Funcoids and Reloids: a Generalization of Proximities and Uniformities*

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Abstract

It is a part of my Algebraic General Topology research.

In this article, I introduce the concepts of **funcoids**, which generalize proximity spaces and **reloids**, which generalize uniform spaces. The concept of a funcoid is a generalized concept of proximity, the concept of a reloid is the concept of uniformity cleared (generalized) from superfluous details. Also funcoids generalize pretopologies and preclosures. Also funcoids and reloids are generalizations of binary relations whose domains and ranges are filters (instead of sets).

Also funcoids and reloids can be considered as a generalization of (directed) graphs, this provides us a common generalization of analysis and discrete mathematics.

The concept of continuity is defined by an algebraic formula (instead of the old messy epsilon-delta notation) for arbitrary morphisms (including funcoids and reloids) of a partially ordered category. In one formula continuity, proximity continuity, and uniform continuity are generalized.

Contents

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1 Common

1.1 Earlier works

Some mathematicians researched generalizations of proximities and uniformities before me but they have failed to reach the right degree of generalization which is presented in this work allowing to represent properties of spaces with algebraic (or categorical) formulas.

Proximity structures were introduced by Smirnov in [5].

Some references to predecessors:

- In [6], [7], [12], [2], [19] are studied generalized uniformities and proximities.
- Proximities and uniformities are also studied in [10], [11], [18], [20], [21].
- [8] and [9] contains recent progress in quasi-uniform spaces. [9] has a very long list of related literature.

Some works ([17]) about proximity spaces consider relationships of proximities and compact topological spaces. In this work the attempt to define or research their generalization, compactness of funcoids or reloids, is not done. It seems potentially productive to attempt to borrow the definitions and procedures from the above mentioned works. I hope to do this study in a separate article.

[4] studies mappings between proximity structures. (In this work no attempt to research mappings between funcoids is done.) [13] researches relationships of quasi-uniform spaces and topological spaces. [1] studies how proximity structures can be treated as uniform structures and compactification regarding proximity and uniform spaces.

1.2 Used concepts, notation and statements

The set of functions from a set A to a set B is denoted as B^A.

I will often skip parentheses and write fx instead of $f(x)$ to denote the result of a function f acting on the argument x.

I will call **small** sets members of some Grothendieck universe. (Let us assume the axiom of existence of a Grothendieck universe.)

Let f is a small binary relation.

I will denote $\langle f\rangle X = \{f\alpha \quad | \quad \alpha \in X\}$ and $X \left[f\right] Y \Leftrightarrow \exists x \in X, y \in Y : x \mathrel{f} y$ for small sets X, Y.

By just $\langle f\rangle$ and $[f]$ I will denote the corresponding function and relation on small sets.

$\lambda x \in D : f(x) = \{(x; f(x)) \mid x \in D\}$ for every formula $f(x)$ depended on a variable x and set D.

I will denote the least and the greatest element of a poset $\mathfrak{A}$ as $0^{\mathfrak{A}}$ and $1^{\mathfrak{A}}$ respectively.

For elements a and b of a lattice with a minimal element I will denote $a \asymp b$ when $a \cap b$ is the minimal element of the lattice and $a \not\asymp b$ otherwise. See [15] for a more general notion.

Proposition 1 *Let f, g, h be binary relations. Then $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}$.*

Proof

$$\begin{aligned}
g \circ f \not\asymp h &\Leftrightarrow \\
\exists a, c : a \left((g \circ f) \cap h\right) c &\Leftrightarrow \\
\exists a, c : (a\ (g \circ f)\ c \wedge a\ h\ c) &\Leftrightarrow \\
\exists a, b, c : (a\ f\ b \wedge b\ g\ c \wedge a\ h\ c) &\Leftrightarrow \\
\exists b, c : \left(b\ g\ c \wedge b \left(h \circ f^{-1}\right) c\right) &\Leftrightarrow \\
\exists b, c : b \left(g \cap \left(h \circ f^{-1}\right)\right) c &\Leftrightarrow \\
g \not\asymp h \circ f^{-1}. &
\end{aligned}$$

□

1.2.1 Filters

In this work the word **filter** will refer to a filter on a set (in contrast to [15] where filters are considered on arbitrary posets). Note that I do not require filters to be proper.

I will call the set of filters on a set A (**base set**) ordered reverse to set-theoretic inclusion of filters **the set of filter objects** on A and denote it $\mathfrak{F}(A)$ or just $\mathfrak{F}$ when the base set is implied and call its element **filter objects** (f.o. for short). I will denote $\operatorname{up}\mathcal{F}$ the filter corresponding to a filter object $\mathcal{F}$. So we have $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \operatorname{up}\mathcal{A} \supseteq \operatorname{up}\mathcal{B}$ for every filter objects $\mathcal{A}$ and $\mathcal{B}$ on the same set.

In this particular manuscript, we will *not* equate principal filter objects with corresponding sets as it is done in [15]. Instead we will have $\operatorname{Base}(\mathcal{A})$ equal to the unique base of a f.o. $\mathcal{A}$. I will denote $\uparrow^A X$ (or just $\uparrow X$ when A is implied) the principal filter object on A corresponding to the set X.

Filters are studied in the work [15].

Every set $\mathfrak{F}(A)$ is a complete lattice and we will apply lattice operations to subsets of such sets without explicitly mentioning $\mathfrak{F}(A)$.

Prior reading of [15] is needed to fully understand this work.

Filter objects corresponding to ultrafilters are atoms of the lattice $\mathfrak{F}(A)$ and will be called **atomic filter objects** (on A).

Also we will need to introduce the concept of *generalized filter base*.

Definition 1 ***Generalized filter base*** *is a set* $S \in \mathscr{P}\mathfrak{F} \setminus \{0^{\mathfrak{F}}\}$ *such that*

$$\forall \mathcal{A}, \mathcal{B} \in S \exists \mathcal{C} \in S : \mathcal{C} \subseteq \mathcal{A} \cap \mathcal{B}.$$

Proposition 2 *Let* S *is a generalized filter base. If* $\mathcal{A}_1, \ldots, \mathcal{A}_n \in S$ $(n \in \mathbb{N})$*, then*

$$\exists \mathcal{C} \in S : \mathcal{C} \subseteq \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_n.$$

Proof Can be easily proved by induction. □

Theorem 1 *If* S *is a generalized filter base, then* $\operatorname{up} \bigcap S = \bigcup \langle \operatorname{up} \rangle S$*.*

Proof Obviously $\operatorname{up} \bigcap S \supseteq \bigcup \langle \operatorname{up} \rangle S$. Reversely, let $K \in \operatorname{up} \bigcap S$; then $K = A_1 \cap \ldots \cap A_n$ where $A_i \in \operatorname{up} \mathcal{A}_i$ where $\mathcal{A}_i \in S$, $i = 1, \ldots, n$, $n \in \mathbb{N}$; so exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \mathcal{A}_1 \cap \ldots \cap \mathcal{A}_n \subseteq \uparrow (A_1 \cap \ldots \cap A_n) = \uparrow K$, $K \in \operatorname{up} \mathcal{C}$, $K \in \bigcup \langle \operatorname{up} \rangle S$. □

Corollary 1 *If* S *is a generalized filter base, then* $\bigcap S = 0^{\mathfrak{F}} \Leftrightarrow 0^{\mathfrak{F}} \in S$*.*

Proof $\bigcap S = 0^{\mathfrak{F}} \Leftrightarrow \emptyset \in \operatorname{up} \bigcap S \Leftrightarrow \emptyset \in \bigcup \langle \operatorname{up} \rangle S \Leftrightarrow \exists \mathcal{X} \in S : \emptyset \in \operatorname{up} \mathcal{X} \Leftrightarrow 0^{\mathfrak{F}} \in S$. □

Obvious 1. If S is a filter base on a set A then $\langle \uparrow^A \rangle S$ is a generalized filter base.

Definition 2 *I will call* ***shifted filtrator*** *a triple* $(\mathfrak{A}; \mathfrak{Z}; \uparrow)$ *where* $\mathfrak{A}$ *and* $\mathfrak{Z}$ *are posets and* $\uparrow$ *is an order embedding from* $\mathfrak{Z}$ *to* $\mathfrak{A}$*.*

Some concepts and notation can be defined for shifted filtrators through similar concepts for filtrators: $\langle \uparrow \rangle \operatorname{up} a = \operatorname{up}^{(\mathfrak{A}; \langle \uparrow \rangle \mathfrak{Z})} a$; $\langle \uparrow \rangle \operatorname{Cor} a = \operatorname{Cor}^{(\mathfrak{A}; \langle \uparrow \rangle \mathfrak{Z})} a$, etc.

For a set $\mathfrak{A}$ and the set of f.o. $\mathfrak{F}$ on this set we will consider the shifted filtrator $(\mathfrak{F}; \mathfrak{A}; \uparrow)$.

2 Partially ordered dagger categories

2.1 Partially ordered categories

Definition 3 *I will call a* ***partially ordered (pre)category*** *a (pre)category together with partial order* $\subseteq$ *on each of its Hom-sets with the additional requirement that*

$$f_1 \subseteq f_2 \wedge g_1 \subseteq g_2 \Rightarrow g_1 \circ f_1 \subseteq g_2 \circ f_2$$

for every morphisms f_1*,* g_1*,* f_2*,* g_2 *such that* $\operatorname{Src} f_1 = \operatorname{Src} f_2 \wedge \operatorname{Dst} f_1 = \operatorname{Dst} f_2 = \operatorname{Src} g_1 = \operatorname{Src} g_2 \wedge \operatorname{Dst} g_1 = \operatorname{Dst} g_2$*.*

2.2 Dagger categories

Definition 4 *I will call a **dagger precategory** a precategory together with an involutive contravariant identity-on-objects prefunctor $x \mapsto x^\dagger$.*

*In other words, a **dagger precategory** is a precategory equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g:*

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$.

Definition 5 *I will call a **dagger category** a category together with an involutive contravariant identity-on-objects functor $x \mapsto x^\dagger$.*

*In other words, a **dagger category** is a category equipped with a function $x \mapsto x^\dagger$ on its set of morphisms which reverses the source and the destination and is subject to the following identities for every morphisms f and g and object A:*

1. $f^{\dagger\dagger} = f$;
2. $(g \circ f)^\dagger = f^\dagger \circ g^\dagger$;
3. $(1_A)^\dagger = 1_A$.

Theorem 2 *If a category is a dagger precategory then it is a dagger category.*

Proof We need to prove only that $(1_A)^\dagger = 1_A$. Really

$$(1_A)^\dagger = (1_A)^\dagger \circ 1_A = (1_A)^\dagger \circ (1_A)^{\dagger\dagger} = ((1_A)^\dagger \circ 1_A)^\dagger = (1_A)^{\dagger\dagger} = 1_A.$$

□

For a partially ordered dagger (pre)category I will additionally require (for every morphisms f and g)

$$f^\dagger \subseteq g^\dagger \Leftrightarrow f \subseteq g.$$

An example of dagger category is the category **Rel** whose objects are sets and whose morphisms are binary relations between these sets with usual composition of binary relations and with $f^\dagger = f^{-1}$.

Definition 6 *A morphism f of a dagger category is called **unitary** when it is an isomorphism and $f^\dagger = f^{-1}$.*

Definition 7 ***Symmetric** (endo)morphism of a dagger precategory is such a morphism f that $f = f^\dagger$.*

Definition 8 ***Transitive*** *(endo)morphism of a precategory is such a morphism f that $f = f \circ f$.*

Theorem 3 *The following conditions are equivalent for a morphism f of a dagger precategory:*

1. *f is symmetric and transitive.*

2. *$f = f^\dagger \circ f$.*

Proof

(1)⇒(2) If f is symmetric and transitive then $f^\dagger \circ f = f \circ f = f$.

(2)⇒(1) $f^\dagger = (f^\dagger \circ f)^\dagger = f^\dagger \circ f^{\dagger\dagger} = f^\dagger \circ f = f$, so f is symmetric. $f = f^\dagger \circ f = f \circ f$, so f is transitive.

□

2.2.1 Some special classes of morphisms

Definition 9 *For a partially ordered dagger category I will call* ***monovalued*** *morphism such a morphism f that $f \circ f^\dagger \subseteq 1_{\operatorname{Dst} f}$.*

Definition 10 *For a partially ordered dagger category I will call* ***entirely defined*** *morphism such a morphism f that $f^\dagger \circ f \supseteq 1_{\operatorname{Src} f}$.*

Definition 11 *For a partially ordered dagger category I will call* ***injective*** *morphism such a morphism f that $f^\dagger \circ f \subseteq 1_{\operatorname{Src} f}$.*

Definition 12 *For a partially ordered dagger category I will call* ***surjective*** *morphism such a morphism f that $f \circ f^\dagger \supseteq 1_{\operatorname{Dst} f}$.*

Remark 1 It's easy to show that this is a generalization of monovalued, entirely defined, injective, and surjective binary relations as morphisms of the category **Rel**.

Obvious 2. "Injective morphism" is a dual of "monovalued morphism" and "surjective morphism" is a dual of "entirely defined morphism".

Definition 13 *For a given partially ordered dagger category C the* ***category of monovalued (entirely defined, injective, surjective) morphisms*** *of C is the category with the same set of objects as of C and the set of morphisms being the set of monovalued (entirely defined, injective, surjective) morphisms of C with the composition of morphisms the same as in C.*

We need to prove that these are really categories, that is that composition of monovalued (entirely defined, injective, surjective) morphisms is monovalued (entirely defined, injective, surjective) and that identity morphisms are monovalued, entirely defined, injective, and surjective.

Proof We will prove only for monovalued morphisms and entirely defined morphisms, as injective and surjective morphisms are their duals.

Monovalued Let f and g are monovalued morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. $(g \circ f) \circ (g \circ f)^\dagger = g \circ f \circ f^\dagger \circ g^\dagger \subseteq g \circ 1_{\operatorname{Dst} f} \circ g^\dagger = g \circ 1_{\operatorname{Src} g} \circ g^\dagger = g \circ g^\dagger \subseteq 1_{\operatorname{Dst} g} = 1_{\operatorname{Dst}(g \circ f)}$. So $g \circ f$ is monovalued.

That identity morphisms are monovalued follows from the following: $1_A \circ (1_A)^\dagger = 1_A \circ 1_A = 1_A = 1_{\operatorname{Dst} 1_A} \subseteq 1_{\operatorname{Dst} 1_A}$.

Entirely defined Let f and g are entirely defined morphisms, $\operatorname{Dst} f = \operatorname{Src} g$. $(g \circ f)^\dagger \circ (g \circ f) = f^\dagger \circ g^\dagger \circ g \circ f \supseteq f^\dagger \circ 1_{\operatorname{Src} g} \circ f = f^\dagger \circ 1_{\operatorname{Dst} f} \circ f = f^\dagger \circ f \supseteq 1_{\operatorname{Src} f} = 1_{\operatorname{Src}(g \circ f)}$. So $g \circ f$ is entirely defined.

That identity morphisms are entirely defined follows from the following: $(1_A)^\dagger \circ 1_A = 1_A \circ 1_A = 1_A = 1_{\operatorname{Src} 1_A} \supseteq 1_{\operatorname{Src} 1_A}$.

□

Definition 14 *I will call a **bijective** morphism a morphism which is entirely defined, monovalued, injective, and surjective.*

Obvious 3. Bijective morphisms form a full subcategory.

Proposition 3 *If a morphism is bijective then it is an isomorphism.*

Proof Let f is bijective. Then $f \circ f^\dagger \subseteq 1_{\operatorname{Dst} f}$, $f^\dagger \circ f \supseteq 1_{\operatorname{Src} f}$, $f^\dagger \circ f \subseteq 1_{\operatorname{Src} f}$, $f \circ f^\dagger \supseteq 1_{\operatorname{Dst} f}$. Thus $f \circ f^\dagger = 1_{\operatorname{Dst} f}$ and $f^\dagger \circ f = 1_{\operatorname{Src} f}$ that is $f^\dagger$ is an inverse of f. □

3 Funcoids

3.1 Informal introduction into funcoids

Funcoids are a generalization of proximity spaces and a generalization of pretopological spaces. Also funcoids are a generalization of binary relations.

That funcoids are a common generalization of “spaces” (proximity spaces, (pre)topological spaces) and binary relations (including monovalued functions) makes them smart for describing properties of functions in regard of spaces. For example the statement “f is a continuous function from a space μ to a space ν” can be described in terms of funcoids as the formula $f \circ \mu \subseteq \nu \circ f$ (see below for details).

Most naturally funcoids appear as a generalization of proximity spaces.

Let δ be a proximity that is certain binary relation so that $A\ \delta\ B$ is defined for every sets A and B. We will extend it from sets to filter objects by the formula:

$$\mathcal{A}\ \delta'\ \mathcal{B} \Leftrightarrow \forall A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B} : A\ \delta\ B.$$

Then (as it will be proved below) there exist two functions $\alpha, \beta \in \mathfrak{F}^{\mathfrak{F}}$ such that

$$\mathcal{A}\ \delta'\ \mathcal{B} \Leftrightarrow \mathcal{B} \cap^{\mathfrak{F}} \alpha\mathcal{A} \neq 0^{\mathfrak{F}} \Leftrightarrow \mathcal{A} \cap^{\mathfrak{F}} \beta\mathcal{B} \neq 0^{\mathfrak{F}}.$$

The pair $(\alpha; \beta)$ is called **funcoid** when $\mathcal{B} \cap^{\mathfrak{F}} \alpha\mathcal{A} \neq 0^{\mathfrak{F}} \Leftrightarrow \mathcal{A} \cap^{\mathfrak{F}} \beta\mathcal{B} \neq 0^{\mathfrak{F}}$. So funcoids are a generalization of proximity spaces.

Funcoids consist of two components the first α and the second β. The first component of a funcoid f is denoted as $\langle f \rangle$ and the second component is denoted as $\langle f^{-1} \rangle$. (The similarity of this notation with the notation for the image of a set under a function is not a coincidence, we will see that in the case of principal funcoids (see below) these coincide.)

One of the most important properties of a funcoid is that it is uniquely determined by just one of its components. That is a funcoid f is uniquely determined by the function $\langle f \rangle$. Moreover a funcoid f is uniquely determined by $\langle f \rangle|_{\mathscr{P} \bigcup \operatorname{dom}\langle f \rangle}$ that is by values of function $\langle f \rangle$ on sets (if we equate principal filters with sets).

Next we will consider some examples of funcoids determined by specified values of the first component on sets.

Funcoids as a generalization of pretopological spaces: Let α be a pretopological space that is a map $\alpha \in \mathfrak{F}^{\mho}$ for some set $\mho$. Then we define $\alpha' X \stackrel{\text{def}}{=} \bigcup^{\mathfrak{F}} \{ \alpha x \mid x \in X \}$ for every set $X \in \mathscr{P}\mho$. We will prove that there exists a unique funcoid f such that $\alpha' = \langle f \rangle|_{\mathscr{P}\mho}$. So funcoids are a generalization of pretopological spaces. Funcoids are also a generalization of preclosure operators: For every preclosure operator p on a set $\mho$ it exists a unique funcoid f such that $\langle f \rangle|_{\mathscr{P}\mho} = \uparrow \circ p$.

For every binary relation p on a set $\mho$ it exists unique funcoid f such that $\forall X \in \mathscr{P}\mho : \langle f \rangle \uparrow X = \uparrow \langle p \rangle X$ (where $\langle p \rangle$ is defined in the introduction), recall that a funcoid is uniquely determined by the values of its first component on sets. I will call such funcoids **principal**. So funcoids are a generalization of binary relations.

Composition of binary relations (i.e. of principal funcoids) complies with the formulas:

$$\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle \quad \text{and} \quad \left\langle (g \circ f)^{-1} \right\rangle = \left\langle f^{-1} \right\rangle \circ \left\langle g^{-1} \right\rangle.$$

By the same formulas we can define composition of every two funcoids. Funcoids with this composition form a category (**the category of funcoids**).

Also funcoids can be reversed (like reversal of X and Y in a binary relation) by the formula $(\alpha; \beta)^{-1} = (\beta; \alpha)$. In particular case if μ is a proximity we have $\mu^{-1} = \mu$ because proximities are symmetric.

Funcoids behave similarly to (multivalued) functions but acting on filter objects instead of acting on sets. Below these will be defined domain and image of a funcoid (the domain and the image of a funcoid are filter objects).

3.2 Basic definitions

Definition 15 *Let's call* ***a funcoid*** *from a set A to a set B a quadruple $(A; B; \alpha; \beta)$ where $\alpha \in \mathfrak{F}(B)^{\mathfrak{F}(A)}$, $\beta \in \mathfrak{F}(A)^{\mathfrak{F}(B)}$ such that*

$$\forall \mathcal{X} \in \mathfrak{F}(A), \mathcal{Y} \in \mathfrak{F}(B) : (\mathcal{Y} \not\asymp \alpha\mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \beta\mathcal{Y}).$$

Further we will assume that all funcoids in consideration are small without mentioning it explicitly.

Definition 16 ***Source*** *and* ***destination*** *of every funcoid $(A; B; \alpha; \beta)$ are defined as*

$$\operatorname{Src}(A; B; \alpha; \beta) = A \quad \text{and} \quad \operatorname{Dst}(A; B; \alpha; \beta) = B.$$

I will denote $\mathsf{FCD}(A; B)$ the set of funcoids from A to B.

I will denote FCD the set of all funcoids (for small sets).

Definition 17 $\langle (A; B; \alpha; \beta) \rangle \stackrel{\text{def}}{=} \alpha$ *for a funcoid $(A; B; \alpha; \beta)$.*

Definition 18 $(A; B; \alpha; \beta)^{-1} = (B; A; \beta; \alpha)$ *for a funcoid $(A; B; \alpha; \beta)$.*

Proposition 4 *If f is a funcoid then f^{-1} is also a funcoid.*

Proof It follows from symmetry in the definition of funcoid. $\square$

Obvious 4. $(f^{-1})^{-1} = f$ for a funcoid f.

Definition 19 *The relation $[f] \in \mathscr{P}(\mathfrak{F}(\operatorname{Src} f) \times \mathfrak{F}(\operatorname{Dst} f))$ is defined (for every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$) by the formula $\mathcal{X}\ [f]\ \mathcal{Y} \stackrel{\text{def}}{=} \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$.*

Obvious 5. $\mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \mathcal{Y} \not\asymp \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X} \not\asymp \langle f^{-1} \rangle \mathcal{Y}$ for every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$.

Obvious 6. $[f^{-1}] = [f]^{-1}$ for a funcoid f.

Theorem 4 *Let A, B are small sets.*

1. *For given value of $\langle f \rangle$ exists no more than one funcoid $f \in \mathsf{FCD}(A; B)$.*
2. *For given value of $[f]$ exists no more than one funcoid $f \in \mathsf{FCD}(A; B)$.*

Proof Let $f, g \in \mathsf{FCD}(A; B)$.

Obviously $\langle f \rangle = \langle g \rangle \Rightarrow [f] = [g]$ and $\langle f^{-1} \rangle = \langle g^{-1} \rangle \Rightarrow [f] = [g]$. So it's enough to prove that $[f] = [g] \Rightarrow \langle f \rangle = \langle g \rangle$.

