak{Z})$.

Definition 60. $\downdownarrows f = f \cap \mathfrak{Z}$ for every $f \in \mathscr{P}\mathfrak{A}$ (*downgrading* f).

Definition 61. $\upuparrows f = \{L \in \mathfrak{A} \mid \operatorname{up} L \subseteq f\}$ for every $f \in \mathscr{P}\mathfrak{Z}$ (*upgrading* f).

Obvious 62. $a \in \upuparrows f \Leftrightarrow \operatorname{up} a \subseteq f$ for every $f \in \mathscr{P}\mathfrak{Z}$ and $a \in \mathfrak{A}$.

Proposition 63. $\downdownarrows \upuparrows f = f$ if f is an upper set.

Proof. $\downdownarrows \upuparrows f = \upuparrows f \cap \mathfrak{Z} = \{L \in \mathfrak{Z} \mid \operatorname{up} L \subseteq f\} = \{L \in \mathfrak{Z} \mid \operatorname{up} L \in f\} = f \cap \mathscr{P}\mathfrak{Z} = f$. □

6.1 Upgrading and downgrading staroids

Let fix a family $(\mathfrak{A}; \mathfrak{Z})$ of filtrators.

For a graph f of a staroid define $\downdownarrows f$ and $\upuparrows f$ taking the filtrator of $(\prod \mathfrak{A}; \prod \mathfrak{Z})$.

For a staroid f define:

$$\operatorname{form} \downdownarrows f = \mathfrak{Z} \quad \text{and} \quad \operatorname{GR} \downdownarrows f = \downdownarrows \operatorname{GR} f;$$
$$\operatorname{form} \upuparrows f = \mathfrak{A} \quad \text{and} \quad \operatorname{GR} \upuparrows f = \upuparrows \operatorname{GR} f.$$

Proposition 64. $(\operatorname{val} \downdownarrows f))_i L = (\operatorname{val} f)_i L \cap \mathfrak{Z}_i$ for every $L \in \prod \mathfrak{Z}|_{(\operatorname{arity} f) \setminus \{i\}}$.

Proof. $(\operatorname{val} \downdownarrows f))_i L = \{X \in (\operatorname{form} f)_i \mid L \cup \{(i; X)\} \in \operatorname{GR} f \cap \prod \mathfrak{Z}\} = \{X \in \mathfrak{Z}_i \mid L \cup \{(i; X)\} \in \operatorname{GR} f\} = (\operatorname{val} f)_i L \cap \mathfrak{Z}_i$. □

Proposition 65. Let $(\mathfrak{A}_i; \mathfrak{Z}_i)$ are finitely join-closed filtrators with both the base and the core being join-semilattices. If f is a staroid of the form $\mathfrak{A}$, then $\downdownarrows f$ is a staroid of the form $\mathfrak{Z}$.

Proof. Let f is a a staroid.

We need to prove that $(\operatorname{val} \downdownarrows f)_i L$ is a free star. It follows from the last proposition and the fact that it is join-closed. □

Proposition 66. $\prod^{\mathsf{Strd}} a = \upuparrows \downdownarrows \prod^{\mathsf{Strd}} a$ if each $a_i \in \mathfrak{A}_i$ (for $i \in n$ where n is some index set) where $\mathfrak{A}_i$ is a separable poset with least element.

Proof. $\upuparrows \downdownarrows \prod^{\mathsf{Strd}} a = \left\{L \in \prod \mathfrak{A} \mid L \subseteq \prod^{\mathsf{Strd}} a\right\} = \{L \in \prod \mathfrak{A} \mid \forall K \in L: K \not\asymp a\} = \{L \in \prod \mathfrak{A} \mid L \not\asymp a\} = \prod^{\mathsf{Strd}} a$ (taken into account that $\prod \mathfrak{A}$ is a separable poset). □

6.2 Displacement

Definition 67. Let f is an indexed family of pointfree funcoids. The *displacement* of the pre-staroid

$$p \in A = \mathsf{pStrd}(\lambda i \in \operatorname{dom} f: \mathsf{FCD}(\operatorname{Src} f_i; \operatorname{Src} g_i))$$

is defined as a staroid

$$q \in B = \mathsf{pStrd}(\lambda i \in \operatorname{dom} f : \mathsf{RLD}(\operatorname{Src} f_i ; \operatorname{Src} g_i))$$

such that

$$q = \upuparrows^{(B; C; \uparrow^B)} \downdownarrows^{(A; C; \uparrow^A)} p$$

where $C = \mathsf{pStrd}\left(\prod_{i \in \operatorname{dom} f} \operatorname{Src} f_i ; \prod_{i \in \operatorname{dom} f} \operatorname{Dst} f_i \right)$.

Definition 68. We will define *displaced product* of a family f of funcoids by the formula: $\prod^{(\mathrm{DP})} f = \mathrm{DP}\left(\prod^{(C)} f \right)$.

Remark 69. The interesting aspect of displaced product of funcoids is that displaced product of pointfree funcoids is a funcoid (not just a pointfree funcoid).

7 Multifuncoids

Definition 70. I call an *pre-multifuncoid sketch* f of the form $\mathfrak{A}$ (where every $\mathfrak{A}_i$ is a poset) the pair $(\mathfrak{A}; \alpha)$ where for every $i \in \operatorname{dom} \alpha$

$$\alpha_i : \prod \mathfrak{A}|_{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}} \rightarrow \mathfrak{A}_i .$$

I denote $\langle f \rangle = \alpha$.

Definition 71. A pre-multifuncoid sketch *on powersets* is a pre-multifuncoid sketch such that every $\mathfrak{A}_i$ is the set of filters on a powerset.

Definition 72. I will call a *pre-multifuncoid* a pre-multifuncoid sketch such that for every i, $j \in \operatorname{dom} \mathfrak{A}$ and $L \in \prod \mathfrak{A}$

$$L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L) \setminus \{i\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L) \setminus \{j\}} . \tag{4}$$

Definition 73. Let $\mathfrak{A}$ is an indexed family of starrish posets. The pre-staroid *corresponding* to a pre-multifuncoid f is $[f]$ defined by the formula:

$$\operatorname{form}[f] = \mathfrak{A} \quad \text{and} \quad L \in \operatorname{GR}[f] \Leftrightarrow L_i \not\asymp \langle f \rangle_i L|_{(\operatorname{dom} L) \setminus \{i\}} .$$

Proposition 74. The pre-staroid corresponding to a pre-multifuncoid is really a pre-staroid.

Proof. By the definition of starrish posets. □

Definition 75. I will call a *multifuncoid* a pre-multifuncoid to which corresponds a staroid.

Definition 76. I will call a *completary multifuncoid* a pre-multifuncoid to which corresponds a completary staroid.

Theorem 77. Fix some indexed family $\mathfrak{A}$ of boolean lattices. The the set of multifuncoids g bijectively corresponds to set of pre-staroids f of form $\mathfrak{A}$ by the formulas:

1. $f = [g]$ for every $i \in \operatorname{dom} \mathfrak{A}$, $L \in \prod \mathfrak{A}$;
2. $\partial \langle g \rangle_i L = (\operatorname{val} f)_i L$.

Proof. Let f is a pre-staroid of the form $\mathfrak{A}$. If α is defined by the formula $\alpha_i L = \langle f \rangle_i L$ then $\partial \alpha_i L = (\operatorname{val} f)_i L$. Then

$$L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L) \setminus \{i\}} \Leftrightarrow L \in f \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L) \setminus \{j\}} .$$

For the staroid f' defined by the formula $L \in f' \Leftrightarrow L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L) \setminus \{i\}}$ we have:

$$L \in f' \Leftrightarrow L_i \in \partial \alpha_i L|_{(\operatorname{dom} L) \setminus \{i\}} \Leftrightarrow L_i \in (\operatorname{val} f)_i L|_{(\operatorname{dom} L) \setminus \{i\}} \Leftrightarrow L \in f;$$

thus $f' = f$.

Let now α is an indexed family of functions $\alpha_i \in \mathfrak{A}_i^{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}}$ conforming to the formula (4). Let relation f between posets is defined by the formula $L \in f \Leftrightarrow L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L) \setminus \{i\}}$. Then

$$(\operatorname{val} f)_i L = \{K \in \mathfrak{A}_i \mid K \not\asymp \alpha_i L|_{(\operatorname{dom} L) \setminus \{i\}}\} = K = \partial \alpha_i L|_{(\operatorname{dom} L) \setminus \{i\}}$$

and thus $(\operatorname{val} f)_i L$ is a core star that is f is a pre-staroid. For the indexed family α' defined by the formula $\alpha'_i L = \langle f \rangle_i L$ we have

$$\partial \alpha'_i L = \partial \langle f \rangle_i L = \{K \in \mathfrak{A}_i \mid K \not\asymp \alpha_i L\} = \partial \alpha_i L;$$

thus $\alpha' = \alpha$.

We have shown that these are bijections. □

Theorem 78. $\langle f \rangle_j (L \cup \{(i; X \cup Y)\}) = \langle f \rangle_j (L \cup \{(i; X)\}) \cup \langle f \rangle_j (L \cup \{(i; Y)\})$ for every staroid f if $(\operatorname{form} f)_j$ is a boolean lattice and $i, j \in \operatorname{arity} f$.

Proof. Let $i \in \operatorname{arity} f$ and $L \in \prod_{k \in L \setminus \{i,j\}} \mathfrak{A}_k$. Let $Z \in \mathfrak{A}_i$.

$Z \not\asymp \langle f \rangle_j (L \cup \{(i; X \cup Y)\}) \Leftrightarrow L \cup \{(i; X \cup Y), (j; Z)\} \in f \Leftrightarrow X \cup Y \in (\operatorname{val} f)_i (L \cup \{(j; Z)\}) \Leftrightarrow X \in (\operatorname{val} f)_i (L \cup \{(j; Z)\} \vee Y \in (\operatorname{val} f)_i (L \cup \{(j; Z)\}) \Leftrightarrow L \cup \{(i; X), (j; Z)\} \in f \vee L \cup \{(i; Y), (j; Z)\} \in f \Leftrightarrow \uparrow^{\mathfrak{A}_i} Z \not\asymp \langle f \rangle_j (L \cup \{(i; X)\}) \vee Z \not\asymp \langle f \rangle_j (L \cup \{(i; Y)\})$

Thus $\langle f \rangle_j (L \cup \{(i; X \cup Y)\}) = \langle f \rangle_j (L \cup \{(i; X)\}) \cup \langle f \rangle_j (L \cup \{(i; Y)\})$. □

Let us consider the filtrator $\left(\prod_{i \in \operatorname{arity} f} \mathfrak{F}((\operatorname{form} f)_i); \prod_{i \in \operatorname{arity} f} (\operatorname{form} f)_i \right)$.

Theorem 79. Let $(\mathfrak{A}_i; \mathfrak{Z}_i)$ is a family of join-closed down-aligned filtrators filtrators whose both base and core are join-semilattices. Let f is a pre-staroid of the form $\mathfrak{Z}$. Then $\upuparrows f$ is a staroid of the form $\mathfrak{A}$.

Proof. First prove that $\operatorname{GR} \upuparrows f$ is a pre-staroid. We need to prove that $0 \notin (\operatorname{GR} \upuparrows f)_i$ (that is up $0 \notin (\operatorname{GR} f)_i$ what is true by the theorem conditions) and that for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{A}_i$ and $\mathcal{L} \in \prod_{i \in (\operatorname{arity} f) \setminus \{i\}} \mathfrak{A}_i$ where $i \in \operatorname{arity} f$

$$\mathcal{L} \cup \{(i; \mathcal{X} \sqcup \mathcal{Y})\} \in \operatorname{GR} \upuparrows f \Leftrightarrow \mathcal{L} \cup \{(i; \mathcal{X})\} \in \operatorname{GR} \upuparrows f \vee \mathcal{L} \cup \{(i; \mathcal{Y})\} \in \operatorname{GR} \upuparrows f.$$

The reverse implication is obvious. Let $\mathcal{L} \cup \{(i; \mathcal{X} \sqcup \mathcal{Y})\} \in \operatorname{GR} \upuparrows f$. Then for every $L \in \mathcal{L}$ and $X \in \mathcal{X}$, $Y \in \mathcal{Y}$ we have and $X \sqcup^{\mathfrak{Z}_i} Y \sqsupseteq \mathcal{X} \sqcup^{\mathfrak{A}_i} \mathcal{Y}$ thus $L \cup \{(i; X \sqcup^{\mathfrak{Z}_i} Y)\} \in \operatorname{GR} f$ and thus

$$L \cup \{(i; X)\} \in \operatorname{GR} f \vee L \cup \{(i; Y)\} \in \operatorname{GR} f$$

consequently $\mathcal{L} \cup \{(i; \mathcal{X})\} \in \operatorname{GR} \upuparrows f \vee \mathcal{L} \cup \{(i; \mathcal{Y})\} \in \operatorname{GR} \upuparrows f$.

It is left to prove that $\upuparrows f$ is an upper set, but this is obvious. □

There is a conjecture similar to the above theorems:

Conjecture 80. $L \in [f] \Rightarrow [f] \cap \prod_{i \in \operatorname{dom} \mathfrak{A}} \operatorname{atoms} L_i \neq \emptyset$ for every multifuncoid f of the form whose elements are atomic posets. (Does this conjecture hold for the special case of form whose elements are posets on filters on a set?)

Conjecture 81. Let $\mho$ be a set, $\mathfrak{F}$ be the set of f.o. on $\mho$, $\mathfrak{P}$ be the set of principal f.o. on $\mho$, let n be an index set. Consider the filtrator $(\mathfrak{F}^n; \mathfrak{P}^n)$. Then if f is a completary staroid of the form $\mathfrak{P}^n$, then $\upuparrows f$ is a completary staroid of the form $\mathfrak{F}^n$.

8 Join of multifuncoids

Pre-multifuncoid sketches are ordered by the formula $f \sqsubseteq g \Leftrightarrow \langle f \rangle \sqsubseteq \langle g \rangle$ where $\sqsubseteq$ in the right part of this formula is the product order. I will denote $\sqcap$, $\sqcup$, $\bigsqcap$, $\bigsqcup$ (without an index) the order poset operations on the poset of pre-multifuncoid sketchs.

Remark 82. To describe this, the definition of order poset is used twice. Let f and g are posets of the same form $\mathfrak{A}$

$$\langle f\rangle \sqsubseteq \langle g\rangle \Leftrightarrow \forall i\in \operatorname{dom}\mathfrak{A}: \langle f\rangle_i \sqsubseteq \langle g\rangle_i \quad \text{and} \quad \langle f\rangle_i \sqsubseteq \langle g\rangle_i \Leftrightarrow \forall L\in \prod \mathfrak{A}|_{(\operatorname{dom}\mathfrak{A})\setminus\{i\}}: \langle f\rangle_i L \sqsubseteq \langle g\rangle_i L.$$

Theorem 83. $f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g = f \sqcup g$ for every pre-multifuncoids f and g of the same form $\mathfrak{A}$ of distributive lattices.

Proof. $\alpha_i x \stackrel{\text{def}}{=} f_i x \sqcup g_i x$. It is enough to prove that α is a multifuncoid.

We need to prove:

$$L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}.$$

Really, $L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_i \not\asymp f_i L|_{(\operatorname{dom} L)\setminus\{i\}} \sqcup g_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_i \not\asymp f_i L|_{(\operatorname{dom} L)\setminus\{i\}} \vee L_i \not\asymp g_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \not\asymp f_j L|_{(\operatorname{dom} L)\setminus\{j\}} \vee L_j \not\asymp g_j L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow L_j \not\asymp f_j L|_{(\operatorname{dom} L)\setminus\{j\}} \sqcup g_j L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}$. □

Theorem 84. $\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F = \bigsqcup F$ for every set F of pre-multifuncoids of the same form $\mathfrak{A}$ of join infinite distributive complete lattices.

Proof. $\alpha_i x \stackrel{\text{def}}{=} \bigsqcup_{f\in F} f_i x$. It is enough to prove that α is a multifuncoid.

We need to prove:

$$L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}.$$

Really, $L_i \not\asymp \alpha_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow L_i \not\asymp \bigsqcup_{f\in F} f_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \exists f\in F: L_i \not\asymp f_i L|_{(\operatorname{dom} L)\setminus\{i\}} \Leftrightarrow \exists f\in F: L_j \not\asymp f_j L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow L_j \not\asymp \bigsqcup_{f\in F} f_j L|_{(\operatorname{dom} L)\setminus\{j\}} \Leftrightarrow L_j \not\asymp \alpha_j L|_{(\operatorname{dom} L)\setminus\{j\}}$. □

Proposition 85. The mapping $f \mapsto [f]$ is an order embedding, for multifuncoids of the form $\mathfrak{A}$ of separable starrish posets.

Proof. The mapping $f \mapsto [f]$ is defined because $\mathfrak{A}$ are starrish poset. The mapping is injective because $\mathfrak{A}$ are separable posets. That $f \mapsto [f]$ is a monotone function is obvious. □

Remark 86. This order embedding is useful to describe properties of posets of pre-staroids.

