Funcoids and Reloids*

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Abstract

It is a part of my Algebraic General Topology research.

In this article I introduce the concepts of *funcoids* which generalize proximity spaces and *reloids* which generalize uniform spaces. The concept of funcoid is generalized concept of proximity space, the concept of reloid is cleared from superfluous details (generalized) concept of uniform space. Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets).

That funcoids and reloids are common generalizations of both (proximity, pretopology, uniform) spaces and of (multivalued) functions, makes this theory smart for analyzing properties (e.g. continuousness) of functions on spaces. Also funcoids and reloids can be considered as a generalization of (oriented) graphs, this provides us with a common generalization of analysis and discrete mathematics.

Keywords: algebraic general topology, generalizations of proximity spaces, generalizations of nearness spaces, generalizations of uniform spaces

A.M.S. subject classification: 54J05, 54A05, 54D99, 54E05, 54E15, 54E17, 54E99

Table of contents

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1 Common

1.1 Draft status

This article is a draft.

This text refers to a preprint edition of [1]. Theorem number clashes may appear due editing both of these manuscripts.

1.2 Used concepts, notation and statements

Set of functions from a set A to a set B is denoted as B^A.

I will often skip parentheses and write fx instead of $f(x)$ to denote the result of a function f acting on the argument x.

I will denote $\langle f\rangle X = \{ f\alpha \mid \alpha \in X\}$.

For simplicity I will assume that all sets in consideration are subsets of universal set $\mho$.

1.2.1 Filters

In this work the word *filter* will refer to a filter on a set $\mho$ (in contrast to [1] where are considered filters on arbitrary posets).

I will call the set of filters ordered reverse to set-theoretic inclusion of filters *the set of filter objects* $\mathfrak{F}$ and its element *filter objects* (f.o. for short). I will denote $\operatorname{up}\mathcal{F}$ the filter corresponding to a filter object $\mathcal{F}$. So we have $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \operatorname{up}\mathcal{A} \supseteq \operatorname{up}\mathcal{B}$ for every filter objects $\mathcal{A}$ and $\mathcal{B}$. We also will equate filter objects corresponding to principal filters with corresponding sets. (Thus we have $\mathscr{P}\mho \subseteq \mathfrak{F}$.) See [1] for formal definition of filter objects in the framework of ZF. Filters (and filter objects) are studied in the work [1].

Filter objects corresponding to ultrafilters are atoms of the lattice $\mathfrak{F}$ and will be called *atomic filter objects*.

Also we will need to introduce the concept of *generalized filter base*.

Definition 1. *Generalized filter base* is a set $S \in \mathscr{P}\mathfrak{F} \setminus \{\emptyset\}$ such that

$$\forall \mathcal{A}, \mathcal{B} \in S \exists \mathcal{C} \in S: \mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}.$$

Proposition 2. Let S is a generalized filter base. If $\mathcal{A}_1, \ldots, \mathcal{A}_n \in S$ ($n \in \mathbb{N}$), then

$$\exists \mathcal{C} \in S: \mathcal{C} \subseteq \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_n.$$

Proof. Can be easily proved by induction. □

Theorem 3. If S is a generalized filter base, then $\operatorname{up} \bigcap^{\mathfrak{F}} S = \bigcup \langle \operatorname{up}\rangle S$.

Proof. Obviously $\operatorname{up}\bigcap^{\mathfrak{F}} S \supseteq \bigcup \langle \operatorname{up}\rangle S$. Reversely, let $K \in \bigcap^{\mathfrak{F}} S$; then $K = A_1 \cap \ldots \cap A_n$ where $A_i \in \operatorname{up}\mathcal{A}_i \in \mathcal{S}$, $i = 1, \ldots, n$, $n \in \mathbb{N}$; so exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_n \subseteq A_1 \cap \ldots \cap A_n = K$, $K \in \operatorname{up}\mathcal{C}$, $K \in \bigcup \langle \operatorname{up}\rangle S$. □

Corollary 4. If S is a generalized filter base, then $\bigcap^{\mathfrak{F}} S = \emptyset \Leftrightarrow \emptyset \in S$.

Proof. $\bigcap^{\mathfrak{F}} S = \emptyset \Leftrightarrow \emptyset \in \operatorname{up}\bigcap^{\mathfrak{F}} S \Leftrightarrow \emptyset \in \bigcup \langle \operatorname{up}\rangle S \Leftrightarrow \exists \mathcal{X} \in S: \emptyset \in \operatorname{up}\mathcal{X} \Leftrightarrow \emptyset \in S$. □

Definition 5. I will call a *partially ordered (pre)category* a (pre)category together with partial order on each of its Hom-sets.

2 Funcoids

2.1 Informal introduction into funcoids

Funcoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also funcoids are a generalization of binary relations.

That funcoids are a common generalization of "spaces" (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement "f is a continuous function from a space μ to a space ν" can be described in terms of funcoids as the formula $f \circ \mu \subseteq \nu \circ f$ (see my yet unpublished article "Generalized continuity" for details).

Most naturally funcoids appear as a generalization of proximity spaces.

Let δ be a proximity that is certain binary relation so that $A \,\delta\, B$ is defined for any sets A and B. We will extend it from sets to filter objects by the formula:

$$\mathcal{A} \,\delta'\, \mathcal{B} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}: A \,\delta\, B.$$

Then (as will be proved below) exist two functions $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\mathcal{A} \,\delta'\, \mathcal{B} \Leftrightarrow \mathcal{B} \cap \alpha\mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap \beta\mathcal{B} \neq \emptyset.$$

The pair $(\alpha; \beta)$ is called *funcoid* when $\mathcal{B} \cap \alpha\mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} \cap \beta\mathcal{B} \neq \emptyset$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first α and the second β. The first component of a funcoid f is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of discrete funcoids (see below) these coincide.)

One of the most important properties of a funcoid is that it is uniquely determined by just one of its components. That is a funcoid f is uniquely determined by the function $\langle f \rangle$. Moreover a funcoid f is uniquely determined by $\langle f \rangle|_{\mathscr{P}\mho}$ that is by values of function $\langle f \rangle$ on sets.

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha \in \mathfrak{F}^{\mho}$. Then we define $\alpha' X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}} \{\alpha X \mid x \in X\}$ for any set X. We will prove that there exists a unique funcoid f such that $\alpha' = \langle f \rangle|_{\mathscr{P}\mho}$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator p exists unique funcoid such that $\langle f \rangle|_{\mathscr{P}\mho} = p$; in this case $\langle f \rangle|_{\mathscr{P}\mho} \in \mathscr{P}\mho^{\mathscr{P}\mho}$.

For any binary relation p exists unique funcoid f such that $\forall X \in \mathscr{P}\mho: \langle f \rangle X = \langle p \rangle X$ (where $\langle p \rangle$ is defined in the introduction), recall that a funcoid is uniquely determined by the values of its first component on sets. I will call such funcoids *discrete*. So funcoids are a generalization of binary relations.

Composition of binary relations (i.e. of discrete funcoids) complies with the formulas:

$$\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle \quad \text{and} \quad \langle (g \circ f)^{-1} \rangle = \langle f^{-1} \rangle \circ \langle g^{-1} \rangle.$$

By the same formulas we can define composition of any two funcoids.

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha; \beta)^{-1} = (\beta; \alpha)$. In particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filter objects instead of acting on sets. Below will be defined domain and image of a funcoid (the domain and the image of a funcoid are filter objects).

2.2 Basic definitions

Definition 6. Let's call *a funcoid* a pair $(\alpha; \beta)$ where $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}: (\mathcal{Y} \cap^{\mathfrak{F}} \alpha\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta\mathcal{Y} \neq \emptyset).$$

Definition 7. $\langle(\alpha;\beta)\rangle \overset{\text{def}}{=} \alpha$ for a funcoid $(\alpha;\beta)$.

Definition 8. $(\alpha;\beta)^{-1} = (\beta;\alpha)$ for a funcoid $(\alpha;\beta)$.

Proposition 9. If f is a funcoid then f^{-1} is also a funcoid.

Proof. Follows from symmetry in the definition of funcoid. □

Obvious 10. $(f^{-1})^{-1} = f$ for a funcoid f.

Definition 11. The relation $[f] \in \mathscr{P}\mathfrak{F}^2$ is defined by the formula (for any filter objects $\mathcal{X}$, $\mathcal{Y}$ and funcoid f)

$$\mathcal{X}[f]\mathcal{Y} \overset{\text{def}}{=} \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset.$$

Obvious 12. $\mathcal{X}[f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y}$ for any filter objects $\mathcal{X}$, $\mathcal{Y}$ and funcoid f.

Obvious 13. $[f^{-1}] = [f]^{-1}$ for a funcoid f.

Theorem 14.

1. For given value of $\langle f\rangle$ exists no more than one funcoid f.
2. For given value of $[f]$ exists no more than one funcoid f.

Proof. Let f and g are funcoids.

Obviously $\langle f\rangle = \langle g\rangle \Rightarrow [f] = [g]$ and $\langle f^{-1}\rangle = \langle g^{-1}\rangle \Rightarrow [f] = [g]$. So enough to prove that $[f] = [g] \Rightarrow \langle f\rangle = \langle g\rangle$.

Provided that $[f] = [g]$ we have $\mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X}[f]\mathcal{Y} \Leftrightarrow \mathcal{X}[g]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle g\rangle\mathcal{X} \neq \emptyset$ and consequently $\langle f\rangle\mathcal{X} = \langle g\rangle\mathcal{X}$ for any f.o. $\mathcal{X}$ and $\mathcal{Y}$ because the set of filter objects is separable [1], thus $\langle f\rangle = \langle g\rangle$. □

Proposition 15. $\langle f\rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f\rangle\mathcal{I} \cup^{\mathfrak{F}} \langle f\rangle\mathcal{J}$ for any funcoid f and $\mathcal{I}, \mathcal{J} \in \mathfrak{F}$.

Proof. [TODO: Point used theorems.]

$$\begin{aligned}
\star\langle f\rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) \neq \emptyset\} &= \text{(by corollary 10 in [1])} \\
\{\mathcal{Y} \in \mathfrak{F} \mid (\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y} \neq \emptyset\} &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid (\mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y}) \cup^{\mathfrak{F}} (\mathcal{J} \cap^{\mathfrak{F}} \langle f^{-1}\rangle\mathcal{Y}) \neq \emptyset\} &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid (\mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{I}) \cup^{\mathfrak{F}} (\mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{J}) \neq \emptyset\} &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \cap^{\mathfrak{F}} (\langle f\rangle\mathcal{I} \cup^{\mathfrak{F}} \langle f\rangle\mathcal{J}) \neq \emptyset\} &= \\
\star(\langle f\rangle\mathcal{I} \cup^{\mathfrak{F}} \langle f\rangle\mathcal{J}).
\end{aligned}$$

Thus $\langle f\rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f\rangle\mathcal{I} \cup^{\mathfrak{F}} \langle f\rangle\mathcal{J}$ because $\mathfrak{F}$ is separable. □

2.2.1 Composition of funcoids

Definition 16. *Composition* of funcoids is defined by the formula

$$(\alpha_2;\beta_2) \circ (\alpha_1;\beta_1) = (\alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

Proposition 17. If f, g are funcoids then $g \circ f$ is funcoid.

Proof. Let $f = (\alpha_1;\beta_1)$, $g = (\alpha_2;\beta_2)$.

$\mathcal{Y} \cap^{\mathfrak{F}} (\alpha_2 \circ \alpha_1)\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \alpha_2\alpha_1\mathcal{X} \neq \emptyset \Leftrightarrow \alpha_1\mathcal{X} \cap^{\mathfrak{F}} \beta_2\mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta_1\beta_2\mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} (\beta_1 \circ \beta_2)\mathcal{Y} \neq \emptyset$.

So $(\alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)$ is a funcoid. □

Obvious 18. $\langle g\circ f\rangle=\langle g\rangle\circ\langle f\rangle$ for any funcoids f and g.

Proposition 19. $(h\circ g)\circ f=h\circ(g\circ f)$ for any funcoids f, g, h.

Proof.
$\langle(h\circ g)\circ f\rangle=\langle h\circ g\rangle\circ\langle f\rangle=(\langle h\rangle\circ\langle g\rangle)\circ\langle f\rangle=\langle h\rangle\circ(\langle g\rangle\circ\langle f\rangle)=\langle h\rangle\circ\langle g\circ f\rangle=\langle h\circ(g\circ f)\rangle$. □

Theorem 20. $(g\circ f)^{-1}=f^{-1}\circ g^{-1}$ for any funcoids f and g.

Proof. $\left\langle(g\circ f)^{-1}\right\rangle=\left\langle f^{-1}\right\rangle\circ\left\langle g^{-1}\right\rangle=\left\langle f^{-1}\circ g^{-1}\right\rangle$. □

2.3 Funcoid as continuation

Theorem 21. For any funcoid f and filter objects $\mathcal{X}$ and $\mathcal{Y}$

1. $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\,\mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}: X[f]Y$.

Proof. 2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall Y\in\mathrm{up}\,\mathcal{Y}: Y\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall Y\in\mathrm{up}\,\mathcal{Y}:\mathcal{X}[f]Y$.
Analogously $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X}: X[f]\mathcal{Y}$. Combining these two equalities we get

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}: X[f]Y.$$

1. $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X}:\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\neq\emptyset$
Let's denote $W=\left\{\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\mid X\in\mathrm{up}\,\mathcal{X}\right\}$. Let $\mathcal{P},\mathcal{Q}\in W$. Then $\mathcal{P}=\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle A$, $\mathcal{Q}=\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle B$ where $A,B\in\mathrm{up}\,\mathcal{X}$; $A\cap B\in\mathrm{up}\,\mathcal{X}$ and $\mathcal{R}\subseteq\mathcal{P}\cap^{\mathfrak{F}}\mathcal{Q}$ for $\mathcal{R}=\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle(A\cap B)\in W$. So W is a generalized filter base. [TODO: Simplify the proof than it is a g.f.b.]
$\emptyset\notin W\Leftrightarrow\bigcap^{\mathfrak{F}}W\not\ni\emptyset$ by the corollary 4 of the theorem 3. That is

$$\forall X\in\mathrm{up}\,\mathcal{X}:\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle X\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\,\mathcal{X}\neq\emptyset.$$

Comparing with the above, $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\,\mathcal{X}\neq\emptyset$. So $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{up}\,\mathcal{X}$. □

Theorem 22.