Provided that $[f] = [g]$ we have $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X} \Leftrightarrow \mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \mathcal{X}\ [g]\ \mathcal{Y} \Leftrightarrow \mathcal{Y} \not\asymp \langle g \rangle \mathcal{X}$ and consequently $\langle f \rangle \mathcal{X} = \langle g \rangle \mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(A)$ and $\mathcal{Y} \in \mathfrak{F}(B)$ because a set of filter objects is separable [15], thus $\langle f \rangle = \langle g \rangle$. $\square$

Proposition 5 *$\langle f\rangle 0^{\mathfrak{F}(\operatorname{Src} f)} = 0^{\mathfrak{F}(\operatorname{Dst} f)}$ for every funcoid f.*

Proof $\mathcal{Y} \not\asymp \langle f\rangle 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow 0^{\mathfrak{F}(\operatorname{Src} f)} \not\asymp \langle f^{-1}\rangle \mathcal{Y} \Leftrightarrow 0 \Leftrightarrow \mathcal{Y} \not\asymp 0^{\mathfrak{F}(\operatorname{Dst} f)}$. Thus $\langle f\rangle 0^{\mathfrak{F}(\operatorname{Src} f)} = 0^{\mathfrak{F}(\operatorname{Dst} f)}$ by separability of filter objects. □

Proposition 6 *$\langle f\rangle (\mathcal{I} \cup \mathcal{J}) = \langle f\rangle \mathcal{I} \cup \langle f\rangle \mathcal{J}$ for every funcoid f and $\mathcal{I}, \mathcal{J} \in \mathfrak{F}(\operatorname{Src} f)$.*

Proof

$$\begin{aligned}
\star \langle f\rangle (\mathcal{I} \cup \mathcal{J}) &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \not\asymp \langle f\rangle (\mathcal{I} \cup \mathcal{J})\} &= \\
\left\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{I} \cup \mathcal{J} \not\asymp \langle f^{-1}\rangle \mathcal{Y}\right\} &= \quad \text{(by corollary 10 in [15])} \\
\left\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{I} \not\asymp \langle f^{-1}\rangle \mathcal{Y} \vee \mathcal{J} \not\asymp \langle f^{-1}\rangle \mathcal{Y}\right\} &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \not\asymp \langle f\rangle \mathcal{I} \vee \mathcal{Y} \not\asymp \langle f\rangle \mathcal{J}\} &= \\
\{\mathcal{Y} \in \mathfrak{F} \mid \mathcal{Y} \not\asymp \langle f\rangle \mathcal{I} \cup \langle f\rangle \mathcal{J}\} &= \\
\star(\langle f\rangle \mathcal{I} \cup \langle f\rangle \mathcal{J}).&
\end{aligned}$$

Thus $\langle f\rangle (\mathcal{I} \cup \mathcal{J}) = \langle f\rangle \mathcal{I} \cup \langle f\rangle \mathcal{J}$ because $\mathfrak{F}(\operatorname{Dst} f)$ is separable. □

Proposition 7 *For every $f \in \mathsf{FCD}(A; B)$ for every small sets A and B we have:*

1. *$\mathcal{K}\,[f]\,\mathcal{I} \cup \mathcal{J} \Leftrightarrow \mathcal{K}\,[f]\,\mathcal{I} \vee \mathcal{K}\,[f]\,\mathcal{J}$ for every $\mathcal{I}, \mathcal{J} \in \mathfrak{F}(B)$, $\mathcal{K} \in \mathfrak{F}(A)$.*

2. *$\mathcal{I} \cup \mathcal{J}\,[f]\,\mathcal{K} \Leftrightarrow \mathcal{I}\,[f]\,\mathcal{K} \vee \mathcal{J}\,[f]\,\mathcal{K}$ for every $\mathcal{I}, \mathcal{J} \in \mathfrak{F}(A)$, $\mathcal{K} \in \mathfrak{F}(B)$.*

Proof 1. $\mathcal{K}\,[f]\,\mathcal{I} \cup \mathcal{J} \Leftrightarrow (\mathcal{I} \cup \mathcal{J}) \cap \langle f\rangle \mathcal{K} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow (\mathcal{I} \cap \langle f\rangle \mathcal{K}) \cup (\mathcal{J} \cap \langle f\rangle \mathcal{K}) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{I} \cap \langle f\rangle \mathcal{K} \neq 0^{\mathfrak{F}(B)} \vee \mathcal{J} \cap \langle f\rangle \mathcal{K} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{K}\,[f]\,\mathcal{I} \vee \mathcal{K}\,[f]\,\mathcal{J}$.

2. Similar. □

3.2.1 Composition of funcoids

Definition 20 *Funcoids f and g are **composable** when $\operatorname{Dst} f = \operatorname{Src} g$.*

Definition 21 ***Composition*** *of composable funcoids is defined by the formula*

$$(B; C; \alpha_2; \beta_2) \circ (A; B; \alpha_1; \beta_1) = (A; C; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2).$$

Proposition 8 *If f, g are composable funcoids then $g \circ f$ is a funcoid.*

Proof Let $f = (A; B; \alpha_1; \beta_1)$, $g = (B; C; \alpha_2; \beta_2)$. For every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(C)$ we have

$$\mathcal{Y} \not\asymp (\alpha_2 \circ \alpha_1)\mathcal{X} \Leftrightarrow \mathcal{Y} \not\asymp \alpha_2\alpha_1\mathcal{X} \Leftrightarrow \alpha_1\mathcal{X} \not\asymp \beta_2\mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp \beta_1\beta_2\mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp (\beta_1 \circ \beta_2)\mathcal{Y}.$$

So $(A; C; \alpha_2 \circ \alpha_1; \beta_1 \circ \beta_2)$ is a funcoid. $\square$

Obvious 7. $\langle g \circ f \rangle = \langle g \rangle \circ \langle f \rangle$ for every composable funcoids f and g.

Proposition 9 *$(h \circ g) \circ f = h \circ (g \circ f)$ for every composable funcoids f, g, h.*

Proof

$\langle (h \circ g) \circ f \rangle = \langle h \circ g \rangle \circ \langle f \rangle = (\langle h \rangle \circ \langle g \rangle) \circ \langle f \rangle = \langle h \rangle \circ (\langle g \rangle \circ \langle f \rangle) = \langle h \rangle \circ \langle g \circ f \rangle = \langle h \circ (g \circ f) \rangle$. $\square$

Theorem 5 *$(g \circ f)^{-1} = f^{-1} \circ g^{-1}$ for every composable funcoids f and g.*

Proof $\left\langle (g \circ f)^{-1} \right\rangle = \left\langle f^{-1} \right\rangle \circ \left\langle g^{-1} \right\rangle = \left\langle f^{-1} \circ g^{-1} \right\rangle$. $\square$

3.3 Funcoid as continuation

Let f is a funcoid.

Definition 22 *$\langle f \rangle^*$ is the function $\mathscr{P}(\operatorname{Src} f) \to \mathfrak{F}(\operatorname{Dst} f)$ defined by the formula*

$$\langle f \rangle^* X = \langle f \rangle \uparrow^{\operatorname{Src} f} X.$$

Definition 23 *$[f]^*$ is the relation between $\mathscr{P}(\operatorname{Src} f)$ and $\mathscr{P}(\operatorname{Dst} f)$ defined by the formula*

$$X \, [f]^* \, Y = \uparrow^{\operatorname{Src} f} X \, [f] \uparrow^{\operatorname{Dst} f} Y.$$

Obvious 8.

1. $\langle f \rangle^* = \langle f \rangle \circ \uparrow^{\operatorname{Src} f}$;

2. $[f]^* = \left(\uparrow^{\operatorname{Dst} f}\right)^{-1} \circ [f] \circ \uparrow^{\operatorname{Src} f}$.

Theorem 6 *For every funcoid f and $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$ and $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$*

1. $\langle f \rangle \mathcal{X} = \bigcap \left\langle \langle f \rangle^* \right\rangle \operatorname{up} \mathcal{X}$;

2. $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \, [f]^* \, Y$.

Proof 2. $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \forall Y \in \operatorname{up} \mathcal{Y} : \uparrow^{\operatorname{Dst} f} Y \cap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \forall Y \in \operatorname{up} \mathcal{Y} : \mathcal{X} \, [f] \uparrow^{\operatorname{Dst} f} Y$.

Analogously $\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X} : \uparrow^{\operatorname{Src} f} X \, [f] \, \mathcal{Y}$. Combining these two equivalences we get

$$\mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : \uparrow^{\operatorname{Src} f} X \, [f] \uparrow^{\operatorname{Dst} f} Y \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : X \, [f]^* \, Y.$$

1. $\mathcal{Y} \cap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{X} \, [f] \, \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X} : \uparrow^{\operatorname{Src} f} X \, [f] \, \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up} \mathcal{X} : \mathcal{Y} \cap \langle f \rangle^* X \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$.

Let's denote $W = \left\{ \mathcal{Y} \cap \langle f\rangle^* X \quad | \quad X \in \operatorname{up} \mathcal{X} \right\}$. We will prove that W is a generalized filter base. To prove this it is enough to show that $V = \left\{ \langle f\rangle^* X \quad | \quad X \in \operatorname{up} \mathcal{X} \right\}$ is a generalized filter base.

Let $\mathcal{P}, \mathcal{Q} \in V$. Then $\mathcal{P} = \langle f\rangle^* A$, $\mathcal{Q} = \langle f\rangle^* B$ where $A, B \in \operatorname{up} \mathcal{X}$; $A \cap B \in \operatorname{up} \mathcal{X}$ and $\mathcal{R} \subseteq \mathcal{P} \cap \mathcal{Q}$ for $\mathcal{R} = \langle f\rangle^* (A \cap B) \in V$. So V is a generalized filter base and thus W is a generalized filter base.

$0^{\mathfrak{F}(\operatorname{Dst} f)} \notin W \Leftrightarrow \bigcap W \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$ by the corollary 1 of the theorem 1. That is

$$\forall X \in \operatorname{up} \mathcal{X} : \mathcal{Y} \cap \langle f\rangle^* X \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{Y} \cap \bigcap \left\langle \langle f\rangle^* \right\rangle \operatorname{up} \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}.$$

Comparing with the above, $\mathcal{Y} \cap \langle f\rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{Y} \cap \bigcap \left\langle \langle f\rangle^* \right\rangle \operatorname{up} \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$. So $\langle f\rangle \mathcal{X} = \bigcap \left\langle \langle f\rangle^* \right\rangle \operatorname{up} \mathcal{X}$ because the lattice of filter objects is separable. □

Proposition 10 *For every $f \in \mathsf{FCD}(A; B)$ we have (for every $I, J \in \mathscr{P} A$)*

$$\langle f\rangle^* \emptyset = 0^{\mathfrak{F}(B)}, \quad \langle f\rangle^* (I \cup J) = \langle f\rangle^* I \cup \langle f\rangle^* J$$

and

$$\begin{array}{ll} \neg(I \,[f]^*\, \emptyset), \quad I \cup J \,[f]^*\, K \Leftrightarrow I \,[f]^*\, K \vee J \,[f]^*\, K & \text{(for every } I, J \in \mathscr{P} A,\ K \in \mathscr{P} B\text{)}, \\ \neg(\emptyset \,[f]^*\, I), \quad K \,[f]^*\, I \cup J \Leftrightarrow K \,[f]^*\, I \vee K \,[f]^*\, J & \text{(for every } I, J \in \mathscr{P} B,\ K \in \mathscr{P} A\text{)}. \end{array}$$

Proof $\langle f\rangle^* \emptyset = \langle f\rangle \uparrow^A \emptyset = \langle f\rangle 0^{\mathfrak{F}(A)} = 0^{\mathfrak{F}(B)}$; $\langle f\rangle^* (I \cup J) = \langle f\rangle \uparrow^A (I \cup J) = \langle f\rangle \left(\uparrow^A I \cup \uparrow^A J\right) = \langle f\rangle \uparrow^A I \cup \langle f\rangle \uparrow^A J = \langle f\rangle^* I \cup \langle f\rangle^* J$.

$I \,[f]^*\, \emptyset \Leftrightarrow 0^{\mathfrak{F}(B)} \not\asymp \langle f\rangle \uparrow^A I \Leftrightarrow 0$; $I \cup J \,[f]^*\, K \Leftrightarrow \uparrow^A (I \cup J) \,[f] \uparrow^B K \Leftrightarrow \uparrow^B K \not\asymp \langle f\rangle^* (I \cup J) \Leftrightarrow \uparrow^B K \not\asymp \langle f\rangle^* I \cup \langle f\rangle^* J \Leftrightarrow \uparrow^B K \not\asymp \langle f\rangle^* I \vee \uparrow^B K \not\asymp \langle f\rangle^* J \Leftrightarrow I \,[f]^*\, K \vee J \,[f]^*\, K$.

The rest follows from symmetry. □

Theorem 7 *Fix small sets A and B. Let $L_F = \lambda f \in \mathsf{FCD}(A; B) : \langle f\rangle^*$ and $L_R = \lambda f \in \mathsf{FCD}(A; B) : [f]^*$.*

1. *L_F is a bijection from the set $\mathsf{FCD}(A; B)$ to the set of functions $\alpha \in \mathfrak{F}(B)^{\mathscr{P} A}$ that obey the conditions (for every $I, J \in \mathscr{P} A$)*

$$\alpha\emptyset = 0^{\mathfrak{F}(B)}, \quad \alpha(I \cup J) = \alpha I \cup \alpha J. \tag{1}$$

   For such α it holds (for every $\mathcal{X} \in \mathfrak{F}(A)$)

$$\left\langle L_F^{-1}\alpha \right\rangle \mathcal{X} = \bigcap \langle \alpha\rangle \operatorname{up} \mathcal{X}. \tag{2}$$

2. *L_R is a bijection from the set $\mathsf{FCD}(A; B)$ to the set of binary relations $\delta \in \mathscr{P}(\mathscr{P} A \times \mathscr{P} B)$ that obey the conditions*

$$\begin{array}{ll} \neg(I \,\delta\, \emptyset), \quad I \cup J \,\delta\, K \Leftrightarrow I \,\delta\, K \vee J \,\delta\, K & \text{(for every } I, J \in \mathscr{P} A,\ K \in \mathscr{P} B\text{)}, \\ \neg(\emptyset \,\delta\, I), \quad K \,\delta\, I \cup J \Leftrightarrow K \,\delta\, I \vee K \,\delta\, J & \text{(for every } I, J \in \mathscr{P} B,\ K \in \mathscr{P} A\text{)}. \end{array} \tag{3}$$

For such δ it holds (for every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(B)$)

$$\mathcal{X} \left[L_R^{-1}\delta\right] \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : X \ \delta\ Y. \tag{4}$$

Proof Injectivity of L_F and L_R, formulas (2) (for $\alpha \in \operatorname{im} L_F$) and (4) (for $\delta \in \operatorname{im} L_R$), formulas (1) and (3) follow from two previous theorems. The only thing remained to prove is that for every α and δ that obey the above conditions a corresponding funcoid f exists.

2. Let define $\alpha \in \mathfrak{F}(B)^{\mathscr{P}A}$ by the formula $\partial(\alpha X) = \{Y \in \mathscr{P}B \mid X\ \delta\ Y\}$ for every $X \in \mathscr{P}A$. (It is obvious that $\{Y \in \mathscr{P}B \mid X\ \delta\ Y\}$ is a free star.) Analogously it can be defined $\beta \in \mathfrak{F}(A)^{\mathscr{P}B}$ by the formula $\partial(\beta Y) = \{X \in \mathscr{P}A \mid X\ \delta\ Y\}$. Let's continue α and β to $\alpha' \in \mathfrak{F}(B)^{\mathfrak{F}(A)}$ and $\beta' \in \mathfrak{F}(A)^{\mathfrak{F}(B)}$ by the formulas

$$\alpha'\mathcal{X} = \bigcap \langle\alpha\rangle \operatorname{up}\mathcal{X} \quad \text{and} \quad \beta'\mathcal{Y} = \bigcap \langle\beta\rangle \operatorname{up}\mathcal{Y}$$

and δ to $\delta' \in \mathscr{P}(\mathfrak{F}(A) \times \mathfrak{F}(B))$ by the formula

$$\mathcal{X}\ \delta'\ \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : X\ \delta\ Y.$$

$\mathcal{Y} \cap \alpha'\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{Y} \cap \bigcap \langle\alpha\rangle \operatorname{up}\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \bigcap \langle\mathcal{Y}\cap\rangle \langle\alpha\rangle \operatorname{up}\mathcal{X} \neq 0^{\mathfrak{F}(B)}$. Let's prove that

$$W = \langle\mathcal{Y}\cap\rangle \langle\alpha\rangle \operatorname{up}\mathcal{X}$$

is a generalized filter base: To prove it is enough to show that $\langle\alpha\rangle \operatorname{up}\mathcal{X}$ is a generalized filter base. If $\mathcal{A}, \mathcal{B} \in \langle\alpha\rangle \operatorname{up}\mathcal{X}$ then exist $X_1, X_2 \in \operatorname{up}\mathcal{X}$ such that $\mathcal{A} = \alpha X_1$ and $\mathcal{B} = \alpha X_2$.

Then $\alpha(X_1 \cap X_2) \in \langle\alpha\rangle \operatorname{up}\mathcal{X}$. So $\langle\alpha\rangle \operatorname{up}\mathcal{X}$ is a generalized filter base and thus W is a generalized filter base.

Accordingly to the corollary 1 of the theorem 1, $\bigcap \langle\mathcal{Y}\cap\rangle \langle\alpha\rangle \operatorname{up}\mathcal{X} \neq 0^{\mathfrak{F}(B)}$ is equivalent to

$$\forall X \in \operatorname{up}\mathcal{X} : \mathcal{Y} \cap \alpha X \neq 0^{\mathfrak{F}(B)},$$

what is equivalent to $\forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : \uparrow^B Y \cap \alpha X \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : Y \in \partial(\alpha X) \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : X\ \delta\ Y$. Combining the equivalencies we get $\mathcal{Y} \cap \alpha'\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X}\ \delta'\ \mathcal{Y}$. Analogously $\mathcal{X} \cap \beta'\mathcal{Y} \neq 0^{\mathfrak{F}(A)} \Leftrightarrow \mathcal{X}\ \delta'\ \mathcal{Y}$. So $\mathcal{Y} \cap \alpha'\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X} \cap \beta'\mathcal{Y} \neq 0^{\mathfrak{F}(A)}$, that is $(A; B; \alpha'; \beta')$ is a funcoid. From the formula $\mathcal{Y} \cap \alpha'\mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X}\ \delta'\ \mathcal{Y}$ it follows that

$$X \left[(A; B; \alpha'; \beta')\right]^* Y \Leftrightarrow \uparrow^B Y \cap \alpha' \uparrow^A X \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^A X\ \delta' \uparrow^B Y \Leftrightarrow X\ \delta\ Y.$$

1. Let define the relation $\delta \in \mathscr{P}(\mathscr{P}A \times \mathscr{P}B)$ by the formula $X\ \delta\ Y \Leftrightarrow \uparrow^B Y \cap \alpha X \neq 0^{\mathfrak{F}(B)}$.

That $\neg(\emptyset\ \delta\ I)$ and $\neg(I\ \delta\ \emptyset)$ is obvious. We have $I \cup J\ \delta\ K \Leftrightarrow \uparrow^B K \cap \alpha(I \cup J) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^B K \cap (\alpha I \cup \alpha J) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^B K \cap \alpha I \neq 0^{\mathfrak{F}(B)} \vee \uparrow^B K \cap \alpha I \neq 0^{\mathfrak{F}(B)} \Leftrightarrow I\ \delta\ K \vee J\ \delta\ K$ and

$K\ \delta\ I \cup J \Leftrightarrow \uparrow^B (I \cup J) \cap \alpha K \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \left(\uparrow^B I \cup \uparrow^B J\right) \cap \alpha K \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \left(\uparrow^B I \cap \alpha K\right) \cup \left(\uparrow^B J \cap \alpha K\right) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^B I \cap \alpha K \neq 0^{\mathfrak{F}(B)} \vee \uparrow^B J \cap \alpha K \neq 0^{\mathfrak{F}(B)} \Leftrightarrow K\ \delta\ I \vee K\ \delta\ J$.

That is the formulas (3) are true.

Accordingly the above there exists a funcoid f such that

$$\mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \forall X \in \operatorname{up}\mathcal{X}, Y \in \operatorname{up}\mathcal{Y} : X\ \delta\ Y.$$

$\forall X \in \mathscr{P}A, Y \in \mathscr{P}B : \left(\uparrow^B Y \cap \langle f\rangle \uparrow^A X \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \uparrow^A X\ [f] \uparrow^B Y \Leftrightarrow X\ \delta\ Y \Leftrightarrow \uparrow^B Y \cap \alpha X \neq 0^{\mathfrak{F}(B)}\right)$, consequently $\forall X \in \mathscr{P}A : \alpha X = \langle f\rangle \uparrow^A X = \langle f\rangle^* X$. $\square$

Note that by the last theorem to every proximity δ corresponds a unique funcoid. So funcoids are a generalization of (quasi-)proximity structures.

Reverse funcoids can be considered as a generalization of conjugate quasi-proximity.

Definition 24 *Any small (multivalued) function $F : A \to B$ corresponds to a funcoid $\uparrow^{\mathsf{FCD}(A;B)} F \in \mathsf{FCD}(A;B)$, where by definition $\left\langle \uparrow^{\mathsf{FCD}(A;B)} F \right\rangle \mathcal{X} = \bigcap \left\langle \uparrow^B \right\rangle \langle\langle F\rangle\rangle \operatorname{up}\mathcal{X}$ for every $\mathcal{X} \in \mathfrak{F}(A)$.*

Using the last theorem it is easy to show that this definition is monovalued and does not contradict to former stuff. (Take $\alpha = \uparrow^B \circ \langle F\rangle$.)

Definition 25 *Funcoids corresponding to a binary relation (= multivalued function) are called **principal funcoids**.*

We may equate principal funcoids with corresponding binary relations by the method of appendix B in [15]. This is useful for describing relationships of funcoids and binary relations, such as for the formulas of continuous functions and continuous funcoids (see below).

Theorem 8 *If S is a generalized filter base on $\operatorname{Src} f$ then $\langle f\rangle \bigcap S = \bigcap \langle\langle f\rangle\rangle S$ for every funcoid f.*

Proof $\langle f\rangle \bigcap S \subseteq \langle f\rangle X$ for every $X \in S$ and thus $\langle f\rangle \bigcap S \subseteq \bigcap \langle\langle f\rangle\rangle S$.

By properties of generalized filter bases:

$\langle f\rangle \bigcap S = \bigcap \left\langle \langle f\rangle^* \right\rangle \operatorname{up} \bigcap S =$
$\bigcap \left\langle \langle f\rangle^* \right\rangle \{X \mid \exists \mathcal{P} \in S : X \in \operatorname{up}\mathcal{P}\} = \bigcap \left\{ \langle f\rangle^* X \mid \exists \mathcal{P} \in S : X \in \operatorname{up}\mathcal{P} \right\} \supseteq$
$\bigcap \{\langle f\rangle \mathcal{P} \mid \mathcal{P} \in S\} = \bigcap \langle\langle f\rangle\rangle S$. $\square$

3.4 Lattices of funcoids

Definition 26 $f \subseteq g \overset{\text{def}}{=} [f] \subseteq [g]$ *for* $f, g \in \mathsf{FCD}$.

Thus every $\mathsf{FCD}(A;B)$ is a poset. (It's taken into account that $[f] \neq [g]$ if $f \neq g$.)

Definition 27 *I will call a **shifted filtrator of funcoids** the shifted filtrator*

$$\left(\mathsf{FCD}(A;B) ; \mathscr{P}(A \times B) ; \uparrow^{\mathsf{FCD}(A;B)}\right)$$

for some small sets A, B.