Theorem 87. If f, g are multifuncoids of the same form $\mathfrak{A}$ of distributive lattices, then $f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g \in \mathsf{FCD}(\mathfrak{A})$.

Proof. Let $A \in \left[f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right]$ and $B \sqsupseteq A$. Then for every $k \in \operatorname{dom}\mathfrak{A}$

$A_k \not\asymp \left(f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right) A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = (f \sqcup g) A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = f(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \sqcup g(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$.

Thus $A_k \not\asymp f(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \vee A_k \not\asymp g(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $A \in [f] \vee A \in [g]$; $B \in [f] \vee B \in [g]$; $B_k \not\asymp f(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \vee B_k \not\asymp g(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $f(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) \sqcup g(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) = (f \sqcup g) B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \left(f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right) B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} \not\asymp B_k$. Thus $B \in \left[f \sqcup^{\mathsf{pFCD}(\mathfrak{A})} g\right]$. □

Theorem 88. If F is a set multifuncoids of the same form $\mathfrak{A}$ of join inifinite distributive complete lattices, then $\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} f \in \mathsf{FCD}(\mathfrak{A})$.

Proof. Let $A \in \left[\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} f\right]$ and $B \sqsupseteq A$. Then for every $k \in \operatorname{dom}\mathfrak{A}$.

$A_k \not\asymp \left(\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right) A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \left(\bigsqcup F\right) A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \bigsqcup_{f\in F} f(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$.

Thus $\exists f\in F: A_k \not\asymp f(A|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $\exists f\in F: A \in [f]$; $B \in [f] \vee B \in [g]$; $\exists f\in F: B_k \not\asymp f(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}})$; $\bigsqcup_{f\in F} f(B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}}) = (f \sqcup g) B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} = \left(\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right) B|_{(\operatorname{dom}\mathfrak{A})\setminus\{k\}} \not\asymp B_k$. Thus $B \in \left[\bigsqcup^{\mathsf{pFCD}(\mathfrak{A})} F\right]$. □

Conjecture 89. The formula $f \sqcup^{\mathsf{FCD}(\mathfrak{A})} g \in \mathsf{cFCD}(\mathfrak{A})$ is not true in general for completary multifuncoids (even for multifuncoids on powersets) f and g of the same form $\mathfrak{A}$.

9 Infinite product of elements and filters

Definition 90. Let A_i is a family of elements of a family $\mathfrak{A}_i$ of posets. The *staroidal product* $\prod^{\mathsf{Strd}(\mathfrak{A})} A_i$ is defined by the formula (for every $L \in \prod \mathfrak{A}$)

$$\operatorname{form} \prod^{\mathsf{Strd}(\mathfrak{A})} A = \mathfrak{A} \quad \text{and} \quad L \in \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i.$$

Theorem 91. Staroidal product is a completary staroid (if our posets are distributive lattices).

Proof. We need to prove

$$\forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp (L_0 i \sqcup L_1 i) \Leftrightarrow \exists c \in \{0,1\}^n \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_{c(i)} i.$$

Really, $\forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp (L_0 i \sqcup L_1 i) \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: (A_i \not\asymp L_0 i \vee A_i \not\asymp L_1 i) \Leftrightarrow \exists c \in \{0,1\}^{\operatorname{dom} \mathfrak{A}} \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_{c(i)} i$. □

Definition 92. Let $\mathfrak{A}$ is an indexed family of posets with least elements. Then *funcoidal product* is defined by the formulas:

$$\operatorname{form} \prod^{\mathsf{FCD}(\mathfrak{A})} A = \mathfrak{A} \quad \text{and} \quad \operatorname{GR}\left(\prod^{\mathsf{FCD}(\mathfrak{A})} A \right)_k L = \begin{cases} A_k & \text{if } \forall i \in (\operatorname{dom} \mathfrak{A}) \setminus \{k\}: A_i \not\asymp L_i \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 93. $\prod^{\mathsf{Strd}(\mathfrak{A})} A = \left[\prod^{\mathsf{FCD}(\mathfrak{A})} A \right]$.

Proof. $L \in \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i \Leftrightarrow \forall i \in (\operatorname{dom} \mathfrak{A}) \setminus \{k\}: A_i \not\asymp L_i \wedge L_k \not\asymp A_k \Leftrightarrow A_k \not\asymp \left(\prod^{\mathsf{FCD}(\mathfrak{A})} A \right)_k L \Leftrightarrow L \in \operatorname{GR} \left[\prod^{\mathsf{FCD}(\mathfrak{A})} A \right]$. □

Corollary 94. Funcoidal product is a completary multifuncoid.

Proof. It is enough to prove that funcoidal product is a pre-multifuncoid. Really,

$$L_i \not\asymp \operatorname{GR}\left(\prod^{\mathsf{FCD}(\mathfrak{A})} A \right)_i L|_{(\operatorname{dom} \mathfrak{A}) \setminus \{i\}} \Leftrightarrow \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i \Leftrightarrow L_j \not\asymp \operatorname{GR}\left(\prod^{\mathsf{FCD}(\mathfrak{A})} A \right)_j L|_{(\operatorname{dom} \mathfrak{A}) \setminus \{j\}}. \ \square$$

Theorem 95. If our filtrator $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is with separable core and $A \in \prod \mathfrak{Z}$, then $\upuparrows \prod^{\mathsf{Strd}(\mathfrak{Z})} A = \prod^{\mathsf{Strd}(\mathfrak{A})} A$.

Proof. $\operatorname{GR} \upuparrows \prod^{\mathsf{Strd}(\mathfrak{Z})} A = \left\{ L \in \mathfrak{A} \mid L \subseteq \prod^{\mathsf{Strd}(\mathfrak{Z})} A \right\} = \{ L \in \mathfrak{A} \mid \forall K \in L, i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp K_i \} = \{ L \in \mathfrak{A} \mid \forall i \in \operatorname{dom} \mathfrak{A}, K \in L_i: A_i \not\asymp K \} = \{ L \in \mathfrak{A} \mid \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i \} = \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A$. □

Proposition 96. Let $(\prod \mathfrak{A}; \prod \mathfrak{Z})$ is a meet-closed filtrator. Then $\downdownarrows \prod^{\mathsf{Strd}(\mathfrak{A})} A = \prod^{\mathsf{Strd}(\mathfrak{Z})} A$.

Proof. $\operatorname{GR} \downdownarrows \prod^{\mathsf{Strd}(\mathfrak{A})} A = \downdownarrows \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{A})} A = \downdownarrows \{ L \in \prod \mathfrak{A} \mid \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i \} = \{ L \in \prod \mathfrak{A} \mid \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i \} \cap \prod \mathfrak{Z} = \{ L \in \prod \mathfrak{Z} \mid \forall i \in \operatorname{dom} \mathfrak{A}: A_i \not\asymp L_i \} = \operatorname{GR} \prod^{\mathsf{Strd}(\mathfrak{Z})} A$. □

Theorem 97. Let $\mathfrak{F}$ is a family of sets of filters on distributive lattices with least elements. Let $a \in \prod \mathfrak{F}$, $S \in \mathscr{P} \prod \mathfrak{F}$ is a generalized filter base, $\bigsqcap S = a$. Then

$$\prod^{\mathsf{Strd}(\mathfrak{F})} a = \bigsqcap \left\{ \prod^{\mathsf{Strd}(\mathfrak{F})} A \mid A \in S \right\}.$$

Proof. That $\prod^{\mathsf{Strd}(\mathfrak{F})} a$ is a lower bound for $\left\{ \prod^{\mathsf{Strd}(\mathfrak{F})} A \mid A \in S \right\}$ is obvious.

Let f is a lower bound for $\left\{ \prod^{\mathsf{Strd}(\mathfrak{F})} A \mid A \in S \right\}$. Thus for every $A \in S$ we have $L \in \mathrm{GR}\, f$ implies $\forall i \in \operatorname{dom}\mathfrak{A}: A_i \not\asymp L_i$. Then, by properties of generalized filter bases, $\forall i \in \operatorname{dom}\mathfrak{A}: a_i \not\asymp L_i$ that is $L \in \mathrm{GR} \prod^{\mathsf{Strd}(\mathfrak{F})} a$.

So $f \subseteq \prod^{\mathsf{Strd}(\mathfrak{F})} a$. □

Theorem 98. Let $\mathfrak{F}$ is a family of sets of filters on distributive lattices with least elements. Let $a \in \prod \mathfrak{F}$, $S \in \mathscr{P} \prod \mathfrak{F}$ is a generalized filter base, $\bigsqcap S = a$, f is a staroid of the form $\prod \mathfrak{F}$. Then

$$\prod^{\mathsf{Strd}(\mathfrak{F})} a \not\asymp f \Leftrightarrow \forall A \in S: \prod^{\mathsf{Strd}(\mathfrak{A})} A \not\asymp f.$$

Proof. It follows from the previous theorem by properties of generalized filter bases. □

9.1 On products of staroids

Definition 99. $\prod^{(D)} F = \{\operatorname{uncurry} z \mid z \in \prod F\}$ (*reindexation product*) for every indexed family F of relations.

Definition 100. *Reindexation product* of an indexed family F of anchored relations is defined by the formulas:

$$\operatorname{form} \prod^{(D)} F = \operatorname{uncurry}(\operatorname{form} \circ F) \quad \text{and} \quad \mathrm{GR} \prod^{(D)} F = \prod (\mathrm{GR} \circ F).$$

Obvious 101.

1. $\operatorname{form} \prod^{(D)} F = \{((i;j); (\operatorname{form} F_i)_j) \mid i \in \operatorname{dom} F, j \in \operatorname{arity} F_i\}$;
2. $\mathrm{GR} \prod^{(D)} F = \{\{((i;j); (zi)j) \mid i \in \operatorname{dom} F, j \in \operatorname{arity} F_i\} \mid z \in \prod (\mathrm{GR} \circ F)\}$.

Proposition 102. $\prod^{(D)} F$ is an anchored relation if every F_i is an anchored relation.

Proof. We need to prove $\mathrm{GR} \prod {}^{(D)}F \in \mathscr{P} \prod \operatorname{form}\left(\prod {}^{(D)}F\right)$ that is

$\mathrm{GR} \prod {}^{(D)}F \subseteq \prod \operatorname{form}\left(\prod {}^{(D)}F\right)$

$\{\operatorname{uncurry} z \mid z \in \prod (\mathrm{GR} \circ F)\} \in \mathscr{P} \prod \{((i;j); (\operatorname{form} F_i)_j) \mid i \in \operatorname{dom} F, j \in \operatorname{arity} F_i\}$;

$\{\operatorname{uncurry} z \mid z \in \prod (\mathrm{GR} \circ F)\} \subseteq \prod \{((i;j); (\operatorname{form} F_i)_j) \mid i \in \operatorname{dom} F, j \in \operatorname{arity} F_i\}$

$\{\{((i;j); (zi)j) \mid i \in \operatorname{dom} F, j \in \operatorname{arity} F_i\} \mid z \in \prod (\mathrm{GR} \circ F)\} \subseteq \prod \{((i;j); (\operatorname{form} F_i)_j) \mid i \in \operatorname{dom} F, j \in \operatorname{arity} F_i\}$;

$\forall z \in \prod (\mathrm{GR} \circ F), i \in \operatorname{dom} F, j \in \operatorname{arity} F_i: (zi)j \in (\operatorname{form} F_i)_j$.

Really, $zi \in \mathrm{GR}\, F_i \subseteq \prod (\operatorname{form} F_i)$ and thus $(zi)j \in (\operatorname{form} F_i)_j$. □

Remark 103. I suspect that the above proof can be simplified.

Obvious 104. $\operatorname{arity} \prod^{(D)} F = \coprod_{i \in \operatorname{dom} F} \operatorname{arity} F_i = \{(i;j) \mid i \in \operatorname{dom} F, j \in \operatorname{arity} F_i\}$.

Definition 105. $f \times^{(D)} g = \prod^{(D)} [\![f; g]\!]$.

Lemma 106. $\prod^{(D)} F$ is an upper set if every F_i is an upper set.

Proof. We need to prove that $\prod^{(D)} F$ is an upper set. Let $a \in \prod^{(D)} F$ and an anchored relation $b \sqsupseteq a$ of the same form as a. We have $a = \operatorname{uncurry} z$ for some $z \in \prod F$ that is $a(i;j) = (zi)j$ for all $i \in \operatorname{dom} F$ and $j \in \operatorname{dom} F_i$ where $zi \in F_i$. Also $b(i;j) \sqsupseteq a(i;j)$. Thus $(\operatorname{curry} b)i \sqsupseteq zi$; $\operatorname{curry} b \in \prod F$ because every F_i is an upper set and so $b \in \prod^{(D)} F$. □

Proposition 107. Let F is an indexed family of anchored relations and every $(\operatorname{form} F)_i$ is a join-semilattice.

1. $\prod^{(D)} F$ is a pre-staroid if every F_i is a pre-staroid.

2. $\prod^{(D)} F$ is a staroid if every F_i is a staroid.

3. $\prod^{(D)} F$ is a completary staroid if every F_i is a completary staroid.

Proof.

1. Let $q \in \operatorname{arity} \prod^{(D)} F$ that is $q = (i; j)$ where $i \in \operatorname{dom} F$, $j \in \operatorname{arity} F_i$; let

$$L \in \prod \left(\left(\operatorname{form} \prod^{(D)} F \right)|_{(\operatorname{arity} \prod^{(D)} F) \setminus \{q\}} \right)$$

that is $L_{(i'; j')} \in \left(\operatorname{form} \prod^{(D)} F \right)_{(i'; j')}$ for every $(i'; j') \in \left(\operatorname{arity} \prod^{(D)} F \right) \setminus \{q\}$, that is $L_{(i'; j')} \in (\operatorname{form} F_i)_j$. We have $X \in \left(\operatorname{form} \prod^{(D)} F \right)_{(i; j)} \Leftrightarrow X \in (\operatorname{form} F_i)_j$. So

$$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\operatorname{form} F_i)_j \mid L \cup \{((i; j); X)\} \in \operatorname{GR} \prod^{(D)} F \right\}$$

$$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\operatorname{form} F_i)_j \mid \exists z \in \prod (\operatorname{GR} \circ F) \colon L \cup \{((i; j); X)\} = \operatorname{uncurry} z \right\}$$

$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\operatorname{form} F_i)_j \mid \exists z \in \prod \left((\operatorname{GR} \circ F)|_{(\operatorname{arity} \prod^{(D)} F) \setminus \{(i; j)\}} \right), v \in \operatorname{GR} F_i \colon (L = \operatorname{uncurry} z \wedge v_j = X) \right\}$

$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \left\{ X \in (\operatorname{form} F_i)_j \mid \exists z \in \prod \left((\operatorname{GR} \circ F)|_{(\operatorname{arity} \prod^{(D)} F) \setminus \{(i; j)\}} \right) \colon L = \operatorname{uncurry} z \wedge \exists v \in \operatorname{GR} F_i \colon v_j = X \right\}$

If $\exists z \in \prod \left((\operatorname{GR} \circ F)|_{(\operatorname{arity} \prod^{(D)} F) \setminus \{(i; j)\}} \right) \colon L = \operatorname{uncurry} z$ is false then $\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \emptyset$ is a free star. We can assume it is true. So

$$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \{ X \in (\operatorname{form} F_i)_j \mid \exists v \in \operatorname{GR} F_i \colon v_j = X \}.$$

Thus

$\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L = \{ X \in (\operatorname{form} F_i)_j \mid \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon K \cup \{(j; X)\} \in \operatorname{GR} F_i \} = \{ X \in (\operatorname{form} F_i)_j \mid \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon K \cup \{(j; X)\} \in \operatorname{GR} F_i \} = \{ X \in (\operatorname{form} F_i)_j \mid \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon X \in (\operatorname{val} F_j) K \}$.

Thus $A \sqcup B \in \left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L \Leftrightarrow \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon A \sqcup B \in (\operatorname{val} F_j) K \Leftrightarrow \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon (A \in (\operatorname{val} F_j) \vee B \in (\operatorname{val} F_j)) \Leftrightarrow \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon A \in (\operatorname{val} F_j) K \vee \exists K \in (\operatorname{form} F_i)|_{(\operatorname{arity} F_i) \setminus \{j\}} \colon A \in (\operatorname{val} F_j) K \Leftrightarrow A \in \left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L \vee B \in \left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L$. Least element 0 is not in $\left(\operatorname{val} \prod^{(D)} F \right)_{(i; j)} L$ because $K \cup \{(j; 0)\} \notin \operatorname{GR} F_i$.