1. A function $\alpha\in\mathfrak{F}^{\mathscr{P}\mho}$ conforming to the formulas (for any $I,J\in\mathscr{P}\mho$)

$$\alpha\emptyset=\emptyset,\quad\alpha(I\cup J)=\alpha I\cup\alpha J$$

can be continued to the function $\langle f\rangle$ for exactly one funcoid f;

$$\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\alpha\rangle\mathrm{up}\,\mathcal{X}\tag{1}$$

for any filter object $\mathcal{X}$.

2. A relation $\delta\in\mathscr{P}(\mathscr{P}\mho)^2$ conforming to the formulas (for any $I,J,K\in\mathscr{P}\mho$)

$$\begin{array}{l}\neg(\emptyset\,\delta\,I),\ \ I\cup J\,\delta\,K\Leftrightarrow I\,\delta\,K\vee J\,\delta\,K,\\ \neg(I\,\delta\,\emptyset),\ \ K\,\delta\,I\cup J\Leftrightarrow K\,\delta\,I\vee K\,\delta\,J\end{array}\tag{2}$$

can be continued to the relation $[f]$ for exactly one funcoid f;

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}: X\,\delta\,Y\tag{3}$$

for any filter objects $\mathcal{X}$, $\mathcal{Y}$.

Proof. Existence of no more than one such funcoids and formulas (1) and (3) follow from the previous theorem.

2. Let define $\alpha\in\mathfrak{F}^{\mathscr{P}\mho}$ by the formula $\partial(\alpha X)=\{Y\in\mathscr{P}\mho\mid X\,\delta\,Y\}$ for any $X\in\mathscr{P}\mho$. Analogously can be defined $\beta\in\mathfrak{F}^{\mathscr{P}\mho}$ by the formula $\partial(\beta X)=\{X\in\mathscr{P}\mho\mid X\,\delta\,Y\}$. Let's continue α and β to $\alpha'\in\mathfrak{F}^{\mathfrak{F}}$ and $\beta'\in\mathfrak{F}^{\mathfrak{F}}$ by the formulas

$$\alpha'\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\alpha\rangle\mathrm{up}\,\mathcal{X}\quad\text{and}\quad\beta'\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\beta\rangle\mathrm{up}\,\mathcal{X}.$$

and δ to $\delta' \in \mathscr{P}\mathfrak{F}^2$ by the formula

$$\mathcal{X}\,\delta'\,\mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y}: X\,\delta\,Y.$$

$\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \langle\alpha\rangle \operatorname{up}\mathcal{X} \neq \emptyset \Leftrightarrow \bigcap^{\mathfrak{F}} \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle\alpha\rangle \operatorname{up}\mathcal{X} \neq \emptyset$. Let's prove that

$$W = \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle\alpha\rangle \operatorname{up}\mathcal{X}$$

is a generalized filter base: [TODO: Simplify this proof.] If $\mathcal{A}, \mathcal{B} \in W$ then exist $X_1, X_2 \in \operatorname{up}\mathcal{X}$ such that

$$\mathcal{A} = \mathcal{Y} \cap^{\mathfrak{F}} \alpha X_1 \quad \text{and} \quad \mathcal{A} = \mathcal{Y} \cap^{\mathfrak{F}} \alpha X_2.$$

Then $\mathcal{Y} \cap^{\mathfrak{F}} \alpha(X_1 \cap X_2) \in W$. So W is a generalized filter base.

Accordingly the corollary 4 of the theorem 3, $\bigcap^{\mathfrak{F}} \langle \mathcal{Y} \cap^{\mathfrak{F}} \rangle \langle\alpha\rangle \operatorname{up}\mathcal{X} \neq \emptyset$ is equivalent to

$$\forall X \in \operatorname{up}\mathcal{X}: \mathcal{Y} \cap^{\mathfrak{F}} \alpha X \neq \emptyset,$$

what is equivalent to $\forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y}: Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y}: Y \in \partial(\alpha X) \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y}: X\,\delta\,Y$. Combining the equivalencies we get $\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow X\,\delta'\,Y$. Analogously $\mathcal{X} \cap^{\mathfrak{F}} \beta'\mathcal{Y} \neq \emptyset \Leftrightarrow X\,\delta'\,Y$. So $\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \beta'\mathcal{Y} \neq \emptyset$, that is $(\alpha'; \beta')$ is a funcoid. From the formula $\mathcal{Y} \cap^{\mathfrak{F}} \alpha'\mathcal{X} \neq \emptyset \Leftrightarrow X\,\delta'\,Y$ follows that $[(\alpha'; \beta')]$ is a continuation of δ.

1. Let define the relation $\delta \in \mathscr{P}(\mathscr{P}\mho)^2$ by the formula $X\,\delta\,Y \Leftrightarrow Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset$. Then the formulas (2) are true.

Accordingly the above δ can be continued to the relation $[f]$ for some funcoid f.

$\forall X, Y \in \mathscr{P}\mho: X[f]Y \Leftrightarrow Y \cap^{\mathfrak{F}} \alpha X \neq \emptyset$, consequently $\forall X \in \mathscr{P}\mho: \alpha X = \langle f\rangle X$. So $\langle f\rangle$ is a continuation of α. □

Note that by the last theorem to every proximity δ corresponds exactly one funcoid. So funcoids is a generalization of proximity structures.

Definition 23. Any (multivalued) function f will be considered as a funcoid, where by definition $\langle f\rangle\mathcal{X} = \bigcap^{\mathfrak{F}} \langle\langle f\rangle\rangle \operatorname{up}\mathcal{X}$ for any $\mathcal{X} \in \mathfrak{F}$.

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff.

So binary relations (= multivalued functions) may be called *discrete funcoids*.

I will denote FCD the set of funcoids or the category of funcoids (see below) dependently on context.

2.4 Lattice of funcoids

Definition 24. $f \subseteq g \overset{\text{def}}{=} [f] \subseteq [g]$ for $f, g \in \mathsf{FCD}$.

Theorem 25. The set of funcoids is a complete lattice. For any $R \in \mathscr{P}\mathsf{FCD}$ and $X, Y \in \mathscr{P}\mho$

1. $X\left[\bigcup^{\mathsf{FCD}} R\right] Y \Leftrightarrow \exists f \in R: X[f]Y$;
2. $\left\langle \bigcup^{\mathsf{FCD}} R \right\rangle X = \bigcup^{\mathfrak{F}} \{\langle f\rangle X \mid f \in R\}$.

Proof.

2. $\langle h\rangle X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}} \{\langle f\rangle X \mid f \in R\}$. $\langle h\rangle\emptyset = \emptyset$;

$$\begin{aligned}
\langle h\rangle(I \cup J) &= \bigcup^{\mathfrak{F}} \{\langle f\rangle(I \cup J) \mid f \in R\} \\
&= \bigcup^{\mathfrak{F}} \{\langle f\rangle(I \cup^{\mathfrak{F}} J) \mid f \in R\} \\
&= \bigcup^{\mathfrak{F}} \{\langle f\rangle I \cup^{\mathfrak{F}} \langle f\rangle J \mid f \in R\} \\
&= \bigcup^{\mathfrak{F}} \{\langle f\rangle I \mid f \in R\} \cup^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \{\langle f\rangle J \mid f \in R\} \\
&= \langle h\rangle I \cup^{\mathfrak{F}} \langle h\rangle J.
\end{aligned}$$

So $\langle h\rangle$ can be continued to a funcoid. Obviously

$$\forall f\in R: h\supseteq f. \tag{4}$$

And h is the least funcoid for which holds the condition (4). So $h=\bigcup^{\mathsf{FCD}}R$.

1. $X\left[\bigcup^{\mathsf{FCD}}R\right]Y \Leftrightarrow Y\cap^{\mathfrak{F}}\left\langle\bigcup^{\mathsf{FCD}}R\right\rangle X\neq\emptyset \Leftrightarrow Y\cap^{\mathfrak{F}}\bigcup^{\mathfrak{F}}\{\langle f\rangle X \mid f\in R\}\neq\emptyset\Leftrightarrow \exists f\in R$: $Y\cap^{\mathfrak{F}}\langle f\rangle X\neq\emptyset\Leftrightarrow\exists f\in R: X[f]Y$. □

In the next theorem, compared to the previous one, the class of infinite unions is replaced with lesser class of finite unions and simultaneously class of sets is changed to more wide class of filter objects.

Theorem 26. For any funcoids f and g and a filter object $\mathcal{X}$

1. $\left\langle f\cup^{\mathsf{FCD}}g\right\rangle\mathcal{X}=\langle f\rangle\mathcal{X}\cup^{\mathfrak{F}}\langle g\rangle\mathcal{X}$;
2. $\left[f\cup^{\mathfrak{F}}g\right]=[f]\cup[g]$.

Proof.

1. Let $\langle h\rangle\mathcal{X}\overset{\text{def}}{=}\langle f\rangle\mathcal{X}\cup^{\mathfrak{F}}\langle g\rangle\mathcal{X}$; $\left\langle h^{-1}\right\rangle\mathcal{Y}\overset{\text{def}}{=}\left\langle f^{-1}\right\rangle\mathcal{Y}\cup^{\mathfrak{F}}\left\langle g^{-1}\right\rangle\mathcal{Y}$ for any $\mathcal{X},\mathcal{Y}\in\mathfrak{F}$. Then

$$\begin{aligned}\mathcal{Y}\cap^{\mathfrak{F}}\langle h\rangle\mathcal{X}\neq\emptyset &\Leftrightarrow \mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\vee\mathcal{Y}\cap^{\mathfrak{F}}\langle g\rangle\mathcal{X}\neq\emptyset\\ &\Leftrightarrow \mathcal{X}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y}\neq\emptyset\vee\mathcal{X}\cap^{\mathfrak{F}}\left\langle g^{-1}\right\rangle\mathcal{Y}\neq\emptyset\\ &\Leftrightarrow \mathcal{X}\cap^{\mathfrak{F}}\left\langle h^{-1}\right\rangle\mathcal{Y}\neq\emptyset.\end{aligned}$$

So h is a funcoid. Consequently $f\cup^{\mathsf{FCD}}g=h$.

2. $\mathcal{X}\left[f\cup^{\mathsf{FCD}}g\right]\mathcal{Y}\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\left\langle f\cup^{\mathsf{FCD}}g\right\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}(\langle f\rangle\mathcal{X}\cup^{\mathfrak{F}}\langle g\rangle\mathcal{X})\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset\vee$ $\mathcal{Y}\cap^{\mathfrak{F}}\langle g\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{X}[f]\mathcal{Y}\vee\mathcal{X}[g]\mathcal{Y}$ for any $\mathcal{X},\mathcal{Y}\in\mathfrak{F}$. □

2.5 More on composition of funcoids

Proposition 27. $[g\circ f]=[g]\circ\langle f\rangle=\left\langle g^{-1}\right\rangle^{-1}\circ[f]$ for $f,g\in\mathsf{FCD}$.

Proof. $\mathcal{X}[g\circ f]\mathcal{Y}\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\langle g\circ f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\mathcal{Y}\cap^{\mathfrak{F}}\langle g\rangle\langle f\rangle\mathcal{X}\neq\emptyset\Leftrightarrow\langle f\rangle\mathcal{X}[g]\mathcal{Y}\Leftrightarrow\mathcal{X}([g]\circ\langle f\rangle)\mathcal{Y}$ for any $\mathcal{X},\mathcal{Y}\in\mathfrak{F}$. $[g\circ f]=\left[(f^{-1}\circ g^{-1})^{-1}\right]=\left[f^{-1}\circ g^{-1}\right]^{-1}=(\left[f^{-1}\right]\circ\left\langle g^{-1}\right\rangle)^{-1}=\left\langle g^{-1}\right\rangle^{-1}\circ[f]$. □

The following theorem is the variant for funcoids of the statement (which defines compositions of relations) that $x(g\circ f)z\Leftrightarrow\exists y(x f y\wedge y g z)$ for any x and z and any binary relations f and g.

Theorem 28. For any $\mathcal{X},\mathcal{Z}\in\mathfrak{F}$ and $f,g\in\mathsf{FCD}$

$$\mathcal{X}[g\circ f]\mathcal{Z}\Leftrightarrow\exists y\in\operatorname{atoms}^{\mathfrak{F}}\mho:(\mathcal{X}[f]y\wedge y[g]\mathcal{Z}).$$

Proof.

$$\begin{aligned}\exists y\in\operatorname{atoms}^{\mathfrak{F}}\mho:(\mathcal{X}[f]y\wedge y[g]\mathcal{Z}) &\Leftrightarrow \exists y\in\operatorname{atoms}^{\mathfrak{F}}\mho:(\mathcal{Z}\cap^{\mathfrak{F}}\langle g\rangle y\neq\emptyset\wedge y\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset)\\ &\Leftrightarrow \exists y\in\operatorname{atoms}^{\mathfrak{F}}\mho:(\mathcal{Z}\cap^{\mathfrak{F}}\langle g\rangle y\neq\emptyset\wedge y\subseteq\langle f\rangle\mathcal{X})\\ &\Rightarrow \mathcal{Z}\cap^{\mathfrak{F}}\langle g\rangle\langle f\rangle\mathcal{X}\neq\emptyset\\ &\Leftrightarrow \mathcal{X}[g\circ f]\mathcal{Z}.\end{aligned}$$

Reversely, if $\mathcal{X}[g\circ f]\mathcal{Z}$ then $\langle f\rangle\mathcal{X}[g]\mathcal{Z}$, consequently exists $y\in\operatorname{atoms}^{\mathfrak{F}}\langle f\rangle\mathcal{X}$ such that $y[g]\mathcal{Z}$; we have $\mathcal{X}[f]y$. □

Theorem 29. If f, g, h are funcoids then

1. $f\circ(g\cup^{\mathsf{FCD}}h)=f\circ g\cup^{\mathsf{FCD}}f\circ h$;

2. $(g \cup^{\mathsf{FCD}} h) \circ f = g \circ f \cup^{\mathsf{FCD}} h \circ f$.