$\operatorname{up} f \overset{\text{def}}{=} \operatorname{up}^{\left(\mathsf{FCD}(A;B) ; \mathscr{P}(A\times B) ; \uparrow^{\mathsf{FCD}(A;B)}\right)} f$ *for every funcoid $f \in \mathsf{FCD}(A;B)$.*

Lemma 1 *$\langle f\rangle^* X = \bigcap\left\{\uparrow^{\mathrm{Dst}\,f} \langle F\rangle X \mid F \in \operatorname{up} f\right\}$ for every funcoid f and set $X \in \mathscr{P}(\mathrm{Src}\,f)$.*

Proof Obviously $\langle f\rangle^* X \subseteq \bigcap\left\{\uparrow^{\mathrm{Dst}\,f} \langle F\rangle X \mid F \in \operatorname{up} f\right\}$.

Let $B \in \operatorname{up}\langle f\rangle^* X$. Let $F_B = X \times B \cup ((\mathrm{Src}\,f) \setminus X) \times (\mathrm{Dst}\,f)$.

$\langle F_B\rangle X = B$.

We have $\emptyset \neq P \subseteq X \Rightarrow \langle F_B\rangle P = B \supseteq \langle f\rangle^* P$ and $\emptyset \neq P \nsubseteq X \Rightarrow \langle F_B\rangle P = \mathrm{Dst}\,f \supseteq \langle f\rangle^* P$. Thus $\langle F_B\rangle P \supseteq \langle f\rangle^* P$ for every set $P \in \mathscr{P}(\mathrm{Src}\,f)$ and so $\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)} F_B \supseteq f$ that is $F_B \in \operatorname{up} f$.

Thus $\forall B \in \operatorname{up}\langle f\rangle^* X : B \in \operatorname{up}\bigcap\left\{\uparrow^{\mathrm{Dst}\,f} \langle F\rangle X \mid F \in \operatorname{up} f\right\}$ because $B \in \operatorname{up}\uparrow^{\mathrm{Dst}\,f} \langle F_B\rangle X$.

So $\bigcap\left\{\uparrow^{\mathrm{Dst}\,f} \langle F\rangle X \mid F \in \operatorname{up} f\right\} \subseteq \langle f\rangle^* X$. □

Theorem 9 *$\langle f\rangle \mathcal{X} = \bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)} F\right\rangle \mathcal{X} \mid F \in \operatorname{up} f\right\}$ for every funcoid f and $\mathcal{X} \in \mathfrak{F}(\mathrm{Src}\,f)$.*

Proof $\bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)} F\right\rangle \mathcal{X} \mid F \in \operatorname{up} f\right\} = \bigcap\left\{\bigcap\left\langle\uparrow^{\mathrm{Dst}\,f}\right\rangle\langle\langle F\rangle\rangle \operatorname{up}\mathcal{X} \mid F \in \operatorname{up} f\right\} = \bigcap\left\{\bigcap\left\{\uparrow^{\mathrm{Dst}\,f} \langle F\rangle X \mid X \in \operatorname{up}\mathcal{X}\right\} \mid F \in \operatorname{up} f\right\} = \bigcap\left\{\bigcap\left\{\uparrow^{\mathrm{Dst}\,f} \langle F\rangle X \mid F \in \operatorname{up} f\right\} \mid X \in \operatorname{up}\mathcal{X}\right\} = \bigcap\left\{\uparrow^{\mathrm{Dst}\,f} \langle f\rangle^* X \mid X \in \operatorname{up}\mathcal{X}\right\} = \langle f\rangle \mathcal{X}$ (the lemma used). □

Conjecture 1 *Every filtrator of funcoids is:*

1. *with separable core;*
2. *with co-separable core.*

Below it is shown that $\mathsf{FCD}(A;B)$ are complete lattices for every small sets A and B. We will apply lattice operations to subsets of such sets without explicitly mentioning $\mathsf{FCD}(A;B)$.

Theorem 10 *$\mathsf{FCD}(A;B)$ is a complete lattice (for every small sets A and B). For every $R \in \mathscr{P}\mathsf{FCD}(A;B)$ and $X \in \mathscr{P}A$, $Y \in \mathscr{P}B$*

1. *$X\left[\bigcup R\right]^* Y \Leftrightarrow \exists f \in R : X [f]^* Y$;*
2. *$\left\langle\bigcup R\right\rangle^* X = \bigcup\left\{\langle f\rangle^* X \mid f \in R\right\}$.*

Proof Accordingly [14] to prove that it is a complete lattice it's enough to prove existence of all joins.

2 $\alpha X \overset{\text{def}}{=} \bigcup\left\{\langle f\rangle^* X \mid f \in R\right\}$. We have $\alpha\emptyset = 0^{\mathfrak{F}(\mathrm{Dst}\,f)}$;

$$\begin{aligned}
\alpha(I \cup J) &= \bigcup\left\{\langle f\rangle^* (I \cup J) \mid f \in R\right\} \\
&= \bigcup\left\{\langle f\rangle^* I \cup \langle f\rangle^* J \mid f \in R\right\} \\
&= \bigcup\left\{\langle f\rangle^* I \mid f \in R\right\} \cup \bigcup\left\{\langle f\rangle^* J \mid f \in R\right\} \\
&= \alpha I \cup \alpha J.
\end{aligned}$$

So $\langle h\rangle \circ \uparrow^A = \alpha$ for some funcoid h. Obviously

$$\forall f \in R : h \supseteq f. \tag{5}$$

And h is the least funcoid for which holds the condition (5). So $h = \bigcup R$.

1 $X \left[\bigcup R\right]^* Y \Leftrightarrow \uparrow^{\mathrm{Dst}\, f} Y \cap \langle \bigcup R\rangle^* X \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \uparrow^{\mathrm{Dst}\, f} Y \cap \bigcup \left\{ \langle f\rangle^* X \quad | \quad f \in R \right\} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \exists f \in R : \uparrow^{\mathrm{Dst}\, f} Y \cap \langle f\rangle^* X \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \Leftrightarrow \exists f \in R : X \left[f\right]^* Y$ (used the theorem 40 in [15]).

□

In the next theorem, compared to the previous one, the class of infinite unions is replaced with lesser class of finite unions and simultaneously class of sets is changed to more wide class of filter objects.

Theorem 11 *For every $f, g \in \mathsf{FCD}(A; B)$ and $\mathcal{X} \in \mathfrak{F}(A)$ (for every small sets A, B)*

1. $\langle f \cup g\rangle \mathcal{X} = \langle f\rangle \mathcal{X} \cup \langle g\rangle \mathcal{X}$;
2. $[f \cup g] = [f] \cup [g]$.

Proof

1. Let $\alpha\mathcal{X} \stackrel{\mathrm{def}}{=} \langle f\rangle \mathcal{X} \cup \langle g\rangle \mathcal{X}$; $\beta\mathcal{Y} \stackrel{\mathrm{def}}{=} \left\langle f^{-1}\right\rangle \mathcal{Y} \cup \left\langle g^{-1}\right\rangle \mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(B)$. Then

$$\begin{aligned} \mathcal{Y} \cap \alpha\mathcal{X} \neq 0^{\mathfrak{F}(B)} &\Leftrightarrow \mathcal{Y} \cap \langle f\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \vee \mathcal{Y} \cap \langle g\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \\ &\Leftrightarrow \mathcal{X} \cap \left\langle f^{-1}\right\rangle \mathcal{Y} \neq 0^{\mathfrak{F}(A)} \vee \mathcal{X} \cap \left\langle g^{-1}\right\rangle \mathcal{Y} \neq 0^{\mathfrak{F}(A)} \\ &\Leftrightarrow \mathcal{X} \cap \beta\mathcal{Y} \neq 0^{\mathfrak{F}(A)}. \end{aligned}$$

   So $h = (A; B; \alpha; \beta)$ is a funcoid. Obviously $h \supseteq f$ and $h \supseteq g$. If $p \supseteq f$ and $p \supseteq g$ for some funcoid p then $\langle p\rangle \mathcal{X} \supseteq \langle f\rangle \mathcal{X} \cup \langle g\rangle \mathcal{X} = \langle h\rangle \mathcal{X}$ that is $p \supseteq h$. So $f \cup g = h$.

2. $\mathcal{X} \, [f \cup g] \, \mathcal{Y} \Leftrightarrow \mathcal{Y} \cap \langle f \cup g\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{Y} \cap (\langle f\rangle \mathcal{X} \cup \langle g\rangle \mathcal{X}) \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{Y} \cap \langle f\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \vee \mathcal{Y} \cap \langle g\rangle \mathcal{X} \neq 0^{\mathfrak{F}(B)} \Leftrightarrow \mathcal{X} \, [f] \, \mathcal{Y} \vee \mathcal{X} \, [g] \, \mathcal{Y}$ for every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(B)$.

□

3.5 More on composition of funcoids

Proposition 11 $[g \circ f] = [g] \circ \langle f\rangle = \left\langle g^{-1}\right\rangle^{-1} \circ [f]$ *for every composable funcoids f and g.*

Proof $\mathcal{X}\ [g\circ f]\ \mathcal{Y} \Leftrightarrow \mathcal{Y}\cap\langle g\circ f\rangle\,\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} g)} \Leftrightarrow \mathcal{Y}\cap\langle g\rangle\,\langle f\rangle\,\mathcal{X}\neq 0^{\mathfrak{F}(\operatorname{Dst} g)} \Leftrightarrow$ $\langle f\rangle\,\mathcal{X}\ [g]\ \mathcal{Y} \Leftrightarrow \mathcal{X}([g]\circ\langle f\rangle)\mathcal{Y}$ for every $\mathcal{X}\in\mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y}\in\mathfrak{F}(\operatorname{Dst} g)$. $[g\circ f]=\left[(f^{-1}\circ g^{-1})^{-1}\right]=\left[f^{-1}\circ g^{-1}\right]^{-1}=$ $(\left[f^{-1}\right]\circ\left\langle g^{-1}\right\rangle)^{-1}=\left\langle g^{-1}\right\rangle^{-1}\circ[f]$. □

The following theorem is a variant for funcoids of the statement (which defines compositions of relations) that $x(g\circ f)z \Leftrightarrow \exists y(x\ f\ y\wedge y\ g\ z)$ for every x and z and every binary relations f and g.

Theorem 12 *For every small sets A, B, C and $f\in\mathsf{FCD}(A;B)$, $g\in\mathsf{FCD}(B;C)$ and $\mathcal{X}\in\mathfrak{F}(A)$, $\mathcal{Z}\in\mathfrak{F}(C)$*

$$\mathcal{X}\ [g\circ f]\ \mathcal{Z} \Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}:(\mathcal{X}\ [f]\ y\wedge y\ [g]\ \mathcal{Z}).$$

Proof

$$\begin{aligned}
\exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}:(\mathcal{X}\ [f]\ y\wedge y\ [g]\ \mathcal{Z}) &\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}:\left(\mathcal{Z}\cap\langle g\rangle\,y\neq 0^{\mathfrak{F}(C)}\wedge y\cap\langle f\rangle\,\mathcal{X}\neq 0^{\mathfrak{F}(B)}\right)\\
&\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}:(\mathcal{Z}\cap\langle g\rangle\,y\neq 0^{\mathfrak{F}(C)}\wedge y\subseteq\langle f\rangle\,\mathcal{X})\\
&\Rightarrow \mathcal{Z}\cap\langle g\rangle\,\langle f\rangle\,\mathcal{X}\neq 0^{\mathfrak{F}(C)}\\
&\Leftrightarrow \mathcal{X}\ [g\circ f]\ \mathcal{Z}.
\end{aligned}$$

Reversely, if $\mathcal{X}\ [g\circ f]\ \mathcal{Z}$ then $\langle f\rangle\,\mathcal{X}\ [g]\ \mathcal{Z}$, consequently exists $y\in\operatorname{atoms}\langle f\rangle\,\mathcal{X}$ such that $y\ [g]\ \mathcal{Z}$; we have $\mathcal{X}\ [f]\ y$. □

Theorem 13 *For every small sets A, B, C*

1. *$f\circ(g\cup h)=f\circ g\cup f\circ h$ for $g,h\in\mathsf{FCD}(A;B)$ and $f\in\mathsf{FCD}(B;C)$;*
2. *$(g\cup h)\circ f=g\circ f\cup h\circ f$ for $g,h\in\mathsf{FCD}(B;C)$ and $f\in\mathsf{FCD}(A;B)$.*

Proof I will prove only the first equality because the other is analogous.

For every $\mathcal{X}\in\mathfrak{F}(A)$, $\mathcal{Z}\in\mathfrak{F}(C)$

$$\begin{aligned}
\mathcal{X}\ [f\circ(g\cup h)]\ \mathcal{Z} &\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}:(\mathcal{X}\ [g\cup h]\ y\wedge y\ [f]\ \mathcal{Z})\\
&\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}:((\mathcal{X}\ [g]\ y\vee\mathcal{X}\ [h]\ y)\wedge y\ [f]\ \mathcal{Z})\\
&\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}:(\mathcal{X}\ [g]\ y\wedge y\ [f]\ \mathcal{Z}\vee\mathcal{X}\ [h]\ y\wedge y\ [f]\ \mathcal{Z})\\
&\Leftrightarrow \exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}:(\mathcal{X}\ [g]\ y\wedge y\ [f]\ \mathcal{Z})\vee\exists y\in\operatorname{atoms}1^{\mathfrak{F}(B)}:(\mathcal{X}\ [h]\ y\wedge y\ [f]\ \mathcal{Z})\\
&\Leftrightarrow \mathcal{X}\ [f\circ g]\ \mathcal{Z}\vee\mathcal{X}\ [f\circ h]\ \mathcal{Z}\\
&\Leftrightarrow \mathcal{X}\ [f\circ g\cup f\circ h]\ \mathcal{Z}.
\end{aligned}$$

□

Conjecture 2 *$g\circ f=\bigcap\left\{\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} g)}(G\circ F)\quad\middle|\quad F\in\operatorname{up} f, G\in\operatorname{up} g\right\}$ for every composable funcoids f and g.*

3.6 Domain and range of a funcoid

Definition 28 *Let A be a small set. The* ***identity funcoid*** $I^{\mathsf{FCD}(A)} = \left(A; A; (=)|_{\mathfrak{F}(A)}; (=)|_{\mathfrak{F}(A)}\right)$.

Obvious 9. The identity funcoid is a funcoid.

Definition 29 *Let A be a small set, $\mathcal{A} \in \mathfrak{F}(A)$. The* ***restricted identity funcoid***

$$I^{\mathsf{FCD}}_{\mathcal{A}} = (A; A; \mathcal{A}\cap; \mathcal{A}\cap).$$

Proposition 12 *The restricted identity funcoid is a funcoid.*

Proof We need to prove that $(\mathcal{A} \cap \mathcal{X}) \cap \mathcal{Y} \neq 0^{\mathfrak{F}(A)} \Leftrightarrow (\mathcal{A} \cap \mathcal{Y}) \cap \mathcal{X} \neq 0^{\mathfrak{F}(A)}$ what is obvious. □

Obvious 10.

1. $\left(I^{\mathsf{FCD}(A)}\right)^{-1} = I^{\mathsf{FCD}(A)}$;
2. $\left(I^{\mathsf{FCD}}_{\mathcal{A}}\right)^{-1} = I^{\mathsf{FCD}}_{\mathcal{A}}$.

Obvious 11. For every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}(A)$

1. $\mathcal{X} \left[I^{\mathsf{FCD}(A)}\right] \mathcal{Y} \Leftrightarrow \mathcal{X} \cap \mathcal{Y} \neq 0^{\mathfrak{F}(A)}$.
2. $\mathcal{X} \left[I^{\mathsf{FCD}}_{\mathcal{A}}\right] \mathcal{Y} \Leftrightarrow \mathcal{A} \cap \mathcal{X} \cap \mathcal{Y} \neq 0^{\mathfrak{F}(A)}$.

Definition 30 *I will define* ***restricting*** *of a funcoid f to a filter object $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$ by the formula*

$$f|_{\mathcal{A}} \stackrel{\text{def}}{=} f \circ I^{\mathsf{FCD}}_{\mathcal{A}}.$$

Definition 31 ***Image*** *of a funcoid f will be defined by the formula* $\operatorname{im} f = \langle f \rangle 1^{\mathfrak{F}(\operatorname{Src} f)}$.

Domain *of a funcoid f is defined by the formula* $\operatorname{dom} f = \operatorname{im} f^{-1}$.

Proposition 13 $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \cap \operatorname{dom} f)$ *for every* $f \in \mathsf{FCD}$, $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$.

Proof For every $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$ we have $\mathcal{Y} \cap \langle f \rangle (\mathcal{X} \cap \operatorname{dom} f) \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{X} \cap \operatorname{dom} f \cap \langle f^{-1} \rangle \mathcal{Y} \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow \mathcal{X} \cap \operatorname{im} f^{-1} \cap \langle f^{-1} \rangle \mathcal{Y} \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow \mathcal{X} \cap \langle f^{-1} \rangle \mathcal{Y} \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow \mathcal{Y} \cap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$. Thus $\langle f \rangle \mathcal{X} = \langle f \rangle (\mathcal{X} \cap \operatorname{dom} f)$ because the lattice of filter objects is separable. □

Proposition 14 $\mathcal{X} \cap \operatorname{dom} f \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$ *for every* $f \in \mathsf{FCD}$, $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$.

Proof $\mathcal{X} \cap \operatorname{dom} f \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow \mathcal{X} \cap \left\langle f^{-1} \right\rangle 1^{\mathfrak{F}(\operatorname{Dst} f)} \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow 1^{\mathfrak{F}(\operatorname{Dst} f)} \cap \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \langle f \rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$. □

Corollary 2 $\operatorname{dom} f = \bigcup \left\{ a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)} \mid \langle f \rangle a \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \right\}$.

Proof This follows from the fact that $\mathfrak{F}(\operatorname{Src} f)$ is an atomistic lattice. □

Proposition 15 $\operatorname{dom} f|_{\mathcal{A}} = \mathcal{A} \cap \operatorname{dom} f$ *for every funcoid* f *and* $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$.

Proof $\operatorname{dom} f|_{\mathcal{A}} = \operatorname{im}\left(I^{\mathsf{FCD}}_{\mathcal{A}} \circ f^{-1}\right) = \left\langle I^{\mathsf{FCD}}_{\mathcal{A}} \right\rangle \left\langle f^{-1} \right\rangle 1^{(\operatorname{Dst} f)} = \mathcal{A} \cap \left\langle f^{-1} \right\rangle 1^{(\operatorname{Dst} f)} = \mathcal{A} \cap \operatorname{dom} f$. □

Theorem 14 $\operatorname{im} f = \bigcap \left\langle \uparrow^{\operatorname{Dst} f} \right\rangle \langle \operatorname{im} \rangle \operatorname{up} f$ *and* $\operatorname{dom} f = \bigcap \left\langle \uparrow^{\operatorname{Src} f} \right\rangle \langle \operatorname{dom} \rangle \operatorname{up} f$ *for every funcoid* f.

Proof $\operatorname{im} f = \langle f \rangle 1^{\mathfrak{F}(\operatorname{Src} f)} = \bigcap \left\{ \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f; \operatorname{Dst} f)} F \right\rangle 1^{\mathfrak{F}(\operatorname{Src} f)} \mid F \in \operatorname{up} f \right\} = \bigcap \left\{ \uparrow^{\operatorname{Dst} f} \operatorname{im} F \mid F \in \operatorname{up} f \right\} = \bigcap \left\langle \uparrow^{\operatorname{Dst} f} \right\rangle \langle \operatorname{im} \rangle \operatorname{up} f$ (used the theorem 9).

The second formula follows from symmetry. □

Proposition 16 *For every composable funcoids* f, g:

1. *If* $\operatorname{im} f \supseteq \operatorname{dom} g$ *then* $\operatorname{im}(g \circ f) = \operatorname{im} g$.
2. *If* $\operatorname{im} f \subseteq \operatorname{dom} g$ *then* $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

Proof

1. $\operatorname{im}(g \circ f) = \langle g \circ f \rangle 1^{\mathfrak{F}(\operatorname{Src} f)} = \langle g \rangle \langle f \rangle 1^{\mathfrak{F}(\operatorname{Src} f)} = \langle g \rangle \operatorname{im} f = \langle g \rangle (\operatorname{im} f \cap \operatorname{dom} g) = \langle g \rangle \operatorname{dom} g = \langle g \rangle 1^{\mathfrak{F}(\operatorname{Src} g)} = \operatorname{im} g$.
2. $\operatorname{dom}(g \circ f) = \operatorname{im}\left(f^{-1} \circ g^{-1}\right)$ what by proved above is equal to $\operatorname{im} f^{-1}$ that is $\operatorname{dom} f$.

□

3.7 Categories of funcoids

I will define two categories, the **category of funcoids** and the **category of funcoid triples**.

The **category of funcoids** is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\mathsf{FCD}(A; B)$.
- The composition is the composition of funcoids.
- Identity morphism for a set is the identity funcoid for that set.

To show it is really a category is trivial.

The **category of funcoid triples** is defined as follows:

- Objects are filter objects on small sets.
- The morphisms from a f.o. $\mathcal{A}$ to a f.o. $\mathcal{B}$ are triples $(f;\mathcal{A};\mathcal{B})$ where $f \in \mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))$ and $\mathrm{dom}\, f \subseteq \mathcal{A} \wedge \mathrm{im}\, f \subseteq \mathcal{B}$.
- The composition is defined by the formula $(g;\mathcal{B};\mathcal{C})\circ(f;\mathcal{A};\mathcal{B}) = (g\circ f;\mathcal{A};\mathcal{C})$.
- Identity morphism for an f.o. $\mathcal{A}$ is $I^{\mathsf{FCD}}_{\mathcal{A}}$.

To prove that it is really a category is trivial.

3.8 Specifying funcoids by functions or relations on atomic filter objects

Theorem 15 *For every funcoid f and $\mathcal{X} \in \mathfrak{F}(\mathrm{Src}\, f)$, $\mathcal{Y} \in \mathfrak{F}(\mathrm{Dst}\, f)$*

1. $\langle f\rangle \mathcal{X} = \bigcup \langle\langle f\rangle\rangle \mathrm{atoms}\,\mathcal{X}$*;*

2. $\mathcal{X}\ [f]\ \mathcal{Y} \Leftrightarrow \exists x \in \mathrm{atoms}\,\mathcal{X}, y \in \mathrm{atoms}\,\mathcal{Y} : x\ [f]\ y$.

Proof 1.

$$\begin{aligned}\mathcal{Y}\cap\langle f\rangle\mathcal{X} \neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)} \quad&\Leftrightarrow\quad \mathcal{X}\cap\left\langle f^{-1}\right\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(\mathrm{Src}\, f)}\\ &\Leftrightarrow\quad \exists x\in\mathrm{atoms}\,\mathcal{X} : x\cap\left\langle f^{-1}\right\rangle\mathcal{Y}\neq 0^{\mathfrak{F}(\mathrm{Src}\, f)}\\ &\Leftrightarrow\quad \exists x\in\mathrm{atoms}\,\mathcal{X} : \mathcal{Y}\cap\langle f\rangle x\neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)}.\end{aligned}$$

$\partial\langle f\rangle\mathcal{X} = \bigcup\langle\partial\rangle\langle\langle f\rangle\rangle\mathrm{atoms}\,\mathcal{X} = \partial\bigcup\langle\langle f\rangle\rangle\mathrm{atoms}\,\mathcal{X}$.