2. From the lemma.

3. We need to prove

$$L_0 \sqcup L_1 \in \operatorname{GR} \prod^{(D)} F \Leftrightarrow \exists c \in \{0, 1\}^{\operatorname{arity} \prod^{(D)} F} \colon \left(\lambda i \in \operatorname{arity} \prod^{(D)} F \colon L_{c(i)} i \right) \in \operatorname{GR} \prod^{(D)} F$$

for every $L_0, L_1 \in \prod \operatorname{form} \prod^{(D)} F$ that is $L_0, L_1 \in \prod \operatorname{uncurry}(\operatorname{form} \circ F)$.

Really $L_0 \sqcup L_1 \in \operatorname{GR} \prod^{(D)} F \Leftrightarrow L_0 \sqcup L_1 \in \{\operatorname{uncurry} z \mid z \in \prod (\operatorname{GR} \circ F)\}$.

$\exists c \in \{0, 1\}^{\operatorname{arity} \prod^{(D)} F}: (\lambda i \in n: L_{c(i)} i) \in \operatorname{GR} \prod^{(D)} F \Leftrightarrow \exists c \in \{0, 1\}^{\operatorname{arity} \prod^{(D)} F}: \left(\lambda i \in \operatorname{arity} \prod^{(D)} F: L_{c(i)} i\right) \in \{\operatorname{uncurry} z \mid z \in \prod (\operatorname{GR} \circ F)\} \Leftrightarrow \exists c \in \{0, 1\}^{\operatorname{arity} \prod^{(D)} F}: \operatorname{curry}\left(\lambda i \in \operatorname{arity} \prod^{(D)} F: L_{c(i)} i\right) \in \prod (\operatorname{GR} \circ F) \Leftrightarrow \exists c \in \{0, 1\}^{\operatorname{arity} \prod^{(D)} F}: \operatorname{curry}\left(\lambda (i; j) \in \operatorname{arity} \prod^{(D)} F: L_{c(i;j)}(i; j)\right) \in \prod (\operatorname{GR} \circ F) \Leftrightarrow \exists c \in \{0, 1\}^{\operatorname{arity} \prod^{(D)} F}: (\lambda i \in \operatorname{dom} F: (\lambda j \in \operatorname{dom} F_i: L_{c(i;j)}(i; j))) \in \prod (\operatorname{GR} \circ F) \Leftrightarrow \exists c \in \{0, 1\}^{\operatorname{arity} \prod^{(D)} F} \forall i \in \operatorname{dom} F: (\lambda j \in \operatorname{dom} F_i: L_{c(i;j)}(i; j)) \in \operatorname{GR} F_i \Leftrightarrow \forall i \in \operatorname{dom} F \exists c \in \{0, 1\}^{\operatorname{dom} F_i}: (\lambda j \in \operatorname{dom} F_i: L_{c(j)}(i; j)) \in \operatorname{GR} F_i \Leftrightarrow \forall i \in \operatorname{dom} F \exists c \in \{0, 1\}^{\operatorname{dom} F_i}: (\lambda j \in \operatorname{dom} F_i: (\operatorname{curry}(L_{c(j)}) i) j) \in \operatorname{GR} F_i \Leftrightarrow \forall i \in \operatorname{dom} F: (\operatorname{curry}(L_0) i \sqcup \operatorname{curry}(L_1) i \in \operatorname{GR} F_i) \Leftrightarrow L_0 \sqcup L_1 \in \{\operatorname{uncurry} z \mid z \in \prod (\operatorname{GR} \circ F)\}$. □

For staroids it is defined *ordinated product* $\prod^{(\operatorname{ord})}$ as defined in [2].

Obvious 108. If f and g are anchored relations and there exists a bijection φ from arity g to arity f such that $\{F \circ \varphi \mid F \in \operatorname{GR} f\} = \operatorname{GR} g$, then:

1. f is a pre-staroid iff g is a pre-staroid.
2. f is a staroid iff g is a staroid.
3. f is a completary staroid iff g is a completary staroid.

Corollary 109. Let F is an indexed family of anchored relations and every $(\operatorname{form} F)_i$ is a join-semilattice.

1. $\prod^{(\operatorname{ord})} F$ is a pre-staroid if every F_i is a pre-staroid.
2. $\prod^{(\operatorname{ord})} F$ is a staroid if every F_i is a staroid.
3. $\prod^{(\operatorname{ord})} F$ is a completary staroid if every F_i is a completary staroid.

Proof. Use the fact that $\operatorname{GR} \prod^{(\operatorname{ord})} F = \left\{F \circ (\bigoplus (\operatorname{dom} \circ F))^{-1} \mid F \in \operatorname{GR} \prod^{(D)} f\right\}$. □

Definition 110. $f \times^{(\operatorname{ord})} g = \prod^{(\operatorname{ord})} [\![f; g]\!]$.

Remark 111. If f and g are binary funcoids, then $f \times^{(\operatorname{ord})} g$ is ternary.

10 Star categories

Definition 112. A *pre-category with star-morphisms* consists of

1. a pre-category C (*the base pre-category*);
2. a set M (*star-morphisms*);
3. a function "arity" defined on M (how many objects are connected by this multimorphism);
4. a function $\operatorname{Obj}_m: \operatorname{arity} m \to \operatorname{Obj}(C)$ defined for every $m \in M$;
5. a function (*star composition*) $(m; f) \mapsto \operatorname{StarComp}(m; f)$ defined for $m \in M$ and f being an $(\operatorname{arity} m)$-indexed family of morphisms of C such that $\forall i \in \operatorname{arity} m: \operatorname{Src} f_i = \operatorname{Obj}_m i$ ($\operatorname{Src} f_i$ is the source object of the morphism f_i) such that $\operatorname{arity} \operatorname{StarComp}(m; f) = \operatorname{arity} m$

such that it holds:

1. $\operatorname{StarComp}(m; f) \in M$;
2. (*associativiy law*)

$$\operatorname{StarComp}(\operatorname{StarComp}(m; f); g) = \operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: g_i \circ f_i).$$

(Here by definition $\lambda x \in D\colon F(x) = \{(x; F(x)) \mid x \in D\}$.)

The meaning of the set M is an extension of C having as morphisms things with arbitrary (possibly infinite) indexed set Obj_m of objects, not just two objects as morphisms of C have only source and destination.

Definition 113. I will call Obj_m the *form* of the star-morphism m.

(Having fixed a pre-category with star-morphisms) I will denote $\mathrm{StarHom}(P)$ the set of star-morphisms of the form P.

Proposition 114. The sets $\mathrm{StarHom}(P)$ are disjoint (for different P).

Proof. If two star-morphisms have different forms, they are clearly not equal. □

Definition 115. A *category with star-morphisms* is a pre-category with star-morphisms whose base is a category and the following equality (*the law of composition with identity*) holds for every multimorphism m:

$$\mathrm{StarComp}(m; \lambda i \in \mathrm{arity}\, m\colon \mathrm{id}_{\mathrm{Obj}_m i}) = m.$$

Definition 116. A *partially ordered pre-category with star-morphisms* is a category with star-morphisms, whose base pre-category is a partially ordered pre-category and every set

$$\{m \in M \mid \mathrm{Obj}_m = X\}$$

is partially ordered for every X, such that:

1. $m_0 \sqsubseteq m_1 \wedge f_0 \sqsubseteq f_1 \Rightarrow \mathrm{StarComp}(m_0; f_0) \sqsubseteq \mathrm{StarComp}(m_1; f_1)$ for every $m_0, m_1 \in M$ such that $\mathrm{Obj}_{m_0} = \mathrm{Obj}_{m_1}$ and indexed families f_0 and f_1 of morphisms such that

$$\forall i \in \mathrm{arity}\, m\colon \mathrm{Src}\, f_0\, i = \mathrm{Src}\, f_1\, i = \mathrm{Obj}_{m_0} i = \mathrm{Obj}_{m_1} i \quad \text{and} \quad \forall i \in \mathrm{arity}\, m\colon \mathrm{Dst}\, f_0\, i = \mathrm{Dst}\, f_1\, i.$$

Definition 117. A *quasi-invertible* pre-category with star-morphisms is a partially ordered pre-category with star-morphisms whose base pre-category is a quasi-invertible pre-category, such that for every index set n, multimorphisms a and b of arity n, and an n-indexed family f of morphisms of the base pre-category it holds

$$b \not\asymp \mathrm{StarComp}(a; f) \Leftrightarrow a \not\asymp \mathrm{StarComp}(b; f^{\dagger}).$$

Definition 118. A *quasi-invertible* category with star-morphisms is a quasi-invertible pre-category with star-morphisms which is a quasi-invertible pre-category with star-morphisms.

Each category with star-morphisms gives rise to a category (*abrupt category*, see a remark below why I call it "abrupt"), as described below. Below for simplicity I assume that the set M and the set of our indexed families of functions are disjoint. The general case (when they are not necessarily disjoint) may be easily elaborated by the reader.

- Objects are indexed (by arity m for some $m \in M$) families of objects of the category C and an (arbitrarily choosen) object None not in this set
- There are the following disjoint sets of morphisms:
    1. indexed (by arity m for some $m \in M$) families of morphisms of C
    2. elements of M
    3. the identity morphism $\mathrm{id}_{\mathrm{None}}$ on None
- Source and destination of morphisms are defined by the formulas:
    - $\mathrm{Src}\, f = \lambda i \in \mathrm{dom}\, f\colon \mathrm{Src}\, f_i$;
    - $\mathrm{Dst}\, f = \lambda i \in \mathrm{dom}\, f\colon \mathrm{Dst}\, f_i$

  - $\operatorname{Src} m = \text{None}$
  - $\operatorname{Dst} m = \operatorname{Obj}_m$.
- Compositions of morphisms are defined by the formulas:
  - $g \circ f = \lambda i \in \operatorname{dom} f: g_i \circ f_i$ for our indexed families f and g of morphisms;
  - $f \circ m = \operatorname{StarComp}(m; f)$ for $m \in M$ and a composable indexed family f;
  - $m \circ \mathrm{id}_{\text{None}} = m$ for $m \in M$;
  - $\mathrm{id}_{\text{None}} \circ \mathrm{id}_{\text{None}} = \mathrm{id}_{\text{None}}$.
- Identity morphisms for an object X are:
  - $\lambda i \in X: \mathrm{id}_{X_i}$ if $X \neq \text{None}$
  - $\mathrm{id}_{\text{None}}$ if $X = \text{None}$

We need to prove it is really a category.

Proof. We need to prove:

1. Composition is associative
2. Composition with identities complies with the identity law.

Really:

1. $(h \circ g) \circ f = \lambda i \in \operatorname{dom} f: (h_i \circ g_i) \circ f_i = \lambda i \in \operatorname{dom} f: h_i \circ (g_i \circ f_i) = h \circ (g \circ f)$;
   $g \circ (f \circ m) = \operatorname{StarComp}(\operatorname{StarComp}(m; f); g) = \operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: g_i \circ f_i) = \operatorname{StarComp}(m; g \circ f) = (g \circ f) \circ m$;
   $f \circ (m \circ \mathrm{id}_{\text{None}}) = f \circ m = (f \circ m) \circ \mathrm{id}_{\text{None}}$.
2. $m \circ \mathrm{id}_{\text{None}} = m$; $\mathrm{id}_{\operatorname{Dst} m} \circ m = \operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: \mathrm{id}_{\operatorname{Obj}_m i}) = m$. □

Remark 119. I call the above defined category *abrupt category* because (excluding identity morphisms) it allows composition with an $m \in M$ only on the left (not on the right) so that the morphism m is "abrupt" on the right.

By $[\![x_0; \ldots; x_{n-1}]\!]$ I denote an n-tuple.

Definition 120. Pre-category with star morphisms *induced* by a dagger pre-category C is:

- The base category is C.
- Star-morphisms are morphisms of C.
- $\operatorname{arity} f = \{0, 1\}$.
- $\operatorname{Obj}_m = [\![\operatorname{Src} m; \operatorname{Dst} m]\!]$.
- $\operatorname{StarComp}(m; [\![f; g]\!]) = g \circ m \circ f^\dagger$.

Let prove it is really a category with star-morphisms.

Proof. We need to prove the associativity law:

$$\operatorname{StarComp}(\operatorname{StarComp}(m; [\![f; g]\!]); [\![p; q]\!]) = \operatorname{StarComp}(m; [\![p \circ f; q \circ g]\!]).$$

Really,

$\operatorname{StarComp}(g \circ m \circ f^\dagger; [\![p; q]\!]) = q \circ g \circ m \circ f^\dagger \circ p^\dagger = q \circ g \circ m \circ (p \circ f)^\dagger = \operatorname{StarComp}(m; [\![p \circ f; q \circ g]\!])$. □

Definition 121. Category with star morphisms *induced* by a dagger category C is the above definined pre-category with star-morphisms.

That it is a category (the law of composition with identity) is trivial.

Remark 122. We can carry definitions (such as below defined cross-composition product) from categories with star-morphisms into plain dagger categories. This allows us to research properties of cross-composition product of indexed families of morphism for categories with star-morphisms without separately considering the special case of dagger categories and just binary star-composition product.

10.1 Abrupt of quasi-invertible categories with star-morphisms

Definition 123. The abrupt partially ordered pre-category of a partially ordered pre-category with star-morphisms is the abrupt pre-category with the following order of morphisms:

- Indexed (by arity m for some $m \in M$) families of morphisms of C are ordered as function spaces of posets.
- Star-morphisms (which are morphisms $\mathrm{None} \rightarrow \mathrm{Obj}_m$ for some $m \in M$) are ordered in the same order as in the pre-category with star-morphisms.
- Morphisms $\mathrm{None} \rightarrow \mathrm{None}$ which are only the identity morphism ordered by the unique order on this one-element set.

We need to prove it is a partially ordered pre-category.

Proof. It trivally follows from the definition of partially ordered pre-category with star-morphisms. □

Theorem 124. When a pre-category with star-morphisms is quasi-invertible, the corresponding abrupt category is also quasi-invertible.

Proof. We need to prove: $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{\dagger}$ (or equivalently $f^{\dagger} \circ g \not\asymp h \Leftrightarrow g \not\asymp f \circ h$) for all kinds of morphisms.

Consider the cases:

$g = \mathrm{id}_{\mathrm{None}}$.
Subcases:

$g = h = \mathrm{id}_{\mathrm{None}}$. Trivial.

$g \in M$. $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{\dagger}$.

$g \in M$.
$f^{\dagger} \circ g \not\asymp h \Leftrightarrow \mathrm{StarComp}(g; f^{\dagger}) \not\asymp h \Leftrightarrow g \not\asymp \mathrm{StarComp}(h; f) \Leftrightarrow g \not\asymp f \circ h$.

g is a family of morphism of C.
$f^{\dagger} \circ g \not\asymp h \Leftrightarrow \exists i \in \mathrm{dom}\, g: f_i^{\dagger} \circ g_i \not\asymp h_i \Leftrightarrow \exists i \in \mathrm{dom}\, g: g_i \not\asymp f_i \circ h_i \Leftrightarrow g \not\asymp f \circ h$. □

11 Product of an arbitrary number of funcoids

In this section it will be defined a product of an arbitrary (possibly infinite) family of funcoids.

11.1 Mapping a morphism into a pointfree funcoid

Definition 125. Let's define the pointfree funcoid χf for every morphism f or a quasi-invertible category:

$$\langle \chi f \rangle a = f \circ a \quad \text{and} \quad \langle (\chi f)^{-1} \rangle b = f^{\dagger} \circ b.$$

We need to prove it is really a pointfree funcoid.

Proof. $b \not\asymp \langle \chi f \rangle a \Leftrightarrow b \not\asymp f \circ a \Leftrightarrow a \not\asymp f^{\dagger} \circ b \Leftrightarrow a \not\asymp \langle (\chi f)^{-1} \rangle b$. □

Remark 126. $\langle \chi f \rangle = (f \circ -)$ is the Hom-functor $\mathrm{Hom}(f, -)$ and we can apply Yoneda lemma to it.

Obvious 127. $\langle \chi(g \circ f) \rangle a = g \circ f \circ a$ for composable morphisms f and g or a quasi-invertible category.

11.2 General cross-composition

Let fix a quasi-invertible category with with star-morphisms. If f is an indexed family of morphisms from its base category, then the pointfree funcoid $\prod^{(C)} f$ from $\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon \mathrm{Src}\, f_i)$ to $\mathrm{StarHom}(\lambda i \in \mathrm{dom}\, f\colon \mathrm{Dst}\, f_i)$ is defined by the formulas (for all star-morphisms a and b of these forms):

$$\left\langle \prod^{(C)} f \right\rangle a = \mathrm{StarComp}(a; f) \quad \text{and} \quad \left\langle \left(\prod^{(C)} f \right)^{-1} \right\rangle b = \mathrm{StarComp}(b; f^{\dagger}).$$

It is really a pointfree funcoid by the definition of quasi-invertible category.

In the terms of abrupt categories, these formulas can be rewritten as:

$$\prod^{(C)} f = \chi f.$$

Theorem 128. $\left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) = \prod^{(C)}_{i \in n} (g_i \circ f_i)$ for every n-indexed families f and g of composable morphisms of a quasi-invertible category with star-morphisms.