Proof. I will prove only the first equality because the other is analogous.

For any $\mathcal{X}, \mathcal{Z} \in \mathfrak{F}$

$$\begin{aligned}\mathcal{X}\left[f \circ (g \cup^{\mathsf{FCD}} h)\right]\mathcal{Z} &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}\left[g \cup^{\mathsf{FCD}} h\right] y \wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: ((\mathcal{X}[g]y \vee \mathcal{X}[h]y) \wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}[g]y \wedge y[f]\mathcal{Z} \vee \mathcal{X}[h]y \wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}[g]y \wedge y[f]\mathcal{Z}) \vee \exists y \in \operatorname{atoms}^{\mathfrak{F}}\mho: (\mathcal{X}[h]y \wedge y[f]\mathcal{Z})\\ &\Leftrightarrow \mathcal{X}[f \circ g]\mathcal{Z} \vee \mathcal{X}[f \circ h]\mathcal{Z}\\ &\Leftrightarrow \mathcal{X}\left[f \circ g \cup^{\mathsf{FCD}} f \circ h\right]\mathcal{Z}.\end{aligned}$$

□

2.6 Domain and range of a funcoid

Definition 30. Let $\mathcal{A} \in \mathfrak{F}$. The identity funcoid $I_{\mathcal{A}}$ will be defined by the formula $\langle I_{\mathcal{A}}\rangle\mathcal{X} \overset{\text{def}}{=} \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A}$.

Proposition 31. This definition is correct and $\mathcal{X}[I_{\mathcal{A}}]\mathcal{Y} \Leftrightarrow \mathcal{A} \cap^{\mathfrak{F}} \mathcal{X} \cap^{\mathfrak{F}} \mathcal{Y} \neq \emptyset$ for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$; $(I_{\mathcal{A}})^{-1} = I_{\mathcal{A}}$.

Proof. $\mathcal{Y} \cap^{\mathfrak{F}} \langle I_{\mathcal{A}}\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \mathcal{A} \cap^{\mathfrak{F}} \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \langle I_{\mathcal{A}}\rangle\mathcal{Y} \neq \emptyset$. □

Definition 32. I will define *restricting* of a funcoid f to a filter object $\mathcal{A}$ by the formula $f|_{\mathcal{A}} \overset{\text{def}}{=} f \circ I_{\mathcal{A}}$.

Obviously the last definition does not contradict to the previous.

Definition 33. *Image* of a funcoid f will be defined by the formula $\operatorname{im} f = \langle f\rangle\mho$.

Domain of a funcoid f is defined by the formula $\operatorname{dom} f = \operatorname{im} f^{-1}$.

Proposition 34. $\langle f\rangle\mathcal{X} = \langle f\rangle(\mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f)$ for any $f \in \mathsf{FCD}$, $\mathcal{X} \in \mathfrak{F}$.

Proof. For any filter object $\mathcal{Y}$ we have $\mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle(\mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f) \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle\mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \operatorname{im} f^{-1} \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle\mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle\mathcal{Y} \neq \emptyset \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset$. □

Proposition 35. $\mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f \neq \emptyset \Leftrightarrow \langle f\rangle\mathcal{X} \neq \emptyset$ for any $f \in \mathsf{FCD}$, $\mathcal{X} \in \mathfrak{F}$.

Proof. $\mathcal{X} \cap^{\mathfrak{F}} \operatorname{dom} f \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle\mho \neq \emptyset \Leftrightarrow \mho \cap^{\mathfrak{F}} \langle f\rangle\mathcal{X} \neq \emptyset \Leftrightarrow \langle f\rangle\mathcal{X} \neq \emptyset$. □

Corollary 36. $\operatorname{dom} f = \bigcup^{\mathfrak{F}}\left\{a \mid a \in \operatorname{atoms}^{\mathfrak{F}}\mho, \langle f\rangle a \neq \emptyset\right\}$.

Proof. This follows from that $\mathfrak{F}$ is an atomistic lattice. □

2.7 Category of funcoids

I will define the category FCD of funcoids:

- The set of objects is $\mathfrak{F}$.
- The set of morphisms from a filter object $\mathcal{A}$ to a filter object $\mathcal{B}$ is the set of triples $(f; \mathcal{A}; \mathcal{B})$ where f is a funcoid such that $\operatorname{dom} f \subseteq \mathcal{A}$, $\operatorname{im} f \subseteq \mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object $\mathcal{A}$ is $(I_{\mathcal{A}}; \mathcal{A}; \mathcal{A})$.

To prove that it is really a category is trivial.

2.8 Specifying funcoids by functions or relations on atomic filter objects

Theorem 37. For any funcoid f and filter objects $\mathcal{X}$ and $\mathcal{Y}$

1. $\langle f\rangle\mathcal{X}=\bigcup^{\mathfrak{F}}\langle\langle f\rangle\rangle\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}$;
2. $\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{X},y\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{Y}:x[f]y$.

Proof. 1.

$$\begin{aligned}\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset &\Leftrightarrow \mathcal{X}\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y}\neq\emptyset\\ &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}:x\cap^{\mathfrak{F}}\left\langle f^{-1}\right\rangle\mathcal{Y}\neq\emptyset\\ &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}:\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle x\neq\emptyset.\end{aligned}$$

$\partial\langle f\rangle\mathcal{X}=\bigcup\langle\partial\rangle\langle\langle f\rangle\rangle\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}=\partial\bigcup\langle\langle f\rangle\rangle\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}$.

2. If $\mathcal{X}[f]\mathcal{Y}$, then $\mathcal{Y}\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset$, consequently exists $y\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{Y}$ such that $y\cap^{\mathfrak{F}}\langle f\rangle\mathcal{X}\neq\emptyset$, $\mathcal{X}[f]y$. Repeating this second time we get that there exist $x\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{X}$ such that $x[f]y$. From this follows

$$\exists x\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{X},y\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{Y}:x[f]y.$$

The reverse is obvious. □

Theorem 38.

1. A function $\alpha\in\mathfrak{F}^{\mathrm{atoms}^{\mathfrak{F}}\mho}$ such that (for any $a\in\mathrm{atoms}^{\mathfrak{F}}\mho$)

$$\alpha a\supseteq\bigcap{}^{\mathfrak{F}}\left\langle\bigcup{}^{\mathfrak{F}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{F}}\right\rangle\mathrm{up}\,a \tag{5}$$

can be continued to the function $\langle f\rangle$ for exactly one funcoid f;

$$\langle f\rangle\mathcal{X}=\bigcup{}^{\mathfrak{F}}\langle\alpha\rangle\mathrm{atoms}^{\mathfrak{F}}\mathcal{X} \tag{6}$$

for any filter object $\mathcal{X}$.

2. A relation $\delta\in\mathscr{P}(\mathrm{atoms}^{\mathfrak{F}}\mho)^2$ such that (for any $a,b\in\mathrm{atoms}^{\mathfrak{F}}\mho$)

$$\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}^{\mathfrak{F}}X,y\in\mathrm{atoms}^{\mathfrak{F}}Y:x\,\delta\,y\Rightarrow a\,\delta\,b \tag{7}$$

can be continued to the relation $[f]$ for exactly one funcoid f;

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\exists x\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{X},y\in\mathrm{atoms}^{\mathfrak{F}}\mathcal{Y}:x\,\delta\,y \tag{8}$$

for any filter objects $\mathcal{X}$, $\mathcal{Y}$.

Proof. Existence of no more than one such funcoids and formulas (6) and (8) follow from the previous theorem.

1. Consider the function $\alpha'\in\mathfrak{F}^{\mho}$ defined by the formula (for any $X\in\mathscr{P}\mho$)

$$\alpha'X=\bigcup{}^{\mathfrak{F}}\langle\alpha\rangle\mathrm{atoms}^{\mathfrak{F}}X.$$

Obviously $\alpha'\emptyset=\emptyset$. For any $I,J\in\mathscr{P}\mho$

$$\begin{aligned}\alpha'(I\cup J) &= \bigcup{}^{\mathfrak{F}}\langle\alpha'\rangle\mathrm{atoms}^{\mathfrak{F}}(I\cup J)\\ &= \bigcup{}^{\mathfrak{F}}\langle\alpha'\rangle(\mathrm{atoms}^{\mathfrak{F}}I\cup\mathrm{atoms}^{\mathfrak{F}}J)\\ &= \bigcup{}^{\mathfrak{F}}\left(\langle\alpha'\rangle\mathrm{atoms}^{\mathfrak{F}}I\cup\langle\alpha'\rangle\mathrm{atoms}^{\mathfrak{F}}J\right)\\ &= \bigcup{}^{\mathfrak{F}}\langle\alpha'\rangle\,\mathrm{atoms}^{\mathfrak{F}}I\cup^{\mathfrak{F}}\bigcup{}^{\mathfrak{F}}\langle\alpha'\rangle\mathrm{atoms}^{\mathfrak{F}}.\\ &= \alpha'I\cup^{\mathfrak{F}}\alpha'J.\end{aligned}$$

Let continue α' till a funcoid f (by the theorem 25): $\langle f\rangle\mathcal{X}=\bigcap^{\mathfrak{F}}\langle\alpha'\rangle\mathrm{up}\,\mathcal{X}$.

Let's prove the reverse of (5):

$$\begin{aligned}\bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{F}}\right\rangle\mathrm{up}\,a &= \bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\right\rangle\langle\mathrm{atoms}^{\mathfrak{F}}\rangle\mathrm{up}\,a\\ &\supseteq \bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\right\rangle\{\{a\}\}\\ &= \bigcap^{\mathfrak{F}}\left\{\left(\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\right)\{a\}\right\}\\ &= \bigcap^{\mathfrak{F}}\left\{\bigcup^{\mathfrak{F}}\langle\alpha\rangle\{a\}\right\}\\ &= \bigcap^{\mathfrak{F}}\left\{\bigcup^{\mathfrak{F}}\{\alpha a\}\right\}=\bigcap^{\mathfrak{F}}\{\alpha a\}=\alpha a.\end{aligned}$$

Finally,

$$\alpha a=\bigcap^{\mathfrak{F}}\left\langle\bigcup^{\mathfrak{F}}\circ\langle\alpha\rangle\circ\mathrm{atoms}^{\mathfrak{F}}\right\rangle\mathrm{up}\,a=\bigcap^{\mathfrak{F}}\langle\alpha'\rangle\mathrm{up}\,a=\langle f\rangle a,$$

so $\langle f\rangle$ is a continuation of α.

2. Consider the relation $\delta'\in\mathscr{P}(\mathscr{P}\mho)^2$ defined by the formula (for any $X,Y\in\mathscr{P}\mho$)

$$X\,\delta'\,Y\Leftrightarrow\exists x\in\mathrm{atoms}^{\mathfrak{F}}X,y\in\mathrm{atoms}^{\mathfrak{F}}Y: x\,\delta\,y.$$

Obviously $\neg(X\,\delta'\,\emptyset)$ and $\neg(\emptyset\,\delta'\,Y)$.

$$\begin{aligned}(I\cup J)\,\delta'\,Y &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{F}}(I\cup J),y\in\mathrm{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\\ &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{F}}I\cup\mathrm{atoms}^{\mathfrak{F}}J,y\in\mathrm{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\\ &\Leftrightarrow \exists x\in\mathrm{atoms}^{\mathfrak{F}}I,y\in\mathrm{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\vee\exists x\in\mathrm{atoms}^{\mathfrak{F}}J,y\in\mathrm{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\\ &\Leftrightarrow I\,\delta'\,Y\vee J\,\delta'\,Y;\end{aligned}$$

analogously $X\,\delta'\,(I\cup J)\Leftrightarrow X\,\delta'\,I\vee X\,\delta'\,J$. Let's continue δ' till a funcoid f (by the theorem 25):

$$\mathcal{X}[f]\mathcal{Y}\Leftrightarrow\forall X\in\mathrm{up}\,\mathcal{X},Y\in\mathrm{up}\,\mathcal{Y}: X\,\delta'\,Y$$

The reverse of (7) implication is trivial, so

$$\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}^{\mathfrak{F}}X,y\in\mathrm{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Leftrightarrow a\,\delta\,b.$$

$\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}^{\mathfrak{F}}X,y\in\mathrm{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Leftrightarrow\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b: X\,\delta'\,Y\Leftrightarrow a[f]b$.

So $a\,\delta\,b\Leftrightarrow a[f]b$, that is $[f]$ is a continuation of δ. □

One of uses of the previous theorem is proof of the following theorem:

Theorem 39. If R is a set of funcoids, $x,y\in\mathrm{atoms}^{\mathfrak{F}}\mho$, then

1. $\left\langle\bigcap^{\mathsf{FCD}}R\right\rangle x=\bigcap^{\mathfrak{F}}\{\langle f\rangle x\mid f\in R\}$;
2. $x\left[\bigcap^{\mathsf{FCD}}R\right]y\Leftrightarrow\forall f\in R: x[f]y$.