2. If $\mathcal{X}\ [f]\ \mathcal{Y}$, then $\mathcal{Y}\cap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$, consequently exists $y\in\mathrm{atoms}\,\mathcal{Y}$ such that $y\cap\langle f\rangle\mathcal{X}\neq 0^{\mathfrak{F}(\mathrm{Dst}\, f)}$, $\mathcal{X}\ [f]\ y$. Repeating this second time we get that there exists $x\in\mathrm{atoms}\,\mathcal{X}$ such that $x\ [f]\ y$. From this follows

$$\exists x\in\mathrm{atoms}\,\mathcal{X}, y\in\mathrm{atoms}\,\mathcal{Y} : x\ [f]\ y.$$

The reverse is obvious. □

Theorem 16 *Let A and B be small sets.*

1. A function $\alpha\in\mathfrak{F}(B)^{\mathrm{atoms}\, 1^{\mathfrak{F}(A)}}$ such that (for every $a\in\mathrm{atoms}\, 1^{\mathfrak{F}(A)}$)

$$\alpha a\subseteq\bigcap\left\langle\bigcup\circ\langle\alpha\rangle\circ\mathrm{atoms}\circ\uparrow^{A}\right\rangle\mathrm{up}\, a \tag{6}$$

can be continued to the function $\langle f\rangle$ for a unique $f\in\mathsf{FCD}(A;B)$;

$$\langle f\rangle\mathcal{X} = \bigcup\langle\alpha\rangle\mathrm{atoms}\,\mathcal{X} \tag{7}$$

for every $\mathcal{X}\in\mathfrak{F}(A)$.

2. A relation $\delta \in \mathscr{P}(\operatorname{atoms} 1^{\mathfrak{F}(A)} \times \operatorname{atoms} 1^{\mathfrak{F}(B)})$ *such that (for every* $a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$*,* $b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}$*)*

$$\forall X \in \operatorname{up} a, Y \in \operatorname{up} b \exists x \in \operatorname{atoms} {\uparrow^A} X, y \in \operatorname{atoms} {\uparrow^B} Y : x \mathrel{\delta} y \Rightarrow a \mathrel{\delta} b \tag{8}$$

can be continued to the relation $[f]$ *for a unique* $f \in \mathsf{FCD}(A; B)$*;*

$$\mathcal{X} \mathrel{[f]} \mathcal{Y} \Leftrightarrow \exists x \in \operatorname{atoms} \mathcal{X}, y \in \operatorname{atoms} \mathcal{Y} : x \mathrel{\delta} y \tag{9}$$

for every $\mathcal{X} \in \mathfrak{F}(A)$*,* $\mathcal{Y} \in \mathfrak{F}(B)$*.*

Proof Existence of no more than one such funcoids and formulas (7) and (9) follow from the previous theorem.

1. Consider the function $\alpha' \in \mathfrak{F}(B)^{\mathscr{P} A}$ defined by the formula (for every $X \in \mathscr{P} A$)

$$\alpha' X = \bigcup \langle \alpha \rangle \operatorname{atoms} {\uparrow^A} X.$$

Obviously $\alpha' \emptyset = 0^{\mathfrak{F}(B)}$. For every $I, J \in \mathscr{P} A$

$$\begin{aligned}
\alpha'(I \cup J) &= \bigcup \langle \alpha \rangle \operatorname{atoms} {\uparrow^A} (I \cup J) \\
&= \bigcup \langle \alpha \rangle (\operatorname{atoms} {\uparrow^A} I \cup \operatorname{atoms} {\uparrow^A} J) \\
&= \bigcup \big(\langle \alpha \rangle \operatorname{atoms} {\uparrow^A} I \cup \langle \alpha \rangle \operatorname{atoms} {\uparrow^A} J \big) \\
&= \bigcup \langle \alpha \rangle \operatorname{atoms} {\uparrow^A} I \cup \bigcup \langle \alpha \rangle \operatorname{atoms} {\uparrow^A} J. \\
&= \alpha' I \cup \alpha' J.
\end{aligned}$$

Let continue α' till a funcoid f (by the theorem 7): $\langle f \rangle \mathcal{X} = \bigcap \langle \alpha' \rangle \operatorname{up} \mathcal{X}$.

Let's prove the reverse of (6):

$$\begin{aligned}
\bigcap \left\langle \bigcup \circ \langle \alpha \rangle \circ \operatorname{atoms} \circ {\uparrow^A} \right\rangle \operatorname{up} a &= \bigcap \left\langle \bigcup \circ \langle \alpha \rangle \right\rangle \langle \operatorname{atoms} \rangle \langle {\uparrow^A} \rangle \operatorname{up} a \\
&\subseteq \bigcap \left\langle \bigcup \circ \langle \alpha \rangle \right\rangle \{\{a\}\} \\
&= \bigcap \left\{ \left(\bigcup \circ \langle \alpha \rangle \right) \{a\} \right\} \\
&= \bigcap \left\{ \bigcup \langle \alpha \rangle \{a\} \right\} \\
&= \bigcap \left\{ \bigcup \{\alpha a\} \right\} = \bigcap \{\alpha a\} = \alpha a.
\end{aligned}$$

Finally,

$$\alpha a = \bigcap \left\langle \bigcup \circ \langle \alpha \rangle \circ \operatorname{atoms} \circ {\uparrow^A} \right\rangle \operatorname{up} a = \bigcap \langle \alpha' \rangle \operatorname{up} a = \langle f \rangle a,$$

so $\langle f \rangle$ is a continuation of α.

2. Consider the relation $\delta' \in \mathscr{P}(\mathscr{P} A \times \mathscr{P} B)$ defined by the formula (for every $X \in \mathscr{P} A$, $Y \in \mathscr{P} B$)

$$X \mathrel{\delta'} Y \Leftrightarrow \exists x \in \operatorname{atoms} {\uparrow^A} X, y \in \operatorname{atoms} {\uparrow^B} Y : x \mathrel{\delta} y.$$

Obviously $\neg(X\ \delta'\ \emptyset)$ and $\neg(\emptyset\ \delta'\ Y)$.

For suitable I and J we have:

$$\begin{aligned}(I\cup J)\ \delta'\ Y &\Leftrightarrow \exists x\in\operatorname{atoms}\uparrow^A(I\cup J),y\in\operatorname{atoms}\uparrow^B Y:x\ \delta\ y\\ &\Leftrightarrow \exists x\in\operatorname{atoms}\uparrow^A I\cup\operatorname{atoms}\uparrow^A J,y\in\operatorname{atoms}\uparrow^B Y:x\ \delta\ y\\ &\Leftrightarrow \exists x\in\operatorname{atoms}\uparrow^A I,y\in\operatorname{atoms}\uparrow^B Y:x\ \delta\ y\vee\exists x\in\operatorname{atoms}\uparrow^A J,y\in\operatorname{atoms}\uparrow^B Y:x\ \delta\ y\\ &\Leftrightarrow I\ \delta'\ Y\vee J\ \delta'\ Y;\end{aligned}$$

similarly $X\ \delta'\ (I\cup J)\Leftrightarrow X\ \delta'\ I\vee X\ \delta'\ J$ for suitable I and J. Let's continue δ' till a funcoid f (by the theorem 7):

$$\mathcal{X}\ [f]\ \mathcal{Y}\Leftrightarrow\forall X\in\operatorname{up}\mathcal{X},Y\in\operatorname{up}\mathcal{Y}:X\ \delta'\ Y.$$

The reverse of (8) implication is trivial, so

$$\forall X\in\operatorname{up}a,Y\in\operatorname{up}b\exists x\in\operatorname{atoms}\uparrow^A X,y\in\operatorname{atoms}\uparrow^B Y:x\ \delta\ y\Leftrightarrow a\ \delta\ b.$$

$\forall X\in\operatorname{up}a,Y\in\operatorname{up}b\exists x\in\operatorname{atoms}\uparrow^A X,y\in\operatorname{atoms}\uparrow^B Y:x\ \delta\ y\Leftrightarrow\forall X\in\operatorname{up}a,Y\in\operatorname{up}b:X\ \delta'\ Y\Leftrightarrow a\ [f]\ b$.

So $a\ \delta\ b\Leftrightarrow a\ [f]\ b$, that is $[f]$ is a continuation of δ. $\square$

One of uses of the previous theorem is the proof of the following theorem:

Theorem 17 *If A, B are small sets, $R\in\mathscr{P}\mathsf{FCD}(A;B)$, $x\in\operatorname{atoms}1^{\mathfrak{F}(A)}$, $y\in\operatorname{atoms}1^{\mathfrak{F}(B)}$, then*

1. $\langle\bigcap R\rangle x=\bigcap\{\langle f\rangle x\mid f\in R\}$;

2. $x\ [\bigcap R]\ y\Leftrightarrow\forall f\in R:x\ [f]\ y$.

Proof 2. Let denote $x\ \delta\ y\Leftrightarrow\forall f\in R:x\ [f]\ y$. For every $a\in\operatorname{atoms}1^{\mathfrak{F}(A)}$, $b\in\operatorname{atoms}1^{\mathfrak{F}(B)}$

$$\begin{gathered}\forall X\in\operatorname{up}a,Y\in\operatorname{up}b\exists x\in\operatorname{atoms}\uparrow^A X,y\in\operatorname{atoms}\uparrow^B Y:x\ \delta\ y\Rightarrow\\ \forall f\in R,X\in\operatorname{up}a,Y\in\operatorname{up}b\exists x\in\operatorname{atoms}\uparrow^A X,y\in\operatorname{atoms}\uparrow^B Y:x\ [f]\ y\Rightarrow\\ \forall f\in R,X\in\operatorname{up}a,Y\in\operatorname{up}b:X\ [f]^*\ Y\Rightarrow\\ \forall f\in R:a\ [f]\ b\Leftrightarrow\\ a\ \delta\ b.\end{gathered}$$

So, by the theorem 16, δ can be continued till $[p]$ for some funcoid $p\in\mathsf{FCD}(A;B)$.

For every funcoid $q\in\mathsf{FCD}(A;B)$ such that $\forall f\in R:q\subseteq f$ we have $x\ [q]\ y\Rightarrow\forall f\in R:x\ [f]\ y\Leftrightarrow x\ \delta\ y\Leftrightarrow x\ [p]\ y$, so $q\subseteq p$. Consequently $p=\bigcap R$.

From this $x\ [\bigcap R]\ y\Leftrightarrow\forall f\in R:x\ [f]\ y$.

1. From the former $y\in\operatorname{atoms}\langle\bigcap R\rangle x\Leftrightarrow y\cap\langle\bigcap R\rangle x\neq 0^{\mathfrak{F}(B)}\Leftrightarrow\forall f\in R:y\cap\langle f\rangle x\neq 0^{\mathfrak{F}(\operatorname{Dst}f)}\Leftrightarrow y\in\bigcap\langle\operatorname{atoms}\rangle\{\langle f\rangle x\mid f\in R\}\Leftrightarrow y\in\operatorname{atoms}\bigcap\{\langle f\rangle x\mid f\in R\}$ for every $y\in\operatorname{atoms}1^{\mathfrak{F}(A)}$. From this follows $\langle\bigcap R\rangle x=\bigcap\{\langle f\rangle x\mid f\in R\}$. $\square$

Theorem 18 *Let A, B, C be sets, $f \in \mathsf{FCD}(A;B)$, $g \in \mathsf{FCD}(B;C)$, $h \in \mathsf{FCD}(A;C)$. Then*
$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

Proof

$$\begin{aligned}
&g \circ f \not\asymp h &\Leftrightarrow\\
&\exists a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : a \left[(g \circ f) \cap h \right] c &\Leftrightarrow\\
&\exists a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : (a \left[g \circ f \right] c \wedge a \left[h \right] c) &\Leftrightarrow\\
&\exists a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : (a \left[f \right] b \wedge b \left[g \right] c \wedge a \left[h \right] c) &\Leftrightarrow\\
&\exists b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : \left(b \left[g \right] c \wedge b \left[h \circ f^{-1} \right] c \right) &\Leftrightarrow\\
&\exists b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, c \in \operatorname{atoms} 1^{\mathfrak{F}(C)} : b \left[g \cap \left(h \circ f^{-1} \right) \right] c &\Leftrightarrow\\
&g \not\asymp h \circ f^{-1}.
\end{aligned}$$

□

3.9 Direct product of filter objects

A generalization of direct (Cartesian) product of two sets is funcoidal product of two filter objects:

Definition 32 ***Funcoidal product*** *of filter objects $\mathcal{A}$ and $\mathcal{B}$ is such a funcoid $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \in \mathsf{FCD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))$ that for every $\mathcal{X} \in \mathfrak{F}(\operatorname{Base}(\mathcal{A}))$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Base}(\mathcal{B}))$*
$$\mathcal{X} \left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right] \mathcal{Y} \Leftrightarrow \mathcal{X} \not\asymp \mathcal{A} \wedge \mathcal{Y} \not\asymp \mathcal{B}.$$

Proposition 17 *$\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ is really a funcoid and*
$$\left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle \mathcal{X} = \begin{cases} \mathcal{B} & \text{if } \mathcal{X} \not\asymp \mathcal{A}; \\ 0^{\mathfrak{F}(\operatorname{Base}(\mathcal{B}))} & \text{if } \mathcal{X} \asymp \mathcal{A}. \end{cases}$$

Proof Obvious. □

Obvious 12. $\uparrow^{\mathsf{FCD}(U;V)} (A \times B) = \uparrow^U A \times^{\mathsf{FCD}} \uparrow^V B$ for sets $A \subseteq U$ and $B \subseteq V$ (for some small sets U and V).

Proposition 18 *$f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ for every $f \in \mathsf{FCD}(A;B)$ and $\mathcal{A} \in \mathfrak{F}(A)$, $\mathcal{B} \in \mathfrak{F}(B)$.*

Proof If $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ then $\operatorname{dom} f \subseteq \operatorname{dom}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{A}$, $\operatorname{im} f \subseteq \operatorname{im}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \subseteq \mathcal{B}$. If $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ then
$$\forall \mathcal{X} \in \mathfrak{F}(A), \mathcal{Y} \in \mathfrak{F}(B) : (\mathcal{X} \left[f \right] \mathcal{Y} \Rightarrow \mathcal{X} \cap \mathcal{A} \neq 0^{\mathfrak{F}(A)} \wedge \mathcal{Y} \cap \mathcal{B} \neq 0^{\mathfrak{F}(B)});$$
consequently $f \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. □

The following theorem gives a formula for calculating an important particular case of intersection on the lattice of funcoids:

Theorem 19 $f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = I^{\mathsf{FCD}}_{\mathcal{B}} \circ f \circ I^{\mathsf{FCD}}_{\mathcal{A}}$ *for every funcoid f and $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.*

Proof $h \stackrel{\text{def}}{=} I^{\mathsf{FCD}}_{\mathcal{B}} \circ f \circ I^{\mathsf{FCD}}_{\mathcal{A}}$. For every $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$

$$\langle h \rangle \mathcal{X} = \left\langle I^{\mathsf{FCD}}_{\mathcal{B}} \right\rangle \langle f \rangle \left\langle I^{\mathsf{FCD}}_{\mathcal{A}} \right\rangle \mathcal{X} = \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap \mathcal{X}).$$

From this, as easy to show, $h \subseteq f$ and $h \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$. If $g \subseteq f \wedge g \subseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ for a $g \in \mathsf{FCD}(\operatorname{Src} f; \operatorname{Dst} f)$ then $\operatorname{dom} g \subseteq \mathcal{A}$, $\operatorname{im} g \subseteq \mathcal{B}$,

$$\langle g \rangle \mathcal{X} = \mathcal{B} \cap \langle g \rangle (\mathcal{A} \cap \mathcal{X}) \subseteq \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap \mathcal{X}) = \left\langle I^{\mathsf{FCD}}_{\mathcal{B}} \right\rangle \langle f \rangle \left\langle I^{\mathsf{FCD}}_{\mathcal{A}} \right\rangle \mathcal{X} = \langle h \rangle \mathcal{X},$$

$g \subseteq h$. So $h = f \cap^{\mathsf{FCD}} (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

Corollary 3 $f|_{\mathcal{A}} = f \cap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{F}(\operatorname{Dst} f)})$ *for every $f \in \mathsf{FCD}$ and $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$.*

Proof $f \cap (\mathcal{A} \times^{\mathsf{FCD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}) = I^{\mathsf{FCD}}_{1^{\mathfrak{F}(\operatorname{Dst} f)}} \circ f \circ I^{\mathsf{FCD}}_{\mathcal{A}} = f \circ I^{\mathsf{FCD}}_{\mathcal{A}} = f|_{\mathcal{A}}$. □

Corollary 4 $f \not\asymp (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \Leftrightarrow \mathcal{A}\ [f]\ \mathcal{B}$ *for every $f \in \mathsf{FCD}$, $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.*

Proof $f \not\asymp (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \Leftrightarrow \left\langle f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) \right\rangle^{*} (\operatorname{Src} f) \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \left\langle I^{\mathsf{FCD}}_{\mathcal{B}} \circ f \circ I^{\mathsf{FCD}}_{\mathcal{A}} \right\rangle^{*} (\operatorname{Src} f) \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \left\langle I^{\mathsf{FCD}}_{\mathcal{B}} \right\rangle \langle f \rangle \left\langle I^{\mathsf{FCD}}_{\mathcal{A}} \right\rangle 1^{\mathfrak{F}(\operatorname{Src} f)} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{B} \cap \langle f \rangle (\mathcal{A} \cap 1^{\mathfrak{F}(\operatorname{Src} f)}) \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{B} \cap \langle f \rangle \mathcal{A} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{A}\ [f]\ \mathcal{B}$. □

Corollary 5 *Every filtrator of funcoids is star-separable.*

Proof The set of funcoidal products of principal filter objects is a separation subset of the lattice of funcoids. □

Theorem 20 *Let A, B be small sets. If $S \in \mathscr{P}(\mathfrak{F}(A) \times \mathfrak{F}(B))$ then*

$$\bigcap \left\{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \quad | \quad (\mathcal{A}; \mathcal{B}) \in S \right\} = \bigcap \operatorname{dom} S \times^{\mathsf{FCD}} \bigcap \operatorname{im} S.$$

Proof If $x \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$ then by the theorem 17

$$\left\langle \bigcap \left\{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \quad | \quad (\mathcal{A}; \mathcal{B}) \in S \right\} \right\rangle x = \bigcap \left\{ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x \quad | \quad (\mathcal{A}; \mathcal{B}) \in S \right\}.$$

If $x \not\asymp \bigcap \operatorname{dom} S$ then

$$\forall (\mathcal{A}; \mathcal{B}) \in S : (x \cap \mathcal{A} \neq 0^{\mathfrak{F}(A)} \wedge \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x = \mathcal{B});$$
$$\left\{ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x \quad | \quad (\mathcal{A}; \mathcal{B}) \in S \right\} = \operatorname{im} S;$$

if $x \asymp \bigcap \operatorname{dom} S$ then

$$\exists (\mathcal{A}; \mathcal{B}) \in S : \left(x \cap \mathcal{A} = 0^{\mathfrak{F}(A)} \wedge \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x = 0^{\mathfrak{F}(B)} \right);$$
$$\left\{ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x \quad | \quad (\mathcal{A}; \mathcal{B}) \in S \right\} \ni 0^{\mathfrak{F}(B)}.$$

So

$$\left\langle \bigcap \left\{ \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \quad | \quad (\mathcal{A}; \mathcal{B}) \in S \right\} \right\rangle x = \begin{cases} \bigcap \operatorname{im} S & \text{if} \quad x \not\asymp \bigcap \operatorname{dom} S; \\ 0^{\mathfrak{F}(B)} & \text{if} \quad x \asymp \bigcap \operatorname{dom} S. \end{cases}$$

From this follows the statement of the theorem. □

Corollary 6 *For every $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{F}(A)$, $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}(B)$ (for every small sets A, B)*

$$(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap \mathcal{B}_1).$$

Proof $(\mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0) \cap (\mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1) = \bigcap \left\{ \mathcal{A}_0 \times^{\mathsf{FCD}} \mathcal{B}_0, \mathcal{A}_1 \times^{\mathsf{FCD}} \mathcal{B}_1 \right\}$ what is by the last theorem equal to $(\mathcal{A}_0 \cap \mathcal{A}_1) \times^{\mathsf{FCD}} (\mathcal{B}_0 \cap \mathcal{B}_1)$. □

Theorem 21 *If A, B are small sets and $\mathcal{A} \in \mathfrak{F}(A)$ then $\mathcal{A} \times^{\mathsf{FCD}}$ is a complete homomorphism from the lattice $\mathfrak{F}(B)$ to the lattice $\mathsf{FCD}(A; B)$, if also $\mathcal{A} \neq 0^{\mathfrak{F}(A)}$ then it is an order embedding.*

Proof Let $S \in \mathscr{P}\mathfrak{F}(B)$, $X \in \mathscr{P}A$, $x \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$.

$$\begin{aligned} \left\langle \bigcup \left\langle \mathcal{A} \times^{\mathsf{FCD}} \right\rangle S \right\rangle^* X &= \bigcup \left\{ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle^* X \quad | \quad \mathcal{B} \in S \right\} \\ &= \begin{cases} \bigcup S & \text{if } X \in \partial \mathcal{A} \\ 0^{\mathfrak{F}(B)} & \text{if } X \notin \partial \mathcal{A} \end{cases} \\ &= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcup S \right\rangle^* X; \end{aligned}$$

$$\begin{aligned} \left\langle \bigcap \left\langle \mathcal{A} \times^{\mathsf{FCD}} \right\rangle S \right\rangle x &= \bigcap \left\{ \left\langle \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right\rangle x \quad | \quad \mathcal{B} \in S \right\} \\ &= \begin{cases} \bigcap S & \text{if } x \not\asymp \mathcal{A} \\ 0^{\mathfrak{F}(B)} & \text{if } x \asymp \mathcal{A} \end{cases} \\ &= \left\langle \mathcal{A} \times^{\mathsf{FCD}} \bigcap S \right\rangle x. \end{aligned}$$

Thus $\bigcup \left\langle \mathcal{A} \times^{\mathsf{FCD}} \right\rangle S = \mathcal{A} \times^{\mathsf{FCD}} \bigcup S$ and $\bigcap \left\langle \mathcal{A} \times^{\mathsf{FCD}} \right\rangle S = \mathcal{A} \times^{\mathsf{FCD}} \bigcap S$.

If $\mathcal{A} \neq 0^{\mathfrak{F}(A)}$ then obviously the function $\mathcal{A} \times^{\mathsf{FCD}}$ is injective. □

The following proposition states that cutting a rectangle of atomic width from a funcoid always produces a rectangular (representable as a funcoidal product of filter objects) funcoid (of atomic width).

