

# Convergence of funcoids\*

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## Abstract

Considered convergence and limit for funcoids (a generalization of proximity spaces).

I also have defined (generalized) limit for arbitrary (not necessarily continuous) functions under certain conditions.

This article is a part of my Algebraic General Topology research.

**Keywords:** limit, convergence, discontinuous analysis, nonsmooth analysis, non smooth analysis

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## 1 Draft status

This is a partial draft.

## 2 Common

See [2] for the definition of funcoid.

## 3 Convergence

**Definition 1.** A filter object  $\mathcal{F}$  converges to a filter object  $\mathcal{A}$  regarding a funcoid  $\mu$  ( $\mathcal{F} \xrightarrow{\mu} \mathcal{A}$ ) iff  $\mathcal{F} \subseteq \langle \mu \rangle \mathcal{A}$ .<sup>1</sup>

**Definition 2.** A funcoid  $f$  converges to a filter object  $\mathcal{A}$  regarding a funcoid  $\mu$  ( $f \xrightarrow{\mu} \mathcal{A}$ ) iff  $\text{im } f \subseteq \langle \mu \rangle \mathcal{A}$  that is iff  $\text{im } f \xrightarrow{\mu} \mathcal{A}$ .

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1. This generalizes the standard definition of filter convergent to a point or to a set.

**Definition 3.** A funcoïd  $f$  converges to a filter object  $\mathcal{A}$  on a filter object  $\mathcal{B}$  regarding a funcoïd  $\mu$  iff  $f|_{\mathcal{B}} \xrightarrow{\mu} \mathcal{A}$ .

**Remark 4.** We can define also convergence for a reloid  $f: f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow \text{im } f \subseteq \langle \mu \rangle \mathcal{A}$  or what is the same  $f \xrightarrow{\mu} \mathcal{A} \Leftrightarrow (\text{FCD}) f \xrightarrow{\mu} \mathcal{A}$ .

**Theorem 5.** Let  $f, g, \mu, \nu$  are funcoïds,  $\mathcal{A}$  is a filter object. If  $f \xrightarrow{\mu} \mathcal{A}$ ,

$$g|_{\langle \mu \rangle \mathcal{A}} \in \text{C}(\mu \cap (\langle \mu \rangle \mathcal{A})^2; \nu)$$

and  $\langle \mu \rangle \mathcal{A} \supseteq \mathcal{A}$  then  $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$ .

**Proof.**  $\text{im } f \subseteq \langle \mu \rangle \mathcal{A}$ ;  $\langle g \rangle \text{im } f \subseteq \langle g \rangle \langle \mu \rangle \mathcal{A}$ ;  $\text{im}(g \circ f) \subseteq \langle g|_{\langle \mu \rangle \mathcal{A}} \rangle \langle \mu \rangle \mathcal{A}$ ;  $\text{im}(g \circ f) \subseteq \langle g|_{\langle \mu \rangle \mathcal{A}} \rangle \langle \mu \cap (\langle \mu \rangle \mathcal{A})^2 \rangle \mathcal{A}$ ;  $\text{im}(g \circ f) \subseteq \langle \nu \circ g|_{\langle \mu \rangle \mathcal{A}} \rangle \mathcal{A}$ ;  $\text{im}(g \circ f) \subseteq \langle \nu \circ g \rangle \mathcal{A}$ ;  $\text{im}(g \circ f) \subseteq \langle \nu \rangle \langle g \rangle \mathcal{A}$ ;  $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$ .  $\square$

**Corollary 6.** Let  $f, g, \mu, \nu$  are funcoïds,  $\mathcal{A}$  is a filter object. If  $f \xrightarrow{\mu} \mathcal{A}$ ,  $g \in \text{C}(\mu; \nu)$  and  $\langle \mu \rangle \mathcal{A} \supseteq \mathcal{A}$  then  $g \circ f \xrightarrow{\nu} \langle g \rangle \mathcal{A}$ .