Proof. 2. Let denote $x\,\delta\,y\Leftrightarrow\forall f\in R: x[f]y$.

$$\begin{gathered}\forall X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}^{\mathfrak{F}}X,y\in\mathrm{atoms}^{\mathfrak{F}}Y: x\,\delta\,y\Leftrightarrow\\ \forall f\in R,X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b\,\exists x\in\mathrm{atoms}^{\mathfrak{F}}X,y\in\mathrm{atoms}^{\mathfrak{F}}Y: x[f]y\Rightarrow\\ \forall f\in R,X\in\mathrm{up}\,a,Y\in\mathrm{up}\,b: X[f]Y\Rightarrow\\ \forall f\in R: a[f]b\Leftrightarrow\\ a\,\delta\,b.\end{gathered}$$

So, by the theorem 38, δ can be continued till $[p]$ for some funcoid p.

For any funcoid q such that $\forall f\in R: q\subseteq f$ we have $x[q]y\Rightarrow\forall f\in R: x[f]y\Leftrightarrow x\,\delta\,y\Leftrightarrow x[p]y$, so $q\subseteq f$. Consequently $p=\bigcap^{\mathsf{FCD}}R$.

From this $x\left[\bigcap^{\mathsf{FCD}}R\right]y\Leftrightarrow\forall f\in R: x[f]y$.

1. From the former $y\cap^{\mathfrak{F}}\left\langle\bigcap^{\mathsf{FCD}}R\right\rangle x\neq\emptyset\Leftrightarrow\forall f\in R: y\cap^{\mathfrak{F}}\langle f\rangle x\neq\emptyset$ for any $y\in\mathrm{atoms}^{\mathfrak{F}}\mho$. From this follows what we need to prove. □

2.9 Direct product of filter objects

A generalization of direct (Cartesian) product of two sets is direct product of two filter objects as defined in the theory of funcoids:

Definition 40. *Direct product* of filter objects $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ that

$$\mathcal{X} [\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}] \mathcal{Y} \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \mathcal{Y} \cap^{\mathfrak{F}} \mathcal{B} \neq \emptyset.$$

Proposition 41. $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ is really a funcoid and

$$\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset; \\ \emptyset & \text{if } \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} = \emptyset. \end{cases}$$

Proof. Obvious. □

Proposition 42. $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ for any $f \in \mathsf{FCD}$ and $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. If $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ then $\operatorname{dom} f \subseteq \operatorname{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{A}$, $\operatorname{im} f \subseteq \operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{B}$. If $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ then

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}: (\mathcal{X} [f] \mathcal{Y} \Rightarrow \mathcal{X} \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \mathcal{Y} \cap^{\mathfrak{F}} \mathcal{B} \neq \emptyset);$$

consequently $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of intersection on the lattice of funcoids:

Theorem 43. $f \cap^{\mathsf{FCD}} \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} = I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$ for any $f \in \mathsf{FCD}$ and $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. $h \stackrel{\text{def}}{=} I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$. For any $\mathcal{X} \in \mathfrak{F}$

$$\langle h \rangle \mathcal{X} = \langle I_{\mathcal{B}} \rangle \langle f \rangle \langle I_{\mathcal{A}} \rangle \mathcal{X} = \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap \mathcal{X}).$$

From this, as easy to show, $h \subseteq f$ and $h \subseteq \mathcal{A} \times \mathcal{B}$. If $g \subseteq f \wedge g \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a funcoid g then $\operatorname{dom} g \subseteq \mathcal{A}$, $\operatorname{im} g \subseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \cap^{\mathfrak{F}} \langle g \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) \subseteq \mathcal{B} \cap^{\mathfrak{F}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mathcal{X}) = \langle I_{\mathcal{B}} \rangle \langle f \rangle \langle I_{\mathcal{A}} \rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

$g \subseteq h$. So $h = f \cap^{\mathsf{FCD}} \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

Corollary 44. $f|_{\mathcal{A}} = f \cap \mathcal{A} \times^{\mathsf{FCD}} \mho$ for any $f \in \mathsf{FCD}$ and $\mathcal{A} \in \mathfrak{F}$.

Proof. $f \cap^{\mathsf{FCD}} \mathcal{A} \times^{\mathsf{FCD}} \mho = I_{\mho} \circ f \circ I_{\mathcal{A}} = f \circ I_{\mathcal{A}} = f|_{\mathcal{A}}$. □

Corollary 45. $f \cap^{\mathsf{FCD}} \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \neq \emptyset \Leftrightarrow \mathcal{A} [f] \mathcal{B}$ for any $f \in \mathsf{FCD}$, $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

Proof. $f \cap^{\mathsf{FCD}} \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \neq \emptyset \Leftrightarrow \langle f \cap^{\mathsf{FCD}} \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle \mho \neq \emptyset \Leftrightarrow \langle I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}} \rangle \mho \neq \emptyset \Leftrightarrow \langle I_{\mathcal{B}} \rangle \langle f \rangle \langle I_{\mathcal{A}} \rangle \mho \neq \emptyset \Leftrightarrow \mathcal{B} \cap^{\mathsf{FCD}} \langle f \rangle (\mathcal{A} \cap^{\mathfrak{F}} \mho) \neq \emptyset \Leftrightarrow \mathcal{B} \cap^{\mathfrak{F}} \langle f \rangle \mathcal{A} \neq \emptyset \Leftrightarrow \mathcal{A} [f] \mathcal{B}$. □

Theorem 46. If $S \in \mathscr{P} \mathfrak{F}^2$ then

$$\bigcap^{\mathsf{FCD}} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} = \bigcap^{\mathfrak{F}} \operatorname{dom} S \times^{\mathsf{FCD}} \bigcap^{\mathfrak{F}} \operatorname{im} S.$$

Proof. If $x \in \operatorname{atoms}^{\mathfrak{F}} \mho$ then by the theorem 39

$$\left\langle \bigcap^{\mathsf{FCD}} \{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S \} \right\rangle x = \bigcap^{\mathfrak{F}} \{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \}.$$

If $x \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \operatorname{dom} S \neq \emptyset$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S: (x \cap^{\mathfrak{F}} \mathcal{A} \neq \emptyset \wedge \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x = \mathcal{B});$$
$$\{ \langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \rangle x \mid (\mathcal{A}; \mathcal{B}) \in S \} = \operatorname{im} S;$$

if $x\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\operatorname{dom} S=\emptyset$ then

$$\exists(\mathcal{A};\mathcal{B})\in S\colon(x\cap^{\mathfrak{F}}\mathcal{A}=\emptyset\wedge\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle x=\emptyset);$$
$$\{\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle x\mid(\mathcal{A};\mathcal{B})\in S\}\ni\emptyset.$$

So

$$\left\langle\bigcap^{\mathsf{FCD}}\{\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\mid(\mathcal{A};\mathcal{B})\in S\}\right\rangle x=\begin{cases}\bigcap^{\mathfrak{F}}\operatorname{im} S & \text{if } x\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\operatorname{dom} S\neq\emptyset;\\ \emptyset & \text{if } x\cap^{\mathfrak{F}}\bigcap^{\mathfrak{F}}\operatorname{dom} S=\emptyset.\end{cases}$$

From this follows the statement of the theorem. □

Corollary 47. $\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0\cap^{\mathsf{FCD}}\mathcal{A}_1\times^{\mathsf{FCD}}\mathcal{B}_1=(\mathcal{A}_0\cap^{\mathsf{FCD}}\mathcal{A}_1)\times^{\mathsf{FCD}}(\mathcal{B}_0\cap^{\mathsf{FCD}}\mathcal{B}_1)$ for any $\mathcal{A}_0,\mathcal{A}_1,\mathcal{B}_0,\mathcal{B}_1\in\mathfrak{F}$.

Proof. $\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0\cap^{\mathsf{FCD}}\mathcal{A}_1\times^{\mathsf{FCD}}\mathcal{B}_1=\bigcap^{\mathfrak{F}}\{\mathcal{A}_0\times^{\mathsf{FCD}}\mathcal{B}_0,\mathcal{A}_1\times^{\mathsf{FCD}}\mathcal{B}_1\}$ what is by the last theorem equal to $(\mathcal{A}_0\cap^{\mathsf{FCD}}\mathcal{A}_1)\times^{\mathsf{FCD}}(\mathcal{B}_0\cap^{\mathsf{FCD}}\mathcal{B}_1)$. □

Theorem 48. If $\mathcal{A}\in\mathfrak{F}$ then $\mathcal{A}\times^{\mathsf{FCD}}$ is a complete homomorphism of the lattice $\mathfrak{F}$ to a complete sublattice of the lattice FCD, if also $\mathcal{A}\neq\emptyset$ then it is an isomorphism.

Proof. Let $S\in\mathscr{P}\mathfrak{F}$, $X\in\mathscr{P}\mho$, $x\in\operatorname{atoms}^{\mathfrak{F}}\mho$.

$$\begin{aligned}\left\langle\bigcup^{\mathsf{FCD}}\langle\mathcal{A}\times^{\mathsf{FCD}}\rangle S\right\rangle X &= \bigcup^{\mathfrak{F}}\{\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle X\mid\mathcal{B}\in S\}\\ &= \begin{cases}\bigcup^{\mathfrak{F}}S & \text{if } X\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset\\ \emptyset & \text{if } X\cap^{\mathfrak{F}}\mathcal{A}=\emptyset\end{cases}\\ &= \left\langle\mathcal{A}\times^{\mathsf{FCD}}\bigcup^{\mathfrak{F}}S\right\rangle X;\\ \left\langle\bigcap^{\mathsf{FCD}}\langle\mathcal{A}\times^{\mathsf{FCD}}\rangle S\right\rangle x &= \bigcap^{\mathfrak{F}}\{\langle\mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\rangle x\mid\mathcal{B}\in S\}\\ &= \begin{cases}\bigcap^{\mathfrak{F}}S & \text{if } x\cap^{\mathfrak{F}}\mathcal{A}\neq\emptyset\\ \emptyset & \text{if } x\cap^{\mathfrak{F}}\mathcal{A}=\emptyset\end{cases}\\ &= \left\langle\mathcal{A}\times^{\mathsf{FCD}}\bigcap^{\mathfrak{F}}S\right\rangle x.\end{aligned}$$

If $\mathcal{A}\neq\emptyset$ then obviously the function $\mathcal{A}\times^{\mathsf{FCD}}$ is injective. □

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a direct product of filter objects) funcoid (of atomic width).

Proposition 49. If a is an atomic filter object, $f\in\mathsf{FCD}$ then $f|_a=a\times^{\mathsf{FCD}}\langle f\rangle a$.

Proof. Let $\mathcal{X}\in\mathfrak{F}$.

$$\mathcal{X}\cap^{\mathfrak{F}}a\neq\emptyset\Rightarrow\langle f|_a\rangle\mathcal{X}=\langle f\rangle a,\quad\mathcal{X}\cap^{\mathfrak{F}}a=\emptyset\Rightarrow\langle f|_a\rangle\mathcal{X}=\emptyset.$$ □

2.10 Atomic funcoids

Theorem 50. A funcoid is an atom of the lattice of funcoids iff it is direct product of two atomic filter objects.

Proof.

Direct implication. Let f is an atomic funcoid. Let's get elements $a\in\operatorname{atoms}^{\mathfrak{F}}\operatorname{dom} f$ and $b\in\operatorname{atoms}^{\mathfrak{F}}\langle f\rangle a$. Then for any $\mathcal{X}\in\mathfrak{F}$

$$\mathcal{X}\cap^{\mathfrak{F}}a=\emptyset\Rightarrow\langle a\times^{\mathsf{FCD}}b\rangle\mathcal{X}=\emptyset\subseteq\langle f\rangle\mathcal{X},\quad\mathcal{X}\cap^{\mathfrak{F}}a\neq\emptyset\Rightarrow\langle a\times^{\mathsf{FCD}}b\rangle\mathcal{X}=b\subseteq\langle f\rangle\mathcal{X}.$$

So $a \times^{\mathsf{FCD}} b \subseteq f$; because f is an atomic funcoid $f = a \times^{\mathsf{FCD}} b$.

Reverse implication. Let a, $b \in \operatorname{atoms}^{\mathfrak{F}} \mho$, $f \in \mathsf{FCD}$. If $b \cap^{\mathfrak{F}} \langle f \rangle a = \emptyset$ then $\neg (a [f] b)$, $f \cap^{\mathfrak{F}} a \times^{\mathsf{FCD}} b = \emptyset$; if $b \subseteq \langle f \rangle a$ then $\forall \mathcal{X} \in \mathfrak{F}$: $(\mathcal{X} \cap^{\mathfrak{F}} a \neq \emptyset \Rightarrow \langle f \rangle \mathcal{X} \supseteq b)$, $f \supseteq a \times^{\mathsf{FCD}} b$. Consequently $f \cap^{\mathsf{FCD}} a \times^{\mathsf{FCD}} b = \emptyset \vee f \supseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atomic filter object. □

Theorem 51. The lattice of funcoids is atomic.

Proof. Let f is a non-empty funcoid. Then $\operatorname{dom} f \neq \emptyset$, thus by the theorem 46 in [1] exists $a \in \operatorname{atoms} \operatorname{dom} f$. So $\langle f \rangle a \neq \emptyset$ thus exists $b \in \operatorname{atoms} \langle f \rangle a$. Finally the atomic funcoid $a \times^{\mathsf{FCD}} b \subseteq f$. □

Theorem 52. The lattice of funcoids is atomically separable.