Proof. $\left\langle \prod^{(C)}_{i \in n} (g_i \circ f_i) \right\rangle a = \mathrm{StarComp}(a; \lambda i \in n\colon g_i \circ f_i) = \mathrm{StarComp}(\mathrm{StarComp}(a; f); g)$ and $\left\langle \left(\prod^{(C)} g \right) \circ \left(\prod^{(C)} f \right) \right\rangle a = \left\langle \prod^{(C)} g \right\rangle \left\langle \prod^{(C)} f \right\rangle a = \mathrm{StarComp}(\mathrm{StarComp}(a; f); g)$. □

Corollary 129. $\left(\prod^{(C)} f_{k-1} \right) \circ \ldots \circ \left(\prod^{(C)} f_0 \right) = \prod^{(C)}_{i \in n} (f_i(k-1) \circ \ldots \circ f_i(k))$ for every n-indexed families $f_0, \ldots, f_{n-1}, g_0, \ldots, g_{n-1}$ composable morphisms of a quasi-invertible category with star-morphisms.

Proof. By math induction. □

11.3 Some properties of staroids

Lemma 130. Let $A_0, A_1 \in (\mathscr{P}\mho)^n$ are two families of sets and $\delta \in \mathscr{P}((\mathscr{P}\mho)^n)$. Then

$$\delta \cap \prod_{i \in n} (A_0\, i \sqcup A_1\, i) \neq \emptyset \Leftrightarrow \exists c \in \{0, 1\}^n\colon \delta \cap \prod_{i \in n} A_{c(i)}\, i \neq \emptyset.$$

Proof. $f \in \prod_{i \in n} (A_0\, i \sqcup A_1\, i) \Leftrightarrow \forall i \in n\colon (f_i \in A_0\, i \cup A_1\, i) \Leftrightarrow \forall i \in n\colon (f_i \in A_0\, i \vee f_i \in A_1\, i) \Leftrightarrow \exists c \in \{0, 1\}^n \forall i \in n\colon f_i \in A_{c(i)} i \Leftrightarrow \exists c \in \{0, 1\}^n\colon f \in \prod_{i \in n} A_{c(i)} i$.

$f \in \delta \cap \prod_{i \in n} (A_0\, i \sqcup A_1\, i) \Leftrightarrow f \in \delta \wedge \exists c \in \{0, 1\}^n\colon f \in \prod_{i \in n} A_{c(i)} i \Leftrightarrow \exists c \in \{0, 1\}^n\colon f \in \delta \cap \prod_{i \in n} A_{c(i)} i \Rightarrow \exists c \in \{0, 1\}^n\colon \delta \cap \prod_{i \in n} A_{c(i)}\, i \neq \emptyset$. The reverse implication is obvious. □

Theorem 131. Let $\mathfrak{A} = \mathfrak{A}_{i \in n}$ is a family of boolean lattices.

A relation $\delta \in \mathscr{P} \prod \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$ such that for every $a \in \prod \mathrm{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$

$$\forall A \in a\colon \delta \cap \prod_{i \in n} \mathrm{atoms} \uparrow^{\mathfrak{A}_i} A_i \neq \emptyset \Rightarrow a \in \delta \tag{5}$$

can be continued till the function $\upuparrows f$ for a unique staroid f of the form $\lambda i \in n\colon \mathfrak{P}(\mathfrak{A}_i)$. The funcoid f is completary.

For every $\mathcal{X} \in \prod_{i \in n} \mathfrak{F}(\mathfrak{A}_i)$

$$\mathcal{X} \in \mathrm{GR} \upuparrows f \Leftrightarrow \delta \cap \prod_{i \in n} \mathrm{atoms}\, \mathcal{X}_i \neq \emptyset. \tag{6}$$

Proof. By the theorem 81 (used that it is a boolean lattice) we have $\mathcal{X} \in \mathrm{GR} \upuparrows f \Leftrightarrow \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \operatorname{atoms} \mathcal{X}_i \neq \emptyset$ and thus (6). From this also follows uniqueness.

It is left to prove that there exists a completary staroid f such that $\upuparrows f$ is a continuation of δ.

Consider the relation f defined by the formula $X \in f \Leftrightarrow \delta \cap \prod_{i \in n} \operatorname{atoms} \uparrow^{\mathfrak{A}_i} X_i \neq \emptyset$.

$I_0 \sqcup I_1 \in f \Leftrightarrow \delta \cap \prod_{i \in n} \operatorname{atoms} \uparrow^{\mathfrak{A}_i} (I_0\, i \sqcup I_1\, i) \neq \emptyset \Leftrightarrow \delta \cap \prod_{i \in n} (\operatorname{atoms} \uparrow^{\mathfrak{A}_i} I_0\, i \cup \operatorname{atoms} \uparrow^{\mathfrak{A}_i} I_1\, i) \neq \emptyset$.

Thus by the lemma $I_0 \sqcup I_1 \in f \Leftrightarrow \exists c \in \{0, 1\}^n \colon \delta \cap \prod_{i \in n} \operatorname{atoms} \uparrow^{\mathfrak{A}_i} I_{c(i)} \neq \emptyset \Leftrightarrow \exists c \in \{0, 1\}^n \colon (\lambda i \in n \colon I_{c(i)} i) \in f$. Trivially if $\exists i \in n \colon X_i = 0$ then $X \notin f$. So f is a completary staroid.

Let $a \in \prod \operatorname{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$.

The reverse of (5) is obvious. So we have $a \in \delta \Leftrightarrow \forall A \in a \colon \delta \cap \prod_{i \in n} \operatorname{atoms} \uparrow^{\mathfrak{A}_i} A_i \neq \emptyset \Leftrightarrow \forall A \in a \colon A \in f \Leftrightarrow \forall A \in a \colon A \in f \Leftrightarrow a \subseteq f \Leftrightarrow a \in \upuparrows f$. Thus $\upuparrows f$ is a continuation of δ. □

Theorem 132. Let R is a set of staroids of the form $\lambda i \in n \colon \mathfrak{F}(\mathfrak{A}_i)$ where every $\mathfrak{A}_i$ is a boolean lattice. If $x \in \prod_{i \in n} \operatorname{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$ then $x \in \mathrm{GR} \upuparrows \bigsqcap R \Leftrightarrow \forall f \in R \colon x \in \upuparrows f$.

Proof. Let denote $x \in \delta \Leftrightarrow \forall f \in R \colon x \in \upuparrows f$ for every $x \in \prod_{i \in n} \operatorname{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$. For every $a \in \prod_{i \in n} \operatorname{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$

$\forall X \in a \colon \delta \cap \prod_{i \in n} \operatorname{atoms} \uparrow^{\mathfrak{A}_i} X_i \neq \emptyset \Leftrightarrow \forall X \in a \exists x \in \prod_{i \in n} \operatorname{atoms} \uparrow^{\mathfrak{A}_i} X_i \colon x \in \delta \Leftrightarrow \forall X \in a \exists x \in \prod_{i \in n} \operatorname{atoms} \uparrow^{\mathfrak{A}_i} X_i \forall f \in R \colon x \in \upuparrows f \Rightarrow \forall X \in a, f \in R \exists x \in \prod_{i \in n} \operatorname{atoms} \uparrow^{\mathfrak{A}_i} X_i \colon x \in \upuparrows f \Rightarrow \forall X \in a, f \in R \colon X \in f \Leftrightarrow \forall f \in R \colon a \subseteq f \Leftrightarrow \forall f \in R \colon a \in \upuparrows f \Leftrightarrow a \in \delta$.

So by the previous theorem δ can be contimued till $\upuparrows p$ for some staroid p of the form $\lambda i \in n \colon \mathfrak{P}(\mho_i)$.

Let's prove $p = \bigsqcap R$.

$x \in \upuparrows p \Leftrightarrow x \in \delta \Rightarrow x \in \upuparrows f$ for every $f \in R$ and $x \in \prod_{i \in n} \operatorname{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$. Thus $\upuparrows p \subseteq \upuparrows f$. Consequently $\forall f \in R \colon p \subseteq f$.

Suppose that q is a staroid of the form $\lambda i \in n \colon \mathfrak{P}(\mathfrak{A}_i)$ such that $\forall f \in R \colon q \subseteq f$. Then for every $x \in \prod_{i \in n} \operatorname{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$ we have $x \in \upuparrows q \Rightarrow \forall f \in R \colon x \in \upuparrows f \Leftrightarrow x \in \delta \Leftrightarrow x \in \upuparrows p$. So $\upuparrows q \subseteq \upuparrows p$ that is $q \subseteq p$.

We have proved $p = \bigsqcap R$. It's remained to prove that $x \in \upuparrows p \Leftrightarrow \forall f \in R \colon x \in \upuparrows f$ for every $x \in \prod_{i \in n} \operatorname{atoms}^{\mathfrak{F}(\mathfrak{A}_i)}$. Really, $x \in \upuparrows p \Leftrightarrow x \in \delta \Leftrightarrow \forall f \in R \colon x \in \upuparrows f$. □

11.4 Star composition of binary relations

First define *star composition* for an n-ary relation a and an n-indexed family f of binary relations as an n-ary relation complying with the formulas:

$$\mathrm{Obj}_{\mathrm{StarComp}(a; f)} = \{*\}^n;$$
$$L \in \mathrm{StarComp}(a; f) \Leftrightarrow \exists y \in a \forall i \in n \colon y_i\, f_i\, L_i$$

where $*$ is a unique object of the semigroup of small binary relations considered as a category.

Proposition 133. $b \not\asymp \mathrm{StarComp}(a; f) \Leftrightarrow \exists x \in a, y \in b \forall j \in n \colon x_j\, f_j\, y_j$.

Proof. We need to prove that $b \not\asymp \mathrm{StarComp}(a; f) \Leftrightarrow a \not\asymp \mathrm{StarComp}(b; f^{\dagger})$.

$b \not\asymp \mathrm{StarComp}(a; f) \Leftrightarrow \exists y \in \prod \mathfrak{A} \colon (y \in b \wedge y \in \mathrm{StarComp}(a; f)) \Leftrightarrow \exists x \in \prod \mathfrak{A} \colon (y \in b \wedge \exists x \in a \forall j \in n \colon x_j\, f_j\, x_j) \Leftrightarrow \exists x \in \prod \mathfrak{A}, x \in a \colon (y \in b \wedge \forall j \in n \colon x_j\, f_j\, y_j) \Leftrightarrow \exists x \in a, y \in b \forall j \in n \colon x_j\, f_j\, y_j$. □

Theorem 134. The semigroup of small binary relations considered as a category together with the set of of all n-ary relations (for every small n) and the above defined star-composition form a category with star-morphisms.

Proof. We need to prove:

1. $\mathrm{StarComp}(\mathrm{StarComp}(m; f); g) = \mathrm{StarComp}(m; \lambda i \in n \colon g_i \circ f_i)$;
2. $\mathrm{StarComp}(m; \lambda i \in \operatorname{arity} m \colon \mathrm{id}_{\mathrm{Obj}_m i}) = m$;
3. $b \not\asymp \mathrm{StarComp}(a; f) \Leftrightarrow a \not\asymp \mathrm{StarComp}(b; f^{\dagger})$

(the rest is obvious).

Really,

1. $L\in\mathrm{StarComp}(a;f)\Leftrightarrow\exists y\in a\forall i\in n: y_i\, f_i\, L_i$.

   Define the relation $R(f)$ by the formula $x\,R(f)\,y\Leftrightarrow\forall i\in n: x_i\, f_i\, y_i$. Obviously

   $$R(\lambda i\in n: g_i\circ f_i)=R(g)\circ R(f).$$

   $L\in\mathrm{StarComp}(a;f)\Leftrightarrow\exists y\in a: y R(f)\, L$.

   $L\in\mathrm{StarComp}(\mathrm{StarComp}(a;f);g)\Leftrightarrow\exists p\in\mathrm{StarComp}(a;f): pR(g)\ L\Leftrightarrow\exists p,\ y\in a: (yR(f)\ p\wedge pR(g)\ L)\Leftrightarrow\exists y\in a: y(R(g)\circ R(f))\ L\Leftrightarrow\exists y\in a: (yR(\lambda i\in n: g_i\circ f_i)\ L)\Leftrightarrow L\in\mathrm{StarComp}(a;\lambda i\in n: g_i\circ f_i)$ because $p\in\mathrm{StarComp}(a;f)\Leftrightarrow\exists y\in a: yR(f)\,p$.

2. Obvious.

3. It follows from the proposition above. □

Theorem 135. $\left\langle\prod^{(C)}f\right\rangle\prod a=\prod_{i\in n}\langle f_i\rangle a_i$ for every families $f=f_{i\in n}$ of binary relations and $a=a_{i\in n}$ where a_i is a small set *(for each $i\in n$).

Proof. $L\in\left\langle\prod^{(C)}f\right\rangle\prod a\Leftrightarrow L\in\mathrm{StarComp}(\prod a;f)\Leftrightarrow\exists y\in\prod a\forall i\in n: y_i\, f_i\, L_i\Leftrightarrow\exists y\in\prod a\forall i\in n: \{y\}\not\asymp\langle f_i{}^{-1}\rangle\{L_i\}\Leftrightarrow\forall i\in n\exists y\in a_i: \{y\}\not\asymp\langle f_i{}^{-1}\rangle\{L_i\}\Leftrightarrow\forall i\in n: a_i\not\asymp\langle f_i{}^{-1}\rangle\{L_i\}\Leftrightarrow\forall i\in n: \{L_i\}\not\asymp\langle f_i\rangle a_i\Leftrightarrow\forall i\in n: L_i\in\langle f_i\rangle a_i\Leftrightarrow L\in\prod_{i\in n}\langle f_i\rangle a_i$. □

11.5 Star composition of Rel-morphisms

Define *star composition* for an n-ary anchored relation a and an n-indexed family f of **Rel**-morphisms as an n-ary anchored relation complying with the formulas:

$$\mathrm{Obj}_{\mathrm{StarComp}(a;f)}=\lambda i\in\operatorname{arity}a: \mathrm{Dst}\, f_i;$$
$$\operatorname{arity}\mathrm{StarComp}(a;f)=\operatorname{arity}a;$$
$$L\in\mathrm{GR}\,\mathrm{StarComp}(a;f)\Leftrightarrow L\in\mathrm{StarComp}(\mathrm{GR}\,a;\mathrm{GR}\circ f).$$

(Here I denote $\mathrm{GR}(A;B;f)=f$ for every **Rel**-morphism f.)

Proposition 136. $b\not\asymp\mathrm{StarComp}(a;f)\Leftrightarrow\exists x\in a, y\in b\forall j\in n: x_j\, f_j\, y_j$.

Proof. From the previous section. □

Theorem 137. Relations with above defined compositions form a quasi-invertible category with star-morphisms.

Proof. We need to prove:

1. $\mathrm{StarComp}(\mathrm{StarComp}(m;f);g)=\mathrm{StarComp}(m;\lambda i\in\operatorname{arity}m: g_i\circ f_i)$;
2. $\mathrm{StarComp}(m;\lambda i\in\operatorname{arity}m: \mathrm{id}_{\mathrm{Obj}_m i})=m$;
3. $b\not\asymp\mathrm{StarComp}(a;f)\Leftrightarrow a\not\asymp\mathrm{StarComp}(b;f^{\dagger})$

(the rest is obvious).

It follows from the previous section. □

Theorem 138. Cross-composition product of a family of **Rel**-morphisms is a discrete funcoid.

Proof. By the proposition and symmetry $\prod^{(C)}f$ is a pointfree funcoid. Obviously it is a funcoid $\prod_{i\in n}\mathrm{Src}\, f_i\rightarrow\prod_{i\in n}\mathrm{Dst}\, f_i$. Its completeness (and dually co-completeness) is obvious. □

11.6 Cross-composition product of funcoids

Let a is a an anchored relation of the form $\mathfrak{A}$ and $\operatorname{dom}\mathfrak{A}=n$.

Let every f_i (for all $i\in n$) is a pointfree funcoid with $\mathrm{Src}\, f_i=\mathfrak{A}_i$.

The star-composition of a with f is an anchored relation of the form $\lambda i\in\operatorname{dom}\mathfrak{A}\colon\operatorname{Dst} f_i$ defined by the formula

$$L\in\operatorname{GR}\operatorname{StarComp}(a;f)\Leftrightarrow\exists y\in\operatorname{GR}a\cap\prod_{i\in n}\operatorname{atoms}\mathfrak{A}_i\forall i\in n\colon y_i\,[f_i]\,L_i.$$

Definition 139. I will call a poset *starrish* when $\star a$ is a free star for every element a of this poset.

Theorem 140.

1. If a is a pre-staroid then $\operatorname{StarComp}(a;f)$ is a staroid.
2. If a is a completary staroid and $\operatorname{Dst} f_i$ is a starrish join-semilattice for every $i\in n$ then $\operatorname{StarComp}(a;f)$ is a completary staroid.

Proof.