**Proof.** From the last theorem and a theorem in [3].  $\square$

The following is the theorem about convergence of a continuous funcoïd:

**Theorem 7.** If  $f \in \text{C}(\mu; \nu)$  then  $f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A}$  (for any funcoïds  $\mu$  and  $\nu$  and a filter object  $\mathcal{A}$ ).

**Proof.**  $f|_{\langle \mu \rangle \mathcal{A}} \xrightarrow{\nu} \langle f \rangle \mathcal{A} \Leftrightarrow \text{im } f|_{\langle \mu \rangle \mathcal{A}} \subseteq \langle \nu \rangle \langle f \rangle \mathcal{A} \Leftrightarrow \langle f \rangle \langle \mu \rangle \mathcal{A} \subseteq \langle \nu \rangle \langle f \rangle \mathcal{A} \Leftrightarrow \langle f \circ \mu \rangle \mathcal{A} \subseteq \langle \nu \circ f \rangle \mathcal{A} \Leftrightarrow f \circ \mu \subseteq \nu \circ f \Leftrightarrow f \in \text{C}(\mu; \nu)$ .  $\square$

## 4 Limit

**Definition 8.**  $\lim^{\mu} f = a$  iff  $f \xrightarrow{\mu} \{a\}$  for a  $T_2$ -separable funcoïd  $\mu$  and a non-empty funcoïd  $f$ .

It is defined correctly, that is  $f$  has no more than one limit.

**Proof.** Let  $\lim^{\mu} f = a$  and  $\lim^{\mu} f = b$ . Then  $\text{im } f \subseteq \langle \mu \rangle \{a\}$  and  $\text{im } f \subseteq \langle \mu \rangle \{b\}$

Because  $f \neq \emptyset$  we have  $\text{im } f \neq \emptyset$ ;  $\langle \mu \rangle \{a\} \cap \langle \mu \rangle \{b\} \neq \emptyset$ ;  $\{b\} \cap \langle \mu^{-1} \rangle \langle \mu \rangle \{a\} \neq \emptyset$ ;  $\{b\} \cap \langle \mu^{-1} \circ \mu \rangle \{a\} \neq \emptyset$ ;  $\{a\}[\mu^{-1} \circ \mu] \{b\}$ . Because  $\mu$  is  $T_2$ -separable we have  $a = b$ .  $\square$

**Definition 9.**  $\lim_{\mathcal{B}}^{\mu} f = \lim^{\mu} (f|_{\mathcal{B}})$ .

**Remark 10.** We can also in an obvious way define limit of a reloid.

## 5 Generalized limit

### 5.1 The definition

Let  $\mu$  and  $\nu$  are funcoïds [2],  $G$  is a group of functions.

Let  $D$  is a set such that  $\forall r \in G; \text{im } r \subseteq D \wedge \forall x, y \in D \exists r \in G: r(x) = y$ .

We require that  $\mu$  and any  $r \in G$  commute, that is  $\mu \circ r = r \circ \mu$ .

We require for every  $y \in \mathcal{U}$

$$\nu \supseteq \langle \nu \rangle \{y\} \times^{\text{FCD}} \langle \nu \rangle \{y\} \quad (1)$$

**Remark 11.** The formula (1) usually works if  $\nu$  is a proximity. It does not work if  $\mu$  is a pretopology or preclosure.

We are going to consider (generalized) limits of arbitrary functions acting from  $\mu$  to  $\nu$ . (The functions in consideration are not required to be continuous.)

**Remark 12.** Most typically  $G$  is the group of translations of some topological vector space.

Generalized limit is defined by the following formula:

**Definition 13.**  $\text{xlim } f \stackrel{\text{def}}{=} \{\nu \circ f \circ r \mid r \in G\}$  for any funcoid  $f$ .

**Remark 14.** Generalized limit technically is a set of funcoids (see [2]).

We will assume that the function  $f$  is defined on  $\langle \mu \rangle \{x\}$ .

