Generalization in ZF*

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Abstract

In the framework of ZF are formally considered generalizations, such as whole numbers generalizing natural numbers, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc. The formal consideration of this may be especially useful for computer proof assistants. The article is accompanied with usable Isabelle/ZF code.

Keywords: ZF, ZFC, generalization, formalistics, bijection, injection

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1 Preface

In this article I define the notion of generalization in ZF set theory.

Examples: whole numbers generalizing natural numbers, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc.

I also have implemented this theory in Isabelle/ZF proof assistant (see http://isabelle.in.tum.de/index.html) formal language. This implementation is a candidate to be actually useful in development of Isabelle/ZF theories. However my implementation may be not ideal and may need further polishing before actual using in practice.

2 Current state of the issue

Whilst in informal mathematics is actively used the notion of generalization, it usually refers to intuition of the reader rather than to a formal consideration. This article provides a formal consideration compatible with usual intuitive notion of generalization.

In Isabelle proof assistant (both Isabelle/ZF and Isabelle/HOL) currently different sets are defined independently. For example in Isabelle/ZF there are sets `nat` (natural numbers) and `int` (integer numbers). It is not assumed that `nat` is a subset of `int`. The only connection between these provided is the function `inf_of` which transforms a natural number into the corresponding integer. Also as an effect of this, operations (such as additions) are defined differently and independently for naturals and integers.

*. This document has been written using the GNU $\TeX_{\text{MACS}}$ text editor (see `www.texmacs.org`).

That state is not good.

3 The rationale and examples

In mathematics it is often encountered that a small set S naturally bijectively corresponds to a subset R of a larger set B. (In other words, there is specified an injection E from S to B.) It is a widespread practice to equate S with R.

Remark 1. I denote the first set S from the first letter of the word "small" and the second set B from the first letter of the word "big", because S is intuitively considered as smaller than B. (However we do not require $\operatorname{card} S < \operatorname{card} B$.) I denote the injection as E from the first letter of the word "embed" because it embeds the set S to the set B.

The set B is considered as a generalization of the set S, for example: whole numbers generalizing natural numbers, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc.

Through these examples we see that B can be considered a generalization of S.

But strictly speaking this equating may contradict to the axioms of ZF/ZFC because we are not insured against $S \cap B \neq \emptyset$ incidents. Not wonderful, as it is often labeled as "without proof".

To work around of this (and formulate things exactly what could benefit computer proof assistants) we will replace the set B with a new set B' having a bijection $M: B \to B'$ such that $M \circ E = \mathrm{id}_S$. (I call this bijection M from the first letter of the word "move" which signifies the move from the old set B to a new set B').

4 Generalization situation

Now to the formalistic: I will call a *generalization situation* sets S and B together with an injection E from S to B. I will call, given a generalization situation, an *arbitrary generalization* a set B' and a bijection $M: B \to B'$ such that $M \circ E = \mathrm{id}_S$.

For every generalization situation I will denote $R = \operatorname{im} E$.

Proposition 2. *For every arbitrary generalization:*

1. $S \subseteq B'$.
2. *E is a bijection from S to R.*
3. $E = M^{-1}|_S$.

Proof.

1. $S = \operatorname{im}(\mathrm{id}_S) = \operatorname{im}(M \circ E) \subseteq \operatorname{im} M = B'$.
2. Obvious.
3. $M \circ E = \mathrm{id}_S \Rightarrow M^{-1} \circ M \circ E = M^{-1} \circ \mathrm{id}_S \Rightarrow E = M^{-1}|_S$. □

Assuming axiom of foundation (one of the axioms of ZF, also known as *axiom of regularity*) I will prove that an arbitrary generalization always exist for every generalization situation. Specifically I will prove that *ZF generalization* (see below) is an arbitrary generalization.

In absence of the axiom of foundation one could reasonably assert existence of a bijection $M: B \to B'$ such that $M \circ E = \mathrm{id}_S$ as an axiom. (Let's call it *the axiom of generalization*.) This axiom does not contradict to ZF because it is a consequence of the axiom of foundation.

