This document contains a list of short ideas of future research in Algebraic General Topology.

I have created branch `devel` in the LaTeXrepository for the book to add new “draft” features there. The `devel` branch isn't distributed by me in PDF format, but you can download and compile it yourself.

This research plan is not formal and may contain vague statements.

Should clearly denote $\mathsf{pFCD}(\mathfrak{A}; \mathfrak{Z})$ or $\mathsf{pFCD}(\mathfrak{A})$.

1. Category theory

Attempting to extend Tychonoff product from topologies to funcoids: —— If i has left adjoint: If r is left adjoint to i, we have $\operatorname{Hom}(A, i(X \times Y)) = \operatorname{Hom}(r(A), X \times Y) = \operatorname{Hom}(r(A), X) \times \operatorname{Hom}(r(A), Y) = \operatorname{Hom}(A, i(X)) \times \operatorname{Hom}(A, i(Y))$. —— If also the left adjoint is full and faithful: $\operatorname{Hom}(A, i(r(X) \times r(Y))) = \operatorname{Hom}(r(A), r(X) \times r(Y)) = \operatorname{Hom}(r(A), r(X)) \times \operatorname{Hom}(r(A), r(Y)) = \operatorname{Hom}(A, X) \times \operatorname{Hom}(A, Y)$. See also http://math.stackexchange.com/q/1982931/4876. However this does not apply because reflection of topologies in funcoids is not full.

Being intersecting is defined for posets (= thin categories). It seems that this can be generalized for any categories. This way we can define (pointfree) funcoids between categories generalizing pointfree funcoids between posets. (However this is probably easily reducible to the case of posets.)

I have defined $\mathsf{RLD}\sharp$ to describe Hom-sets of the category or reloids but without source and destination and without composition. RLD should be replaced with $\mathsf{RLD}\sharp$ where possible, in order to make the theorems throughout the book a little more general. Also introduce similar features like $\Gamma\sharp$ and $\mathfrak{F}\Gamma\sharp$ (the last notation may need to be changed).

Misc properties of continuous functions between endofuncoids and endoreloids.

http://nforum.ncatlab.org/discussion/6765/please-help-with-a-proof-that-a-category-is-monoidal/ proves that finitary staroids are isomorphic to an ideal on a poset (for semilattices only).

[1] defines two categories with objects being filters. Another article on the same topic:
https://eudml.org/doc/16352 (Koubek, Václav, and Reiterman, Jan. ”On the category of filters.”)

FiXme: https://en.wikipedia.org/wiki/Cauchyspace says “The category of Cauchy spaces and Cauchy continuous maps is cartesian closed.” Generalize. http://www.sciencedirect.com/science/article/pii/0166864187900988

2. Compact funcoids

Generalize the theorem that compact topology corresponds to only one uniformity.

For compact funcoids the Cantor's theorem that a function continuous on a compact is uniformly continuous.

Every closed subset of a compact space is compact. A compact subset of a Hausdorff space is closed. 17.5 theorem in Willard.

17.6 theorem in Willard.

17.7 theorem in Willard: The continuous image of a compact space is compact.

17.10 Theorem in Willard: A compact Hausdorff space X is a $T4$-space. Also 17.11 Corollary, 17.13, 17.14 theorem.

"Locally compact" for funcoids. See also 18 "Locally compact spaces: in Willard. Compactification.

3. Misc

Example that $\operatorname{Compl} f \sqcup \operatorname{CoCompl} f \sqsubset f$ (for both funcoids and reloids). Proof for funcoids (for reloids it's similar): Take $f = \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. Then (write an explicit proof) $\operatorname{Compl} f = (\operatorname{Cor} \mathcal{A}) \times^{\mathsf{FCD}} \mathcal{B}$ and $\operatorname{CoCompl} f = \mathcal{A} \times^{\mathsf{FCD}} (\operatorname{Cor} \mathcal{B})$. Thus $\operatorname{Compl} f \sqcup \operatorname{CoCompl} f \neq f$ (if $\mathcal{A}$, $\mathcal{B}$ are non-principal).

Every funcoid (reloid) is a join of monovalued funcoids (reloids). For funcoids it's obvious (because it's a join of atomic funcoids). For reloids?

"Vicinity" and "neighborhood" mean different things, e.g. in [2].

Micronization μ and S^* are in some sense related as Galois connection. To formalize this we need to extend μ to arbitrary reloids (not only binary relations).

https://en.wikipedia.org/wiki/Transitive_reduction is a special case of micronization. Note it.

We need (it is especially important for studying compactness) to find a product of funcoids which coincides with product of topological spaces. (Cross-composition product doesn't because it is even not a funcoid (but a pointfree funcoid).) Neither subatomic product.

Subspace topology for space μ and set X is equal to $\mu \sqcap (X \times^{\mathsf{FCD}} X)$.

Change terminology: *montone* → *increasing*.

What are necessary and sufficient conditions for $\operatorname{up} f$ to be a filter for a funcoid f?