Proposition 19 *If $f \in \mathsf{FCD}$ and a is an atomic filter object on* $\operatorname{Src} f$ *then*

$$f|_a = a \times^{\mathsf{FCD}} \langle f \rangle a.$$

Proof Let $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$.

$$\mathcal{X} \not\asymp a \Rightarrow \langle f|_a \rangle \mathcal{X} = \langle f \rangle a, \quad \mathcal{X} \asymp a \Rightarrow \langle f|_a \rangle \mathcal{X} = 0^{\mathfrak{F}(\operatorname{Dst} f)}.$$

□

3.10 Atomic funcoids

Theorem 22 *An $f \in \mathsf{FCD}(A;B)$ is an atom of the lattice $\mathsf{FCD}(A;B)$ (for small sets A, B) iff it is funcoidal product of two atomic filter objects.*

Proof

$\Rightarrow$ Let $f \in \mathsf{FCD}(A;B)$ be an atom of the lattice $\mathsf{FCD}(A;B)$. Let's get elements $a \in \operatorname{atoms}\operatorname{dom} f$ and $b \in \operatorname{atoms}\langle f\rangle a$. Then for every $\mathcal{X} \in \mathfrak{F}(A)$

$$\mathcal{X} \asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b\right\rangle \mathcal{X} = 0^{\mathfrak{F}(B)} \subseteq \langle f\rangle \mathcal{X}, \quad \mathcal{X} \not\asymp a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b\right\rangle \mathcal{X} = b \subseteq \langle f\rangle \mathcal{X}.$$

So $a \times^{\mathsf{FCD}} b \subseteq f$; because f is atomic we have $f = a \times^{\mathsf{FCD}} b$.

$\Leftarrow$ Let $a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$, $b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}$, $f \in \mathsf{FCD}(A;B)$. If $b \asymp \langle f\rangle a$ then $\neg(a\ [f]\ b)$, $f \asymp a \times^{\mathsf{FCD}} b$; if $b \subseteq \langle f\rangle a$ then $\forall \mathcal{X} \in \mathfrak{F}(A) : (\mathcal{X} \not\asymp a \Rightarrow \langle f\rangle \mathcal{X} \supseteq b)$, $f \supseteq a \times^{\mathsf{FCD}} b$. Consequently $f \asymp a \times^{\mathsf{FCD}} b \vee f \supseteq a \times^{\mathsf{FCD}} b$; that is $a \times^{\mathsf{FCD}} b$ is an atom.

□

Theorem 23 *The lattice $\mathsf{FCD}(A;B)$ is atomic (for every small sets A, B).*

Proof Let f is a non-empty funcoid from A to B. Then $\operatorname{dom} f \neq 0^{\mathfrak{F}(A)}$, thus by the theorem 47 in [15] there exists $a \in \operatorname{atoms}\operatorname{dom} f$. So $\langle f\rangle a \neq 0^{\mathfrak{F}(B)}$ thus exists $b \in \operatorname{atoms}\langle f\rangle a$. Finally the atomic funcoid $a \times^{\mathsf{FCD}} b \subseteq f$. □

Theorem 24 *The lattice $\mathsf{FCD}(A;B)$ is separable (for every small sets A, B).*

Proof Let $f, g \in \mathsf{FCD}(A;B)$, $f \subset g$. Then exists $a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$ such that $\langle f\rangle a \subset \langle g\rangle a$. So because the lattice $\mathfrak{F}(B)$ is atomically separable then exists $b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}$ such that $\langle f\rangle a \cap b = 0^{\mathfrak{F}(B)}$ and $b \subseteq \langle g\rangle a$. For every $x \in \operatorname{atoms} 1^{\mathfrak{F}(A)}$

$$\langle f\rangle a \cap \left\langle a \times^{\mathsf{FCD}} b\right\rangle a = \langle f\rangle a \cap b = 0^{\mathfrak{F}(B)},$$
$$x \neq a \Rightarrow \langle f\rangle x \cap \left\langle a \times^{\mathsf{FCD}} b\right\rangle x = \langle f\rangle x \cap 0^{\mathfrak{F}(B)} = 0^{\mathfrak{F}(B)}.$$

Thus $\langle f\rangle x \cap \left\langle a \times^{\mathsf{FCD}} b\right\rangle x = 0^{\mathfrak{F}(B)}$ and consequently $f \asymp a \times^{\mathsf{FCD}} b$.

$$\left\langle a \times^{\mathsf{FCD}} b\right\rangle a = b \subseteq \langle g\rangle a,$$
$$x \neq a \Rightarrow \left\langle a \times^{\mathsf{FCD}} b\right\rangle x = 0^{\mathfrak{F}(B)} \subseteq \langle g\rangle x.$$

Thus $\left\langle a \times^{\mathsf{FCD}} b\right\rangle x \subseteq \langle g\rangle x$ and consequently $a \times^{\mathsf{FCD}} b \subseteq g$.

So the lattice $\mathsf{FCD}(A;B)$ is separable by the theorem 19 in [15]. □

Corollary 7 *The lattice $\mathsf{FCD}(A;B)$ is:*

1. *separable;*
2. *atomically separable;*
3. *conforming to Wallman's disjunction property.*

Proof By the theorem 22 in [15]. □

Remark 2 For more ways to characterize (atomic) separability of the lattice of funcoids see [15], subsections "Separation subsets and full stars" and "Atomically separable lattices".

Corollary 8 *The lattice* $\mathsf{FCD}(A;B)$ *is an atomistic lattice.*

Proof Let $f \in \mathsf{FCD}(A;B)$. Suppose contrary to the statement to be proved that $\bigcup \operatorname{atoms} f \subset f$. Then it exists $a \in \operatorname{atoms} f$ such that $a \cap \bigcup \operatorname{atoms} f = 0^{\mathsf{FCD}(A;B)}$ what is impossible. □

Proposition 20 $\operatorname{atoms}(f \cup g) = \operatorname{atoms} f \cup \operatorname{atoms} g$ *for every funcoids* $f, g \in \mathsf{FCD}(A;B)$ *(for every small sets* A *and* B*).*

Proof $a \times^{\mathsf{FCD}} b \not\asymp f \cup g \Leftrightarrow a\ [f \cup g]\ b \Leftrightarrow a\ [f]\ b \vee a\ [g]\ b \Leftrightarrow a \times^{\mathsf{FCD}} b \not\asymp f \vee a \times^{\mathsf{FCD}} b \not\asymp g$ for every atomic filter objects a and b. □

Theorem 25 *For every* $f, g, h \in \mathsf{FCD}(A;B)$, $R \in \mathscr{P}\mathsf{FCD}(A;B)$ *(for every small sets* A *and* B*)*

1. $f \cap (g \cup h) = (f \cap g) \cup (f \cap h)$;
2. $f \cup \bigcap R = \bigcap \langle f \cup \rangle R$.

Proof We will take in account that the lattice of funcoids is an atomistic lattice.

1. $\operatorname{atoms}(f \cap (g \cup h)) = \operatorname{atoms} f \cap \operatorname{atoms}(g \cup h) = \operatorname{atoms} f \cap (\operatorname{atoms} g \cup \operatorname{atoms} h) = (\operatorname{atoms} f \cap \operatorname{atoms} g) \cup (\operatorname{atoms} f \cap \operatorname{atoms} h) = \operatorname{atoms}(f \cap g) \cup \operatorname{atoms}(f \cap h) = \operatorname{atoms}((f \cap g) \cup (f \cap h))$.

2. $\operatorname{atoms}(f \cup \bigcap R) = \operatorname{atoms} f \cup \operatorname{atoms} \bigcap R = \operatorname{atoms} f \cup \bigcap \langle \operatorname{atoms} \rangle R = \bigcap \langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R = \bigcap \langle \operatorname{atoms} \rangle \langle f \cup \rangle R = \operatorname{atoms} \bigcap \langle f \cup \rangle R$. (Used the following equality.)

$$\begin{aligned}
\langle (\operatorname{atoms} f) \cup \rangle \langle \operatorname{atoms} \rangle R &= \\
\{ (\operatorname{atoms} f) \cup A \mid A \in \langle \operatorname{atoms} \rangle R \} &= \\
\{ (\operatorname{atoms} f) \cup A \mid \exists C \in R : A = \operatorname{atoms} C \} &= \\
\{ (\operatorname{atoms} f) \cup (\operatorname{atoms} C) \mid C \in R \} &= \\
\{ \operatorname{atoms}(f \cup C) \mid C \in R \} &= \\
\{ \operatorname{atoms} B \mid \exists C \in R : B = f \cup C \} &= \\
\{ \operatorname{atoms} B \mid B \in \langle f \cup \rangle R \} &= \\
\langle \operatorname{atoms} \rangle \langle f \cup \rangle R.
\end{aligned}$$

$\square$

Note that distributivity of the lattice of funcoids is proved through using atoms of this lattice. I have never seen such method of proving distributivity.

Corollary 9 *The lattice* $\mathsf{FCD}(A; B)$ *is co-brouwerian (for every small sets* A *and* B*).*

The next proposition is one more (among the theorem 12) generalization for funcoids of composition of relations.

Proposition 21 *For every composable funcoids* f*,* g

$$\operatorname{atoms}(g \circ f) = \left\{ x \times^{\mathsf{FCD}} z \quad \middle| \quad \begin{array}{c} x \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)}, z \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} g)}, \\ \exists y \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} : (x \times^{\mathsf{FCD}} y \in \operatorname{atoms} f \wedge y \times^{\mathsf{FCD}} z \in \operatorname{atoms} g) \end{array} \right\}.$$

Proof $(x \times^{\mathsf{FCD}} z) \cap (g \circ f) \neq 0 \Leftrightarrow x \; [g \circ f] \; z \Leftrightarrow \exists y \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} : (x \; [f] \; y \wedge y \; [g] \; z) \Leftrightarrow \exists y \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} : ((x \times^{\mathsf{FCD}} y) \cap f \neq 0 \wedge (y \times^{\mathsf{FCD}} z) \cap g \neq 0)$ (it was used the theorem 12). $\square$

Theorem 26 *Let* f *be a funcoid.*

1. $\mathcal{X} \; [f] \; \mathcal{Y} \Leftrightarrow \exists F \in \operatorname{atoms} f : \mathcal{X} \; [F] \; \mathcal{Y}$*;*

2. $\langle f \rangle \mathcal{X} = \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ *for every* $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$*.*

Proof 1. $\exists F \in \operatorname{atoms} f : \mathcal{X} \; [F] \; \mathcal{Y} \Leftrightarrow \exists a \in \mathfrak{F}(\operatorname{Src} f), b \in \mathfrak{F}(\operatorname{Dst} f) : \left(a \times^{\mathrm{FCD}} b \not\asymp f \wedge \mathcal{X} \; [a \times^{\mathrm{FCD}} b] \; \mathcal{Y}\right) \Leftrightarrow \exists a \in \mathfrak{F}(\operatorname{Src} f), b \in \mathfrak{F}(\operatorname{Dst} f) : (a \times^{\mathrm{FCD}} b \not\asymp f \wedge a \times^{\mathrm{FCD}} b \not\asymp \mathcal{X} \times^{\mathrm{FCD}} \mathcal{Y}) \Leftrightarrow \exists F \in \operatorname{atoms} f : (F \not\asymp f \wedge F \not\asymp \mathcal{X} \times^{\mathrm{FCD}} \mathcal{Y}) \Leftrightarrow f \not\asymp \mathcal{X} \times^{\mathrm{FCD}} \mathcal{Y} \Leftrightarrow \mathcal{X} \; [f] \; \mathcal{Y}$.

2. Let $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$. Suppose $\mathcal{Y} \not\asymp \langle f \rangle \mathcal{X}$. Then $\mathcal{X} \; [f] \; \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{X} \; [F] \; \mathcal{Y}$; $\exists F \in \operatorname{atoms} f : \mathcal{Y} \not\asymp \langle F \rangle \mathcal{X}$; $\mathcal{Y} \not\asymp \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. So $\langle f \rangle \mathcal{X} \sqsubseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$. The contrary $\langle f \rangle \mathcal{X} \sqsupseteq \bigsqcup_{F \in \operatorname{atoms} f} \langle F \rangle \mathcal{X}$ is obvious. $\square$

3.11 Complete funcoids

Definition 33 *I will call* ***co-complete*** *such a funcoid* f *that* $\langle f \rangle^* X$ *is a principal f.o. for every* $X \in \mathscr{P}(\operatorname{Src} f)$*.*

Remark 3 I will call **generalized closure** such a function $\alpha \in \mathscr{P} B^{\mathscr{P} A}$ (for some small sets A, B) that

1. $\alpha \emptyset = \emptyset$;

2. $\forall I, J \in \mathscr{P} A : \alpha(I \cup J) = \alpha I \cup \alpha J$.

Obvious 13. A funcoid f is co-complete iff $\langle f\rangle^* = \uparrow^{\mathrm{Dst}\, f} \circ\alpha$ for a generalized closure α.

Remark 4 Thus funcoids can be considered as a generalization of generalized closures. A topological space in Kuratowski sense is the same as reflexive and transitive generalized closure. So topological spaces can be considered as a special case of funcoids.

Definition 34 *I will call a* ***complete funcoid*** *a funcoid whose reverse is co-complete.*

Theorem 27 *The following conditions are equivalent for every funcoid f:*

1. *funcoid f is complete;*
2. $\forall S \in \mathscr{P}\mathfrak{F}(\mathrm{Src}\, f), J \in \mathscr{P}(\mathrm{Dst}\, f) : (\bigcup S\ [f] \uparrow^{\mathrm{Dst}\, f} J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I}\ [f] \uparrow^{\mathrm{Dst}\, f} J)$;
3. $\forall S \in \mathscr{P}\mathscr{P}(\mathrm{Src}\, f), J \in \mathscr{P}(\mathrm{Dst}\, f) : (\bigcup S\ [f]^* J \Leftrightarrow \exists I \in S : I\ [f]^* J)$;
4. $\forall S \in \mathscr{P}\mathfrak{F}(\mathrm{Src}\, f) : \langle f\rangle \bigcup S = \bigcup \langle\langle f\rangle\rangle S$;
5. $\forall S \in \mathscr{P}\mathscr{P}(\mathrm{Src}\, f) : \langle f\rangle^* \bigcup S = \bigcup \left\langle \langle f\rangle^* \right\rangle S$;
6. $\forall A \in \mathscr{P}(\mathrm{Src}\, f) : \langle f\rangle^* A = \bigcup \left\{ \langle f\rangle^* \{a\} \mid a \in A \right\}$.

Proof

(3)⇒(1) For every $S \in \mathscr{P}\mathscr{P}(\mathrm{Src}\, f)$, $J \in \mathscr{P}(\mathrm{Dst}\, f)$

$$\uparrow^{\mathrm{Src}\, f} \bigcup S \cap \left\langle f^{-1}\right\rangle^* J \neq 0^{\mathfrak{F}(\mathrm{Src}\, f)} \Leftrightarrow \exists I \in S : \uparrow^{\mathrm{Src}\, f} I \cap \left\langle f^{-1}\right\rangle^* J \neq 0^{\mathfrak{F}(\mathrm{Src}\, f)}, \tag{10}$$

consequently by the theorem 53 in [15] we have that $\left\langle f^{-1}\right\rangle^* J$ is a principal f.o.

(1)⇒(2) For every $S \in \mathscr{P}\mathfrak{F}(\mathrm{Src}\, f)$, $J \in \mathscr{P}(\mathrm{Dst}\, f)$ we have $\left\langle f^{-1}\right\rangle^* J$ a principal f.o., consequently

$$\bigcup S \cap \left\langle f^{-1}\right\rangle^* J \neq 0^{\mathfrak{F}(\mathrm{Src}\, f)} \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I} \cap \left\langle f^{-1}\right\rangle^* J \neq 0^{\mathfrak{F}(\mathrm{Src}\, f)}.$$

From this follows (2).

(6)⇒(5) $\langle f\rangle^* \bigcup S = \bigcup \left\{ \langle f\rangle^* \{a\} \mid a \in \bigcup S \right\} = \bigcup \left\{ \bigcup \left\{ \langle f\rangle^* \{a\} \mid a \in A \right\} \mid A \in S \right\} = \bigcup \left\{ \langle f\rangle^* A \mid A \in S \right\} = \bigcup \left\langle \langle f\rangle^* \right\rangle S$.

(2)⇒(4) $\uparrow^{\mathrm{Dst}\, f} J \not\asymp \langle f\rangle \bigcup S \Leftrightarrow \bigcup S\ [f] \uparrow^{\mathrm{Dst}\, f} J \Leftrightarrow \exists \mathcal{I} \in S : \mathcal{I}\ [f] \uparrow^{\mathrm{Dst}\, f} J \Leftrightarrow \exists \mathcal{I} \in S : \uparrow^{\mathrm{Dst}\, f} J \not\asymp \langle f\rangle \mathcal{I} \Leftrightarrow \uparrow^{\mathrm{Dst}\, f} J \not\asymp \bigcup \langle\langle f\rangle\rangle S$ (used the theorem 53 in [15]).

(2)⇒(3), (4)⇒(5), (5)⇒(3), (5)⇒(6) Obvious.

□

The following proposition shows that complete funcoids are a direct generalization of pre-topological spaces.

Proposition 22 *To specify a complete funcoid f it is enough to specify $\langle f\rangle^*$ on one-element sets, values of $\langle f\rangle^*$ on one element sets can be specified arbitrarily.*

Proof From the above theorem is clear that knowing $\langle f\rangle^*$ on one-element sets $\langle f\rangle^*$ can be found on every set and then the value of $\langle f\rangle$ can be inferred for every filter objects.

Choosing arbitrarily the values of $\langle f\rangle^*$ on one-element sets we can define a complete funcoid the following way: $\langle f\rangle^* X \stackrel{\text{def}}{=} \bigcup\left\{\langle f\rangle^* \{\alpha\} \quad | \quad \alpha \in X\right\}$ for every $X \in \mathscr{P}(\operatorname{Src} f)$. Obviously it is really a complete funcoid. □

Theorem 28 *A funcoid is principal iff it is both complete and co-complete.*

Proof

⇒ Obvious.

⇐ Let f is both a complete and co-complete funcoid. Consider the relation g defined by that $\uparrow^{\operatorname{Dst} f} \langle g\rangle \{\alpha\} = \langle f\rangle^* \{\alpha\}$ (g is correctly defined because f corresponds to a generalized closure). Because f is a complete funcoid f is the funcoid corresponding to g.

□

Theorem 29 *If $R \in \mathscr{P}\mathsf{FCD}(A;B)$ is a set of (co-)complete funcoids then $\bigcup R$ is a (co-)complete funcoid (for every small sets A and B).*

Proof It is enough to prove only for co-complete funcoids. Let $R \in \mathscr{P}\mathsf{FCD}(A;B)$ is a set of co-complete funcoids. Then for every $X \in \mathscr{P}(\operatorname{Src} f)$

$$\left\langle \bigcup R\right\rangle^* X = \bigcup\left\{\langle f\rangle^* X \quad | \quad f \in R\right\}$$

is a principal f.o. (used the theorem 10). □

Corollary 10 *If R is a set of binary relations between small sets A and B then $\bigcup \left\langle \uparrow^{\mathsf{FCD}(A;B)}\right\rangle R = \uparrow^{\mathsf{FCD}(A;B)} \bigcup R$.*

Proof From two last theorems. □

Theorem 30 *Filtrators of funcoids are filtered.*

Proof It's enough to prove that every funcoid is representable as an (infinite) meet (on the lattice $\mathsf{FCD}(A;B)$) of some set of principal funcoids.

Let $f \in \mathsf{FCD}(A;B)$, $X \in \mathscr{P}A$, $Y \in \operatorname{up}\langle f\rangle X$, $g(X;Y) \stackrel{\text{def}}{=} \uparrow^A X \times^{\mathsf{FCD}} \uparrow^B Y \cup \uparrow^A \overline{X} \times^{\mathsf{FCD}} 1^{\mathfrak{F}(B)}$. For every $K \in \mathscr{P}A$

$$\langle g(X;Y)\rangle^* K = \left\langle \uparrow^A X \times^{\mathsf{FCD}} \uparrow^B Y\right\rangle^* K \cup \left\langle \uparrow^A \overline{X} \times^{\mathsf{FCD}} 1^{\mathfrak{F}(B)}\right\rangle^* K = \left(\begin{cases} 0^{\mathfrak{F}(B)} & \text{if } K = \emptyset \\ Y & \text{if } \emptyset \neq K \subseteq X \\ 1^{\mathfrak{F}(B)} & \text{if } K \nsubseteq X \end{cases}\right) \supseteq \langle f\rangle^* K;$$

so $g(X;Y) \supseteq f$. For every $X \in \mathscr{P}A$

$$\bigcap \left\{ \langle g(X;Y)\rangle^* X \quad | \quad Y \in \operatorname{up}\langle f\rangle X \right\} = \bigcap \left\{ Y \quad | \quad Y \in \operatorname{up}\langle f\rangle^* X \right\} = \langle f\rangle^* X;$$

consequently

$$\left\langle \bigcap \left\{ g(X;Y) \quad | \quad X \in \mathscr{P}A, Y \in \operatorname{up}\langle f\rangle^* X \right\} \right\rangle^* X \subseteq \langle f\rangle^* X$$

that is

$$\bigcap \left\{ g(X;Y) \quad | \quad X \in \mathscr{P}A, Y \in \operatorname{up}\langle f\rangle^* X \right\} \subseteq f$$

and finally

$$f = \bigcap \left\{ g(X;Y) \quad | \quad X \in \mathscr{P}A, Y \in \operatorname{up}\langle f\rangle^* X \right\}.$$

□

Conjecture 3 *If* $f \in \mathsf{FCD}(B;C)$ *is a complete funcoid and* $R \in \mathscr{P}\mathsf{FCD}(A;B)$ *then* $f \circ \bigcup R = \bigcup \langle f\circ\rangle R$.

This conjecture can be weakened:

Conjecture 4 *If* f *is a principal funcoid from* B *to* C *and* $R \in \mathscr{P}\mathsf{FCD}(A;B)$ *then* $f \circ \bigcup R = \bigcup \langle f\circ\rangle R$.

I will denote $\operatorname{Compl}\mathsf{FCD}$ and $\operatorname{CoCompl}\mathsf{FCD}$ the sets of complete and co-complete funcoids correspondingly. $\operatorname{Compl}\mathsf{FCD}(A;B)$ are complete funcoids from A to B and likewise with $\operatorname{CoCompl}\mathsf{FCD}(A;B)$.

Obvious 14. $\operatorname{Compl}\mathsf{FCD}$ and $\operatorname{CoCompl}\mathsf{FCD}$ are closed regarding composition of funcoids.

Proposition 23 $\operatorname{Compl}\mathsf{FCD}(A;B)$ *and* $\operatorname{CoCompl}\mathsf{FCD}(A;B)$ *(with induced order) are complete lattices.*

Proof It follows from the theorem 29. □

Theorem 31 *Atoms of the lattice* $\operatorname{Compl}\mathsf{FCD}(A;B)$ *are exactly funcoidal products of the form* $\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ *where* $\alpha \in A$ *and* b *is an atomic f.o. on* B

Proof First, it's easy to see that $\{\alpha\} \times^{\mathsf{FCD}} b$ are elements of $\operatorname{Compl} \mathsf{FCD}(A;B)$. Also $0^{\mathsf{FCD}(A;B)}$ is an element of $\operatorname{Compl} \mathsf{FCD}(A;B)$.

$\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ are atoms of $\operatorname{Compl} \mathsf{FCD}(A;B)$ because these are atoms of $\mathsf{FCD}(A;B)$.

It remains to prove that if f is an atom of $\operatorname{Compl} \mathsf{FCD}(A;B)$ then $f = \{\alpha\} \times^{\mathsf{FCD}} b$ for some $\alpha \in A$ and an atomic f.o. b on B.