1. First prove that $\operatorname{StarComp}(a;f)$ is a pre-staroid. We need to prove that $(\operatorname{val} f)_j L$ is a free star, that is $\{X\in(\operatorname{form} f)_j\mid L\cup\{(j;X)\}\in\operatorname{GR}f\}$ is a free star, that is the following is a free star

   $$\{X\in(\operatorname{form} f)_j\mid R(X)\}$$

   where $R(X)=\exists y\in\prod_{i\in n}\operatorname{atoms}\mathfrak{A}_i\colon(\forall i\in n\colon(i\neq j\Rightarrow y_i\,[f_i]\,L_i)\wedge y_j\,[f_i]\,X\wedge y\in a)$.

   $R=\exists y\in\prod_{i\in n}\operatorname{atoms}\mathfrak{A}_i\colon(\forall i\in n\colon(i\neq j\Rightarrow y_i\,[f_i]\,L_i)\wedge y_j\,[f_j]\,X\wedge y_j\in(\operatorname{val})_j(a|_{n\setminus\{j\}}))=$ $\exists y\in\prod_{i\in n\setminus\{j\}}\operatorname{atoms}\mathfrak{A}_i,y'\in\operatorname{atoms}\mathfrak{A}_j\colon(\forall i\in n\colon y_i\,[f_i]\,L_i\wedge y'\,[f_j]\,X\wedge y'\in(\operatorname{val})_j(a|_{n\setminus\{j\}}))=$ $\exists y\in\prod_{i\in n\setminus\{j\}}\operatorname{atoms}\mathfrak{A}_i\forall i\in n\colon y_i\,[f_i]\,L_i\wedge\exists y'\in\operatorname{atoms}\mathfrak{A}_j\colon(y'\,[f_j]\,X\wedge y'\in(\operatorname{val})_j(a|_{n\setminus\{j\}}))$

   If $\exists y\in\prod_{i\in n\setminus\{j\}}\operatorname{atoms}\mathfrak{A}_i\forall i\in n\colon y_i\,[f_i]\,L_i$ is false our statement is obvious. We can assume it is true.

   So it is enough to prove that

   $$\{X\in(\operatorname{form} f)_j\mid\exists y'\in\operatorname{atoms}\mathfrak{A}_j\colon(y'\,[f_j]\,X\wedge y'\in(\operatorname{val})_j(a|_{n\setminus\{j\}}))\}$$

   is a free star. That is

   $$Q=\{X\in(\operatorname{form} f)_j\mid\exists y'\in(\operatorname{atoms}\mathfrak{A}_j)\cap(\operatorname{val})_j(a|_{n\setminus\{j\}})\colon y'\,[f_j]\,X\}$$

   is a free star. $0^{(\operatorname{form} f)_j}\notin Q$ is obvious. That Q is an upper set is obvious. It remains to prove that $X_0\sqcup X_1\in Q\Rightarrow X_0\in Q\vee X_1\in Q$ for every $X_0,X_1\in(\operatorname{form} f)_j$. Let $X_0\sqcup X_1\in Q$. Then there exist $y'\in(\operatorname{atoms}\mathfrak{A}_j)\cap(\operatorname{val})_j(a|_{n\setminus\{j\}})$ such that $y'\,[f_j]\,X_0\sqcup X_1$. Consequently $y'\,[f_j]\,X_0\vee y'\,[f_j]\,X_1$. But then $X_0\in Q\vee X_1\in Q$.

   To finish the proof we need to show that $\operatorname{GR}\operatorname{StarComp}(a;f)$ is an upper set, but this is obvious.

2. Let a is a completary staroid. Let $L_0\sqcup L_1\in\operatorname{GR}\operatorname{StarComp}(a;f)$ that is $\exists y\in\prod_{i\in n}\operatorname{atoms}\mathfrak{A}_i\colon(\forall i\in n\colon y_i\,[f_i]\,L_0\,i\sqcup L_1\,i\wedge y\in a)$ that is $\exists c\in\{0,1\}^n,y\in\prod_{i\in n}\operatorname{atoms}\mathfrak{A}_i\colon(\forall i\in n\colon y_i\,[f_i]\,L_{c(i)}\,i\wedge y\in a)$ (taken into account that $\operatorname{Dst} f_i$ is starrish) that is $\exists c\in\{0,1\}^n\colon(\lambda i\in n\colon L_{c(i)}i)\in\operatorname{GR}\operatorname{StarComp}(a;f)$. So $\operatorname{GR}\operatorname{StarComp}(a;f)$ is a completary staroid. □

Lemma 141. $b\not\asymp^{\operatorname{Anch}(\mathfrak{A})}\operatorname{StarComp}(a;f)\Leftrightarrow\forall A\in a,B\in b,i\in n\colon A_i\,[f_i]\,B_i$ for anchored relations a and b.

Proof.

$$\begin{aligned}
b\not\asymp\operatorname{StarComp}(a;f)&\Leftrightarrow\\
\exists x\in\operatorname{Anch}(\mathfrak{A})\colon(x\sqsubseteq b\wedge x\sqsubseteq\operatorname{StarComp}(a;f))&\Leftrightarrow\\
\exists x\in\operatorname{Anch}(\mathfrak{A})\colon(x\sqsubseteq b\wedge\forall B\in x\colon B\in\operatorname{StarComp}(a;f))&\Leftrightarrow\\
\exists x\in\operatorname{Anch}(\mathfrak{A})\colon\left(x\sqsubseteq b\wedge\forall B\in x\exists A\in\prod_{i\in\operatorname{dom}\mathfrak{A}}\mathfrak{A}_i\colon(\forall i\in n\colon A_i\,[f_i]\,B_i\wedge A\in a)\right)&\Leftrightarrow\\
\exists x\in\operatorname{Anch}(\mathfrak{A})\colon(x\sqsubseteq b\wedge\forall B\in x,A\in a,i\in n\colon A_i\,[f_i]\,B_i)&\Leftrightarrow\\
\forall B\in b,A\in a,i\in n\colon A_i\,[f_i]\,B_i.&
\end{aligned}$$

□

Theorem 142. $a\left[\prod^{(C)} f\right] b \Leftrightarrow \forall A \in a, B \in b, i \in n: A_i\, [f_i]\, B_i$ for anchored relations a and b.

Proof. From the lemma. □

Proposition 143. $b \not\asymp^{\mathsf{pStrd}(\mathfrak{A})} \operatorname{StarComp}(a; f) \Leftrightarrow b \not\asymp^{\mathsf{pStrd}(\mathfrak{B})} \operatorname{StarComp}(a; f)$ for staroids a and b.

Proof. Because $\operatorname{StarComp}(a; f)$ is a staroid. □

Theorem 144. Anchored relations with above defined compositions form a quasi-invertible category with star-morphisms.

Proof. We need to prove:

1. $\operatorname{StarComp}(\operatorname{StarComp}(m; f); g) = \operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: g_i \circ f_i)$;
2. $\operatorname{StarComp}(m; \lambda i \in \operatorname{arity} m: \mathrm{id}_{\mathrm{Obj}_m i}) = m$;
3. $b \not\asymp \operatorname{StarComp}(a; f) \Leftrightarrow a \not\asymp \operatorname{StarComp}(b; f^\dagger)$

(the rest is obvious).

Really,

1. $L \in \mathrm{GR}\operatorname{StarComp}(a; f) \Leftrightarrow \exists y \in \mathrm{GR}\, a \cap \prod_{i\in n} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: y_i\, [f_i]\, L_i$.

   Define the relation $R(f)$ by the formula $x\, R(f)\, y \Leftrightarrow \forall i \in n: x_i\, [f_i]\, y_i$. Obviously

   $$R(\lambda i \in n: g_i \circ f_i) = R(g) \circ R(f).$$

   $L \in \mathrm{GR}\operatorname{StarComp}(a; f) \Leftrightarrow \exists y \in \mathrm{GR}\, a \cap \prod_{i\in n} \operatorname{atoms} \mathfrak{A}_i: y\, R(f)\, L$.

   $L \in \mathrm{GR}\operatorname{StarComp}(\operatorname{StarComp}(a; f); g) \Leftrightarrow \exists p \in \mathrm{GR}\operatorname{StarComp}(a; f) \cap \prod_{i\in n} \operatorname{atoms} \mathfrak{A}_i: p\, R(g)\, L \Leftrightarrow \exists p, y \in \mathrm{GR}\, a \cap \prod_{i\in n} \operatorname{atoms} \mathfrak{A}_i: (y\, R(f)\, p \wedge p\, R(g)\, L) \Leftrightarrow \exists y \in \mathrm{GR}\, a \cap \prod_{i\in n} \operatorname{atoms} \mathfrak{A}_i: y (R(g) \circ R(f))\, L \Leftrightarrow \exists y \in \mathrm{GR}\, a \cap \prod_{i\in n} \operatorname{atoms} \mathfrak{A}_i: y\, R(\lambda i \in n: g_i \circ f_i)\, L \Leftrightarrow \exists y \in \mathrm{GR}\, a \cap \prod_{i\in n} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: y_i\, [g_i \circ f_i]\, L_i \Leftrightarrow L \in \mathrm{GR}\operatorname{StarComp}(a; \lambda i \in n: g_i \circ f_i)$ because $p \in \mathrm{GR}\operatorname{StarComp}(a; f) \Leftrightarrow \exists y \in \mathrm{GR}\, a \cap \prod_{i\in n} \operatorname{atoms} \mathfrak{A}_i: y\, R(f)\, p$.
2. Obvious.
3. It follows from the lemma above. □

Theorem 145. $\left\langle \prod^{(C)} f \right\rangle \prod^{\mathsf{Strd}} a = \prod^{\mathsf{Strd}}_{i\in n} \langle f_i \rangle a_i$ for every families $f = f_{i\in n}$ of pointfree funcoids and $a = a_{i\in n}$ where $a_i \in \operatorname{Src} f_i$, if $\operatorname{Src} f_i$ (for every $i \in n$) is an atomic lattice.

Proof. $L \in \left\langle \prod^{(C)} f \right\rangle \prod^{\mathsf{Strd}} a \Leftrightarrow L \in \operatorname{StarComp}\left(\prod^{\mathsf{Strd}} a; f\right) \Leftrightarrow \exists y \in \prod_{i\in\operatorname{dom}\mathfrak{A}} \operatorname{atoms} \mathfrak{A}_i \forall i \in n: (y_i\, [f_i]\, L_i \wedge y_i \not\asymp a_i) \Leftrightarrow \forall i \in n \exists y \in \operatorname{atoms} \mathfrak{A}_i: (y\, [f_i]\, L_i \wedge y \not\asymp a_i) \Leftrightarrow \forall i \in n: a_i\, [f_i]\, L_i \Leftrightarrow \forall i \in n: L_i \not\asymp \langle f_i \rangle a_i \Leftrightarrow L \in \prod^{\mathsf{Strd}}_{i\in n} \langle f_i \rangle a_i$. □

Theorem 146. For every filters a_0, a_1, b_0, b_1 we have

$$a_0 \times^{\mathsf{FCD}} b_0 \left[f \times^{(C)} g \right] a_1 \times^{\mathsf{FCD}} b_1 \Leftrightarrow a_0 \times^{\mathsf{RLD}} b_0 \left[f \times^{(\mathrm{DP})} g \right] a_1 \times^{\mathsf{RLD}} b_1.$$

Proof. $a_0 \times^{\mathsf{RLD}} b_0 \left[f \times^{(\mathrm{DP})} g \right] a_1 \times^{\mathsf{RLD}} b_1 \Leftrightarrow \forall A_0 \in a_0, B_0 \in b_0, A_1 \in a_1, B_1 \in b_1: A_0 \times B_0 \left[f \times^{(\mathrm{DP})} g \right] A_1 \times B_1$.

$A_0 \times B_0 \left[f \times^{(\mathrm{DP})} g \right] A_1 \times B_1 \Leftrightarrow A_0 \times B_0 \left[f \times^{(C)} g \right] A_1 \times B_1 \Leftrightarrow A_0\, [f]\, A_1 \wedge B_0\, [g]\, B_1$.

Thus it is equivalent to $a_0\, [f]\, a_1 \wedge b_0\, [g]\, b_1$ that is $a_0 \times^{\mathsf{FCD}} b_0 \left[f \times^{(C)} g \right] a_1 \times^{\mathsf{FCD}} b_1$.

(It was used the theorem 142.) □

Can the above theorem be generalized for the infinitary case?

Proposition 147. $\mathrm{GR}\operatorname{StarComp}(a; \lambda i \in n: f_i \sqcup g_i) = \mathrm{GR}\operatorname{StarComp}(a; f) \sqcup^{\mathsf{pFCD}} \mathrm{GR}\operatorname{StarComp}(a; g)$ if f, g are pointfree funcoids and every $\operatorname{Src} f_i = \operatorname{Src} g_i$ and $\operatorname{Dst} f_i = \operatorname{Dst} g_i$ are distributive lattices with least elements, and a is a multifuncoid of the form $\lambda i \in n: \operatorname{Src} f_i$.

Proof. It follows from the theorem ?? in [3]. □

Conjecture 148. $\mathrm{GR}\,\mathrm{StarComp}(a \sqcup^{\mathsf{pFCD}} b; f) = \mathrm{GR}\,\mathrm{StarComp}(a; f) \sqcup^{\mathsf{pFCD}} \mathrm{GR}\,\mathrm{StarComp}(b; f)$ if f is a pointfree funcoid and a, b are multifuncoids of the same form, composable with f.

12 More on cross-composition of funcoids

Lemma 149. Let f is a staroid such that $(\mathrm{form}\, f)_i$ is a boolean lattice for each $i \in \mathrm{arity}\, f$. Let $a \in \prod_{i \in \mathrm{arity}\, f} \mathfrak{F}^{(\mathrm{form}\, f)_i}$.

If $\upuparrows f \sqsubseteq \prod^{\mathsf{Strd}} a$ then $\upuparrows f = \mathrm{StarComp}\left(\upuparrows f; \lambda i \in \mathrm{dom}\, \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right)$.

Proof. Let $\upuparrows f \sqsubseteq \prod^{\mathsf{Strd}} a$. Then $L \in \mathrm{GR} \upuparrows f \Rightarrow L \not\asymp a$.

$L \in \mathrm{GR}\ \mathrm{StarComp}\left(\upuparrows f;\ \lambda i \in \mathrm{dom}\ \mathfrak{A}:\ I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \Leftrightarrow \exists y \in \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \mathrm{atoms}\, \mathfrak{A}_i \forall i \in n$: $y_i \left[I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right] L_i \Leftrightarrow \exists y \in \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \mathrm{atoms}\, \mathfrak{A}_i \forall i \in n$: $(y_i \sqsubseteq L_i \wedge y_i \sqsubseteq a_i) \Leftrightarrow \exists y \in \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \mathrm{atoms}\, \mathfrak{A}_i \forall i \in n$: $(y_i \not\asymp L_i \wedge y_i \not\asymp a_i) \Leftrightarrow \exists y \in \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \mathrm{atoms}\, \mathfrak{A}_i \forall i \in n$: $y_i \not\asymp L_i$ because $\upuparrows f \in \mathrm{GR}\, g \Rightarrow y \not\asymp a$.

If $L \in \upuparrows f$ then there exists $y \in \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \mathrm{atoms}\, \mathfrak{A}_i$ such as $y \sqsubseteq L$ and thus $\forall i \in n: y_i \not\asymp L_i$ (by the theorem 81).

We have $L \in \mathrm{GR}\,\mathrm{StarComp}\left(\upuparrows f; \lambda i \in \mathrm{dom}\, \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \Leftarrow L \in \upuparrows f$ that is $\mathrm{GR}\,\mathrm{StarComp}\left(\upuparrows f; \lambda i \in \mathrm{dom}\, \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \sqsupseteq \upuparrows f$. The other directoin is obvious. □

Theorem 150. Let f is a staroid such that $(\mathrm{form}\, f)_i$ is a boolean lattice for each $i \in \mathrm{arity}\, f$. Let $a \in \prod_{i \in \mathrm{arity}\, f} \mathfrak{F}^{(\mathrm{form}\, f)_i}$. Then

$$\upuparrows f \sqcap^{\mathsf{FCD}(\mathrm{form}\, f)} \prod^{\mathsf{Strd}} a = \mathrm{StarComp}\left(\upuparrows f; \lambda i \in \mathrm{dom}\, \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right).$$

Proof. $h \overset{\mathrm{def}}{=} \mathrm{StarComp}\left(\upuparrows f; \lambda i \in \mathrm{dom}\, \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right)$.

Obviously $h \sqsubseteq \upuparrows f$ and $h \sqsubseteq \prod^{\mathsf{Strd}} a$.

Suppose $g \sqsubseteq \upuparrows f$ and $g \sqsubseteq \prod^{\mathsf{Strd}} a$.

$x \in g \Leftrightarrow x \in \mathrm{StarComp}\left(g; \lambda i \in \mathrm{dom}\, \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \Rightarrow x \in \mathrm{StarComp}\left(f; \lambda i \in \mathrm{dom}\, \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \Leftrightarrow x \in h$ (used the proposition above).