Article "Neighborhood Spaces" by D. C. KENT and WON KEUN MIN. ftp://ftp.math.ethz.ch/EMIS/journals/IJMMS/Volume327/239107.pdf

$g \sqsubseteq f^\circ \circ h \Leftrightarrow f \circ g \sqsubseteq h$?

$\lim x \to a f(x) = b$ iff $x \to a$ implies $\langle f \rangle x \to b$ for all filters x.

http://mathoverflow.net/q/36999/4086 "A good place to read about uniform spaces".

Research the posets of all proximity spaces and all uniform spaces (and also possibly reflexive and transitive funcoids/reloids).

Are filters on all Heyting or all co-Heyting lattices star-separable? http://math.stackexchange.com/q/1326266/4876

Define generalized pointfree reloids as filters on systems of sides.

Galois connections primer – study to ensure that we considered all Galois connections properties.

Germs seems to be equivalent to monovalued reloids.

$\mathcal{A} = \min\left\{ \frac{X}{\forall K \in \partial \mathcal{A} : K \not\asymp K} \right\}$, so we can restore $\mathcal{A}$ from $\partial \mathcal{A}$.

Boolean funcoid is a join-semilattice morphism from a boolean lattice to a boolean lattice. Generalize for pointfree funcoids.

Another way to define pointfree reloid as filters on Galois connections between two posets.

$L \in \operatorname{GR} \prod^{\mathsf{Strd}*} A \Leftrightarrow \forall \text{finite } M \subseteq \operatorname{dom} A \forall i \in M : Ai \not\asymp Li$?

Star-composition with identity staroids?

Does upgrading/downgrading of the ideal which represents a prestaroid coincide with upgrading/downgrading of the prestaroid?

It seems that equivalence of filters on different bases can be generalized: filters $\mathcal{A} \in \mathfrak{A}$ and $\mathcal{B} \in \mathfrak{B}$ are *equivalent* iff there exists an $X \in \mathfrak{A} \cap \mathfrak{B}$ which is greater than both $\mathcal{A}$ and $\mathcal{B}$. This however works only in the case if order of the orders $\mathfrak{A}$ and $\mathfrak{B}$ agree, that is if then are both a suborders of a greater fixed order.

Under which conditions a function spaces of posets is strongly separable?

Generalize both funcoids and reloids as filters on a superset of the lattice Γ (see "Funcoids are filters" chapter).

When the set of filters closed regarding a funcoid is a (co-)frame?

If a formula $F(x0, \ldots, xn)$ holds for every poset $\mathfrak{A}i$ then it also holds for product order $\prod \mathfrak{A}$. (What about infinite formulas like complete lattice joins and meets?) Moreover $F(x0, \ldots, xn) = \lambda i \in n : F(x0, i, \ldots, xn, i)$ (confused logical forms and functions). It looks like a promising approach, but how to define it exactly? For example, F may be a form always true for boolean lattices or for Heyting lattices, or whatsoever. How one theorem can encompass all kinds of lattices and posets? We may attempt to restrict to (partial) functions determined by order. (This is not enough, because we can define an operation restricting $\setminus$ defined only for posets of cardinality above or below some cardinal κ. For such restricted $\setminus$ the above formula does not work.) See also https://portonmath.wordpress.com/2016/01/12/a-conjecture-about-product-order-and-logic/. It seems that Todd Trimble shows a general category-theoretic way to describe this: https://nforum.ncatlab.org/discussion/6887/operations-on-product-order/.

Get results from http://ncatlab.org/toddtrimble/published/topogeny.

What about distributivity of quasicomplements over meets and joins for the filtrator of funcoids? Seems like nontrivial conjectures.

Conjecture: Each filtered filtrator is isomorphic to a primary filtrator. (If it holds, then primary and filtered filtrators are the same!)

Add analog of the last item of the theorem about co-complete funcoids for point-free funcoids.

Generalize theorems about $\mathsf{RLD}(A; B)$ as $\mathscr{F}(A \times B)$ in order to clean up the notation (for example in the chapter "Funcoids are filters").

Define reloids as a filtrator whose core is an ordered semigroup. This way reloids can be described in several isomorphic ways (just like primary filtrators are both filtrators of filters, of ideals, etc.) Is it enough to describe all properties of reloids? Well, it is not a semigroup, it is a precategory. It seems that we also need functions dom and im into partially ordered sets and "reversion" (dagger).

http://mathoverflow.net/a/191381/4086 says that n-staroids can be identified with certain ideals!

To relax theorem conditions and definition, we can define *protofuncoids* as arbitrary pairs $(\alpha; \beta)$ of functions between two posets. For protofuncoids composition and reverse are defined.

Add examples of funcoids to demonstrate their power: $D \sqcup T$ (D is a digraph T is a topological space), $T \sqcap \left\{ \frac{(x;y)}{y \geq x} \right\}$ as "one-side topology" and also a circle made from its π-length segment.

Say explicitly that pseudodifference is a special case of difference.