**Definition 15.**  $\text{xlim}_x f \stackrel{\text{def}}{=} \lim f|_{\langle \mu \rangle \{x\}}$ .

**Obvious 16.**  $\text{xlim}_x f = \{\nu \circ f|_{\langle \mu \rangle \{x\}} \circ r \mid r \in G\}$ .

**Remark 17.**  $\text{xlim}_x f$  is the same for funcoids  $\mu$  and  $\text{Compl } \mu$ .

**Lemma 18.**  $(\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \circ f = \langle f^{-1} \rangle \mathcal{A} \times^{\text{FCD}} \mathcal{B}$  for every  $f \in \text{FCD}$ ,  $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$ .

**Proof.** For every filter object  $\mathcal{X}$

$$\begin{aligned} \langle (\mathcal{A} \times^{\text{FCD}} \mathcal{B}) \circ f \rangle \mathcal{X} &= \\ \langle \langle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle \langle f \rangle \rangle \mathcal{X} &= \\ \begin{cases} \mathcal{B} & \text{if } \langle f \rangle \mathcal{X} \cap \mathcal{A} \neq \emptyset; \\ \emptyset & \text{if } \langle f \rangle \mathcal{X} \cap \mathcal{A} = \emptyset; \end{cases} &= \\ \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \cap \langle f^{-1} \rangle \mathcal{A} \neq \emptyset; \\ \emptyset & \text{if } \mathcal{X} \cap \langle f^{-1} \rangle \mathcal{A} = \emptyset; \end{cases} &= \\ \langle \langle f^{-1} \rangle \mathcal{A} \times^{\text{FCD}} \mathcal{B} \rangle \mathcal{X}. & \end{aligned}$$

□

The function  $\tau$  will define an injection from the set of points of the space  $\nu$  (“numbers”, “points”, or “vectors”) to the set of all (generalized) limits (i.e. values which  $\text{xlim}_x f$  may take).

**Definition 19.**  $\tau(y) \stackrel{\text{def}}{=} \{\langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\} \mid x \in D\}$ .

**Proposition 20.**  $\tau(y) \stackrel{\text{def}}{=} \{(\langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\}) \circ r \mid r \in G\}$  for every  $x \in D$ .

**Proof.**  $(\langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\}) \circ r = \langle r^{-1} \rangle \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\} = \langle \mu \rangle \langle r^{-1} \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\} = \langle \mu \rangle \{r^{-1}x\} \times^{\text{FCD}} \langle \nu \rangle \{y\} \in \{\langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\} \mid x \in D\}$  where  $x' \in D$ .

Reversely  $\langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\} = (\langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\}) \circ e$  where  $e$  is the identify element of  $G$ . □

**Theorem 21.** If  $f|_{\langle \mu \rangle \{x\}} \in \text{C}(\mu; \nu)$  and  $\langle \mu \rangle \{x\} \supseteq \{x\}$  then  $\text{xlim}_x f = \tau(fx)$ .

**Proof.**  $f|_{\langle \mu \rangle \{x\}} \circ \mu \subseteq \nu \circ f|_{\langle \mu \rangle \{x\}} \subseteq \nu \circ f$ ; thus  $\langle f \rangle \langle \mu \rangle \{x\} \subseteq \langle \nu \rangle \langle f \rangle \{x\}$ ; consequently we have

$$\nu \supseteq \langle \nu \rangle \langle f \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \{x\} \supseteq \langle f \rangle \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \{x\}.$$

$$\begin{aligned} \nu \circ f|_{\langle \mu \rangle \{x\}} &\supseteq \\ (\langle f \rangle \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \{x\}) \circ f|_{\langle \mu \rangle \{x\}} &= \\ \langle (f|_{\langle \mu \rangle \{x\}})^{-1} \rangle \langle f \rangle \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \{x\} &\supseteq \\ \text{dom } f|_{\langle \mu \rangle \{x\}} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \{x\} &= \\ \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \{x\}. & \end{aligned}$$

$$\text{im}(\nu \circ f|_{\langle \mu \rangle \{x\}}) = \langle \nu \rangle \langle f \rangle \langle \mu \rangle \{x\};$$

$$\begin{aligned} & \nu \circ f|_{\langle \mu \rangle \{x\}} \subseteq \\ & \langle \mu \rangle \{x\} \times^{\text{FCD}} \text{im}(\nu \circ f|_{\langle \mu \rangle \{x\}}) = \\ & \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \langle \mu \rangle \{x\} \subseteq \\ & \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \{x\}. \end{aligned}$$

So  $\nu \circ f|_{\langle \mu \rangle \{x\}} = \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \{x\}$ .