So $g \sqsubseteq h$. □

Corollary 151. Let f is a completary staroid such that $(\mathrm{form}\, f)_i$ is a boolean lattice for each $i \in \mathrm{arity}\, f$. Let $a \in \prod_{i \in \mathrm{arity}\, f} \mathfrak{F}^{(\mathrm{form}\, f)_i}$. Then

$$\upuparrows f \sqcap^{\mathsf{cStrd}(\mathrm{form}\, f)} \prod^{\mathsf{Strd}} a = \mathrm{StarComp}\left(\upuparrows f; \lambda i \in \mathrm{dom}\, \mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right).$$

Proof. Using the theorem 140. □

Theorem 152. Let f is a staroid such that $(\mathrm{form}\, f)_i$ is a boolean lattice for each $i \in \mathrm{arity}\, f$. Let $a \in \prod_{i \in \mathrm{arity}\, f} \mathfrak{F}^{(\mathrm{form}\, f)_i}$. Then $\upuparrows f \not\asymp^{\mathsf{FCD}(\mathrm{form}\, f)} \prod^{\mathsf{Strd}} a \Leftrightarrow a \in \upuparrows f$.

Proof. $\upuparrows f \not\asymp^{\mathsf{FCD}(\mathrm{form}\, f)} \prod^{\mathsf{Strd}} a \Leftrightarrow \upuparrows f \sqcap^{\mathsf{FCD}(\mathrm{form}\, f)} \prod^{\mathsf{Strd}} a \neq 0 \Leftrightarrow \mathrm{StarComp}\left(\upuparrows f;\ \lambda i \in \mathrm{arity}\ f: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right) \neq 0^{\mathsf{FCD}(\mathrm{form}\, f)} \Leftrightarrow \exists L \in \mho^n, y \in \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \mathrm{atoms}\, \mathfrak{A}_i \forall i \in n: y_i \left[I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)} \right] L_i \Leftrightarrow \exists L \in \mho^n$, $y \in \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \mathrm{atoms}\, \mathfrak{A}_i \forall i \in n$: $(y_i \sqsubseteq a_i \wedge y_i \sqsubseteq L_i) \Leftrightarrow \exists y \in \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \mathrm{atoms}\, \mathfrak{A}_i \forall i \in n$: $y_i \sqsubseteq a_i \Leftrightarrow \mathrm{GR} \upuparrows f \cap \prod_{i \in n} \mathrm{atoms}\, a_i \neq \emptyset \Leftrightarrow a \in f$. □

Corollary 153. Let f is a completary staroid such that $(\mathrm{form}\, f)_i$ is a boolean lattice for each $i \in \mathrm{arity}\, f$. Let $a \in \prod_{i \in \mathrm{arity}\, f} \mathfrak{F}^{(\mathrm{form}\, f)_i}$. Then $\upuparrows f \not\asymp^{\mathsf{cStrd}(\mathrm{form}\, f)} \prod^{\mathsf{Strd}} a \Leftrightarrow a \in \upuparrows f$.

Proof. Using the fact that $\upuparrows f\sqcap^{\mathsf{pStrd}(\mathrm{form}\, f)} \prod^{\mathsf{Strd}} a = \mathrm{StarComp}\Big(\upuparrows f; \lambda i\in\mathrm{dom}\,\mathfrak{A}: I_{a_i}^{\mathsf{FCD}(\mathfrak{A}_i)}\Big)$ is a completary staroid (theorem 140). $\square$

Theorem 154. $\prod^{\mathsf{Strd}} a \not\asymp^{\mathsf{pStrd}} \prod^{\mathsf{Strd}} b \Leftrightarrow \prod^{\mathsf{Strd}} a \not\asymp^{\mathsf{cStrd}} \prod^{\mathsf{Strd}} b \Leftrightarrow b \in \prod^{\mathsf{Strd}} a \Leftrightarrow a \in \prod^{\mathsf{Strd}} b \Leftrightarrow a \not\asymp b$ for every indexed families a and b of filters on boolean algebras.

Proof. By corollary 66 we have $\prod^{\mathsf{Strd}} b = \upuparrows f$ for some f. Thus as our filtrator is with separable core we can apply the theorem 152 and its corollary. So $\prod^{\mathsf{Strd}} a \not\asymp^{\mathsf{cStrd}} \prod^{\mathsf{Strd}} b \Leftrightarrow a \in \prod^{\mathsf{Strd}} b$ and $\prod^{\mathsf{Strd}} a \not\asymp^{\mathsf{cStrd}} \prod^{\mathsf{Strd}} b \Leftrightarrow a \in \prod^{\mathsf{Strd}} b$. Similarly $\prod^{\mathsf{Strd}} a \not\asymp^{\mathsf{cStrd}} \prod^{\mathsf{Strd}} b \Leftrightarrow b \in \prod^{\mathsf{Strd}} a$. This by the definition of staroidal product is equivalent to $a \not\asymp b$. We are done. $\square$

13 Multireloids

Definition 155. I will call a *multireloid* of the form $A = A_{i\in n}$, where every each A_i is a set, a pair $(f; A)$ where f is a filter on the set $\prod A$.

Definition 156. I will denote $\mathrm{Obj}(f; A) = A$ and $\mathrm{GR}(f; A) = f$ for every multireloid $(f; A)$.

I will denote $\mathsf{RLD}(A)$ the set of multireloids of the form A.

The multireloid $\uparrow^{\mathsf{RLD}(A)} F$ for a binary relation F is defined by the formulas:

$$\mathrm{Obj}\uparrow^{\mathsf{RLD}(A)} F = A \quad \text{and} \quad \mathrm{GR}\uparrow^{\mathsf{RLD}(A)} F = \uparrow^{\prod A}\mathrm{GR}\, F.$$

Let a is a multireloid of the form A and $\mathrm{dom}\, A = n$.

Let every f_i is a reloid with $\mathrm{Src}\, f_i = A_i$.

The star-composition of a with f is a multireloid of the form $\lambda i\in\mathrm{dom}\, A: \mathrm{Src}\, f_i$ defined by the formulas:

$$\mathrm{arity}\,\mathrm{StarComp}(a; f) = n;$$
$$\mathrm{GR}\,\mathrm{StarComp}(a; f) = \bigsqcap\left\{\uparrow^{\mathsf{RLD}(A)}\mathrm{StarComp}(A; F) \mid \forall A\in a, F\in\prod_{i\in n} f_i\right\};$$
$$\mathrm{Obj}_m\,\mathrm{StarComp}(a; f) = \lambda i\in n: \mathrm{Dst}\, f_i.$$

Theorem 157. Multireloids with above defined compositions form a quasi-invertible category with star-morphisms.

Proof. We need to prove:

1. $\mathrm{StarComp}(\mathrm{StarComp}(m; f); g) = \mathrm{StarComp}(m; \lambda i\in\mathrm{arity}\, m: g_i\circ f_i)$;
2. $\mathrm{StarComp}(m; \lambda i\in\mathrm{arity}\, m: \mathrm{id}_{\mathrm{Obj}_m i}) = m$;
3. $b\not\asymp\mathrm{StarComp}(a; f)\Leftrightarrow a\not\asymp\mathrm{StarComp}(b; f^{\dagger})$

(the rest is obvious).

Really,

1. $\mathrm{StarComp}(\mathrm{StarComp}(A; f); g) = \bigsqcap\left\{\uparrow^{\mathsf{RLD}(A)}\mathrm{StarComp}(B; G) \mid \forall B\in\mathrm{StarComp}(A; f), G\in\prod_{i\in n} g_i\right\} = \bigsqcap\left\{\uparrow^{\mathsf{RLD}(A)}\mathrm{StarComp}(\mathrm{StarComp}(A; F); G) \mid \forall A\in a, F\in\prod_{i\in n} f_i, G\in\prod_{i\in n} g_i\right\} = \bigsqcap\left\{\uparrow^{\mathsf{RLD}(A)}\mathrm{StarComp}(A; G\circ F) \mid \forall A\in a, F\in\prod_{i\in n} f_i, G\in\prod_{i\in n} g_i\right\} = \bigsqcap\left\{\uparrow^{\mathsf{RLD}(A)}\mathrm{StarComp}(A; H) \mid \forall A\in a, H\in\prod_{i\in n}\lambda i\in n: g_i\circ f_i\right\} = \mathrm{StarComp}(a; \lambda i\in n: g_i\circ f_i)$ (used properties of generalized filter bases) [TODO: More detailed proof.]
2. $\mathrm{StarComp}(m; \lambda i\in\mathrm{arity}\, m: \mathrm{id}_{\mathrm{Obj}_m i}) = \bigsqcap\left\{\uparrow^{\mathsf{RLD}(A)}\mathrm{StarComp}(A; \mathrm{id}_X) \mid \forall A\in m, X\in\bigcup_{i\in n}\mathscr{P}\mathrm{Obj}_m i\right\} = \bigsqcap\left\{\uparrow^{\mathsf{RLD}(A)} A \mid \forall A\in a\right\} = m.$
3. Using properties of generalized filter bases,

$b\not\asymp\operatorname{StarComp}(a;f)\Leftrightarrow\forall A\in a,B\in B,F\in\prod_{i\in n}f_i: B\not\asymp\operatorname{StarComp}(A;F)\Leftrightarrow\forall A\in a,B\in B,$ $F\in\prod_{i\in n}f_i: B\not\asymp\left\langle\prod^{(C)}F\right\rangle A\Leftrightarrow\forall A\in a,B\in B,F\in\prod_{i\in n}f_i: A\not\asymp\left\langle\left(\prod^{(C)}F\right)^{-1}\right\rangle B\Leftrightarrow$ $\forall A\in a,B\in B,F\in\prod_{i\in n}f_i: A\not\asymp\operatorname{StarComp}(B;F^\dagger)\Leftrightarrow a\not\asymp\operatorname{StarComp}(b;f^\dagger).$ □

Definition 158. Let f is a multireloid of the form A. Then for $i\in\operatorname{dom}A$

$$\operatorname{Pr}_i^{\mathsf{RLD}}f=\bigsqcap\langle\uparrow^{A_i}\rangle\langle\operatorname{Pr}_i\rangle f.$$

Definition 159. $\prod^{\mathsf{RLD}}\mathcal{X}=\bigsqcap\left\{\uparrow^{\mathsf{RLD}(\lambda i\in\operatorname{dom}\mathcal{X}:\operatorname{Base}(\mathcal{X}_i))}\prod X\mid X\in\mathcal{X}\right\}$ for every indexed family $\mathcal{X}$ of filters on powersets.

Proposition 160. $\operatorname{Pr}_k^{\mathsf{RLD}}\prod^{\mathsf{RLD}}x=x_k$ for every indexed family x of proper filters.

Proof. It follows from $\langle\operatorname{Pr}_k\rangle\left\{\uparrow^{\mathsf{RLD}(\lambda i\in\operatorname{dom}\mathcal{X}:\operatorname{Base}(\mathcal{X}_i))}\prod X\mid X\in x\right\}=\bigsqcap\{X\mid X\in x\}=x$. □

Conjecture 161. $\operatorname{GR}\operatorname{StarComp}(a;\lambda i\in n: f_i\sqcup g_i)=\operatorname{GR}\operatorname{StarComp}(a;f)\sqcup\operatorname{GR}\operatorname{StarComp}(a;g)$ for a multireloid a and indexed families f and g of multireloids where $\operatorname{Src}f_i=\operatorname{Src}g_i$ and $\operatorname{Dst}f_i=\operatorname{Dst}g_i$.

Conjecture 162. $\operatorname{GR}\operatorname{StarComp}(a\sqcup b;f)=\operatorname{GR}\operatorname{StarComp}(a;f)\sqcup\operatorname{GR}\operatorname{StarComp}(b;f)$ if f is a reloid and a, b are multireloids of the same form, composable with f.

Theorem 163. $\prod^{\mathsf{RLD}}A=\bigsqcup\left\{\prod^{\mathsf{RLD}}a\mid a\in\prod_{i\in\operatorname{dom}A}\operatorname{atoms}A_i\right\}$ for every indexed family A of filters on powersets.

Proof. Obviously $\prod^{\mathsf{RLD}}A\sqsupseteq\bigsqcup\left\{\prod^{\mathsf{RLD}}a\mid a\in\prod_{i\in\operatorname{dom}A}\operatorname{atoms}A_i\right\}$.

Reversely, let $K\in\bigsqcup\left\{\prod^{\mathsf{RLD}}a\mid a\in\prod_{i\in\operatorname{dom}A}\operatorname{atoms}A_i\right\}$. Then for every $i\in\operatorname{dom}A$ we have $K\in\prod^{\mathsf{RLD}}a_i$ for every $a_i\in\prod_{j\in\operatorname{dom}A}\operatorname{atoms}A_j$ and so $K\sqsupseteq\prod X_i$ for some $X_i\in\prod_{j\in\operatorname{dom}A}A_j$. Consequently $K\sqsupseteq\bigsqcup_{i\in\operatorname{dom}A}\prod X_i=\bigsqcup_{i\in\operatorname{dom}A}\prod_{j\in\operatorname{dom}A}X_{i,j}=\prod_{j\in\operatorname{dom}A}\bigsqcup_{i\in\operatorname{dom}A}X_{i,j}\sqsupseteq\prod_{j\in\operatorname{dom}A}Z_j$ for some $Z_j\in A_j$. So $K\in\prod^{\mathsf{RLD}}A$. □

Theorem 164. Let a, b be indexed families of filters on powersets of the same form $\mathfrak{A}$. Then

$$\prod^{\mathsf{RLD}}a\sqcap\prod^{\mathsf{RLD}}b=\prod^{\mathsf{RLD}}_{i\in\operatorname{dom}\mathfrak{A}}(a_i\sqcap b_i).$$

Proof.

$$\begin{aligned}\prod^{\mathsf{RLD}}a\sqcap\prod^{\mathsf{RLD}}b &=\\ \left\{\uparrow^{\mathsf{RLD}(\mathfrak{A})}(P\cap Q)\mid P\in\prod^{\mathsf{RLD}}a,Q\in\prod^{\mathsf{RLD}}b\right\} &=\\ \left\{\uparrow^{\mathsf{RLD}(\mathfrak{A})}\left(\prod p\cap\prod q\right)\mid p\in\prod a,q\in\prod b\right\} &=\\ \left\{\uparrow^{\mathsf{RLD}(\mathfrak{A})}\left(\prod_{i\in\operatorname{dom}\mathfrak{A}}(p_i\cap q_i)\right)\mid p\in\prod a,q\in\prod b\right\} &=\\ \left\{\uparrow^{\mathsf{RLD}(\mathfrak{A})}\prod r\mid r\in\prod_{i\in\operatorname{dom}\mathfrak{A}}(a_i\sqcap b_i)\right\} &=\\ \prod^{\mathsf{RLD}}_{i\in\operatorname{dom}\mathfrak{A}}(a_i\sqcap b_i).\end{aligned}$$

□

Theorem 165. If $S\in\mathscr{P}\prod_{i\in\operatorname{dom}\mathfrak{A}}\mathfrak{F}(\mathfrak{A}_i)$ where $\mathfrak{A}$ is an indexed family of sets, then

$$\bigsqcap\left\{\prod^{\mathsf{RLD}}a\mid a\in S\right\}=\prod^{\mathsf{RLD}}_{i\in\operatorname{dom}\mathfrak{A}}\bigsqcap\langle\uparrow^{\mathfrak{F}(\mathfrak{A}_i)}\rangle\operatorname{Pr}_i S.$$

Proof. Special case when S is empty is obvious. Let $S \neq \emptyset$.

$\bigsqcap \langle \uparrow^{\mathfrak{F}(\mathfrak{A}_i)} \rangle \mathrm{Pr}_i S \sqsubseteq \bigsqcap \langle \uparrow^{\mathfrak{F}(\mathfrak{A}_i)} \rangle \{a_i\} = a_i$ for every $a \in S$ because $a_i \in \mathrm{Pr}_i S$. Thus $\prod^{\mathsf{RLD}}_{i \in \operatorname{dom} \mathfrak{A}} \bigsqcap \langle \uparrow^{\mathfrak{F}(\mathfrak{A}_i)} \rangle \mathrm{Pr}_i S \sqsubseteq \prod^{\mathsf{RLD}} a$;

$$\bigsqcap \left\{ \prod^{\mathsf{RLD}} a \mid a \in S \right\} \sqsupseteq \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} \mathfrak{A}} \bigsqcap \langle \uparrow^{\mathfrak{F}(\mathfrak{A}_i)} \rangle \mathrm{Pr}_i S.$$

Now suppose $F \in \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} \mathfrak{A}} \bigsqcap \langle \uparrow^{\mathfrak{F}(\mathfrak{A}_i)} \rangle \mathrm{Pr}_i S$. Then there exist $X \in \left(\lambda i \in \operatorname{dom} \mathfrak{A} \colon \bigsqcap \langle \uparrow^{\mathfrak{F}(\mathfrak{A}_i)} \rangle \mathrm{Pr}_i S \right)$ such that $F \supseteq \prod X$. It is enough to prove that there exist $a \in S$ such that $F \in \prod^{\mathsf{RLD}} a$. For this it is enough $\prod X \in \prod^{\mathsf{RLD}} a$.