Proof. Let $f, g \in \mathsf{FCD}$, $f \subset g$. Then exists $a \in \operatorname{atoms}^{\mathfrak{F}} \mho$ such that $\langle f \rangle a \subset \langle g \rangle a$. So because the lattice $\mathfrak{F}$ is atomically separable then exists $b \in \operatorname{atoms}^{\mathfrak{F}} \mho$ such that $\langle f \rangle a \cap^{\mathfrak{F}} b = \emptyset$ and $b \subseteq \langle g \rangle a$. For any $x \in \operatorname{atoms}^{\mathfrak{F}} \mho$

$$\langle f \rangle a \cap^{\mathfrak{F}} \left\langle a \times^{\mathsf{FCD}} b \right\rangle a = \langle f \rangle a \cap^{\mathfrak{F}} b = \emptyset,$$
$$x \neq a \Rightarrow \langle f \rangle x \cap^{\mathfrak{F}} \left\langle a \times^{\mathsf{FCD}} b \right\rangle x = \langle f \rangle x \cap^{\mathfrak{F}} \emptyset = \emptyset$$

Thus $\langle f \rangle x \cap^{\mathfrak{F}} \langle a \times b \rangle x = \emptyset$ and consequently $f \cap^{\mathsf{FCD}} a \times^{\mathsf{FCD}} b = \emptyset$.

$$\left\langle a \times^{\mathsf{FCD}} b \right\rangle a = b \subseteq \langle g \rangle a,$$
$$x \neq a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b \right\rangle x = \emptyset \subseteq \langle g \rangle a.$$

Thus $\left\langle a \times^{\mathsf{FCD}} b \right\rangle x = b \subseteq \langle g \rangle x$ and consequently $a \times^{\mathsf{FCD}} b \subseteq g$.

So the lattice of funcoids is separable by the theorem 19 in [1]. □

Corollary 53. The lattice of funcoids is:

1. separable;
2. conforming to Wallman's disjunction property.

Proof. By the theorem 22 in [1]. □

Remark 54. For more ways to characterize (atomic) separability of the lattice of funcoids see [1], subsections "Separation subsets and full stars" and "Atomically separable lattices".

Corollary 55. The lattice of funcoids is an atomistic lattice.

Proof. Let f is a funcoid. Suppose contrary to the statement to be proved that $\bigcup^{\mathfrak{F}} \operatorname{atoms}^{\mathsf{FCD}} f \subset f$. Then exists $a \in \operatorname{atoms}^{\mathsf{FCD}} f$ such that $a \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} \operatorname{atoms}^{\mathsf{FCD}} f = \emptyset$ what is impossible. □

Proposition 56. $\operatorname{atoms}^{\mathsf{FCD}} (f \cup^{\mathfrak{F}} g) = \operatorname{atoms}^{\mathsf{FCD}} f \cup \operatorname{atoms}^{\mathsf{FCD}} g$ for any funcoids f and g.

Proof. $a \times^{\mathsf{FCD}} b \cap^{\mathsf{FCD}} (f \cup^{\mathsf{FCD}} g) \neq \emptyset \Leftrightarrow a \left[f \cup^{\mathsf{FCD}} g \right] b \Leftrightarrow a [f] b \vee a [g] b \Leftrightarrow a \times^{\mathsf{FCD}} b \cap^{\mathsf{FCD}} f \neq \emptyset \vee a \times^{\mathsf{FCD}} b \cap^{\mathsf{FCD}} g \neq \emptyset$ for any atomic filter objects a and b. □

Corollary 57. For any $f, g, h \in \mathsf{FCD}$, $R \in \mathscr{P} \mathsf{FCD}$

1. $f \cap^{\mathsf{FCD}} (g \cup^{\mathsf{FCD}} h) = (f \cap^{\mathsf{FCD}} g) \cup^{\mathsf{FCD}} (f \cap^{\mathsf{FCD}} h)$;
2. $f \cup^{\mathsf{FCD}} \bigcap^{\mathsf{FCD}} R = \bigcap^{\mathsf{FCD}} \left\langle f \cup^{\mathsf{FCD}} \right\rangle R$.

Proof. We will take in account that the lattice of funcoids is an atomistic lattice. To be concise I will write atoms instead of $\operatorname{atoms}^{\mathsf{FCD}}$ and $\cap$ and $\cup$ instead of $\cap^{\mathsf{FCD}}$ and $\cup^{\mathsf{FCD}}$.

1. $\operatorname{atoms}(f \cap (g \cup h)) = \operatorname{atoms} f \cap \operatorname{atoms}(g \cup h) = \operatorname{atoms} f \cap (\operatorname{atoms} g \cup \operatorname{atoms} h) = (\operatorname{atoms} f \cap \operatorname{atoms} g) \cup (\operatorname{atoms} f \cap \operatorname{atoms} h) = \operatorname{atoms}(f \cap g) \cup \operatorname{atoms}(f \cap h) = \operatorname{atoms}((f \cap g) \cup (f \cap h))$.

2. $\operatorname{atoms}\left(f \cup \bigcap^{\mathsf{FCD}} R\right) = \operatorname{atoms} f \cup \operatorname{atoms} \bigcap^{\mathsf{FCD}} R = \operatorname{atoms} f \cup \bigcap^{\mathsf{FCD}} \langle \operatorname{atoms} \rangle R = \bigcap^{\mathsf{FCD}} \langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R = \bigcap^{\mathsf{FCD}} \langle \operatorname{atoms} \rangle \langle f \cup \rangle R = \operatorname{atoms} \bigcap^{\mathsf{FCD}} \langle f \cup \rangle R$. □

Note that distributivity of the lattice of funcoids is proved through using atoms of this lattice. I have never seen such method of proving distributivity.

The next proposition is one more (among the theorem 28) generalization for funcoids of composition of relations.

Proposition 58. For any $f, g \in \mathsf{FCD}$

$\operatorname{atoms}^{\mathsf{FCD}}(g \circ f) = \left\{ x \times^{\mathsf{FCD}} z \mid x, z \in \operatorname{atoms}^{\mathfrak{F}} \mho, \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms}^{\mathsf{FCD}} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms}^{\mathsf{FCD}} g) \right\}$.

Proof. $x \times^{\mathsf{FCD}} z \cap^{\mathsf{FCD}} g \circ f \neq \emptyset \Leftrightarrow x [g \circ f] z \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (x [f] y \wedge y [g] z) \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho : (x \times^{\mathsf{FCD}} y \cap^{\mathsf{FCD}} f \neq \emptyset \wedge y \times^{\mathsf{FCD}} z \cap^{\mathsf{FCD}} g \neq \emptyset)$ (were used the theorem 28). □

2.11 Complete funcoids

Definition 59. I will call *generalized closure* such a funcoid f that $\forall X \in \mathscr{P} \mho : \langle f \rangle X \in \mathscr{P} \mho$.

Remark 60. Generalized closures can be expressed without using the concept of filter objects the following way: Let's call generalized closure such a function $\langle f \rangle \in \mathscr{P} \mho^{\mathscr{P} \mho}$ that

1. $\langle f \rangle \emptyset = \emptyset$;
2. $\forall I, J \in \mathscr{P} \mho : \langle f \rangle (I \cup J) = \langle f \rangle I \cup^{\mathfrak{F}} \langle f \rangle J$.

Remark 61. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of funcoids.

Definition 62. I will call a *complete funcoid* a funcoid which is a reverse of a generalized closure.

Theorem 63. The following conditions are equivalent:

1. funcoid f is complete.
2. $\forall S \in \mathscr{P} \mathfrak{F}, J \in \mathscr{P} \mho : (\bigcup^{\mathfrak{F}} S [f] J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I} [f] J)$;
3. $\forall S \in \mathscr{P} \mathscr{P} \mho, J \in \mathscr{P} \mho : (\bigcup S [f] J \Leftrightarrow \exists I \in S : I [f] J)$;
4. $\forall S \in \mathscr{P} \mathfrak{F} : \langle f \rangle \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$;
5. $\forall S \in \mathscr{P} \mathscr{P} \mho : \langle f \rangle \bigcup S = \bigcup^{\mathfrak{F}} \langle \langle f \rangle \rangle S$.

Proof.

(3) ⇒ (1). For any $S \in \mathscr{P} \mathscr{P} \mho$, $J \in \mathscr{P} \mho$

$$\bigcup S \cap^{\mathfrak{F}} \left\langle f^{-1} \right\rangle J \neq \emptyset \Leftrightarrow \exists I \in S : I \cap^{\mathfrak{F}} \left\langle f^{-1} \right\rangle J \neq \emptyset, \tag{9}$$

consequently by the theorem 52 in [1] we have $\left\langle f^{-1} \right\rangle J \in \mathscr{P} \mho$.

(1) ⇒ (2). For any $S \in \mathscr{P} \mathfrak{F}$, $J \in \mathscr{P} \mho$ we have $\left\langle f^{-1} \right\rangle J \in \mathscr{P} \mho$, consequently the formula (9) is true. From this follows (2).

(2) ⇒ (3), (4) ⇒ (5), (5) ⇒ (3), (2) ⇒ (4). Obvious. □

The following proposition shows that complete funcoids are a direct generalization of pre-topological spaces.

Proposition 64. To specify a complete funcoid f it is enough to specify $\langle f \rangle$ on one-element sets, values of $\langle f \rangle$ on one element sets can be specified arbitrarily.

Proof. From the above theorem is clear that knowing $\langle f\rangle$ on one-element sets $\langle f\rangle$ can be found on any sets and then its value can be inferred for any filter objects.

Choosing arbitrarily the values of $\langle f\rangle$ on one-element sets we can define a complete funcoid the following way: $\langle f\rangle X \overset{\text{def}}{=} \bigcup^{\mathfrak{F}}\{\langle f\rangle\{\alpha\} \mid \alpha \in X\}$ for any $X \in \mathscr{P}\mho$. Obviously it is really a complete funcoid. □

Theorem 65. A funcoid is discrete iff it is both generalized closure and complete funcoid.

Proof.

Direct implication. Obvious.

Reverse implication. Let f is both a generalized closure and a complete funcoid. Consider the relation g defined by that $\langle g\rangle\{\alpha\} = \langle f\rangle\{\alpha\}$ (g is correctly defined because f is a generalized closure). Because f is a complete funcoid $f = g$. □

Theorem 66. If R is a set of complete funcoids (generalized closures) then $\bigcup^{\mathsf{FCD}} R$ is a complete funcoid (generalized closure).

Proof. It is enough to prove only for generalized closures. Let R is a set of generalized closures. Then for any $X \in \mathscr{P}\mho$

$$\left\langle \bigcup^{\mathsf{FCD}} R\right\rangle X = \bigcup \{\langle f\rangle X \mid f \in R\} \in \mathscr{P}\mho$$

(used the theorem 25). □

Corollary 67. If R is a set of binary relations then $\bigcup^{\mathsf{FCD}} R = \bigcup R$.

Proof. From two last theorems. □

Conjecture 68. If f, g are complete funcoids (generalized closures) then $f \cap^{\mathsf{FCD}} g$ is a complete funcoid (generalized closure).

Conjecture 69. If f, g are discrete funcoids then $f \cap^{\mathsf{FCD}} g$ is a discrete funcoid.

The conjecture 69 easily follows from the conjecture 68. I'm almost sure that these two important conjectures are true.

The next is an important theorem about representation of funcoids.

Theorem 70. Any funcoid is representable as (infinite) intersection (on the lattice of funcoids) of some set of relations.

Proof. Let $f \in \mathsf{FCD}$, $A \in \mathscr{P}\mho$, $B \in \operatorname{up}\langle f\rangle A$, $g(A; B) \overset{\text{def}}{=} A \times B \cup^{\mathsf{FCD}} \overline{A} \times \mho$. For any $X \in \mathscr{P}\mho$

$$\langle g(A; B)\rangle X = \left\langle A \times^{\mathsf{FCD}} B\right\rangle X \cup \left\langle \overline{A} \times^{\mathsf{FCD}} \mho\right\rangle X = \left(\begin{cases} \emptyset & \text{if } X = \emptyset \\ B & \text{if } \emptyset \neq X \subseteq A \\ \mho & \text{if } X \nsubseteq A \end{cases}\right) \supseteq \langle f\rangle X;$$

so $g(A; B) \supseteq f$. For any $A \in \mathscr{P}\mho$

$$\bigcap^{\mathfrak{F}}\{\langle g(A; B)\rangle A \mid B \in \operatorname{up}\langle f\rangle A\} = \bigcap^{\mathfrak{F}}\{B \mid B \in \operatorname{up}\langle f\rangle A\} = \langle f\rangle A;$$

consequently

$$\bigcap^{\mathsf{FCD}}\{g(A; B) \mid A \in \mathscr{P}\mho, B \in \operatorname{up}\langle f\rangle A\} = f.$$

□

In certain cases the theorem 29 can be generalized for infinite unions.

Theorem 71. Let $f \in \mathsf{FCD}$. If R is a set of generalized closures then

$$f \circ \bigcup^{\mathsf{FCD}} R = \bigcup^{\mathsf{FCD}} \langle f \circ\rangle R.$$

Proof. If R is a set of generalized closures then for $X, Z \in \mathscr{P}\mho$

$$
\begin{aligned}
X\left[f \circ \bigcup^{\mathsf{FCD}} R\right] Z & \Leftrightarrow \\
\text{(by the theorem 28)} & \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho \colon \left(X\left[\bigcup^{\mathsf{FCD}} R\right] y \wedge y[f]Z\right) \\
& \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho \colon \left(y \cap^{\mathfrak{F}} \left\langle \bigcup^{\mathsf{FCD}} R\right\rangle X \neq \emptyset \wedge y[f]Z\right) \\
\text{(by the theorem 25)} & \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho \colon \left(y \cap^{\mathfrak{F}} \bigcup \{\langle u \rangle X \mid u \in R\} \neq \emptyset \wedge y[f]Z\right) \\
\text{(by the theorem 52 in [1])} & \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho \colon (\exists u \in R \colon y \cap^{\mathfrak{F}} \langle u \rangle X \neq \emptyset \wedge y[f]Z) \\
& \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho \colon (\exists u \in R \colon X[u]y \wedge y[f]Z) \\
& \Leftrightarrow \exists y \in \operatorname{atoms}^{\mathfrak{F}} \mho, u \in R \colon (X[u]y \wedge y[f]Z) \\
\text{(by the theorem 28)} & \Leftrightarrow \exists u \in R \colon X[f \circ u]Z \\
\text{(by the theorem 25)} & \Leftrightarrow X\left[\bigcup^{\mathsf{FCD}} \langle f \circ \rangle R\right] Z.
\end{aligned}
$$

□

2.12 Monovalued funcoids

Following the idea of definition of monovalued morphism in [2] let's call *monovalued* such a funcoid f that $f \circ f^{-1} \subseteq I_{\operatorname{im} f}$.