Thus  $\text{xlim}_x f = \{(\langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle f \rangle \{x\}) \circ r \mid r \in G\} = \tau(fx)$ .  $\square$

**Remark 22.** Without the requirement of  $\langle \mu \rangle \{x\} \supseteq \{x\}$  the last theorem would not work in the case of removable singularity.

**Theorem 23.** Let  $\nu \subseteq \nu \circ \nu$ . If  $f|_{\langle \mu \rangle \{x\}} \xrightarrow{\nu} \{y\}$  then  $\text{xlim}_x f = \tau(y)$ .

**Proof.**  $\text{im} f|_{\langle \mu \rangle \{x\}} \subseteq \langle \nu \rangle \{y\}$ ;  $\langle f \rangle \langle \mu \rangle \{x\} \subseteq \langle \nu \rangle \{y\}$ ;

$$\begin{aligned} & \nu \circ f|_{\langle \mu \rangle \{x\}} \supseteq \\ & (\langle \nu \rangle \{y\} \times^{\text{FCD}} \langle \nu \rangle \{y\}) \circ f|_{\langle \mu \rangle \{x\}} = \\ & \langle (f|_{\langle \mu \rangle \{x\}})^{-1} \rangle \langle \nu \rangle \{y\} \times^{\text{FCD}} \langle \nu \rangle \{y\} = \\ & \langle I_{\langle \mu \rangle \{x\}} \circ f^{-1} \rangle \langle \nu \rangle \{y\} \times^{\text{FCD}} \langle \nu \rangle \{y\} \supseteq \\ & \langle I_{\langle \mu \rangle \{x\}} \circ f^{-1} \rangle \langle f \rangle \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\} = \\ & \langle I_{\langle \mu \rangle \{x\}} \rangle \langle f^{-1} \circ f \rangle \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\} \supseteq \\ & \langle I_{\langle \mu \rangle \{x\}} \rangle \langle I_{\langle \mu \rangle \{x\}} \rangle \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\} = \\ & \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\}. \end{aligned}$$

On the other hand,  $f|_{\langle \mu \rangle \{x\}} \subseteq \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\}$ ;

$$\nu \circ f|_{\langle \mu \rangle \{x\}} \subseteq \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \langle \nu \rangle \{y\} \subseteq \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\}.$$

So  $\nu \circ f|_{\langle \mu \rangle \{x\}} = \langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\}$ .

$$\text{xlim}_x f = \{\nu \circ f|_{\langle \mu \rangle \{x\}} \circ r \mid r \in G\} \supseteq \{(\langle \mu \rangle \{x\} \times^{\text{FCD}} \langle \nu \rangle \{y\}) \circ r \mid r \in G\} = \tau(y). \quad \square$$

**Corollary 24.** If  $\lim_{\langle \mu \rangle \{x\}}^{\nu} f = y$  then  $\text{xlim}_x f = \tau(y)$ .

We have bijective  $\tau$  if  $\langle \nu \rangle \{y_1\} \cap^{\tilde{\delta}} \langle \nu \rangle \{y_2\} \neq \emptyset$  that is if  $\nu$  is  $T_1$ -separable.

## 5.2 Generalized limits as a generalization of limits

When  $\tau$  is bijective, using the procedure described in appending B in [1] we can equate points of the space with certain generalized limits.

Thus we can use only “lim” to denote all kinds of limits and eliminate “xlim” notation.

## 5.3 Yet to do

We need to study generalized limits of composition of functions.

We should introduce  $n$ -ary functions extended on values which generalized limits take, so that we could be able for example to add two generalized limits.

We should study differential equations generalized for the derivative of non-smooth functions.

Need examples with metric spaces.

## Bibliography

- [1] Victor Porton. Filters on posets and generalizations. At <http://www.mathematics21.org/binaries/filters.pdf>.
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