Really, $X_i \in \bigsqcap \langle \uparrow^{\mathfrak{F}(\mathfrak{A}_i)} \rangle \mathrm{Pr}_i S$ thus $X_i \in a_i$ for every $a \in S$ because $\mathrm{Pr}_i S \supseteq \{a_i\}$.

Thus $\prod X \in \prod^{\mathsf{RLD}} a$. □

Definition 166. I call a multireloid *convex* iff it is a join of reloidal products.

Conjecture 167. $f \sqsubseteq \prod^{\mathsf{RLD}} a \Leftrightarrow \forall i \in \operatorname{arity} f \colon \mathrm{Pr}^{\mathsf{RLD}}_i f \sqsubseteq a_i$ for every multireloid f and $a_i \in \mathfrak{F}((\operatorname{form} f)_i)$ for every $i \in \operatorname{arity} f$.

14 Subatomic product of funcoids

Lemma 168. $\bigsqcap \langle \uparrow^A \rangle \langle \mathrm{Pr}_i \rangle a = \langle \mathrm{Pr}_i \rangle a$ for every multireloid a and $i \in \operatorname{arity} a$.

Proof. $\bigsqcap \uparrow^A \langle \mathrm{Pr}_i \rangle a \supseteq \langle \mathrm{Pr}_i \rangle a$ is obvious.

$\langle \mathrm{Pr}_i \rangle a$ is a filter base. Really, let $P, Q \in \langle \mathrm{Pr}_i \rangle a$. Then $P = \operatorname{dom} X_0$, $Q = \operatorname{dom} X_1$ where $X_0, X_1 \in a$. Then $P \cap Q = \operatorname{dom} X_0 \cap \operatorname{dom} X_1 \supseteq \operatorname{dom}(X_0 \cap X_1) \in \langle \mathrm{Pr}_i \rangle a$.

Let $K \in \bigsqcap \langle \uparrow^A \rangle \langle \mathrm{Pr}_i \rangle a$. Then by properties of generalized filter bases there exists $X \in a$ such that $K \supseteq \langle \uparrow^A \rangle \langle \mathrm{Pr}_i \rangle X$ that is $K \in \mathrm{Pr}_i X$ and consequently $K \in \langle \mathrm{Pr}_i \rangle a$. □

Definition 169. Let f is an indexed family of funcoids. Then $\prod^{(A)} f$ (*subatomic product*) is a funcoid $\prod_{i \in \operatorname{dom} f} \operatorname{Src} f_i \to \prod_{i \in \operatorname{dom} f} \operatorname{Dst} f_i$ such that for every $a \in \operatorname{atoms} 1^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)}$, $b \in \operatorname{atoms} 1^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Dst} f_i)}$

$$a \left[\prod^{(A)} f \right] b \Leftrightarrow \forall i \in \operatorname{dom} f \colon \mathrm{Pr}_i a \, [f_i] \, \mathrm{Pr}_i b.$$

Proposition 170. The funcoid $\prod^{(A)} f$ exists.

Proof. To prove that $\prod^{(A)} f$ exists we need to prove (for every $a \in \operatorname{atoms} 1^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)}$, $b \in \operatorname{atoms} 1^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Dst} f_i)}$)

$\forall X \in a, Y \in b \exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Dst} f_i)} Y \colon x \left[\prod^{(A)} f \right] y \Rightarrow a \left[\prod^{(A)} f \right] b.$

Let $\forall X \in a, Y \in b \exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Dst} f_i)} Y \colon x \left[\prod^{(A)} f \right] y$.

Then

$\forall X \in a, Y \in b \exists x \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Src} f_i)} X, y \in \operatorname{atoms} \uparrow^{\mathsf{RLD}(\lambda i \in \operatorname{dom} f : \operatorname{Dst} f_i)} Y \forall i \in \operatorname{dom} f \colon \mathrm{Pr}_i x \, [f_i] \, \mathrm{Pr}_i y$.

Then because $\mathrm{Pr}_i x \in \operatorname{atoms} \uparrow^{\operatorname{Src} f_i} \mathrm{Pr}_i X$ and likewise for y:

Then $\forall X \in a, Y \in b \forall i \in \operatorname{dom} f \exists x \in \operatorname{atoms} \uparrow^{\operatorname{Src} f_i} \mathrm{Pr}_i X, y \in \operatorname{atoms} \uparrow^{\operatorname{Dst} f_i} \mathrm{Pr}_i Y \colon x \, [f_i] \, y$.

Thus $\forall X \in a, Y \in b \forall i \in \operatorname{dom} f \colon \uparrow^{\operatorname{Src} f_i} \mathrm{Pr}_i X \, [f_i] \uparrow^{\operatorname{Dst} f_i} \mathrm{Pr}_i Y$;

$\forall X \in a, Y \in b \forall i \in \operatorname{dom} f: \Pr_i X [f_i]^* \Pr_i Y$.
Then $\forall X \in \langle \Pr_i \rangle a, Y \in \langle \Pr_i \rangle b: X [f_i]^* Y$.
Thus by the lemma $\forall X \in \prod \langle \uparrow^{\operatorname{Src} f_i} \rangle \langle \Pr_i \rangle a, Y \in \prod \langle \uparrow^{\operatorname{Dst} f_i} \rangle \langle \Pr_i \rangle b: X [f_i]^* Y$.
$\forall X \in \Pr_i a, Y \in \Pr_i b: X [f_i]^* Y$.
Thus $\Pr_i a [f_i] \Pr_i b$. So $\forall i \in \operatorname{dom} f: \Pr_i a [f_i] \Pr_i b$ and thus $a \left[f \times^{(A)} g \right] b$. □

Remark 171. It seems that the proof of the above theorem can be simplified using cross-composition product.

Theorem 172. $\prod^{(A)}_{i \in n} (g_i \circ f_i) = \prod^{(A)} g \circ \prod^{(A)} f$ for indexed (by an index set n) families f and g of funcoids such that $\forall i \in n: \operatorname{Dst} f_i = \operatorname{Src} g_i$.

Proof. Let a, b are ultrafilters on $\prod_{i \in n} \operatorname{Src} f_i$ and $\prod_{i \in n} \operatorname{Dst} g_i$ correspondingly,
$a \left[\prod^{(A)}_{i \in n} (g_i \circ f_i) \right] b \Leftrightarrow \forall i \in \operatorname{dom} f: \Pr_i a [g_i \circ f_i] \Pr_i b \Leftrightarrow \forall i \in \operatorname{dom} f \exists C \in \operatorname{atoms}^{\mathfrak{F}^{\prod_{i \in n} \operatorname{Dst} f_i}}$: $(\Pr_i a [f_i] C \wedge C [g_i] \Pr_i b) \Leftrightarrow \forall i \in \operatorname{dom} f \exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n: \operatorname{Dst} f)}: (\Pr_i a [f_i] \Pr_i c \wedge \Pr_i c [g_i] \Pr_i b) \Leftarrow \exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n: \operatorname{Dst} f)} \forall i \in \operatorname{dom} f: (\Pr_i a [f_i] \Pr_i c \wedge \Pr_i c [g_i] \Pr_i b) \Leftrightarrow \exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n: \operatorname{Dst} f)}$: $\left(a \left[\prod^{(A)} f \right] c \wedge c \left[\prod^{(A)} g \right] b \right) \Leftrightarrow a \left[\prod^{(A)} g \circ \prod^{(A)} f \right] b$.

Let

$$\forall i \in \operatorname{dom} f \exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n: \operatorname{Dst} f)}: (\Pr_i a [f_i] \Pr_i c \wedge \Pr_i c [g_i] \Pr_i b).$$

Then there exists $c' \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n: \operatorname{Dst} f)}$ such that

$$\forall i \in \operatorname{dom} f: (\Pr_i a [f_i] \Pr_i c'_i \wedge \Pr_i c'_i [g_i] \Pr_i b).$$

Then take $c'' = \prod^{\mathsf{RLD}} c'$. Then $\forall i \in \operatorname{dom} f: (\Pr_i a [f_i] \Pr_i c''_i \wedge \Pr_i c''_i [g_i] \Pr_i b)$. Thus

$$\exists c \in \operatorname{atoms}^{\mathsf{RLD}(\lambda i \in n: \operatorname{Dst} f)} \forall i \in \operatorname{dom} f: (\Pr_i a [f_i] \Pr_i c \wedge \Pr_i c [g_i] \Pr_i b).$$

We have $a \left[\prod^{(A)}_{i \in n} (g_i \circ f_i) \right] b \Leftrightarrow a \left[\prod^{(A)} g \circ \prod^{(A)} f \right] b$. □

Proposition 173. $\prod^{\mathsf{RLD}} a \left[\prod^{(A)} f \right] \prod^{\mathsf{RLD}} b \Leftrightarrow \forall i \in \operatorname{dom} f: a_i [f_i] b_i$ for an indexed family f of funcoids and indexed families a abd b of filters where $a_i \in \mathfrak{F}(\operatorname{Src} f)$, $b_i \in \mathfrak{F}(\operatorname{Dst} f)$ for every $i \in \operatorname{dom} f$.

Proof. $\prod^{\mathsf{RLD}} a \left[\prod^{(A)} f \right] \prod^{\mathsf{RLD}} b \Leftrightarrow \exists x \in \operatorname{atoms} \prod^{\mathsf{RLD}} a, y \in \operatorname{atoms} \prod^{\mathsf{RLD}} b: x \left[\prod^{(A)} f \right] y \Leftrightarrow \exists x \in \operatorname{atoms} \prod^{\mathsf{RLD}} a, y \in \operatorname{atoms} \prod^{\mathsf{RLD}} b \forall i \in \operatorname{dom} f: \Pr_i x [f_i] \Pr_i y \Leftrightarrow \exists x \in \operatorname{atoms} \prod^{\mathsf{RLD}} a$, $y \in \operatorname{atoms} \prod^{\mathsf{RLD}} b \forall i \in \operatorname{dom} f: a_i [f_i] b_i \Leftrightarrow \forall i \in \operatorname{dom} f: a_i [f_i] b_i$. □

15 On products and projections

Conjecture 174. For discrete funcoids $\prod^{(C)}$ and $\prod^{(A)}$ coincide with the conventional product of binary relations.

15.1 Staroidal product

Let f is a staroid components of whose form are boolean lattices.

Definition 175. *Staroidal projection* of a staroid

$$\Pr^{\mathsf{Strd}}_k f = \langle f \rangle_k \left(\lambda i \in (\operatorname{arity} f) \setminus \{k\}: 1^{(\operatorname{form} f)_i} \right).$$

Proposition 176. $\Pr_k \operatorname{GR} \prod^{\mathsf{Strd}} x = \star x_k$.

Proof. $\Pr_k \operatorname{GR} \prod^{\mathsf{Strd}} x = \Pr_k \{L \in \mho^{\operatorname{dom} x} \mid \forall i \in \operatorname{dom} x : x_i \not\asymp L_i\} = ?? = \{l \mid x_k \not\asymp l\} = \star x_k$. □

Proposition 177. $\Pr^{\mathsf{Strd}}_k \prod^{\mathsf{Strd}} x = x_k$ if x is an indexed family of proper filters, and $k \in \operatorname{dom} x$.

Proof. $\Pr^{\mathsf{Strd}}_k \prod^{\mathsf{Strd}} x = \langle \prod^{\mathsf{Strd}} x \rangle_k (\lambda i \in (\operatorname{dom} x) \setminus \{k\} : 1^{(\operatorname{form} x)_i})$.

Thus $\partial \Pr^{\mathsf{Strd}}_k \prod^{\mathsf{Strd}} x = (\operatorname{val} \prod^{\mathsf{Strd}} x)_k (\lambda i \in (\operatorname{dom} x) \setminus \{k\} : 1^{(\operatorname{form} x)_i}) = \{X \in (\operatorname{form} \prod^{\mathsf{Strd}} x)_k \mid (\lambda i \in (\operatorname{dom} x) \setminus \{k\} : 1^{(\operatorname{form} x)_i}) \cup \{(k; X)\} \in \operatorname{GR} \prod^{\mathsf{Strd}} x\} = \{X \in \operatorname{Base} x_k \mid (\forall i \in (\operatorname{dom} x) \setminus \{k\} : 1^{(\operatorname{form} x)_i} \not\asymp x_i) \wedge X \not\asymp x_k\} = \{X \in \operatorname{Base} x_k \mid X \not\asymp x_k\} = \partial x_k$.

Consequently $\Pr^{\mathsf{Strd}}_k \prod^{\mathsf{Strd}} x = x_k$. □

15.2 Cross-composition product of pointfree funcoids

Zero morphisms of the category of pointfree funcoids are ??.

Proposition 178. Values x_i (for every $i \in \operatorname{dom} x$) can be restored from the value of $\prod^{(C)} x$ provided that x is an indexed family of non-zero pointfree funcoids if $\operatorname{Src} f_i$ (for every $i \in n$) is an atomic lattice and every $\operatorname{Dst} f_i$ has greatest element.

Proof. $\left\langle \prod^{(C)} x \right\rangle \prod^{\mathsf{Strd}} p = \prod^{\mathsf{FCD}}_{i \in n} \langle x_i \rangle p_i$ by the theorem 145.

Since $x_i \neq 0$ there exist p such that $\langle x_i \rangle p_i \neq 0$. Take $k \in n$, $p'_i = p_i$ for $i \neq k$ and $p'_k = q$ for an arbitrary value q; then (using the staroidal projections from the previous subsection)

$$\langle x_k \rangle q = \Pr^{\mathsf{Strd}}_k \prod^{\mathsf{FCD}}_{i \in n} \langle x_i \rangle p'_i = \Pr^{\mathsf{Strd}}_k \left\langle \prod^{(C)} x \right\rangle \prod^{\mathsf{Strd}} p'.$$

So the value of x can be restored from $\prod^{(C)} x$ by this formula. □

15.3 Subatomic product

Proposition 179. Values x_i (for every $i \in \operatorname{dom} x$) can be restored from the value of $\prod^{(A)} x$ provided that x is an indexed family of non-zero funcoids.

Proof. Fix $k \in \operatorname{dom} f$. Let for some filters x and y

$$a = \begin{cases} 1^{\mathfrak{F}(\operatorname{Base}(x))} & \text{if } i \neq k; \\ x & \text{if } i = k \end{cases} \quad \text{and} \quad b = \begin{cases} 1^{\mathfrak{F}(\operatorname{Base}(y))} & \text{if } i \neq k; \\ y & \text{if } i = k. \end{cases}$$

Then $a_k [x_k] b_k \Leftrightarrow \forall i \in \operatorname{dom} f : a_i [x_i] b_i \Leftrightarrow \prod^{\mathsf{RLD}} a \left[\prod^{(A)} x \right] \prod^{\mathsf{RLD}} b$. So we have restored x_k from $\prod^{(A)} x$. □

Conjecture 180. For every funcoid $f : \prod A \rightarrow \prod B$ (where A and B are indexed families of sets) there exists a funcoid $\Pr^{(A)}_k f$ defined by the formula

$$x \left[\Pr^{(A)}_k f \right] y \Leftrightarrow \prod^{\mathsf{RLD}} \left(\begin{cases} 1^{\mathfrak{F}(\operatorname{Base}(x))} & \text{if } i \neq k; \\ x & \text{if } i = k \end{cases} \right) [f] \prod^{\mathsf{RLD}} \left(\begin{cases} 1^{\mathfrak{F}(\operatorname{Base}(y))} & \text{if } i \neq k; \\ y & \text{if } i = k. \end{cases} \right)$$

for:

1. every filters x and y;
2. every principal filters x and y;
3. every atomic filters x and y.

15.4 Other

Conjecture 181. Values x_i (for every $i \in \operatorname{dom} x$) can be restored from the value of $\prod^{(C)} x$ provided that x is an indexed family of non-zero reloids.

Conjecture 182. Values x_i (for every $i \in \operatorname{dom} x$) can be restored from the value of $\prod^{(\mathrm{DP})} x$ provided that x is an indexed family of non-zero funcoids.

Definition 183. Let $f \in \mathscr{P}\left(Z^{\coprod Y}\right)$ where Z is a set and Y is a function.

$$\Pr_k^{(D)} f = \Pr_k \{\operatorname{curry} z \mid z \in f\}.$$

Proposition 184. $\Pr_k^{(D)} \prod^{(D)} F = F_k$ for every indexed family F of non-empty relations.

Proof. Obvious. □

Corollary 185. $\operatorname{GR}\Pr_k^{(D)} \prod^{(D)} F = \operatorname{GR} F_k$ and $\operatorname{form}\Pr_k^{(D)} \prod^{(D)} F = \operatorname{form} F_k$ for every indexed family F of non-empty anchored relations.