Suppose $f \in \mathsf{FCD}(A;B)$ is a non-empty complete funcoid. Then exists $\alpha \in A$ such that $\langle f \rangle^* \{\alpha\} \neq 0^{\mathfrak{F}(B)}$. Thus $\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b \subseteq f$ for some atomic f.o. b on B. If f is an atom then $f = \uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$. □

Theorem 32

1. *A funcoid $f \in \mathsf{FCD}(A;B)$ is complete iff there exists a function $G : A \to \mathfrak{F}(B)$ such that*

$$f = \bigcup \left\{ \uparrow^A \{\alpha\} \times^{\mathsf{FCD}} G(\alpha) \quad | \quad \alpha \in A \right\}. \tag{11}$$

2. *A funcoid $f \in \mathsf{FCD}(A;B)$ is co-complete iff there exists a function $G : B \to \mathfrak{F}(A)$ such that*

$$f = \bigcup \left\{ G(\alpha) \times^{\mathsf{FCD}} \uparrow^B \{\alpha\} \quad | \quad \alpha \in B \right\}.$$

Proof We will prove only the first as the second is symmetric.

⇒ Let f is complete. Then take

$$G(\alpha) = \bigcup \left\{ b \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} \quad | \quad \uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b \subseteq f \right\}$$

and we have (11) obviously.

⇐ Let (11) holds. Then $G(\alpha) = \bigcup \operatorname{atoms} G(\alpha)$ and thus

$$f = \bigcup \left\{ \uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b \quad | \quad \alpha \in \operatorname{Src} f, b \in \operatorname{atoms} G(\alpha) \right\}$$

and so f is complete.

□

Theorem 33

1. *For a complete funcoid f there exists exactly one function $F \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$ such that*

$$f = \bigcup \left\{ \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} F(\alpha) \quad | \quad \alpha \in \operatorname{Src} f \right\}.$$

2. *For a co-complete funcoid f there exists exactly one function $F \in \mathfrak{F}(\operatorname{Src} f)^{\operatorname{Dst} f}$ such that*

$$f = \bigcup \left\{ F(\alpha) \times^{\mathsf{FCD}} \uparrow^{\operatorname{Dst} f} \{\alpha\} \quad | \quad \alpha \in \operatorname{Dst} f \right\}.$$

Proof We will prove only the first as the second is similar. Let

$$f = \bigcup \left\{ \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} F(\alpha) \quad | \quad \alpha \in \operatorname{Src} f \right\} = \bigcup \left\{ \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} G(\alpha) \quad | \quad \alpha \in \operatorname{Src} f \right\}$$

for some $F, G \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$. We need to prove $F = G$. Let $\beta \in \operatorname{Src} f$.

$$\langle f \rangle \{\beta\} = \bigcup \left\{ \left\langle \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{FCD}} F(\alpha) \right\rangle \{\beta\} \quad | \quad \alpha \in \operatorname{Src} f \right\} = F(\beta).$$

Similarly $\langle f \rangle \{\beta\} = G(\beta)$. So $F(\beta) = G(\beta)$. □

3.12 Completion of funcoids

Theorem 34 $\operatorname{Cor} f = \operatorname{Cor}' f$ *for an element* f *of a filtrator of funcoids. (Core part is taken for the shifted filtrator of funcoids.)*

Proof From the theorem 26 in [15] and the corollary 10 and theorem 30. □

Definition 35 ***Completion*** *of a funcoid* $f \in \mathsf{FCD}(A; B)$ *is the complete funcoid* $\operatorname{Compl} f \in \mathsf{FCD}(A; B)$ *defined by the formula* $\langle \operatorname{Compl} f \rangle^* \{\alpha\} = \langle f \rangle^* \{\alpha\}$ *for* $\alpha \in \operatorname{Src} f$.

Definition 36 ***Co-completion*** *of a funcoid* f *is defined by the formula*

$$\operatorname{CoCompl} f = (\operatorname{Compl} f^{-1})^{-1}.$$

Obvious 15. $\operatorname{Compl} f \subseteq f$ and $\operatorname{CoCompl} f \subseteq f$ for every funcoid f.

Proposition 24 *The filtrator* $(\mathsf{FCD}(A; B); \operatorname{Compl} \mathsf{FCD}(A; B))$ *is filtered.*

Proof Because the shifted filtrator $(\mathsf{FCD}(A; B); \mathscr{P}(A \times B); \uparrow^{\mathsf{FCD}(A;B)})$ is filtered. □

Theorem 35 $\operatorname{Compl} f = \operatorname{Cor}^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f = \operatorname{Cor}'^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f$ *for every funcoid* $f \in \mathsf{FCD}(A; B)$.

Proof $\operatorname{Cor}^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f = \operatorname{Cor}'^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f$ since (the theorem 26 in [15]) the filtrator $(\mathsf{FCD}(A; B); \operatorname{Compl} \mathsf{FCD}(A; B))$ is filtered and with join closed core (the theorem 29).

Let $g \in \operatorname{up}^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f$. Then $g \in \operatorname{Compl} \mathsf{FCD}(A; B)$ and $g \supseteq f$. Thus $g = \operatorname{Compl} g \supseteq \operatorname{Compl} f$.

Thus $\forall g \in \operatorname{up}^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f : g \supseteq \operatorname{Compl} f$.

Let $\forall g \in \operatorname{up}^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f : h \subseteq g$ for some $h \in \operatorname{Compl} \mathsf{FCD}(A; B)$.

Then $h \subseteq \bigcap^{\mathsf{FCD}(A;B)} \operatorname{up}^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f = f$ and consequently $h = \operatorname{Compl} h \subseteq \operatorname{Compl} f$.

Thus

$$\operatorname{Compl} f = \bigcap^{\operatorname{Compl} \mathsf{FCD}(A;B)} \operatorname{up}^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f = \operatorname{Cor}^{(\mathsf{FCD}(A;B);\operatorname{Compl} \mathsf{FCD}(A;B))} f.$$

□

Theorem 36 $\langle \operatorname{CoCompl} f\rangle^* X = \operatorname{Cor}\langle f\rangle^* X$ *for every funcoid f and set $X \in \mathscr{P}(\operatorname{Src} f)$.*

Proof $\operatorname{CoCompl} f \subseteq f$ thus $\langle \operatorname{CoCompl} f\rangle^* X \subseteq \langle f\rangle^* X$, but $\langle \operatorname{CoCompl} f\rangle^* X$ is a principal f.o. thus $\langle \operatorname{CoCompl} f\rangle^* X \subseteq \operatorname{Cor}\langle f\rangle^* X$.

Let $\alpha X = \operatorname{Cor}\langle f\rangle^* X$. Then $\alpha\emptyset = 0^{\mathfrak{F}(\operatorname{Dst} f)}$ and

$$\alpha(X\cup Y) = \operatorname{Cor}\langle f\rangle^* (X\cup Y) = \operatorname{Cor}(\langle f\rangle^* X\cup\langle f\rangle^* Y) = \operatorname{Cor}\langle f\rangle^* X\cup\operatorname{Cor}\langle f\rangle^* Y = \alpha X\cup\alpha Y.$$

(used the theorem 65 from [15]). Thus α can be continued till $\langle g\rangle$ for some funcoid g. This funcoid is co-complete.

Evidently g is the greatest co-complete element of $\mathsf{FCD}(\operatorname{Src} f; \operatorname{Dst} f)$ which is lower than f.

Thus $g = \operatorname{CoCompl} f$ and so $\operatorname{Cor}\langle f\rangle^* X = \alpha X = \langle g\rangle^* X = \langle \operatorname{CoCompl} f\rangle^* X$. □

Theorem 37 $\operatorname{Compl}\mathsf{FCD}(A;B)$ *is an atomistic lattice.*

Proof Let $f \in \operatorname{Compl}\mathsf{FCD}(A;B)$. $\langle f\rangle^* X = \bigcup\left\{\langle f\rangle^*\{x\} \mid x \in X\right\} = \bigcup\left\{\left\langle f|_{\uparrow^{\operatorname{Src} f}\{x\}}\right\rangle^*\{x\} \mid x \in X\right\} = \bigcup\left\{\left\langle f|_{\uparrow^{\operatorname{Src} f}\{x\}}\right\rangle^* X \mid x \in X\right\}$, thus $f = \bigcup\left\{f|_{\uparrow^{\operatorname{Src} f}\{x\}} \mid x \in X\right\}$. It is trivial that every $f|_{\uparrow^{\operatorname{Src} f}\{x\}}$ is a join of atoms of $\operatorname{Compl}\mathsf{FCD}(A;B)$. □

Theorem 38 *A funcoid f is complete iff it is a join (on the lattice $\mathsf{FCD}(\operatorname{Src} f; \operatorname{Dst} f)$) of atomic complete funcoids.*

Proof It follows from the theorem 29 and the previous theorem. □

Corollary 11 $\operatorname{Compl}\mathsf{FCD}(A;B)$ *is join-closed.*

Theorem 39 $\operatorname{Compl}(\bigcup R) = \bigcup\langle\operatorname{Compl}\rangle R$ *for every $R \in \mathscr{P}\mathsf{FCD}(A;B)$ (for every small sets A, B).*

Proof $\langle\operatorname{Compl}(\bigcup R)\rangle^* X = \bigcup\left\{\langle\bigcup R\rangle^*\{\alpha\} \mid \alpha \in X\right\} = \bigcup\left\{\bigcup\left\{\langle f\rangle^*\{\alpha\} \mid f \in R\right\} \mid \alpha \in X\right\} = \bigcup\left\{\bigcup\left\{\langle f\rangle^*\{\alpha\} \mid \alpha \in X\right\} \mid f \in R\right\} = \bigcup\left\{\langle\operatorname{Compl} f\rangle^* X \mid f \in R\right\} = \langle\bigcup\langle\operatorname{Compl}\rangle R\rangle^* X$ for every set X. □

Corollary 12 Compl *is a lower adjoint.*

Conjecture 5 Compl *is not an upper adjoint (in general).*

Proposition 25 $\operatorname{Compl} f = \bigcup\left\{f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} \mid \alpha \in \operatorname{Src} f\right\}$ *for every funcoid f.*

Proof Let denote R the right part of the equality to prove.

$\langle R\rangle^* \{\beta\} = \bigcup \left\{ \langle f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}\rangle^* \{\beta\} \quad | \quad \alpha \in \operatorname{Src} f \right\} = \langle f\rangle^* \{\beta\}$ for every $\beta \in \operatorname{Src} f$ and R is complete as a join of complete funcoids.

Thus R is the completion of f. □

Conjecture 6 $\operatorname{Compl} f = f \setminus^* (\Omega \times^{\mathsf{FCD}} \mho)$ *for every funcoid f.*

This conjecture may be proved by considerations similar to these in the section "Fréchet filter" in [15].

Lemma 2 *Co-completion of a complete funcoid is complete.*

Proof Let f is a complete funcoid.

$\langle \operatorname{CoCompl} f\rangle^* X = \operatorname{Cor} \langle f\rangle^* X = \operatorname{Cor} \bigcup \left\{ \langle f\rangle^* \{x\} \quad | \quad x \in X \right\} = \bigcup \left\{ \operatorname{Cor} \langle f\rangle^* \{x\} \quad | \quad x \in X \right\} = \bigcup \left\{ \langle \operatorname{CoCompl} f\rangle^* \{x\} \quad | \quad x \in X \right\}$ for every set X. Thus $\operatorname{CoCompl} f$ is complete. □

Theorem 40 $\operatorname{Compl} \operatorname{CoCompl} f = \operatorname{CoCompl} \operatorname{Compl} f = \operatorname{Cor} f$ *for every funcoid f.*

Proof $\operatorname{Compl} \operatorname{CoCompl} f$ is co-complete since (used the lemma) $\operatorname{CoCompl} f$ is co-complete. Thus $\operatorname{Compl} \operatorname{CoCompl} f$ is a principal funcoid. $\operatorname{CoCompl} f$ is the the greatest co-complete funcoid under f and $\operatorname{Compl} \operatorname{CoCompl} f$ is the greatest complete funcoid under $\operatorname{CoCompl} f$. So $\operatorname{Compl} \operatorname{CoCompl} f$ is greater than any principal funcoid under $\operatorname{CoCompl} f$ which is greater than any principal funcoid under f. Thus $\operatorname{Compl} \operatorname{CoCompl} f$ it is the greatest principal funcoid under f. Thus $\operatorname{Compl} \operatorname{CoCompl} f = \operatorname{Cor} f$. Similarly $\operatorname{CoCompl} \operatorname{Compl} f = \operatorname{Cor} f$. □

Question 16. Is $\operatorname{Compl} \mathsf{FCD}(A; B)$ a co-brouwerian lattice for every small sets A, B?

3.13 Monovalued and injective funcoids

Following the idea of definition of monovalued morphism let's call **monovalued** such a funcoid f that $f \circ f^{-1} \subseteq I^{\mathsf{FCD}}_{\operatorname{im} f}$.

Similarly, I will call a funcoid **injective** when $f^{-1} \circ f \subseteq I^{\mathsf{FCD}}_{\operatorname{dom} f}$.

Obvious 17. A funcoid f is

- monovalued iff $f \circ f^{-1} \subseteq I^{\mathsf{FCD}(\operatorname{Dst} f)}$;
- injective iff $f^{-1} \circ f \subseteq I^{\mathsf{FCD}(\operatorname{Src} f)}$.

In other words, a funcoid is monovalued (injective) when it is a monovalued (injective) morphism of the category of funcoids.

Monovaluedness is dual of injectivity.

Obvious 18.

1. A morphism $(f;\mathcal{A};\mathcal{B})$ of the category of funcoid triples is monovalued iff the funcoid f is monovalued.
2. A morphism $(f;\mathcal{A};\mathcal{B})$ of the category of funcoid triples is injective iff the funcoid f is injective.

Theorem 41 *The following statements are equivalent for a funcoid f:*

1. *f is monovalued.*
2. *$\forall a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)} : \langle f\rangle a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} \cup \left\{0^{\mathfrak{F}(\operatorname{Dst} f)}\right\}$.*
3. *$\forall \mathcal{I},\mathcal{J} \in \mathfrak{F}(\operatorname{Dst} f) : \left\langle f^{-1}\right\rangle (\mathcal{I}\cap\mathcal{J}) = \left\langle f^{-1}\right\rangle \mathcal{I} \cap \left\langle f^{-1}\right\rangle \mathcal{J}$.*
4. *$\forall I,J \in \mathscr{P}(\operatorname{Dst} f) : \left\langle f^{-1}\right\rangle^* (I\cap J) = \left\langle f^{-1}\right\rangle^* I \cap \left\langle f^{-1}\right\rangle^* J$.*

Proof

(2)⇒(3) Let $a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)}$, $\langle f\rangle a = b$. Then because $b \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} \cup \left\{0^{\mathfrak{F}(\operatorname{Dst} f)}\right\}$

$$(\mathcal{I}\cap\mathcal{J})\cap b \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{I}\cap b \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \wedge \mathcal{J}\cap b \neq 0^{\mathfrak{F}(\operatorname{Dst} f)};$$
$$a\,[f]\,(\mathcal{I}\cap\mathcal{J}) \Leftrightarrow a\,[f]\,\mathcal{I} \wedge a\,[f]\,\mathcal{J};$$
$$(\mathcal{I}\cap\mathcal{J})\left[f^{-1}\right] a \Leftrightarrow \mathcal{I}\left[f^{-1}\right] a \wedge \mathcal{J}\left[f^{-1}\right] a;$$
$$a\cap\left\langle f^{-1}\right\rangle(\mathcal{I}\cap\mathcal{J}) \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \Leftrightarrow a\cap\left\langle f^{-1}\right\rangle\mathcal{I} \neq 0^{\mathfrak{F}(\operatorname{Src} f)} \wedge a\cap\left\langle f^{-1}\right\rangle\mathcal{J} \neq 0^{\mathfrak{F}(\operatorname{Src} f)};$$
$$\left\langle f^{-1}\right\rangle(\mathcal{I}\cap\mathcal{J}) = \left\langle f^{-1}\right\rangle\mathcal{I}\cap\left\langle f^{-1}\right\rangle\mathcal{J}.$$

(3)⇒(1) $\left\langle f^{-1}\right\rangle a \cap \left\langle f^{-1}\right\rangle b = \left\langle f^{-1}\right\rangle (a\cap b) = \left\langle f^{-1}\right\rangle 0^{\mathfrak{F}(\operatorname{Dst} f)} = 0^{\mathfrak{F}(\operatorname{Src} f)}$ for every two distinct atomic filter objects a and b on $\operatorname{Dst} f$. This is equivalent to $\neg(\left\langle f^{-1}\right\rangle a\;[f]\;b)$; $b \asymp \langle f\rangle\left\langle f^{-1}\right\rangle a$; $b \asymp \left\langle f\circ f^{-1}\right\rangle a$; $\neg(a\left[f\circ f^{-1}\right] b)$. So $a\left[f\circ f^{-1}\right] b \Rightarrow a = b$ for every atomic filter objects a and b. This is possible only when $f\circ f^{-1} \subseteq I^{\mathsf{FCD}(\operatorname{Dst} f)}$.

(4)⇒(3) $\left\langle f^{-1}\right\rangle(\mathcal{I}\cap\mathcal{J}) = \bigcap\left\langle\left\langle f^{-1}\right\rangle^*\right\rangle \operatorname{up}(\mathcal{I}\cap\mathcal{J}) = \bigcap\left\langle\left\langle f^{-1}\right\rangle^*\right\rangle \{I\cap J \mid I\in\operatorname{up}\mathcal{I}, J\in\operatorname{up}\mathcal{J}\} = \bigcap\left\{\left\langle f^{-1}\right\rangle^*(I\cap J) \mid I\in\operatorname{up}\mathcal{I}, J\in\operatorname{up}\mathcal{J}\right\} = \bigcap\left\{\left\langle f^{-1}\right\rangle^* I\cap\left\langle f^{-1}\right\rangle^* J \mid I\in\operatorname{up}\mathcal{I}, J\in\operatorname{up}\mathcal{J}\right\} = \bigcap\left\{\left\langle f^{-1}\right\rangle^* I \mid I\in\operatorname{up}\mathcal{I}\right\} \cap \bigcap\left\{\left\langle f^{-1}\right\rangle^* J \mid J\in\operatorname{up}\mathcal{J}\right\} = \left\langle f^{-1}\right\rangle\mathcal{I}\cap\left\langle f^{-1}\right\rangle\mathcal{J}$.

(3)⇒(4) Obvious.

¬(2)⇒ ¬(1) Suppose $\langle f\rangle a \notin \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} \cup \left\{0^{\mathfrak{F}(\operatorname{Dst} f)}\right\}$ for some $a\in\operatorname{atoms}\mathcal{A}$. Then there exist two atomic filter objects p and q on $\operatorname{Dst} f$ such that $p\neq q$ and $\langle f\rangle a \supseteq p \wedge \langle f\rangle a \supseteq q$. Consequently $p \not\asymp \langle f\rangle a$; $a \not\asymp \left\langle f^{-1}\right\rangle p$; $a \subseteq \left\langle f^{-1}\right\rangle p$; $\left\langle f\circ f^{-1}\right\rangle p = \langle f\rangle\left\langle f^{-1}\right\rangle p \supseteq \langle f\rangle a \supseteq q$; $\left\langle f\circ f^{-1}\right\rangle p \nsubseteq p$ and $\left\langle f\circ f^{-1}\right\rangle p \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$. So it cannot be $f\circ f^{-1} \subseteq I^{\mathsf{FCD}(\operatorname{Dst} f)}$.

$\square$

Corollary 13 *A binary relation corresponds to a monovalued funcoid iff it is a function.*

Proof Because $\forall I, J \in \mathscr{P}(\operatorname{im} f) : \left\langle f^{-1} \right\rangle^* (I \cap J) = \left\langle f^{-1} \right\rangle^* I \cap \left\langle f^{-1} \right\rangle^* J$ is true for a funcoid f corresponding to a binary relation if and only if it is a function. $\square$

Remark 5 This corollary can be reformulated as follows: For binary relations (principal funcoids) the classic concept of monovaluedness and monovaluedness in the above defined sense of monovaluedness of a funcoid are the same.

3.14 T_0-, T_1- and T_2-separable funcoids

For funcoids it can be generalized T_0-, T_1- and T_2- separability. Worthwhile note that T_0 and T_2 separability is defined through T_1 separability.

Definition 37 *Let call T_1**-separable** such funcoid f that for every $\alpha \in \operatorname{Src} f$, $\beta \in \operatorname{Dst} f$ is true*

$$\alpha \neq \beta \Rightarrow \neg(\{\alpha\} \left[f\right]^* \{\beta\}).$$

Definition 38 *Let call T_0**-separable** such funcoid $f \in \mathsf{FCD}(A; A)$ that $f \cap f^{-1}$ is T_1-separable.*

Definition 39 *Let call T_2**-separable** such funcoid f that the funcoid $f^{-1} \circ f$ is T_1-separable.*

For symmetric transitive funcoids T_1- and T_2-separability are the same (see theorem 3).

Obvious 19. A funcoid f is T_2**-separable** iff $\alpha \neq \beta \Rightarrow \langle f \rangle^* \{\alpha\} \asymp \langle f \rangle^* \{\beta\}$ for every $\alpha, \beta \in \operatorname{Src} f$.

3.15 Filter objects closed regarding a funcoid

Definition 40 *Let's call **closed** regarding a funcoid $f \in \mathsf{FCD}(A; A)$ such filter object $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$ that $\langle f \rangle \mathcal{A} \subseteq \mathcal{A}$.*

This is a generalization of closedness of a set regarding an unary operation.

Proposition 26 *If $\mathcal{I}$ and $\mathcal{J}$ are closed (regarding some funcoid f), S is a set of closed filter objects on* $\operatorname{Src} f$*, then*

1. *$\mathcal{I} \cup \mathcal{J}$ is a closed filter object;*

2. *$\bigcap S$ is a closed filter object.*

Proof Let denote the given funcoid as f. $\langle f\rangle(\mathcal{I}\cup\mathcal{J})=\langle f\rangle\mathcal{I}\cup\langle f\rangle\mathcal{J}\subseteq\mathcal{I}\cup\mathcal{J}$, $\langle f\rangle\bigcap S\subseteq\bigcap\langle\langle f\rangle\rangle S\subseteq\bigcap S$. Consequently the filter objects $\mathcal{I}\cup\mathcal{J}$ and $\bigcap S$ are closed. □

Proposition 27 *If S is a set of filter objects closed regarding a complete funcoid, then the filter object $\bigcup S$ is also closed regarding our funcoid.*

Proof $\langle f\rangle\bigcup S=\bigcup\langle\langle f\rangle\rangle S\subseteq\bigcup S$ where f is the given funcoid. □

4 Reloids

Definition 41 *I will call a **reloid** from a small set A to a small set B a triple $(A;B;F)$ where $F\in\mathfrak{F}(A\times B)$.*

Definition 42 ***Source** and **destination** of every reloid $(A;B;F)$ are defined as*

$$\operatorname{Src}(A;B;F)=A\quad\text{and}\quad\operatorname{Dst}(A;B;F)=B.$$

I will denote $\mathsf{RLD}(A;B)$ the set of reloids from A to B.

I will denote RLD the set of all reloids (for small sets).

Further we will assume that all reloids in consideration are small.

Reloids are a generalization of uniform spaces. Also reloids are generalization of binary relations (I will call a reloid $(A;B;F)$ **principal** when F is a principal filter on $A\times B$.)

I will denote $\operatorname{up}(A;B;F)=\operatorname{up}F$ for every reloid $(A;B;F)$.