For pointfree funcoids, if $f : \mathfrak{A} \to \mathfrak{B}$ exists, then existence of least element of $\mathfrak{A}$ is equivalent to existence of least element of $\mathfrak{B}$: $y \not\asymp \langle f \rangle \bot^{\mathfrak{A}} \Leftrightarrow \bot^{\mathfrak{A}} \not\asymp \langle f^{-1} \rangle y \Leftrightarrow 0$. Thus $\langle f \rangle y \asymp \langle f \rangle y$ and so $\langle f \rangle y = \bot^{\mathfrak{B}}$. Can a similar statement be made that $\mathfrak{A}$ being join-semilattice implies $\mathfrak{B}$ being join-semilattice (at least for separable posets)? If yes, this could allow to shorten some theorem conditions. It seems we can produce a counter-example for non-separable posets by replacing an element with another element with the same full star.

Develop Todd Trimble's idea to represent funcoids as a relation ξ further: Define funcoid as a function from sets to sets of sets $\xi(A \sqcup B) = \xi A \cap \xi B$ and $\xi \bot = \emptyset$.

Denote the set of least elements as Least. (It is either an one-element set or empty set.)

Show that cross-composition product is a special case of infimum product.

Analog of order topology for funcoids/reloids.

A set is connected if every function from it to a discrete space is constant. Can this be generalized for generalized connectedness and generalized continuity? I have no idea how to relate these two concepts in general.

Develop theory of *funcoidal groups* by analogy with topological groups. Attempt to use this theory to solve this open problem:
http://garden.irmacs.sfu.ca/?q=op/iseveryregularparatopologicalgrouptychonoff
Is it useful as topological group determines not only a topology but even a uniformity?

A space μ is $T2$- iff the diagonal Δ is closed in $\mu \times \mu$.

The β-th projection map is not only continuous but also open (Willard, theorem 8.6).

Tx-separation axioms for products of spaces.

Willard 13.13 and its important corollary 13.14.

Willard 15.10.

About real-valued functions on endofuncoids: Urysohn's Lemma (and consequences: Tietze's extension theorem) for funcoids.

About product of reloids:
http://portonmath.wordpress.com/2012/05/23/unfirunded-questions/

Generalized Fréchet filter on a poset (generalize for filtrators) $\mathfrak{A}$ is a filter Ω such that

$$\partial\Omega = \left\{ \frac{x \in \mathfrak{A}}{\operatorname{atoms} x \text{ is infinite}} \right\}.$$

Research their properties (first, whether they exists for every poset). Also consider Fréchet element of $\mathsf{FCD}(A; B)$. Another generalization of Fréchet filter is meet of all coatoms.

Manifolds.

http://www.sciencedirect.com/science/article/pii/0304397585900623
(free download, also Google for "pre-adjunction", also "semi" instead of "pre") Are (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ adjunct?

Check how multicategories are related with categories with star-morphisms.

At https://en.wikipedia.org/wiki/Semilattice they are defined distributive semilattices. A join-semilattice is distributive if and only if the lattice of its ideals (under inclusion) is distributive.

The article http://arxiv.org/abs/1410.1504 has solved "Every paratopological group is Tychonoff" conjecture positively. Rewrite this article in terms of funcoids and reloids (especially with the algebraic formulas characterizing regular funcoids).

Generalize interior in topological spaces as the *interior funcoid* of a co-complete funcoid f, defined as a pointfree funcoid $f^\circ : \mathscr{F} \operatorname{dual} \operatorname{Src} f \to \mathscr{F} \operatorname{dual} \operatorname{Dst} f$ conforming to the formula: $\langle f^\circ \rangle^* (I \sqcap J) = \overline{\langle f \rangle^* \overline{I \sqcap J}} = \overline{\langle f \rangle^* (\overline{I} \sqcup \overline{J})}$. However composition of an interior funcoid with a funcoid is neither a funcoid nor an interior funcoid. It can be generalized using pseudocomplement.

http://math.sun.ac.za/cattop/Output/Kunzi/quasiintr.pdf "An Introduction to the Theory of Quasi-uniform Spaces".

http://www.mscand.dk/article/download/10581/8602 ("On equivalence between proximity structures and totally bounded uniform structures")

Characterize the set $\left\{ \frac{f \in \mathsf{FCD}}{(\mathsf{RLD})_{\mathrm{in}} f = (\mathsf{RLD})_{\mathrm{out}} f} \right\}$. (This seems a difficult problem.) Another (possibly related) problem: when up f is a filter for a funcoid f.

4. Common generalizations of funcoids and convergence spaces

I propose the following (possible) common generalizations of funcoids and convergence spaces ([2]):

- To every set we associate an isotone (and in some sense preserving finite joins) collection of filters.
- To every filter we associate an isotone (and in some sense preserving finite joins) collection of filters.
- Consider pointfree funcoids between isotone families of filters.

References

[1] Andreas Blass. Two closed categories of filters. *Fundamenta Mathematicae*, 94(2):129–143, 1977.

[2] Szymon Dolecki and Frédéric Mynard. *Convergence Foundations of Topology*. World Scientific, 2016.