Obvious 72. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of funcoids is monovalued iff the funcoid f is monovalued.

Theorem 73. The following statements are equivalent for a funcoid f:

1. f is monovalued.
2. $\forall a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A} \colon \langle f \rangle a \in \operatorname{atoms}^{\mathfrak{F}} \mho \cup \{\emptyset\}$.
3. $\forall \mathcal{I}, \mathcal{J} \in \mathfrak{F} \colon \langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) = \langle f^{-1} \rangle \mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{J}$.
4. $\forall I, J \in \mathscr{P}\mho \colon \langle f^{-1} \rangle (I \cap J) = \langle f^{-1} \rangle I \cap^{\mathfrak{F}} \langle f^{-1} \rangle J$.

Proof.

(2) ⇒ (3). Let $a \in \operatorname{atoms}^{\mathfrak{F}} \mho$, $\langle f \rangle a = b$. Then because $b \in \operatorname{atoms}^{\mathfrak{F}} \mho \cup \{\emptyset\}$

$$
\begin{gathered}
(\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \cap^{\mathfrak{F}} b \neq \emptyset \Leftrightarrow \mathcal{I} \cap^{\mathfrak{F}} b \neq \emptyset \wedge \mathcal{J} \cap^{\mathfrak{F}} b \neq \emptyset; \\
a[f](\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \Leftrightarrow a[f]\mathcal{I} \wedge a[f]\mathcal{J}; \\
(\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J})[f^{-1}]a \Leftrightarrow \mathcal{I}[f^{-1}]a \wedge \mathcal{J}[f^{-1}]a; \\
a \cap^{\mathfrak{F}} \langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) \neq \emptyset \Leftrightarrow a \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{I} \neq \emptyset \wedge a \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{J} \neq \emptyset; \\
\langle f^{-1} \rangle (\mathcal{I} \cap^{\mathfrak{F}} \mathcal{J}) = \langle f^{-1} \rangle \mathcal{I} \cap^{\mathfrak{F}} \langle f^{-1} \rangle \mathcal{J}.
\end{gathered}
$$

(4) ⇒ (1). $\langle f^{-1} \rangle a \cap^{\mathfrak{F}} \langle f^{-1} \rangle b = \emptyset$ for any two distinct atomic filter objects a and b. This is equivalent to $\neg(b[f^{-1}]\langle f^{-1} \rangle a)$; $\neg(\langle f^{-1} \rangle a[f]b)$; $b \cap^{\mathfrak{F}} \langle f \rangle \langle f^{-1} \rangle a = \emptyset$; $b \cap^{\mathfrak{F}} \langle f \circ f^{-1} \rangle a = \emptyset$; $\neg(a[f \circ f^{-1}]b)$. So $a[f \circ f^{-1}]b \Rightarrow a = b$ for any atomic filter objects a and b. This is possible only when $f \circ f^{-1} \subseteq I_{\operatorname{Dst} f}$.

(3) ⇒ (4). Obvious.

¬(2) ⇒ ¬(1). Suppose $\langle f \rangle a \notin \operatorname{atoms}^{\mathfrak{F}} \mathcal{B} \cup \{\emptyset\}$ for some $a \in \operatorname{atoms}^{\mathfrak{F}} \mathcal{A}$. Then there exist two atomic filter objects $p \neq q$ such that $\langle f \rangle a \supseteq p \wedge \langle f \rangle a \supseteq q$. Consequently $p \cap^{\mathfrak{F}} \langle f \rangle a \neq \emptyset$; $a \cap^{\mathfrak{F}} \langle f^{-1} \rangle p \neq \emptyset$; $a \subseteq \langle f^{-1} \rangle p$; $\langle f \circ f^{-1} \rangle p = \langle f \rangle \langle f^{-1} \rangle p \supseteq \langle f \rangle a \supseteq q$; $\langle f \circ f^{-1} \rangle p \nsubseteq p$. So it cannot be $f \circ f^{-1} \subseteq I_{\operatorname{Dst} f}$. □

Corollary 74. A function is a monovalued funcoid.

Remark 75. This corollary can be reformulated as follows: For binary relations the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a funcoid are the same.

Proof. Because $\forall I, J \in \mathscr{P}\mho$: $\left\langle f^{-1}\right\rangle(I \cap J) = \left\langle f^{-1}\right\rangle I \cap^{\mathfrak{F}} \left\langle f^{-1}\right\rangle J$ is true for any function f. □

2.13 T_1- and T_2-separable funcoids

For funcoids can be generalized T_0-, T_1- and T_2- separability. Worthwhile note that T_0 and T_2 separability is defined through T_1 separability.

Definition 76. Let call T_1-*separable* such funcoid f that for any $\alpha, \beta \in \mho$ is true

$$\alpha \neq \beta \Rightarrow \neg(\{\alpha\}[f]\{\beta\})$$

Definition 77. Let call T_0-*separable* such funcoid f that $f \cap^{\mathsf{FCD}} f^{-1}$ is T_1-separable.

Definition 78. Let call T_2-*separable* such funcoid f that the funcoid $f^{-1} \circ f$ is T_1-separable.

For symmetric transitive funcoids T_1- and T_2-separability are the same (see theorem 11).

2.14 Filter objects closed regarding a funcoid

Definition 79. Let's call *closed* regarding a funcoid f such filter object $\mathcal{A}$ that $\langle f\rangle\mathcal{A} \subseteq \mathcal{A}$.

This is a generalization of closedness of a set regarding an unary operation.

Proposition 80. If $\mathcal{I}$ and $\mathcal{J}$ are closed (regarding some funcoid), S is a set of closed filter objects, then

1. $\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$ is a closed filter object;
2. $\bigcap^{\mathfrak{F}} S$ is a closed filter object.

Proof. Let denote the given funcoid as f. $\langle f\rangle(\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}) = \langle f\rangle\mathcal{I} \cup^{\mathfrak{F}} \langle f\rangle\mathcal{J} \subseteq \mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$, $\langle f\rangle \bigcap^{\mathfrak{F}} S \subseteq \bigcap^{\mathfrak{F}} \langle\langle f\rangle\rangle S \subseteq \bigcap^{\mathfrak{F}} S$. Consequently the filter objects $\mathcal{I} \cup^{\mathfrak{F}} \mathcal{J}$ and $\bigcap^{\mathfrak{F}} S$ are closed. □

Proposition 81. If S is a set of closed regarding a complete funcoid filter objects, then the filter object $\bigcup^{\mathfrak{F}} S$ is also closed regarding our funcoid.

Proof. $\langle f\rangle \bigcup^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle\langle f\rangle\rangle S \subseteq \bigcup^{\mathfrak{F}} S$ where f is the given funcoid. □

3 Reloids

Definition 82. I will call a *reloid* a filter object on the set of binary relations.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations.

Definition 83. The *reverse* reloid of a reloid f is defined by the formula

$$\operatorname{up} f^{-1} = \left\{F^{-1} \mid F \in \operatorname{up} f^{-1}\right\}.$$

3.1 Composition of reloids

Definition 84. Composition of reloids is defined by the formula

$$g \circ f = \bigcap^{\mathsf{RLD}}\{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}.$$

Composition of reloids is a reloid.

Lemma 85. $(h\circ g)\circ f = h\circ(g\circ f)$ for any reloids f, g, h.

Proof. For two nonempty collections A and B of sets I will denote

$$A\sim B\Leftrightarrow(\forall K\in A\exists L\in B\colon L\subseteq K)\wedge(\forall K\in B\exists L\in A\colon L\subseteq K).$$

It is easy to see that $\sim$ is a transitive relation.

I will denote $B\circ A=\{L\circ K|K\in A, L\in B\}$.

Let first prove that for any nonempty collections of relations A, B, C

$$A\sim B\Rightarrow A\circ C\sim B\circ C.$$

Suppose $A\sim B$ and $P\in A\circ C$ that is $K\in A$ and $M\in C$ such that $P=K\circ M$. $\exists K'\in B\colon K'\subseteq K$ because $A\sim B$. We have $P'=K'\circ M\in B\circ C$. Obviously $P'\subseteq P$. So for any $P\in A\circ C$ exist $P'\in B\circ C$ such that $P'\subseteq P$; vice verse is analogous. So $A\circ C\sim B\circ C$.

$\operatorname{up}((h\circ g)\circ f)\sim\operatorname{up}(h\circ g)\circ\operatorname{up}f$, $\operatorname{up}(h\circ g)\sim(\operatorname{up}h)\circ(\operatorname{up}g)$. By proven above $\operatorname{up}((h\circ g)\circ f)\sim(\operatorname{up}h)\circ(\operatorname{up}g)\circ(\operatorname{up}f)$.

Analogously $\operatorname{up}(h\circ(g\circ f))\sim(\operatorname{up}h)\circ(\operatorname{up}g)\circ(\operatorname{up}f)$.

So $\operatorname{up}((h\circ g)\circ f)\sim\operatorname{up}(h\circ(g\circ f))$ what is possible only if $\operatorname{up}((h\circ g)\circ f)=\operatorname{up}(h\circ(g\circ f))$. □

Conjecture 86. If f, g, h are reloids then

1. $f\circ(g\cup^{\mathsf{RLD}}h)=f\circ g\cup^{\mathsf{RLD}}f\circ h$;
2. $(g\cup^{\mathsf{RLD}}h)\circ f=g\circ f\cup^{\mathsf{RLD}}h\circ f$.

3.2 Direct product of filter objects

In theory of reloids direct product of filter objects $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula

$$\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\overset{\text{def}}{=}\bigcap^{\mathfrak{F}}\{A\times B\mid A\in\operatorname{up}\mathcal{A}, B\in\operatorname{up}\mathcal{B}\}.$$

Theorem 87. $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}=\bigcup^{\mathfrak{F}}\left\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{A}, b\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{B}\right\}$ for any $\mathcal{A},\mathcal{B}\in\mathfrak{F}$.

Proof. Obviously

$$\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\supseteq\bigcup^{\mathfrak{F}}\left\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{A}, b\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{B}\right\}$$

Reversely, let $K\in\operatorname{up}\bigcup^{\mathfrak{F}}\left\{a\times^{\mathsf{RLD}}b\mid a\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{A}, b\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{B}\right\}$. Then $K\in\operatorname{up}(a\times^{\mathsf{RLD}}b)$ for every $a\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{A}$, $b\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{B}$; $K\supseteq X_a\times^{\mathsf{RLD}}Y_b$ for some $X_a\in\operatorname{up}a$, $Y_b\in\operatorname{up}b$; $K\supseteq\bigcup\left\{X_a\times Y_b\mid a\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{A}, b\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{B}\right\}=\bigcup\left\{X_a\mid a\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{A}\right\}\times\bigcup\left\{Y_b\mid b\in\operatorname{atoms}^{\mathfrak{F}}\mathcal{A}\right\}=A\times B$ where $A\in\operatorname{up}\mathcal{A}$, $B\in\operatorname{up}\mathcal{B}$; $K\in\operatorname{up}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B})$. □

Theorem 88. $\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0\cap^{\mathsf{RLD}}\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1=(\mathcal{A}_0\cap^{\mathsf{RLD}}\mathcal{A}_1)\times^{\mathsf{RLD}}(\mathcal{B}_0\cap^{\mathsf{RLD}}\mathcal{B}_1)$ for any $\mathcal{A}_0,\mathcal{A}_1,\mathcal{B}_0,\mathcal{B}_1\in\mathfrak{F}$.

Proof.

$$\begin{aligned}
\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0\cap^{\mathsf{RLD}}\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1 &= \bigcap^{\mathsf{RLD}}\left\{P\cap Q\mid P\in\operatorname{up}(\mathcal{A}_0\times^{\mathsf{RLD}}\mathcal{B}_0), Q\in\operatorname{up}(\mathcal{A}_1\times^{\mathsf{RLD}}\mathcal{B}_1)\right\}\\
&= \bigcap^{\mathsf{RLD}}\{A_0\times B_0\cap A_1\times B_1\mid A_0\in\operatorname{up}\mathcal{A}_0, B_0\in\operatorname{up}\mathcal{B}_0, A_1\in\operatorname{up}\mathcal{A}_1,\\
&\quad B_1\in\operatorname{up}\mathcal{B}_1\}\\
&= \bigcap^{\mathsf{RLD}}\{(A_0\cap A_1)\times(B_0\cap B_1)\mid A_0\in\operatorname{up}\mathcal{A}_0, B_0\in\operatorname{up}\mathcal{B}_0, A_1\in\operatorname{up}\mathcal{A}_1,\\
&\quad B_1\in\operatorname{up}\mathcal{B}_1\}\\
&= \bigcap^{\mathsf{RLD}}\{K\times L\mid K\in\operatorname{up}(\mathcal{A}_0\cap\mathcal{A}_1), L\in\operatorname{up}(\mathcal{B}_0\cap\mathcal{B}_1)\}\\
&= (\mathcal{A}_0\cap^{\mathsf{RLD}}\mathcal{A}_1)\times^{\mathsf{RLD}}(\mathcal{B}_0\cap^{\mathsf{RLD}}\mathcal{B}_1).
\end{aligned}$$

□

Theorem 89. If $S\in\mathscr{P}\mathfrak{F}^2$ then

$$\bigcap^{\mathsf{RLD}}\left\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B}\mid(\mathcal{A};\mathcal{B})\in S\right\}=\bigcap^{\mathfrak{F}}\operatorname{dom}S\times^{\mathsf{RLD}}\bigcap^{\mathfrak{F}}\operatorname{im}S.$$

Proof. Let $\mathcal{P} = \bigcap^{\mathfrak{F}} \operatorname{dom} S$, $\mathcal{Q} = \bigcap^{\mathfrak{F}} \operatorname{im} S$; $l = \bigcap^{\mathsf{RLD}} \{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid (\mathcal{A}; \mathcal{B}) \in S\}$.