16 Coordinate-wise continuity

Theorem 186. Let μ and ν are indexed (by some index set n) families of endo-morphisms for a partially ordered dagger category with star-morphisms, and $f_i \in \operatorname{Hom}(\operatorname{Ob}\mu_i; \operatorname{Ob}\nu_i)$ for every $i \in n$. Then:

1. $\forall i \in n: f_i \in \mathrm{C}(\mu_i; \nu_i) \Rightarrow \prod^{(C)} f \in \mathrm{C}\left(\prod^{(C)} \mu; \prod^{(C)} \nu\right)$;
2. $\forall i \in n: f_i \in \mathrm{C}'(\mu_i; \nu_i) \Rightarrow \prod^{(C)} f \in \mathrm{C}'\left(\prod^{(C)} \mu; \prod^{(C)} \nu\right)$;
3. $\forall i \in n: f_i \in \mathrm{C}''(\mu_i; \nu_i) \Rightarrow \prod^{(C)} f \in \mathrm{C}''\left(\prod^{(C)} \mu; \prod^{(C)} \nu\right)$.

Proof. Using the corollary 129:

1. $\forall i \in n: f_i \in \mathrm{C}(\mu_i; \nu_i) \Leftrightarrow \forall i \in n: f_i \circ \mu_i \sqsubseteq \nu_i \circ f_i \Rightarrow \prod^{(C)}_{i \in n} (f_i \circ \mu_i) \sqsubseteq \prod^{(C)}_{i \in n} (\nu_i \circ f_i) \Leftrightarrow \left(\prod^{(C)} f\right) \circ \left(\prod^{(C)} \mu\right) \sqsubseteq \left(\prod^{(C)} \nu\right) \circ \left(\prod^{(C)} f\right) \Leftrightarrow \prod^{(C)} f \in \mathrm{C}\left(\prod^{(C)} \mu; \prod^{(C)} \nu\right)$.
2. $\forall i \in n: f_i \in \mathrm{C}'(\mu_i; \nu_i) \Leftrightarrow \forall i \in n: \mu_i \sqsubseteq f_i^\dagger \circ \nu_i \circ f_i \Rightarrow \prod^{(C)} \mu \sqsubseteq \prod^{(C)}_{i \in n} \left(f_i^\dagger \circ \nu_i \circ f_i\right) \Leftrightarrow \prod^{(C)} \mu \sqsubseteq \left(\prod^{(C)}_{i \in n} f_i^\dagger\right) \circ \left(\prod^{(C)}_{i \in n} \nu_i\right) \circ \left(\prod^{(C)}_{i \in n} f_i\right) \Leftrightarrow \prod^{(C)} \mu \sqsubseteq \left(\prod^{(C)}_{i \in n} f_i\right)^\dagger \circ \left(\prod^{(C)}_{i \in n} \nu_i\right) \circ \left(\prod^{(C)}_{i \in n} f_i\right) \Leftrightarrow \prod^{(C)} f \in \mathrm{C}'\left(\prod^{(C)} \mu; \prod^{(C)} \nu\right)$.
3. $\forall i \in n: f_i \in \mathrm{C}''(\mu_i; \nu_i) \Leftrightarrow \forall i \in n: f_i \circ \mu_i \circ f_i^\dagger \sqsubseteq \nu_i \Rightarrow \prod^{(C)}_{i \in n} \left(f_i \circ \mu_i \circ f_i^\dagger\right) \sqsubseteq \prod^{(C)}_{i \in n} \nu_i \Leftrightarrow \prod^{(C)}_{i \in n} f_i \circ \prod^{(C)}_{i \in n} \mu_i \circ \prod^{(C)}_{i \in n} f_i^\dagger \sqsubseteq \prod^{(C)}_{i \in n} \nu_i \Leftrightarrow \prod^{(C)}_{i \in n} f_i \circ \prod^{(C)}_{i \in n} \mu_i \circ \left(\prod^{(C)}_{i \in n} f_i\right)^\dagger \sqsubseteq \prod^{(C)}_{i \in n} \nu_i \Leftrightarrow \prod^{(C)}_{i \in n} f_i \in \mathrm{C}''\left(\prod^{(C)} \mu; \prod^{(C)} \nu\right)$. □

Theorem 187. Let μ and ν are indexed (by some index set n) families of endo-funcoids, and $f_i \in \mathsf{FCD}(\operatorname{Ob}\mu_i; \operatorname{Ob}\nu_i)$ for every $i \in n$. Then:

1. $\forall i \in n: f_i \in \mathrm{C}(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}\left(\prod^{(A)} \mu; \prod^{(A)} \nu\right)$;
2. $\forall i \in n: f_i \in \mathrm{C}'(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}'\left(\prod^{(A)} \mu; \prod^{(A)} \nu\right)$;
3. $\forall i \in n: f_i \in \mathrm{C}''(\mu_i; \nu_i) \Rightarrow \prod^{(A)} f \in \mathrm{C}''\left(\prod^{(A)} \mu; \prod^{(A)} \nu\right)$.

Proof. Similar to the previous theorem. □

17 Counter-examples

Example 188. $\uparrow\uparrow\downarrow\downarrow f\neq f$ for some staroid f whose form is a family of filters on a set.

Proof. Let $\operatorname{GR} f=\{\mathcal{A}\in\mathfrak{F}(\mho)\mid\uparrow^{\mho}\operatorname{Cor}\mathcal{A}\not\asymp\Delta\}$ for some infinite set $\mho$ where Δ is some non-principal f.o. on $\mho$.

$A\sqcup B\in\operatorname{GR} f\Leftrightarrow\uparrow^{\mho}\operatorname{Cor}(A\sqcup B)\not\asymp\Delta\Leftrightarrow\uparrow^{\mho}\operatorname{Cor}A\sqcup\uparrow^{\mho}\operatorname{Cor}B\not\asymp\Delta\Leftrightarrow(\operatorname{Cor}A\sqcup\operatorname{Cor}B)\sqcap\Delta\neq 0^{\mathfrak{F}(\mho)}\Leftrightarrow\uparrow^{\mho}\operatorname{Cor}A\sqcap\Delta\neq 0^{\mathfrak{F}(\mho)}\vee\uparrow^{\operatorname{Base}(B)}\operatorname{Cor}B\sqcap\Delta\neq 0^{\mathfrak{F}(\mho)}\Leftrightarrow A\in f\vee B\in f$.

Obviously $0^{\mathcal{F}(\mho)}\notin\operatorname{GR} f$. So f is a free star. But free stars ere essentially the same as 1-staroids.

$\operatorname{GR}\downarrow\downarrow f=\partial\Delta$. $\operatorname{GR}\uparrow\uparrow\downarrow\downarrow f=\star\Delta\neq f$. □

For the below counter-examples we will define a staroid ϑ with $\operatorname{arity}\vartheta=\mathbb{N}$ and $\operatorname{GR}\vartheta\in\mathscr{P}(\mathbb{N}^{\mathbb{N}})$ (based on a suggestion by Andreas Blass):

$$A\in\operatorname{GR}\vartheta\Leftrightarrow\sup_{i\in\mathbb{N}}\operatorname{card}(A_i\cap i)=\mathbb{N}\wedge\forall i\in\mathbb{N}\colon A_i\neq\emptyset.$$

Proposition 189. ϑ is a staroid.

Proof. $(\operatorname{val}\vartheta)_i L=\mathscr{P}\mathbb{N}\setminus\{\emptyset\}$ for every $L\in(\mathscr{P}\mathbb{N})^{\mathbb{N}\setminus\{i\}}$ if $\forall i\in\mathbb{N}\colon L_i\neq\emptyset$. Otherwise $(\operatorname{val}\vartheta)_i L=\emptyset$. Thus $(\operatorname{val}\vartheta)_i L$ is a free star. So ϑ is a staroid. □

Proposition 190. ϑ is a completary staroid.

Proof. $A_0\sqcup A_1\in\operatorname{GR}\vartheta\Leftrightarrow A_0\cup A_1\in\operatorname{GR}\vartheta\Leftrightarrow\sup_{i\in\mathbb{N}}\operatorname{card}((A_0 i\cup A_1 i)\cap i)=\mathbb{N}\wedge\forall i\in\mathbb{N}\colon A_0 i\cup A_1 i\neq\emptyset\Leftrightarrow\sup_{i\in\mathbb{N}}\operatorname{card}((A_0 i\cap i)\cup(A_1 i\cap i))=\mathbb{N}\wedge\forall i\in\mathbb{N}\colon A_0 i\cup A_1 i\neq\emptyset$.

If $A_0 i=\emptyset$ then $A_0 i\cap i=\emptyset$ and thus $A_1 i\cap i\sqsupseteq A_0 i\cap i$. Thus we can select $c(i)=1$ in such a way that $\forall d\in\{0,1\}\colon\operatorname{card}(A_{c(i)}\cap i)\sqsupseteq\operatorname{card}(A_d\cap i)$ and $A_{c(i)} i\neq\emptyset$. (Consider the case $A_0 i, A_1 i\neq\emptyset$ and the similar cases $A_0 i=\emptyset$ and $A_1 i=\emptyset$.)

So $A_0\sqcup A_1\in f\Leftrightarrow\sup_{i\in\mathbb{N}}\operatorname{card}(A_{c(i)} i\cap i)=\mathbb{N}\wedge A_{c(i)} i\neq\emptyset\Leftrightarrow(\lambda i\in n\colon A_{c(i)} i)\in\vartheta$.

Thus ϑ is completary. □

Obvious 191. ϑ is non-zero.

Example 192. For every family $a=a_{i\in\mathbb{N}}$ of atomic f.o. $\prod a$ is not an atom nor of the poset of staroids neither of the poset of completary staroids of the form $\lambda i\in\mathbb{N}\colon\operatorname{Base}(a_i)$.

Proof. It's enough to prove $\vartheta\not\sqsupseteq\prod a$.

Let $\uparrow^{\mathbb{N}} R_i=a_i$ is a_i is principal and $R_i=\mathbb{N}\setminus i$ if a_i is non-principal.

We have $\forall i\in\mathbb{N}\colon R_i\in a_i$.

We have $R\notin\vartheta$ because $\sup_{i\in\mathbb{N}}\operatorname{card}(R_i\cap i)=0$.

$R\in\prod a$ because $\forall X\in a_i\colon X\cap R_i\neq\emptyset$.

So $\vartheta\not\sqsupseteq\prod a$. □

Remark 193. At http://mathoverflow.net/questions/60925/special-infinitary-relations-and-ultrafilters there are a proof for arbitary infinite form, not just for $\mathbb{N}$.

Conjecture 194. There exists a non-completary staroid.

Conjecture 195. There exists a pre-staroid which is not a staroid.

Conjecture 196. The set of staroids of the form A^B where A and B are sets is atomic.

Conjecture 197. The set of staroids of the form A^B where A and B are sets is atomistic.

Conjecture 198. The set of completary staroids of the form A^B where A and B are sets is atomic.

Conjecture 199. The set of completary staroids of the form A^B where A and B are sets is atomistic.

18 Conjectures

Remark 200. Below I present special cases of possible theorems. The theorems may be generalized after the below special cases are proved.

Conjecture 201. For every two a. funcoids; b. of reloids f and g we have:

1. $(\mathsf{RLD})_{\mathrm{in}} a \left[f \times^{(\mathrm{DP})} g \right] (\mathsf{RLD})_{\mathrm{in}} b \Leftrightarrow a \left[f \times^{(C)} g \right] b$ for every funcoids $a \in \mathsf{FCD}(\mathrm{Src}\, f; \mathrm{Src}\, g)$, $b \in \mathsf{FCD}(\mathrm{Dst}\, f; \mathrm{Dst}\, g)$;
2. $(\mathsf{RLD})_{\mathrm{out}} a \left[f \times^{(\mathrm{DP})} g \right] (\mathsf{RLD})_{\mathrm{out}} b \Leftrightarrow a \left[f \times^{(C)} g \right] b$ for every funcoids $a \in \mathsf{FCD}(\mathrm{Src}\, f; \mathrm{Src}\, g)$, $b \in \mathsf{FCD}(\mathrm{Dst}\, f; \mathrm{Dst}\, g)$;
3. $(\mathsf{FCD}) a \left[f \times^{(C)} g \right] (\mathsf{FCD}) b \Leftrightarrow a \left[f \times^{(\mathrm{DP})} g \right] b$ for every reloids $a \in \mathsf{RLD}(\mathrm{Src}\, f; \mathrm{Src}\, g)$, $b \in \mathsf{RLD}(\mathrm{Dst}\, f; \mathrm{Dst}\, g)$.

Definition 202. A *staroid on power sets* is such a staroid f that every $(\mathrm{form}\, f)_i$ is a lattice of all subsets of some set.

Conjecture 203. $\prod^{\mathsf{Strd}} a \not\asymp \prod^{\mathsf{Strd}} b \Leftrightarrow b \in \prod^{\mathsf{Strd}} a \Leftrightarrow a \in \prod^{\mathsf{Strd}} b \Leftrightarrow a \not\asymp b$ for every indexed families a and b of filters on powersets of some sets.

Conjecture 204. Let f is a staroid on powersets and $a \in \prod_{i \in \mathrm{arity}\, f} \mathrm{Src}\, f_i$, $b \in \prod_{i \in \mathrm{arity}\, f} \mathrm{Dst}\, f_i$. Then

$$\prod^{\mathsf{Strd}} a \left[\prod^{(C)} f \right] \prod^{\mathsf{Strd}} b \Leftrightarrow \forall i \in n : a_i \left[f_i \right] b_i .$$

Proposition 205. The conjecture 203 is a consequence of the conjecture 177.

Proof. Applying the definition of staroidal product and the theorem 177 we get:

$$\prod^{\mathsf{Strd}} a \not\asymp \prod^{\mathsf{Strd}} b \Leftrightarrow (\text{theorem } 177) \Leftrightarrow b \in \prod^{\mathsf{Strd}} a \Leftrightarrow a \not\asymp b.$$

Similarly $\prod^{\mathsf{Strd}} a \not\asymp \prod^{\mathsf{Strd}} b \Leftrightarrow a \in \prod^{\mathsf{Strd}} b$. □

Proposition 206. The conjecture 204 is a consequence of the conjecture 203.

Proof. $\prod^{\mathsf{Strd}} a \left[\prod^{(C)} f \right] \prod^{\mathsf{Strd}} b \Leftrightarrow \prod^{\mathsf{Strd}} b \not\asymp \left\langle \prod^{(C)} f \right\rangle \prod^{\mathsf{Strd}} a \Leftrightarrow \prod^{\mathsf{Strd}} b \not\asymp \prod^{\mathsf{Strd}}_{i \in n} \langle f_i \rangle a_i \Leftrightarrow \forall i \in n : b_i \not\asymp \langle f_i \rangle a_i \Leftrightarrow \forall i \in n : a_i \left[f_i \right] b_i$. □

Conjecture 207. For every indexed families a and b of filters and an indexed family f of pointfree funcoids we have

$$\prod^{\mathsf{Strd}} a \left[\prod^{(C)} f \right] \prod^{\mathsf{Strd}} b \Leftrightarrow \prod^{\mathsf{RLD}} a \left[\prod^{(\mathrm{DP})} f \right] \prod^{\mathsf{RLD}} b.$$

Conjecture 208. Displaced product of funcoids is a quasi-cartesian functions. (Consider also a similar conjecture for reloids.)

Strenghtening of an above result:

Conjecture 209. If a is a completary staroid and $\mathrm{Dst}\, f_i$ is a starrish poset for every $i \in n$ then $\mathrm{StarComp}(a; f)$ is a completary staroid.

Strenghtenings of above results:

Conjecture 210.

1. $\prod^{(D)} F$ is a pre-staroid if every F_i is a pre-staroid.

2. $\prod^{(D)} F$ is a completary staroid if every F_i is a completary staroid.

Conjecture 211. If f_1 and f_2 are funcoids, then there exists a pointfree funcoid $f_1 \times f_2$ such that

$$\langle f_1 \times f_2 \rangle x = \bigsqcup \{ \langle f_1 \rangle X \times^{\mathsf{FCD}} \langle f_2 \rangle X \mid X \in \operatorname{atoms} x \}$$

for every ultrafilter x.

18.1 Informal questions

Are the above defined products categorical direct products for some category?

Do products of funcoids and reloids coincide with Tychonov topology?

Limit and generalized limit for multiple arguments.

Is product of connected spaces connected?

Product of T_0-separable is T_0, of T_1 is T_1?

Relationships between multireloids and staroids.

Generalize the section "Specifying funcoids by functions or relations on atomic filter objects" from [3].

Generalize "Relationships between funcoids and reloids" in [1].

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[1] Victor Porton. Funcoids and reloids. At `http://www.mathematics21.org/binaries/funcoids-reloids.pdf`.

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[4] Victor Porton. Filters on posets and generalizations. *International Journal of Pure and Applied Mathematics*, 74(1):55–119, 2012.