Definition 43 *The **reverse** reloid of a reloid f is defined by the formula*

$$(A;B;F)^{-1}=\left(B;A;\left\{F^{-1}\quad|\quad F\in\operatorname{up}f^{-1}\right\}\right).$$

Reverse reloid is a generalization of conjugate quasi-uniformity.

I will denote $\uparrow^{\mathsf{RLD}(A;B)}f=\left(A;B;\uparrow^{A\times B}f\right)$ for every small sets A, B and a binary relation $f\subseteq A\times B$.

The order (in fact a complete lattice) on $\mathsf{RLD}(A;B)$ is defined by the formula

$$(A;B;F)\subseteq(A;B;G)\Leftrightarrow F\subseteq G.$$

We will apply lattice operations to subsets of $\mathsf{RLD}(A;B)$ without explicitly mentioning $\mathsf{RLD}(A;B)$.

4.1 Composition of reloids

Definition 44 *Reloids f and g are* ***composable*** *when* $\operatorname{Dst} f = \operatorname{Src} g$.

Definition 45 *Composition of (composable) reloids is defined by the formula*

$$g \circ f = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} (G \circ F) \quad \mid \quad F \in \operatorname{up} f, G \in \operatorname{up} g \right\}.$$

Composition of reloids is a reloid.

Theorem 42 $(h \circ g) \circ f = h \circ (g \circ f)$ *for every composable reloids f, g, h.*

Proof For two nonempty collections A and B of sets I will denote

$$A \sim B \Leftrightarrow (\forall K \in A \exists L \in B : L \subseteq K) \wedge (\forall K \in B \exists L \in A : L \subseteq K).$$

It is easy to see that $\sim$ is a transitive relation.

I will denote $B \circ A = \{ L \circ K | K \in A, L \in B \}$.

Let first prove that for every nonempty collections of relations A, B, C

$$A \sim B \Rightarrow A \circ C \sim B \circ C.$$

Suppose $A \sim B$ and $P \in A \circ C$ that is $K \in A$ and $M \in C$ such that $P = K \circ M$. $\exists K' \in B : K' \subseteq K$ because $A \sim B$. We have $P' = K' \circ M \in B \circ C$. Obviously $P' \subseteq P$. So for every $P \in A \circ C$ exist $P' \in B \circ C$ such that $P' \subseteq P$; the vice versa is analogous. So $A \circ C \sim B \circ C$.

$\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ g) \circ \operatorname{up} f$, $\operatorname{up}(h \circ g) \sim (\operatorname{up} h) \circ (\operatorname{up} g)$. By proven above $\operatorname{up}((h \circ g) \circ f) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

Analogously $\operatorname{up}(h \circ (g \circ f)) \sim (\operatorname{up} h) \circ (\operatorname{up} g) \circ (\operatorname{up} f)$.

So $\operatorname{up}((h \circ g) \circ f) \sim \operatorname{up}(h \circ (g \circ f))$ what is possible only if $\operatorname{up}((h \circ g) \circ f) = \operatorname{up}(h \circ (g \circ f))$. □

Theorem 43 *For every reloid f:*

1. $f \circ f = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (F \circ F) \quad \mid \quad F \in \operatorname{up} f \right\}$ *if* $\operatorname{Src} f = \operatorname{Dst} f$*;*
2. $f^{-1} \circ f = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Src} f)} \left(F^{-1} \circ F \right) \quad \mid \quad F \in \operatorname{up} f \right\}$*;*
3. $f \circ f^{-1} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Dst} f)} \left(F \circ F^{-1} \right) \quad \mid \quad F \in \operatorname{up} f \right\}$.

Proof I will prove only (1) and (2) because (3) is analogous to (2).

1. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f : H \circ H \subseteq G \circ F$. To prove it take $H = F \cap G$.

2. It's enough to show that $\forall F, G \in \operatorname{up} f \exists H \in \operatorname{up} f : H^{-1} \circ H \subseteq G^{-1} \circ F$. To prove it take $H = F \cap G$. Then $H^{-1} \circ H = (F \cap G)^{-1} \circ (F \cap G) \subseteq G^{-1} \circ F$.

□

Theorem 44 *For every small sets A, B, C if $g, h \in \mathsf{RLD}(A; B)$ then*

1. $f \circ (g \cup h) = f \circ g \cup f \circ h$ *for every* $f \in \mathsf{RLD}(B; C)$*;*

2. $(g \cup h) \circ f = g \circ f \cup h \circ f$ *for every* $f \in \mathsf{RLD}(C; A)$*.*

Proof We'll prove only the first as the second is dual.

By the infinite distributivity law for filters we have

$$\begin{aligned} f \circ g \cup f \circ h &= \begin{array}{c} \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} (F \circ G) \quad | \quad F \in \operatorname{up} f, G \in \operatorname{up} g \right\} \cup \\ \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} (F \circ H) \quad | \quad F \in \operatorname{up} f, H \in \operatorname{up} h \right\} \end{array} \\ &= \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} (F_1 \circ G) \cup \uparrow^{\mathsf{RLD}(A;C)} (F_2 \circ H) \quad | \quad F_1, F_2 \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h \right\} \\ &= \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} ((F_1 \circ G) \cup (F_2 \circ H)) \quad | \quad F_1, F_2 \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h \right\}. \end{aligned}$$

Obviously

$$\begin{aligned} \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} ((F_1 \circ G) \cup (F_2 \circ H)) \quad | \quad F_1, F_2 \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h \right\} &\supseteq \\ \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} (((F_1 \cap F_2) \circ G) \cup ((F_1 \cap F_2) \circ H)) \quad | \quad F_1, F_2 \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h \right\} &= \\ \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} ((F \circ G) \cup (F \circ H)) \quad | \quad F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h \right\} &= \\ \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} (F \circ (G \cup H)) \quad | \quad F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h \right\}&. \end{aligned}$$

Because $G \in \operatorname{up} g \wedge H \in \operatorname{up} h \Rightarrow G \cup H \in \operatorname{up}(g \cup h)$ we have

$$\begin{aligned} \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} (F \circ (G \cup H)) \quad | \quad F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h \right\} &\supseteq \\ \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} (F \circ K) \quad | \quad F \in \operatorname{up} f, K \in \operatorname{up}(g \cup h) \right\} &= \\ f \circ (g \cup h)&. \end{aligned}$$

Thus we proved $f \circ g \cup f \circ h \supseteq f \circ (g \cup h)$. But obviously $f \circ (g \cup h) \supseteq f \circ g$ and $f \circ (g \cup h) \supseteq f \circ h$ and so $f \circ (g \cup h) \supseteq f \circ g \cup f \circ h$. Thus $f \circ (g \cup h) = f \circ g \cup f \circ h$.

□

Conjecture 7 *If f and g are reloids, then*

$$g \circ f = \bigcup \left\{ G \circ F \quad | \quad F \in \operatorname{atoms} f, G \in \operatorname{atoms} g \right\}.$$

Theorem 45 *Let A, B, C be sets, $f \in \mathsf{RLD}(A; B)$, $g \in \mathsf{RLD}(B; C)$, $h \in \mathsf{RLD}(A; C)$. Then*

$$g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}.$$

Proof $g \circ f \not\asymp h \Leftrightarrow \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;C)} (G \circ F) \quad | \quad F \in \operatorname{up} f, G \in \operatorname{up} g \right\} \cap \bigcap \left\langle \uparrow^{\mathsf{RLD}(A;C)} \right\rangle \operatorname{up} h \neq 0^{\mathsf{RLD}(A;C)} \Leftrightarrow \bigcap \left\{ \left\langle \uparrow^{\mathsf{RLD}(A;C)} \right\rangle ((G \circ F) \cap H) \quad | \quad F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h \right\} \neq 0^{\mathsf{RLD}(A;C)} \Leftrightarrow \forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : \uparrow^{\mathsf{RLD}(A;C)} ((G \circ F) \cap H) \neq$

$0^{\mathsf{RLD}(A;C)} \Leftrightarrow \forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : G \circ F \not\asymp H$ (used properties of generalized filter bases).

Similarly $g \not\asymp h \circ f^{-1} \Leftrightarrow \forall F \in \operatorname{up} f, G \in \operatorname{up} g, H \in \operatorname{up} h : G \not\asymp H \circ F^{-1}$.

Thus $g \circ f \not\asymp h \Leftrightarrow g \not\asymp h \circ f^{-1}$ because $G \circ F \not\asymp H \Leftrightarrow G \not\asymp H \circ F^{-1}$ by the proposition 1. □

4.2 Direct product of filter objects

Definition 46 ***Reloidal product*** *of filter objects $\mathcal{A}$ and $\mathcal{B}$ is defined by the formula*

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \stackrel{\text{def}}{=} \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} (A \times B) \quad \middle| \quad A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B} \right\}.$$

Obvious 20. $\uparrow^U A \times^{\mathsf{RLD}} \uparrow^V B = \uparrow^{\mathsf{RLD}(U;V)} (A \times B)$ for every small sets $A \subseteq U$ and $B \subseteq V$.

Theorem 46 $\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} = \bigcup \left\{ a \times^{\mathsf{RLD}} b \quad \middle| \quad a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B} \right\}$ *for every filter objects $\mathcal{A}$, $\mathcal{B}$.*

Proof Obviously

$$\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \supseteq \bigcup \left\{ a \times^{\mathsf{RLD}} b \quad \middle| \quad a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B} \right\}.$$

Reversely, let

$$K \in \operatorname{up} \bigcup \left\{ a \times^{\mathsf{RLD}} b \quad \middle| \quad a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B} \right\}.$$

Then $K \in \operatorname{up}(a \times^{\mathsf{RLD}} b)$ for every $a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B}$; $K \supseteq X_a \times Y_b$ for some $X_a \in \operatorname{up} a$, $Y_b \in \operatorname{up} b$; $K \supseteq \bigcup \{ X_a \times Y_b \quad | \quad a \in \operatorname{atoms} \mathcal{A}, b \in \operatorname{atoms} \mathcal{B} \} = \bigcup \{ X_a \quad | \quad a \in \operatorname{atoms} \mathcal{A} \} \times \bigcup \{ Y_b \quad | \quad b \in \operatorname{atoms} \mathcal{B} \} \supseteq A \times B$ where $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$; $K \in \operatorname{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$. □

Theorem 47 *If $\mathcal{A}_0, \mathcal{A}_1 \in \mathfrak{F}(A)$, $\mathcal{B}_0, \mathcal{B}_1 \in \mathfrak{F}(B)$ for some small sets A, B then*

$$(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0) \cap (\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) = (\mathcal{A}_0 \cap \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap \mathcal{B}_1).$$

Proof

$$\begin{aligned}
& (\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0) \cap (\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) \\
= \; & \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;B)} (P \cap Q) \quad \middle| \quad P \in \operatorname{up}(\mathcal{A}_0 \times^{\mathsf{RLD}} \mathcal{B}_0), Q \in \operatorname{up}(\mathcal{A}_1 \times^{\mathsf{RLD}} \mathcal{B}_1) \right\} \\
= \; & \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;B)} ((A_0 \times B_0) \cap (A_1 \times B_1)) \quad \middle| \quad A_0 \in \operatorname{up} \mathcal{A}_0, B_0 \in \operatorname{up} \mathcal{B}_0, A_1 \in \operatorname{up} \mathcal{A}_1, B_1 \in \operatorname{up} \mathcal{B}_1 \right\} \\
= \; & \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;B)} ((A_0 \cap A_1) \times (B_0 \cap B_1)) \quad \middle| \quad A_0 \in \operatorname{up} \mathcal{A}_0, B_0 \in \operatorname{up} \mathcal{B}_0, A_1 \in \operatorname{up} \mathcal{A}_1, B_1 \in \operatorname{up} \mathcal{B}_1 \right\} \\
= \; & \bigcap \left\{ \uparrow^{\mathsf{RLD}(A;B)} (K \times L) \quad \middle| \quad K \in \operatorname{up}(\mathcal{A}_0 \cap \mathcal{A}_1), L \in \operatorname{up}(\mathcal{B}_0 \cap \mathcal{B}_1) \right\} \\
= \; & (\mathcal{A}_0 \cap \mathcal{A}_1) \times^{\mathsf{RLD}} (\mathcal{B}_0 \cap \mathcal{B}_1).
\end{aligned}$$

□

Theorem 48 *If $S \in \mathscr{P}(\mathfrak{F}(A) \times \mathfrak{F}(B))$ for some small sets A, B then*

$$\bigcap\left\{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \quad | \quad (\mathcal{A};\mathcal{B}) \in S\right\} = \bigcap \operatorname{dom} S \times^{\mathsf{RLD}} \bigcap \operatorname{im} S.$$

Proof Let $\mathcal{P} = \bigcap \operatorname{dom} S$, $\mathcal{Q} = \bigcap \operatorname{im} S$; $l = \bigcap\left\{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \quad | \quad (\mathcal{A};\mathcal{B}) \in S\right\}$.

$\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q} \subseteq l$ is obvious.

Let $F \in \operatorname{up}(\mathcal{P} \times^{\mathsf{RLD}} \mathcal{Q})$. Then exist $P \in \operatorname{up} \mathcal{P}$ and $Q \in \operatorname{up} \mathcal{Q}$ such that $F \supseteq P \times Q$.

$P = P_1 \cap \ldots \cap P_n$ where $P_i \in \langle \operatorname{up} \rangle \operatorname{dom} S$ and $Q = Q_1 \cap \ldots \cap Q_m$ where $Q_j \in \langle \operatorname{up} \rangle \operatorname{im} S$.

$P \times Q = \bigcap_{i,j}(P_i \times Q_j)$.

$P_i \times Q_j \in \operatorname{up}\left(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}\right)$ for some $(\mathcal{A};\mathcal{B}) \in S$. $P \times Q = \bigcap_{i,j}(P_i \times Q_j) \in \operatorname{up} l$. So $F \in \operatorname{up} l$. □

Conjecture 8 *If $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \times^{\mathsf{RLD}}$ is a complete homomorphism from every lattice $\mathfrak{F}(B)$ to the lattice $\mathsf{RLD}(A;B)$, if also $\mathcal{A} \neq 0^{\mathfrak{F}}$ then it is an order embedding.*

Definition 47 *I will call a reloid* ***convex*** *iff it is a join of direct products.*

4.3 Restricting reloid to a filter object. Domain and image

Definition 48 ***Identity reloid*** *for a small set A is defined by the formula $I^{\mathsf{RLD}(A)} = \uparrow^{\mathsf{RLD}(A;A)} I_A$.*

Definition 49 *I call* ***restricting*** *a reloid f to a filter object $\mathcal{A}$ as $f|_{\mathcal{A}} = f \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right)$.*

Definition 50 ***Domain*** *and* ***image*** *of a reloid f are defined as follows:*

$$\operatorname{dom} f = \bigcap \left\langle \uparrow^{\operatorname{Src} f} \right\rangle \langle \operatorname{dom} \rangle \operatorname{up} f; \quad \operatorname{im} f = \bigcap \left\langle \uparrow^{\operatorname{Dst} f} \right\rangle \langle \operatorname{im} \rangle \operatorname{up} f.$$

Proposition 28 *$f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \Leftrightarrow \operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$ for every reloid f and filter objects $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.*

Proof

$\Rightarrow$ It follows from $\operatorname{dom}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{A} \wedge \operatorname{im}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \subseteq \mathcal{B}$.

$\Leftarrow$ $\operatorname{dom} f \subseteq \mathcal{A} \Leftrightarrow \forall A \in \operatorname{up} \mathcal{A} \exists F \in \operatorname{up} f : \operatorname{dom} F \subseteq A$. Analogously

$$\operatorname{im} f \subseteq \mathcal{B} \Leftrightarrow \forall B \in \operatorname{up} \mathcal{B} \exists G \in \operatorname{up} f : \operatorname{im} G \subseteq B.$$

Let $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$, $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$. Then exist $F \in \operatorname{up} f, G \in \operatorname{up} f$ such that $\operatorname{dom} F \subseteq A \wedge \operatorname{im} G \subseteq B$. Consequently $F \cap G \in \operatorname{up} f$, $\operatorname{dom}(F \cap G) \subseteq A$, $\operatorname{im}(F \cap G) \subseteq B$ that is $F \cap G \subseteq A \times B$. So exists $H \in \operatorname{up} f$ such that $H \subseteq A \times B$ for every $A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}$. So $f \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.

□

Definition 51 *I call **restricted identity reloid** for a filter object $\mathcal{A}$ the reloid*

$$I^{\mathsf{RLD}}_{\mathcal{A}} \stackrel{\text{def}}{=} \left(I^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}))}\right)|_{\mathcal{A}}.$$

Theorem 49 $I^{\mathsf{RLD}}_{\mathcal{A}} = \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} I_A \mid A \in \operatorname{up}\mathcal{A}\right\}$ *where I_A is the identity relation on a set A.*

Proof Let $K \in \operatorname{up}\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} I_A \mid A \in \operatorname{up}\mathcal{A}\right\}$, then exists $A \in \operatorname{up}\mathcal{A}$ such that $K \supseteq I_A$. Then

$I^{\mathsf{RLD}}_{\mathcal{A}} \subseteq \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} \left(I_{\operatorname{Base}(\mathcal{A})}\right) \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Base}(\mathcal{A}))}\right) \subseteq \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} \left(I_{\operatorname{Base}(\mathcal{A})}\right) \cap \left(\uparrow^{\operatorname{Base}(\mathcal{A})} A \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Base}(\mathcal{A}))}\right) = \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} \left(I_{\operatorname{Base}(\mathcal{A})}\right) \cap \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} (A \times \operatorname{Base}(\mathcal{A})) = \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} \left(I_{\operatorname{Base}(\mathcal{A})} \cap (A \times \operatorname{Base}(\mathcal{A}))\right) = \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} I_A \subseteq \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} K$,

Thus $K \in \operatorname{up} I^{\mathsf{RLD}}_{\mathcal{A}}$.

Reversely let $K \in \operatorname{up} I^{\mathsf{RLD}}_{\mathcal{A}} = \operatorname{up}\left(I^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A}))} \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Base}(\mathcal{A}))}\right)\right)$, then exists $A \in \operatorname{up}\mathcal{A}$ such that $K \in \operatorname{up} \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} \left(I_{\operatorname{Base}(\mathcal{A})} \cap (A \times \operatorname{Base}(\mathcal{A}))\right) = \operatorname{up} \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} I_A \subseteq \operatorname{up}\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{A}))} I_A \mid A \in \operatorname{up}\mathcal{A}\right\}$. □

Proposition 29 $\left(I^{\mathsf{RLD}}_{\mathcal{A}}\right)^{-1} = I^{\mathsf{RLD}}_{\mathcal{A}}$.

Proof It follows from the previous theorem. □

Theorem 50 $f|_{\mathcal{A}} = f \circ I^{\mathsf{RLD}}_{\mathcal{A}}$ *for every reloid f and $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$.*

Proof We need to prove that $f \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right) = f \circ \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Src} f)} I_A \mid A \in \operatorname{up}\mathcal{A}\right\}$.

We have $f \circ \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Src} f)} I_A \mid A \in \operatorname{up}\mathcal{A}\right\} = \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} (F \circ I_A) \mid F \in \operatorname{up} f, A \in \operatorname{up}\mathcal{A}\right\} = \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} (F|_A) \mid F \in \operatorname{up} f, A \in \operatorname{up}\mathcal{A}\right\} = \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} (F \cap (A \times \operatorname{Dst} f)) \mid F \in \operatorname{up} f, A \in \operatorname{up}\mathcal{A}\right\} = \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} F \mid F \in \operatorname{up} f\right\} \cap \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} (A \times \operatorname{Dst} f) \mid A \in \operatorname{up}\mathcal{A}\right\} = f \cap \left(\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right)$. □

Theorem 51 $(g \circ f)|_{\mathcal{A}} = g \circ (f|_{\mathcal{A}})$ *for every composable reloids f and g and $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$.*

Proof $(g \circ f)|_{\mathcal{A}} = (g \circ f) \circ I^{\mathsf{RLD}}_{\mathcal{A}} = g \circ (f \circ I^{\mathsf{RLD}}_{\mathcal{A}}) = g \circ (f|_{\mathcal{A}})$. □

Theorem 52 $f \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = I^{\mathsf{RLD}}_{\mathcal{B}} \circ f \circ I^{\mathsf{RLD}}_{\mathcal{A}}$ *for every reloid f and $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.*

Proof $f \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = f \cap (\mathcal{A} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}) \cap (1^{\mathfrak{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B}) = f|_{\mathcal{A}} \cap (1^{\mathfrak{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B}) = (f \circ I^{\mathsf{RLD}}_{\mathcal{A}}) \cap (1^{\mathfrak{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B}) = \left(\left(f \circ I^{\mathsf{RLD}}_{\mathcal{A}}\right)^{-1} \cap (1^{\mathfrak{F}(\operatorname{Src} f)} \times^{\mathsf{RLD}} \mathcal{B})^{-1}\right)^{-1} = \left((I^{\mathsf{RLD}}_{\mathcal{A}} \circ f^{-1}) \cap (\mathcal{B} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Src} f)})\right)^{-1} = \left(I^{\mathsf{RLD}}_{\mathcal{A}} \circ f^{-1} \circ I^{\mathsf{RLD}}_{\mathcal{B}}\right)^{-1} = I^{\mathsf{RLD}}_{\mathcal{B}} \circ f \circ I^{\mathsf{RLD}}_{\mathcal{A}}$.
□

Theorem 53 *$f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}} = \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} \operatorname{im}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}\right)$ for every reloid f and $\alpha \in \operatorname{Src} f$.*

Proof First,

$$\begin{aligned}
\operatorname{im}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}\right) &= \\
\bigcap \left\langle \uparrow^{\operatorname{Dst} f} \right\rangle \langle \operatorname{im} \rangle \operatorname{up}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}\right) &= \\
\bigcap \left\langle \uparrow^{\operatorname{Dst} f} \right\rangle \langle \operatorname{im} \rangle \operatorname{up}\left(f \cap \left(\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right)\right) &= \\
\bigcap \left\{ \uparrow^{\operatorname{Dst} f} \operatorname{im}\left(F \cap (\{\alpha\} \times \operatorname{Dst} f)\right) \mid F \in \operatorname{up} f \right\} &= \\
\bigcap \left\{ \uparrow^{\operatorname{Dst} f} \operatorname{im}\left(F|_{\{\alpha\}}\right) \mid F \in \operatorname{up} f \right\}.
\end{aligned}$$

Taking this into account we have:

$$\begin{aligned}
\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} \operatorname{im}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}\right) &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (\{\alpha\} \times K) \mid K \in \operatorname{up} \operatorname{im}\left(f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}\right) \right\} &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} \left(\{\alpha\} \times \operatorname{im}\left(F|_{\{\alpha\}}\right)\right) \mid F \in \operatorname{up} f \right\} &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} \left(F|_{\{\alpha\}}\right) \mid F \in \operatorname{up} f \right\} &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} \left(F \cap (\{\alpha\} \times \operatorname{Dst} f)\right) \mid F \in \operatorname{up} f \right\} &= \\
\bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} F \mid F \in \operatorname{up} f \right\} \cap \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (\{\alpha\} \times \operatorname{Dst} f) &= \\
f \cap \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)} (\{\alpha\} \times \operatorname{Dst} f) &= \\
f|_{\uparrow^{\operatorname{Src} f}\{\alpha\}}.
\end{aligned}$$

□

4.4 Categories of reloids

I will define two categories, the **category of reloids** and the **category of reloid triples**.