$\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q} \subseteq l$ is obvious.

Let $F \in \operatorname{up}(\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q})$. Then exist $P \in \operatorname{up} \mathcal{P}$ and $Q \in \operatorname{up} \mathcal{Q}$ such that $F \supseteq P \times Q$.

$P = P_1 \cap \ldots \cap P_n$ where $P_i \in \langle \operatorname{up} \rangle \operatorname{dom} S$ and $Q = Q_1 \cap \ldots \cap Q_m$ where $Q_i \in \langle \operatorname{up} \rangle \operatorname{im} S$.

$P \times Q = \bigcap_{i,j} (P_i \times Q_i)$.

$P_i \times Q_i \supseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some $(\mathcal{A}; \mathcal{B}) \in S$. $P \times Q = \bigcap_{i,j} (P_i \times Q_i) \supseteq l$. $F \in \operatorname{up} l$. □

Conjecture 90. If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\mathsf{RLD}}$ is a complete homomorphism of the lattice $\mathfrak{F}$ to a complete sublattice of the lattice RLD, if also $\mathcal{A} \neq \emptyset$ then it is an isomorphism.

Definition 91. I will call a reloid *convex* iff it is a union of direct products.

I will call two filter objects *isomorphic* when the corresponding filters are isomorphic (in the sense defined in [1]).

Theorem 92. The reloid $\{a\} \times^{\mathsf{RLD}} \mathcal{F}$ is isomorphic to the filter object $\mathcal{F}$ for every $a \in \mho$.

Proof. Consider $B = \{a\} \times \mho$ and $f = \{(x; (a; x)) \mid x \in \mho\}$. Then f is a bijection from $\mho$ to B.

If $X \in \operatorname{up} \mathcal{F}$ then $\langle f \rangle X \in B$ and $\langle f \rangle X = \{a\} \times X \in \operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F})$.

For every $Y \in \operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P} B$ we have $Y = \{a\} \times X$ for some $X \in \operatorname{up} \mathcal{F}$ and thus $Y = \langle f \rangle X$.

So $\langle f \rangle|_{\operatorname{up} \mathcal{F} \cap \mathscr{P} \mho} = \langle f \rangle|_{\operatorname{up} \mathcal{F}}$ is a bijection from $\operatorname{up} \mathcal{F} \cap \mathscr{P} \mho$ to $\operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P} B$.

We have $\operatorname{up} \mathcal{F} \cap \mathscr{P} \mho$ and $\operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P} B$ directly isomorphic and thus $\operatorname{up} \mathcal{F}$ is isomorphic to $\operatorname{up}(\{a\} \times^{\mathsf{RLD}} \mathcal{F})$. □

3.3 Restricting reloid to a filter object. Domain and image

Definition 93. I call restricting a reloid f to a filter object $\mathcal{A}$ as $f|_{\mathcal{A}} = f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mho$.

Definition 94. *Domain* and *image* of a reloid f are defined as follows:

$$\operatorname{dom} f = \bigcap^{\mathfrak{F}} \langle \operatorname{dom} \rangle \operatorname{up} f; \quad \operatorname{im} f = \bigcap^{\mathfrak{F}} \langle \operatorname{im} \rangle \operatorname{up} f.$$

Proposition 95. $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$.

Proof.

Direct implication. Follows from $\operatorname{dom}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{A} \wedge \operatorname{im}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{B}$.

Reverse implication. $\operatorname{dom} f \subseteq \mathcal{A} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A} \exists F \in \operatorname{up} f: \operatorname{dom} F \subseteq A$. Analogously

$$\operatorname{im} f \subseteq \mathcal{B} \Leftrightarrow \forall B \in \operatorname{up} \mathcal{B} \exists G \in \operatorname{up} f: \operatorname{im} G \subseteq B.$$

Let $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$, $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$. Then exist $F \in \operatorname{up} f$, $G \in \operatorname{up} f$ such that $\operatorname{dom} F \subseteq A \wedge \operatorname{im} G \subseteq B$. Consequently $F \cap G \in \operatorname{up} f$, $\operatorname{dom}(F \cap G) \subseteq A$, $\operatorname{im}(F \cap G) \subseteq B$ that is $F \cap G \subseteq A \times B$. We have exists $H \in \operatorname{up} f$ such that $H \subseteq A \times B$ for any $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$. So $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

Definition 96. I call *identity reloid* for a filter object $\mathcal{A}$ the reloid $I_{\mathcal{A}} \overset{\text{def}}{=} (=)|_{\mathcal{A}}$.

Theorem 97. $I_{\mathcal{A}} = \bigcap^{\mathfrak{F}} \{I_A \mid A \in \operatorname{up} \mathcal{A}\}$ where I_A is the identity relation on a set A.

Proof. Let $K \in \operatorname{up} \bigcap^{\mathfrak{F}} \{I_A \mid A \in \operatorname{up} \mathcal{A}\}$, then exists $A \in \operatorname{up} \mathcal{A}$ such that $K \supseteq I_A$. Then $(=)|_{\mathcal{A}} = (=) \cap^{\mathsf{RLD}} \mathcal{A} \times \mho \subseteq (=) \cap A \times \mho = I|_A \subseteq K$; $K \in \operatorname{up} I_{\mathcal{A}}$. Reversely let $K \in \operatorname{up} I_{\mathcal{A}} = \operatorname{up}((=) \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mho)$, then exists $A \in \operatorname{up} \mathcal{A}$ such that $K \in \operatorname{up}((=) \cap A \times \mho) = \operatorname{up} I_A \subseteq \operatorname{up} I_{\mathcal{A}}$. □

Proposition 98. $I_{\mathcal{A}}^{-1} = I_{\mathcal{A}}$.

Proof. Follows from the previous theorem. □

Theorem 99. $f|_{\mathcal{A}} = f \circ I_{\mathcal{A}}$ for any reloid f and filter object $\mathcal{A}$.

Proof. We need to prove that $f \cap^{\mathsf{RLD}} \mathcal{A} \times \mho = f \circ \bigcap^{\mathsf{RLD}} \{I_A \mid A \in \mathrm{up}\mathcal{A}\}$. $f \circ \bigcap^{\mathsf{RLD}} \{I_A \mid A \in \mathrm{up}\mathcal{A}\} = \bigcap^{\mathsf{RLD}} \{F \circ I_A \mid F \in \mathrm{up}\, f, A \in \mathrm{up}\mathcal{A}\} = \bigcap^{\mathsf{RLD}} \{F|_A \mid F \in \mathrm{up}\, f, A \in \mathrm{up}\mathcal{A}\} = \bigcap^{\mathsf{RLD}} \{F \cap A \times \mho \mid F \in \mathrm{up}\, f, A \in \mathrm{up}\mathcal{A}\} = \bigcap^{\mathsf{RLD}} \{F \mid F \in \mathrm{up}\, f\} \cap \bigcap^{\mathsf{RLD}} \{A \times \mho \mid A \in \mathrm{up}\mathcal{A}\} = f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mho$. □

Theorem 100. $(g \circ f)|_{\mathcal{A}} = g \circ (f|_{\mathcal{A}})$ for any reloids f and g and filter object $\mathcal{A}$.

Proof. $(g \circ f)|_{\mathcal{A}} = (g \circ f) \circ I_{\mathcal{A}} = g \circ (f \circ I_{\mathcal{A}}) = g \circ (f|_{\mathcal{A}})$. □

Theorem 101. $f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$ for any reloid f and filter objects $\mathcal{A}$ and $\mathcal{B}$.

Proof. $f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = f \cap^{\mathsf{RLD}} \mathcal{A} \times^{\mathsf{RLD}} \mho \cap^{\mathsf{RLD}} \mho \times^{\mathsf{RLD}} \mathcal{B} = f|_{\mathcal{A}} \cap^{\mathsf{RLD}} \mho \times \mathcal{B} = f \circ I_{\mathcal{A}} \cap^{\mathsf{RLD}} \mho \times \mathcal{B} = ((f \circ I_{\mathcal{A}})^{-1} \cap^{\mathsf{RLD}} (\mho \times^{\mathsf{RLD}} \mathcal{B})^{-1})^{-1} = (I_{\mathcal{A}} \circ f^{-1} \cap^{\mathsf{RLD}} \mathcal{B} \times^{\mathsf{RLD}} \mho)^{-1} = (I_{\mathcal{A}} \circ f^{-1} \circ I_{\mathcal{B}})^{-1} = I_{\mathcal{B}} \circ f \circ I_{\mathcal{A}}$. □

3.4 Category of reloids

I will define the category RLD of reloids:

- The set of objects is $\mathfrak{F}$.
- The set of morphisms from a filter object $\mathcal{A}$ to a filter object $\mathcal{B}$ is the set of triples $(f; \mathcal{A}; \mathcal{B})$ where f is a reloid such that $\mathrm{dom}\, f \subseteq \mathcal{A}$, $\mathrm{im}\, f \subseteq \mathcal{B}$.
- Composition of morphisms is defined in the natural way.
- Identity morphism of a filter object $\mathcal{A}$ is $(I_{\mathcal{A}}; \mathcal{A}; \mathcal{A})$.

To prove that it is really a category is trivial.

3.4.1 Monovalued reloids

Following the idea of definition of monovalued morphism in [2] let's call *monovalued* such a reloid f that $f \circ f^{-1} \subseteq I_{\mathrm{im}\, f}$.

Obvious 102. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of reloids is monovalued iff the reloid f is monovalued.

Conjecture 103. If a reloid is monovalued then it is a monovalued function restricted to some filter object.

Conjecture 104. A reloid f is monovalued iff $\forall g \in \mathsf{RLD}: (g \subseteq f \Rightarrow \exists \mathcal{A} \in \mathfrak{F}: g = f|_{\mathcal{A}})$.

Conjecture 105. A monovalued reloid restricted to an atomic filter object is atomic or empty.

A weaker conjecture:

Conjecture 106. A (monovalued) function restricted to an atomic filter object is atomic or empty.

4 Relationships of funcoids and reloids

4.1 Funcoid induced by a reloid

Every reloid f induces a funcoid $(\mathsf{FCD}) f$ by the following formulas:

$$\mathcal{X} [(\mathsf{FCD}) f] \mathcal{Y} \Leftrightarrow \forall F \in \mathrm{up}\, f: \mathcal{X} [F] \mathcal{Y}$$
$$\langle (\mathsf{FCD}) f \rangle \mathcal{X} = \bigcap^{\mathfrak{F}} \{\langle F \rangle \mathcal{X} \mid F \in \mathrm{up}\, f\}.$$

We should prove that $(\mathsf{FCD})f$ is really a funcoid. For this purpose we will additionally define

$$\left\langle (\mathsf{FCD})f^{-1}\right\rangle \mathcal{Y} = \bigcap^{\mathfrak{F}} \left\{ \left\langle F^{-1}\right\rangle \mathcal{Y} \mid F \in \operatorname{up} f \right\}.$$

Proof. We need to prove that

$$\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \langle (\mathsf{FCD})f\rangle \mathcal{X} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \left\langle (\mathsf{FCD})f^{-1}\right\rangle \mathcal{Y} \neq \emptyset.$$

The above formula is equivalent to:

$$\forall F \in \operatorname{up} f: \mathcal{X}[F]\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \{\langle F\rangle \mathcal{X} \mid F \in \operatorname{up} f\} \neq \emptyset \Leftrightarrow \mathcal{X} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \left\{ \left\langle F^{-1}\right\rangle \mathcal{Y} \mid F \in \operatorname{up} f \right\} \neq \emptyset.$$

We have $\mathcal{Y} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \{\langle F\rangle \mathcal{X} \mid F \in \operatorname{up} f\} = \bigcap^{\mathfrak{F}} \left\{ \mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle \mathcal{X} \mid F \in \operatorname{up} f \right\}$.

Let's denote $W = \left\{ \mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle \mathcal{X} \mid F \in \operatorname{up} f \right\}$.

We need to prove that $\bigcap^{\mathfrak{F}} W \neq \emptyset \Leftrightarrow \forall F \in \operatorname{up} f: \mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle \mathcal{X} \neq \emptyset$. (The rest follows from symmetry.)

Let's prove that W is a generalized filter base. [TODO: Simplify this proof.] Let $\mathcal{A}, \mathcal{B} \in W$ that is $\mathcal{A} = \mathcal{Y} \cap^{\mathfrak{F}} \langle P\rangle \mathcal{X}$, $\mathcal{B} = \mathcal{Y} \cap^{\mathfrak{F}} \langle Q\rangle \mathcal{X}$ where $P, Q \in \operatorname{up} f$. Then for $\mathcal{C} = \mathcal{Y} \cap^{\mathfrak{F}} \langle P \cap Q\rangle \mathcal{X}$ is true both $\mathcal{C} \in W$ and $\mathcal{C} \subseteq \mathcal{A}, \mathcal{B}$. So W is a generalized filter base.

From this by the corollary 4 follows that $\bigcap^{\mathfrak{F}} W \neq \emptyset \Leftrightarrow \emptyset \notin W \Leftrightarrow \forall F \in \operatorname{up} f: \mathcal{Y} \cap^{\mathfrak{F}} \langle F\rangle \mathcal{X} \neq \emptyset$. □

Theorem 107. $\mathcal{X}[(\mathsf{FCD})f]\mathcal{Y} \Leftrightarrow \mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \cap f \neq \emptyset$ for any $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$ and $f \in \mathsf{RLD}$.