5 ZF generalization

Let S and B are sets. Let E is an injection from S to B. (So we have a generalization situation.) Let $R = \operatorname{im} E$.

Let $t = \mathscr{P} \bigcup \bigcup S$.

Let $M(x) = \begin{cases} E^{-1}x & \text{if } x \in R; \\ (t; x) & \text{if } x \notin R. \end{cases}$

Recall that in standard ZF $(t; x) = \{\{t\}, \{t, x\}\}$ by definition.

Theorem 3. $(t; x) \notin S$.

Proof. Suppose $(t;\, x) \in S$. Then $\{\{t\},\, \{t,\, x\}\} \in S$. Consequently $\{t\} \in \bigcup S$; $\{t\} \subseteq \bigcup \bigcup S$; $\{t\} \in \mathscr{P} \bigcup \bigcup S$; $\{t\} \in t$ what contradicts to the axiom of foundation. □

Definition 4. *Let* $B' = \operatorname{im} M$.

Theorem 5. *M is a bijection from B to B'.*

Proof. Surjectivity of M is obvious. Let's prove injectivity.

Let $a, b \in B$ and $M(a) = M(b)$. Consider all cases:

$\boldsymbol{a, b \in R}$**.** $M(a) = E^{-1}a$; $M(b) = E^{-1}b$; $E^{-1}a = E^{-1}b$. Thus $a = b$ because E^{-1} is a bijection.

$\boldsymbol{a \in R, b \notin R}$**.** $M(a) = E^{-1}a$; $M(b) = (t; b)$; $M(a) \in S$; $M(b) \notin S$. Thus $M(a) \neq M(b)$.

$\boldsymbol{a \notin R, b \in R}$**.** Analogous.

$\boldsymbol{a, b \notin R}$**.** $M(a) = (t; a)$; $M(b) = (t; b)$. Thus $M(a) = M(b)$ implies $a = b$. □

Theorem 6. $M \circ E = \mathrm{id}_S$.

Proof. Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$. □

I will call the above defined B' and M the *ZF generalization* for a given generalization situation. From the above follows that a ZF generalization is an arbitrary generalization.

6 Isabelle/ZF code

You can download the code from http://www.mathematics21.org/binaries/gen/isabelle-ZF.zip.

I formulate and prove the properties of generalization formally in the language ISAR of computer proof assistant Isabelle.

First I define the theory file `ZF_Addons.thy` which contains some useful lemmas missing in the current version of Isabelle/ZF core.

The main theory file `Generalization.thy` contains (among other) three locales:

- `generalization_situation` describing generalization situations;
- `arbitrary_generalization` describing arbitrary generalizations;
- `ZF_generalization` describing the ZF generalization.

Both `arbitrary_generalization` and `ZF_generalization` locales are derived from `generalization_situation` locale.

Strictly speaking, `ZF_generalization` contains no new assumptions compared to its base, `generalization_situation` and all definitions from `ZF_generalization` could be put directly into `generalization_situation` but I prefer to differentiate these two locales conceptually: `generalization_situation` being only about arbitrary generalization and not about ZF generalization in particular. Also this makes porting to other logics having no axiom of foundation easier as only `ZF_generalization` locale is depended on the axiom of foundation.

Finally, I prove that `ZF_generalization` is a special case (sublocale) of `arbitrary_generalization`.

After that follows an example theory `int_obj_ex`. In this theory I define the set `int_obj` which is a set of integers considered as a generalization of the set `nat` of natural numbers. In this example theory I prove the theorem that `int_obj` is a superset of `nat`. I suggest the convention to use in Isabelle code the suffix `_obj` for sets M' as in this example. (This convention is questionable however, we may consider other possible conventions.)

7 Future directions

There are advantages and disadvantages of both untyped (such as Isabelle/ZF) and strongly typed (such as Isabelle/HOL) logical systems.

In this work I have shown that in untyped systems can be mastered the notion of generalization.

I think that we should invent something having advantages of both untyped and typed systems. We need a good idea how to crossbred different type systems to provide common advantages. Defining generalization in ZF seems being a step in this direction.

As a smaller challenge we also should polish my implementation and usage of generalizations in Isabelle/ZF because my implementation is not ideal.