Proof.

$$\begin{aligned}
\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \cap^{\mathsf{RLD}} f \neq \emptyset &\Leftrightarrow \forall P \in \operatorname{up}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}): P \cap^{\mathsf{RLD}} f \neq \emptyset \\
&\Leftrightarrow \forall F \in \operatorname{up} f, P \in \operatorname{up}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}): P \cap F \neq \emptyset \\
&\Leftrightarrow \forall F \in \operatorname{up} f, X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y}: X \times^{\mathsf{RLD}} Y \cap^{\mathsf{RLD}} F \neq \emptyset \\
&\Leftrightarrow \forall F \in \operatorname{up} f, X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y}: X[F]Y \\
&\Leftrightarrow \forall F \in \operatorname{up} f: \mathcal{X}[F]\mathcal{Y} \\
&\Leftrightarrow \mathcal{X}[(\mathsf{FCD})f]\mathcal{Y}.
\end{aligned}$$

□

Theorem 108. $(\mathsf{FCD})f = \bigcap^{\mathsf{FCD}} \operatorname{up} f$ for any reloid f.

Proof. Let a is an atomic filter object.

$((\mathsf{FCD})f)a = \bigcap^{\mathfrak{F}} \{\langle F\rangle a \mid F \in \operatorname{up} f\}$ by the definition of (FCD).

$\left\langle \bigcap^{\mathsf{FCD}} \operatorname{up} f \right\rangle a = \bigcap^{\mathfrak{F}} \{\langle F\rangle a \mid F \in \operatorname{up} f\}$ by the theorem 39.

So $\langle (\mathsf{FCD})f\rangle a = \left\langle \bigcap^{\mathsf{FCD}} \operatorname{up} f \right\rangle a$ for any atomic filter object a. □

Lemma 109. $\langle g\rangle \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle \langle g\rangle \rangle S$ if g is a funcoid and S is a filter base.

Proof. $\operatorname{up} \bigcap^{\mathfrak{F}} S = \bigcup \langle \operatorname{up}\rangle S$ by the theorem 3.

$\langle g\rangle \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle \langle g\rangle \rangle \operatorname{up} \bigcap^{\mathfrak{F}} S$ by the theorem 21.

$\bigcap^{\mathfrak{F}} \langle \langle g\rangle \rangle \operatorname{up} \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle \langle g\rangle \rangle \bigcup \langle \operatorname{up}\rangle S$.

Easy to see that $\bigcap^{\mathfrak{F}} \langle \langle g\rangle \rangle \bigcup \langle \operatorname{up}\rangle S = \bigcap^{\mathfrak{F}} \langle \langle g\rangle \rangle S$ because $S \subseteq \bigcup \langle \operatorname{up}\rangle S$.

Combining these equalities we produce $\langle g\rangle \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle \langle g\rangle \rangle S$. □

Lemma 110. For two sets of binary relations S and T and a set A

$$\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T \Rightarrow \bigcap^{\mathfrak{F}} \{\langle F\rangle A \mid F \in S\} = \bigcap^{\mathfrak{F}} \{\langle G\rangle A \mid G \in T\}$$

Proof. Let $\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} T$. Suppose $X \in \operatorname{up} \bigcap^{\mathfrak{F}} \{\langle F\rangle A \mid F \in S\}$. Then $X' \in \{\langle F\rangle A \mid F \in S\}$ where $X \supseteq X'$. That is $X' = \langle F\rangle A$ for some $F \in S$. There exists $G \in T$ such that $G \subseteq F$. So $Y' = \langle G\rangle A \subseteq X' \subseteq X$. $Y' \in \{\langle G\rangle A \mid G \in T\}$; $Y' \in \operatorname{up} \bigcap^{\mathfrak{F}} \{\langle G\rangle A \mid G \in T\}$; $X \in \operatorname{up} \bigcap^{\mathfrak{F}} \{\langle G\rangle A \mid G \in T\}$. The reverse is symmetric. □

Theorem 111. $(\mathsf{FCD})(g\circ f)=((\mathsf{FCD})g)\circ((\mathsf{FCD})f)$ for any reloids f and g.

Proof.

$$\begin{aligned}\langle(\mathsf{FCD})(g\circ f)\rangle X &= \bigcap^{\mathfrak{F}}\{\langle H\rangle X \mid H\in\operatorname{up}(g\circ f)\}\\ &= \bigcap^{\mathfrak{F}}\left\{\langle H\rangle X \mid H\in\operatorname{up}\bigcap^{\mathsf{RLD}}\{G\circ F \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}\right\}.\end{aligned}$$

Obviously

$$\bigcap^{\mathsf{RLD}}\{G\circ F \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}=\bigcap^{\mathsf{RLD}}\operatorname{up}\bigcap^{\mathsf{RLD}}\{G\circ F \mid F\in\operatorname{up}f, G\in\operatorname{up}g\};$$

from this by the lemma 110

$$\bigcap^{\mathfrak{F}}\left\{\langle H\rangle X \mid H\in\operatorname{up}\bigcap^{\mathsf{RLD}}\{G\circ F \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}\right\}=\bigcap^{\mathfrak{F}}\{\langle G\circ F\rangle X \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}.$$

On the other side

$$\begin{aligned}\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle X &= \langle(\mathsf{FCD})g\rangle\langle(\mathsf{FCD})f\rangle X\\ &= \langle(\mathsf{FCD})g\rangle\bigcap^{\mathfrak{F}}\{\langle F\rangle X \mid F\in\operatorname{up}f\}\\ &= \bigcap^{\mathfrak{F}}\left\{\langle G\rangle\bigcap^{\mathfrak{F}}\{\langle F\rangle X \mid F\in\operatorname{up}f\} \mid G\in\operatorname{up}g\right\}.\end{aligned}$$

Let's prove that $\{\langle F\rangle X \mid F\in\operatorname{up}f\}$ is a filter base. If $A,B\in\{\langle F\rangle X \mid F\in\operatorname{up}f\}$ then $A=\langle F_1\rangle X$ and $B=\langle F_2\rangle X$ where $F_1,F_2\in\operatorname{up}f$. $A\cap B\supseteq\langle F_1\cap F_2\rangle X\in\{\langle F\rangle X \mid F\in\operatorname{up}f\}$. So $\{\langle F\rangle X \mid F\in\operatorname{up}f\}$ is really a filter base.

By the lemma 109 $\langle G\rangle\bigcap^{\mathfrak{F}}\{\langle F\rangle X \mid F\in\operatorname{up}f\}=\bigcap^{\mathfrak{F}}\{\langle G\rangle\langle F\rangle X \mid F\in\operatorname{up}f\}$. So continuing the above equalities,

$$\begin{aligned}\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle X &= \bigcap^{\mathfrak{F}}\left\{\bigcap^{\mathfrak{F}}\{\langle G\rangle\langle F\rangle X \mid F\in\operatorname{up}f\} \mid G\in\operatorname{up}g\right\}\\ &= \bigcap^{\mathfrak{F}}\{\langle G\rangle\langle F\rangle X \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}\\ &= \bigcap^{\mathfrak{F}}\{\langle G\circ F\rangle X \mid F\in\operatorname{up}f, G\in\operatorname{up}g\}.\end{aligned}$$

Combining these equalities we get $\langle(\mathsf{FCD})(g\circ f)\rangle X=\langle((\mathsf{FCD})g)\circ((\mathsf{FCD})f)\rangle X$ for any set X. □

4.2 Reloids induced by funcoid

Every funcoid f induces a reloid in two ways, intersection of *outward* relations and union of *inward* direct products of filter objects:

$$(\mathsf{RLD})_{\mathrm{out}}f\overset{\mathrm{def}}{=}\bigcap^{\mathsf{RLD}}\left\{F \mid F\in\mathscr{P}\mho^2, F\supseteq f\right\}$$
$$(\mathsf{RLD})_{\mathrm{in}}f\overset{\mathrm{def}}{=}\bigcup^{\mathsf{RLD}}\left\{\mathcal{A}\times^{\mathsf{RLD}}\mathcal{B} \mid \mathcal{A},\mathcal{B}\in\mathfrak{F}, \mathcal{A}\times^{\mathsf{FCD}}\mathcal{B}\subseteq f\right\}$$

Proposition 112. $\operatorname{up}(\mathsf{RLD})_{\mathrm{out}}f=\left\{F \mid F\in\mathscr{P}\mho^2, F\supseteq f\right\}$.

Proof. Easy to prove. □

Theorem 113. $(\mathsf{RLD})_{\mathrm{in}}f=\bigcup^{\mathsf{RLD}}\left\{a\times^{\mathsf{RLD}}b \mid a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho, a\times^{\mathsf{FCD}}b\subseteq f\right\}$.

Proof. Follows from the theorem 87. □

Lemma 114. $F\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f\Leftrightarrow\forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a[f]b\Rightarrow F\supseteq a\times^{\mathsf{RLD}}b)$ for a funcoid f.

Proof.

$$\begin{aligned}F\in\operatorname{up}(\mathsf{RLD})_{\mathrm{in}}f &\Leftrightarrow F\in\operatorname{up}\bigcup^{\mathfrak{F}}\left\{a\times^{\mathsf{RLD}}b \mid a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho, a\times^{\mathsf{FCD}}b\subseteq f\right\}\\ &\Leftrightarrow \forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a\times^{\mathsf{FCD}}b\subseteq f\Rightarrow F\in\operatorname{up}(a\times^{\mathsf{RLD}}b))\\ &\Leftrightarrow \forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a\times^{\mathsf{FCD}}b\cap^{\mathsf{FCD}}f\neq\emptyset\Rightarrow F\supseteq a\times^{\mathsf{RLD}}b)\\ &\Leftrightarrow \forall a,b\in\operatorname{atoms}^{\mathfrak{F}}\mho\colon(a[f]b\Rightarrow F\supseteq a\times^{\mathsf{RLD}}b).\end{aligned}$$

□

Surprisingly a funcoid is greater inward than outward:

Theorem 115. $(\mathsf{RLD})_{\mathrm{out}} f \subseteq (\mathsf{RLD})_{\mathrm{in}} f$ for a funcoid f.

Proof. We need to prove

$$\bigcap^{\mathsf{RLD}} \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \} \subseteq \bigcup^{\mathsf{RLD}} \{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \}.$$

Let

$$K \in \mathrm{up} \bigcup^{\mathfrak{F}} \{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \}.$$

Then

$$\begin{aligned} K &= \bigcup \{ X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \} \\ &= \bigcup^{\mathsf{RLD}} \{ X_{\mathcal{A}} \times Y_{\mathcal{B}} \mid \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \} \\ &\supseteq f \end{aligned}$$

where $X_{\mathcal{A}} \in \mathrm{up}\,\mathcal{A}$, $Y_{\mathcal{B}} \in \mathrm{up}\,\mathcal{B}$. $K \in \mathrm{up} \bigcap^{\mathsf{RLD}} \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \}$. □

Theorem 116. $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}} f = f$ for any funcoid f.

Proof. $\mathrm{up}\,(\mathsf{RLD})_{\mathrm{out}} f = \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \}$

$(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}} f = \bigcap^{\mathsf{FCD}} \mathrm{up} (\mathsf{RLD})_{\mathrm{out}} f = \bigcap^{\mathsf{FCD}} \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \}$.

$\bigcap^{\mathsf{FCD}} \{ F \mid F \in \mathscr{P}\mho^2, F \supseteq f \} = f$ by the theorem 70. So $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}} f = f$. □

Conjecture 117. $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$ for any funcoid f.

Conjecture 118. For any funcoid f and reloid g

$$(\mathsf{RLD})_{\mathrm{out}} f \subseteq g \subseteq (\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow (\mathsf{FCD}) g = f.$$

Conjecture 119. For a convex reloid f

1. $(\mathsf{RLD})_{\mathrm{out}} (\mathsf{FCD}) f = f$;
2. $(\mathsf{RLD})_{\mathrm{in}} (\mathsf{FCD}) f = f$.

Appendix A Some counter-examples

[TODO: More counter-examples similar to examples in [1].]

Theorem 120. For a f.o. a we have $a \times^{\mathsf{RLD}} a \subseteq (=)|_{\mho}$ only in the case if $a = \emptyset$ or a is a trivial atomic f.o. (that is an one-element set).

Proof. If $a \times^{\mathsf{RLD}} a \subseteq (=)|_{\mho}$ then exists $m \in \mathrm{up}(a \times^{\mathsf{RLD}} a)$ such that $m \subseteq (=)|_{\mho}$. Consequently exist $A, B \in \mathrm{up}\,a$ such that $A \times B \subseteq (=)|_{\mho}$ what is possible only in the case when $A = B = a$ is an one-element set or empty set. □

Corollary 121. Direct product (in the sense of reloids) of non-trivial atomic filter objects is non-atomic.

Proof. Obviously $(a \times^{\mathsf{RLD}} a) \cap^{\mathsf{RLD}} (=)|_{\mho} \neq \emptyset$ and $(a \times^{\mathsf{RLD}} a) \cap^{\mathsf{RLD}} (=)|_{\mho} \subset a \times^{\mathsf{RLD}} a$. □

Example 122. There exist two atomic reloids whose composition is non-atomic and non-empty.

Proof. Let a is a non-trivial atomic filter object and $x \in \mho$. Then

$$(a \times \{x\}) \circ (\{x\} \times a) = \bigcap^{\mathfrak{F}} \{ (A \times \{x\}) \circ (\{x\} \times A) \mid A \in \mathrm{up}\,a \} = \bigcap^{\mathfrak{F}} \{ A \times A \mid A \in \mathrm{up}\,a \} = a \times a$$

is non-atomic despite of $a \times \{x\}$ and $\{x\} \times a$ are atomic. □

Example 123. There exists non-monovalued atomic reloid.

Proof. From the previous example follows that the atomic reloid $\{x\} \times a$ is not monovalued. □

Bibliography

[1] Victor Porton. Filters on posets and generalizations. At http://www.mathematics21.org/binaries/filters.pdf.

[2] Victor Porton. Partially ordered dagger categories. At http://www.mathematics21.org/binaries/ordered-cats-with-inv.pdf.