The **category of reloids** is defined as follows:

- Objects are small sets.
- The set of morphisms from a set A to a set B is $\mathsf{RLD}(A; B)$.
- The composition is the composition of reloids.
- Identity morphism for a set is the identity reloid for that set.

To show it is really a category is trivial.

The **category of reloid triples** is defined as follows:

- Objects are filter objects on small sets.
- The morphisms from a f.o. $\mathcal{A}$ to a f.o. $\mathcal{B}$ are triples $(f; \mathcal{A}; \mathcal{B})$ where $f \in \mathsf{RLD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))$ and $\operatorname{dom} f \subseteq \mathcal{A} \wedge \operatorname{im} f \subseteq \mathcal{B}$.
- The composition is defined by the formula $(g; \mathcal{B}; \mathcal{C}) \circ (f; \mathcal{A}; \mathcal{B}) = (g \circ f; \mathcal{A}; \mathcal{C})$.
- Identity morphism for an f.o. $\mathcal{A}$ is $I^{\mathsf{RLD}}_{\mathcal{A}}$.

To prove that it is really a category is trivial.

4.5 Monovalued and injective reloids

Following the idea of definition of monovalued morphism let's call **monovalued** such a reloid f that $f \circ f^{-1} \subseteq I^{\mathsf{RLD}}_{\operatorname{im} f}$.

Similarly, I will call a reloid **injective** when $f^{-1} \circ f \subseteq I^{\mathsf{RLD}}_{\operatorname{dom} f}$.

Obvious 21. A reloid f is

- monovalued iff $f \circ f^{-1} \subseteq I^{\mathsf{RLD}(\operatorname{Dst} f)}$;
- injective iff $f^{-1} \circ f \subseteq I^{\mathsf{RLD}(\operatorname{Src} f)}$.

In other words, a funcoid is monovalued (injective) when it is a monovalued (injective) morphism of the category of funcoids.

Monovaluedness is dual of injectivity.

Obvious 22.

1. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of reloid triples is monovalued iff the reloid f is monovalued.
2. A morphism $(f; \mathcal{A}; \mathcal{B})$ of the category of reloid triples is injective iff the reloid f is injective.

Theorem 54

1. *A reloid f is a monovalued iff it exists a function (monovalued binary relation) $F \in \operatorname{up} f$.*
2. *A reloid f is a injective iff it exists an injective binary relation $F \in \operatorname{up} f$.*
3. *A reloid f is a both monovalued and injective iff exists an injection (a monovalued and injective binary relation = injective function) $F \in \operatorname{up} f$.*

Proof The reverse implications are obvious. Let's prove the direct implications:

1. Let f is a monovalued reloid. Then $f \circ f^{-1} \subseteq I^{\mathsf{RLD}(\operatorname{Dst} f)}$. So exists

$$h \in \operatorname{up}(f \circ f^{-1}) = \operatorname{up} \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Dst} f)} \left(F \circ F^{-1} \right) \quad \middle| \quad F \in \operatorname{up} f \right\}$$

such that $h \subseteq I^{\mathsf{RLD}(\operatorname{Dst} f)}$. It's simple to show that $\left\{ F \circ F^{-1} \quad \middle| \quad F \in \operatorname{up} f \right\}$ is a filter base. Consequently it exists $F \in \operatorname{up} f$ such that $F \circ F^{-1} \subseteq I_{\operatorname{Dst} f}$ that is F is a function.

2. Similar.

3. Let f is a both monovalued and injective reloid. Then by proved above there exist $F, G \in \operatorname{up} f$ such that F is monovalued and G is injective. Thus $F \cap G \in \operatorname{up} f$ is both monovalued and injective.

□

Conjecture 9 *A reloid f is monovalued iff*

$$\forall g \in \mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f) : (g \subseteq f \Rightarrow \exists \mathcal{A} \in \mathfrak{F}(\operatorname{Src} f) : g = f|_{\mathcal{A}}).$$

4.6 Complete reloids and completion of reloids

Definition 52 *A* ***complete*** *reloid is a reloid representable as join of direct products $\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in A$ and b is an atomic f.o. on B for some small sets A and B.*

Definition 53 *A* ***co-complete*** *reloid is a reloid representable as join of direct products $a \times^{\mathsf{RLD}} \uparrow^B \{\beta\}$ where $\beta \in B$ and a is an atomic f.o. on A for some small sets A and B.*

I will denote the sets of complete and co-complete reloids correspondingly as $\operatorname{Compl}\mathsf{RLD}$ and $\operatorname{CoCompl}\mathsf{RLD}$.
Obvious 23. Complete and co-complete are dual.

Theorem 55

1. *A reloid $f \in \mathsf{RLD}(A; B)$ is complete iff there exists a function $G : A \to \mathfrak{F}(B)$ such that*
$$f = \bigcup \left\{ \uparrow^A \{\alpha\} \times^{\mathsf{RLD}} G(\alpha) \quad \middle| \quad \alpha \in A \right\}. \tag{12}$$

2. *A reloid $f \in \mathsf{RLD}(A; B)$ is co-complete iff there exists a function $G : B \to \mathfrak{F}(A)$ such that*
$$f = \bigcup \left\{ G(\alpha) \times^{\mathsf{RLD}} \uparrow^B \{\alpha\} \quad \middle| \quad \alpha \in B \right\}.$$

Proof We will prove only the first as the second is symmetric.

⇒ Let f is complete. Then take

$$G(\alpha)=\bigcup\left\{b\in\operatorname{atoms}1^{\mathfrak{F}(\operatorname{Dst}f)}\quad|\quad\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}b\subseteq f\right\}$$

and we have (12) obviously.

⇐ Let (12) holds. Then $G(\alpha)=\bigcup\operatorname{atoms}G(\alpha)$ and thus

$$f=\bigcup\left\{\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}b\quad|\quad\alpha\in\operatorname{Src}f,b\in\operatorname{atoms}G(\alpha)\right\}$$

and so f is complete.

□

Obvious 24. Complete and co-complete reloids are convex.

Obvious 25. Principal reloids are complete and co-complete.

Obvious 26. Join (on the lattice of reloids) of complete reloids is complete.

Corollary 14 $\operatorname{Compl}\mathsf{RLD}$ *(with the induced order) is a complete lattice.*

Theorem 56 *A reloid which is both complete and co-complete is principal.*

Proof Let f is a complete and co-complete reloid. We have

$$f=\bigcup\left\{\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}G(\alpha)\quad|\quad\alpha\in\operatorname{Src}f\right\}\quad\text{and}\quad f=\bigcup\left\{H(\beta)\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\}\quad|\quad\beta\in\operatorname{Dst}f\right\}$$

for some functions $G:\operatorname{Src}f\to\mathfrak{F}(\operatorname{Dst}f)$, $H:\operatorname{Dst}f\to\mathfrak{F}(\operatorname{Src}f)$. For every $\alpha\in\operatorname{Src}f$ we have

$$\begin{array}{rcl}
G(\alpha) & = & \\
\operatorname{im}f|_{\uparrow^{\operatorname{Src}f}\{\alpha\}} & = & \\
\operatorname{im}\left(f\cap\left(\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}1^{\mathfrak{F}(\operatorname{Dst}f)}\right)\right) & = & (*)\\
\operatorname{im}\bigcup\left\{\left(H(\beta)\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\}\right)\cap\left(\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}1^{\mathfrak{F}(\operatorname{Dst}f)}\right)\quad|\quad\beta\in\operatorname{Dst}f\right\} & = & \\
\operatorname{im}\bigcup\left\{\left(H(\beta)\cap\uparrow^{\operatorname{Src}f}\{\alpha\}\right)\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\}\quad|\quad\beta\in\operatorname{Dst}f\right\} & = & \\
\operatorname{im}\bigcup\left\{\left(\begin{cases}\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\} & \text{if }H(\beta)\not\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}\\ 0^{\mathsf{RLD}(\operatorname{Src}f;\operatorname{Dst}f)} & \text{if }H(\beta)\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}\end{cases}\right)\quad|\quad\beta\in\operatorname{Dst}f\right\} & = & \\
\operatorname{im}\bigcup\left\{\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}\uparrow^{\operatorname{Dst}f}\{\beta\}\quad|\quad\beta\in\operatorname{Dst}f,H(\beta)\not\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}\right\} & = & \\
\operatorname{im}\bigcup\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src}f;\operatorname{Dst}f)}\{(\alpha;\beta)\}\quad|\quad\beta\in\operatorname{Dst}f,H(\beta)\not\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}\right\} & = & \\
\bigcup\left\{\uparrow^{\operatorname{Dst}f}\{\beta\}\quad|\quad\beta\in\operatorname{Dst}f,H(\beta)\not\asymp\uparrow^{\operatorname{Src}f}\{\alpha\}\right\}. & &
\end{array}$$

* the theorem 40 from [15] was used.

Thus $G(\alpha)$ is a principal f.o. that is $G(\alpha)=\uparrow^{\operatorname{Dst}f}g(\alpha)$ for some $g:\operatorname{Src}f\to\operatorname{Dst}f$; $\uparrow^{\operatorname{Src}f}\{\alpha\}\times^{\mathsf{RLD}}G(\alpha)=\uparrow^{\mathsf{RLD}(\operatorname{Src}f;\operatorname{Dst}f)}(\{\alpha\}\times g(\alpha))$; f is principal as a join of principal reloids. □

Conjecture 10 *Composition of complete reloids is complete.*

Theorem 57

1. *For a complete reloid f there exists exactly one function $F \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$ such that*
$$f = \bigcup \left\{ \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} F(\alpha) \quad | \quad \alpha \in \operatorname{Src} f \right\}.$$

2. *For a co-complete reloid f there exists exactly one function $F \in \mathfrak{F}(\operatorname{Src} f)^{\operatorname{Dst} f}$ such that*
$$f = \bigcup \left\{ F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f} \{\alpha\} \quad | \quad \alpha \in \operatorname{Dst} f \right\}.$$

Proof We will prove only the first as the second is similar. Let

$$f = \bigcup \left\{ \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} F(\alpha) \quad | \quad \alpha \in \operatorname{Src} f \right\} = \bigcup \left\{ \uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} G(\alpha) \quad | \quad \alpha \in \operatorname{Src} f \right\}$$

for some $F, G \in \mathfrak{F}(\operatorname{Dst} f)^{\operatorname{Src} f}$. We need to prove $F = G$. Let $\beta \in \operatorname{Src} f$.

$$\begin{aligned} f \cap \left(\uparrow^{\operatorname{Src} f} \{\beta\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)} \right) &= \text{(theorem 40 in [15])} \\ \bigcup^{\mathsf{RLD}} \left\{ \left(\uparrow^{\operatorname{Src} f} \{\alpha\} \times^{\mathsf{RLD}} F(\alpha) \right) \cap^{\mathsf{RLD}} \left(\uparrow^{\operatorname{Src} f} \{\beta\} \times 1^{\mathfrak{F}(\operatorname{Dst} f)} \right) \quad | \quad \alpha \in \operatorname{Src} f \right\} &= \\ \uparrow^{\operatorname{Src} f} \{\beta\} \times^{\mathsf{RLD}} F(\beta). & \end{aligned}$$

Similarly $f \cap \left(\uparrow^{\operatorname{Src} f} \{\beta\} \times 1^{\mathfrak{F}(\operatorname{Dst} f)} \right) = \uparrow^{\operatorname{Src} f} \{\beta\} \times^{\mathsf{RLD}} G(\beta)$. Thus $\uparrow^{\operatorname{Src} f} \{\beta\} \times^{\mathsf{RLD}} F(\beta) = \uparrow^{\operatorname{Src} f} \{\beta\} \times^{\mathsf{RLD}} G(\beta)$ and so $F(\beta) = G(\beta)$. □

Definition 54 ***Completion*** *and* ***co-completion*** *of a reloid $f \in \mathsf{RLD}(A; B)$ are defined by the formulas:*

$$\operatorname{Compl} f = \operatorname{Cor}^{(\mathsf{RLD}(A;B); \operatorname{Compl} \mathsf{RLD}(A;B))} f \quad \text{and} \quad \operatorname{CoCompl} f = \operatorname{Cor}^{(\mathsf{RLD}(A;B); \operatorname{CoCompl} \mathsf{RLD}(A;B))} f.$$

Theorem 58 *Atoms of the lattice* $\operatorname{Compl} \mathsf{RLD}(A; B)$ *are exactly direct products of the form $\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ where $\alpha \in A$ and b is an atomic f.o. on B.*

Proof First, it's easy to see that $\uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$ are elements of $\operatorname{Compl} \mathsf{RLD}(A; B)$. Also $0^{\mathsf{RLD}(A;B)}$ is an element of $\operatorname{Compl} \mathsf{RLD}$.

$\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ are atoms of $\operatorname{Compl} \mathsf{FCD}$ because these are atoms of RLD.

It remains to prove that if f is an atom of $\operatorname{Compl} \mathsf{RLD}(A; B)$ then $f = \uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b$ for some $\alpha \in A$ and an atomic f.o. b on B.

Suppose f is a non-empty complete reloid. Then $\uparrow^A \{\alpha\} \times^{\mathsf{RLD}} b \subseteq f$ for some $\alpha \in A$ and atomic f.o. b on B. If f is an atom then $f = \uparrow^A \{\alpha\} \times^{\mathsf{FCD}} b$. □

Obvious 27. $\operatorname{Compl} \mathsf{RLD}(A; B)$ is an atomistic lattice.

Proposition 30 $\operatorname{Compl} f = \bigcup \left\{ f|_{\uparrow^{\operatorname{Src} f} \{\alpha\}} \quad | \quad \alpha \in \operatorname{Src} f \right\}$ *for every reloid f.*

Proof Let's denote R the right part of the equality to be proven.

That R is a complete reloid follows from the equality

$$f|_{\uparrow^{\mathrm{Src}\,f}\{\alpha\}} = \uparrow^{\mathrm{Src}\,f} \{\alpha\} \times^{\mathsf{RLD}} \operatorname{im}\left(f|_{\uparrow^{\mathrm{Src}\,f}\{\alpha\}}\right).$$

The only thing left to prove is that $g \subseteq R$ for every complete reloid g such that $g \subseteq f$.

Really let g is a complete reloid such that $g \subseteq f$. Then

$$g = \bigcup \left\{ \uparrow^{\mathrm{Src}\,f} \{\alpha\} \times^{\mathsf{RLD}} G(\alpha) \quad | \quad \alpha \in \mathrm{Src}\,f \right\}$$

for some function $G : \mathrm{Src}\,f \to \mathfrak{F}(\mathrm{Dst}\,f)$.

We have $\uparrow^{\mathrm{Src}\,f} \{\alpha\} \times^{\mathsf{RLD}} G(\alpha) = g|_{\uparrow^{\mathrm{Src}\,f}\{\alpha\}} \subseteq f|_{\uparrow^{\mathrm{Src}\,f}\{\alpha\}}$. Thus $g \subseteq R$. □

Conjecture 11 Compl $f \cap$ Compl g = Compl$(f \cap g)$ for every reloids f and g.

Theorem 59 $\mathrm{Compl}\left(\bigcup R\right) = \bigcup \langle \mathrm{Compl} \rangle R$ *for every set* $R \in \mathscr{P}\mathsf{RLD}(A; B)$ *for every small sets* A, B.

Proof

$$\begin{aligned}
\mathrm{Compl}\left(\bigcup R\right) &= \\
\bigcup \left\{ \left(\bigcup R\right)|_{\uparrow^A\{\alpha\}} \quad | \quad \alpha \in A \right\} &= \text{(theorem 40 in [15])} \\
\bigcup \left\{ \bigcup \left\{ f|_{\uparrow^A\{\alpha\}} \quad | \quad \alpha \in A \right\} \quad | \quad f \in R \right\} &= \\
\bigcup \langle \mathrm{Compl} \rangle R.
\end{aligned}$$

□

Lemma 3 *Completion of a co-complete reloid is principal.*

Proof Let f is a co-complete reloid. Then there is a function $F : \mathrm{Dst}\,f \to \mathfrak{F}(\mathrm{Src}\,f)$ such that

$$f = \bigcup \left\{ F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\mathrm{Dst}\,f} \{\alpha\} \quad | \quad \alpha \in \mathrm{Dst}\,f \right\}.$$

So

$$\begin{aligned}
\mathrm{Compl}\,f &= \\
\bigcup \left\{ \left(\bigcup \left\{ F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\mathrm{Dst}\,f} \{\alpha\} \quad | \quad \alpha \in \mathrm{Dst}\,f \right\} \right)|_{\uparrow^{\mathrm{Src}\,f}\{\beta\}} \quad | \quad \beta \in \mathrm{Src}\,f \right\} &= \\
\bigcup \left\{ \left(\bigcup \left\{ F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\mathrm{Dst}\,f} \{\alpha\} \quad | \quad \alpha \in \mathrm{Dst}\,f \right\} \right) \cap \left(\uparrow^{\mathrm{Src}\,f} \{\beta\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Dst}\,f)} \right) \quad | \quad \beta \in \mathrm{Src}\,f \right\} &= (*) \\
\bigcup \left\{ \bigcup \left\{ \left(F(\alpha) \times^{\mathsf{RLD}} \uparrow^{\mathrm{Dst}\,f} \{\alpha\} \right) \cap \left(\uparrow^{\mathrm{Src}\,f} \{\beta\} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\mathrm{Dst}\,f)} \right) \quad | \quad \alpha \in \mathrm{Dst}\,f \right\} \quad | \quad \beta \in \mathrm{Src}\,f \right\} &= \\
\bigcup \left\{ \bigcup \left\{ \uparrow^{\mathrm{Src}\,f} \{\beta\} \times^{\mathsf{RLD}} \uparrow^{\mathrm{Dst}\,f} \{\alpha\} \quad | \quad \alpha \in \mathrm{Dst}\,f \right\} \quad | \quad \beta \in \mathrm{Src}\,f, \uparrow^{\mathrm{Src}\,f} \{\beta\} \subseteq F(\alpha) \right\}.
\end{aligned}$$

* theorem 40 in [15].

Thus $\operatorname{Compl} f$ is principal. $\square$

Theorem 60 $\operatorname{Compl}\operatorname{CoCompl} f = \operatorname{CoCompl}\operatorname{Compl} f = \operatorname{Cor} f$ *for every reloid* f.

Proof We will prove only $\operatorname{Compl}\operatorname{CoCompl} f = \operatorname{Cor} f$. The rest follows from symmetry.

From the lemma $\operatorname{Compl}\operatorname{CoCompl} f$ is principal. It is obvious $\operatorname{Compl}\operatorname{CoCompl} f \subseteq f$. So to finish the proof we need to show only that for every principal reloid $F \subseteq f$ we have $F \subseteq \operatorname{Compl}\operatorname{CoCompl} f$.

Really, obviously $F \subseteq \operatorname{CoCompl} f$ and thus $F = \operatorname{Compl} F \subseteq \operatorname{Compl}\operatorname{CoCompl} f$. $\square$

Question 28. Is $\operatorname{Compl}\mathsf{RLD}(A;B)$ a distributive lattice? Is $\operatorname{Compl}\mathsf{RLD}(A;B)$ a co-brouwerian lattice?

Conjecture 12 *Let A, B, C are small sets. If $f \in \mathsf{RLD}(B;C)$ is a complete reloid and $R \in \mathscr{P}\mathsf{RLD}(A;B)$ then*

$$f \circ \bigcup R = \bigcup \langle f\circ\rangle R.$$

This conjecture can be weakened:

Conjecture 13 *Let A, B, C are small sets. If $f \in \mathsf{RLD}(B;C)$ is a principal reloid and $R \in \mathscr{P}\mathsf{RLD}(A;B)$ then*

$$f \circ \bigcup R = \bigcup \langle f\circ\rangle R.$$

Conjecture 14 $\operatorname{Compl} f = f \setminus^{*} \left(\Omega^{\operatorname{Src} f} \times^{\mathsf{RLD}} 1^{\mathfrak{F}(\operatorname{Dst} f)}\right)$ *for every reloid* f.

5 Relationships between funcoids and reloids

5.1 Funcoid induced by a reloid

Every reloid f induces a funcoid $(\mathsf{FCD})f \in \mathsf{FCD}(\operatorname{Src} f; \operatorname{Dst} f)$ by the following formulas (for every $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$):

$$\mathcal{X}\,[(\mathsf{FCD})f]\,\mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up} f : \mathcal{X}\left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F\right]\mathcal{Y};$$
$$\langle(\mathsf{FCD})f\rangle\,\mathcal{X} = \bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F\right\rangle\mathcal{X} \quad \middle| \quad F \in \operatorname{up} f\right\}.$$

We should prove that $(\mathsf{FCD})f$ is really a funcoid.

Proof We need to prove that

$$\mathcal{X}\,[(\mathsf{FCD})f]\,\mathcal{Y} \Leftrightarrow \mathcal{Y} \cap \langle(\mathsf{FCD})f\rangle\,\mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow \mathcal{X} \cap \left\langle(\mathsf{FCD})f^{-1}\right\rangle\mathcal{Y} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}.$$

The above formula is equivalent to:

$$\begin{aligned}\forall F \in \operatorname{up} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F\right] \mathcal{Y} \quad \Leftrightarrow \\ \mathcal{Y} \cap \bigcap \left\{ \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F \right\rangle \mathcal{X} \quad | \quad F \in \operatorname{up} f \right\} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \quad \Leftrightarrow \\ \mathcal{X} \cap \bigcap \left\{ \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F^{-1} \right\rangle \mathcal{Y} \quad | \quad F \in \operatorname{up} f \right\} \neq 0^{\mathfrak{F}(\operatorname{Src} f)}.\end{aligned}$$

We have $\mathcal{Y} \cap \bigcap \left\{ \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F \right\rangle \mathcal{X} \quad | \quad F \in \operatorname{up} f \right\} = \bigcap \left\{ \mathcal{Y} \cap \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F \right\rangle \mathcal{X} \quad | \quad F \in \operatorname{up} f \right\}$.

Let's denote $W = \left\{ \mathcal{Y} \cap \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F \right\rangle \mathcal{X} \quad | \quad F \in \operatorname{up} f \right\}$.

$\forall F \in \operatorname{up} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F\right] \mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up} f : \mathcal{Y} \cap \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F \right\rangle \mathcal{X} \neq 0^{\mathfrak{F}(\operatorname{Dst} f)} \Leftrightarrow 0^{\mathfrak{F}(\operatorname{Dst} f)} \notin W$.

We need to prove only that $0^{\mathfrak{F}(\operatorname{Dst} f)} \notin W \Leftrightarrow \bigcap W \neq 0^{\mathfrak{F}(\operatorname{Dst} f)}$. (The rest follows from symmetry.)

This follows from the fact that W is a generalized filter base.

Let's prove that W is a generalized filter base. For this it's enough to prove that $V = \left\{ \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F \right\rangle \mathcal{X} \quad | \quad F \in \operatorname{up} f \right\}$ is a generalized filter base. Let $\mathcal{A}, \mathcal{B} \in V$ that is $\mathcal{A} = \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} P \right\rangle \mathcal{X}$, $\mathcal{B} = \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} Q \right\rangle \mathcal{X}$ where $P, Q \in \operatorname{up} f$. Then for $\mathcal{C} = \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} (P \cap Q) \right\rangle \mathcal{X}$ is true both $\mathcal{C} \in V$ and $\mathcal{C} \subseteq \mathcal{A}, \mathcal{B}$. So V is a generalized filter base and thus W is a generalized filter base. □

Proposition 31 $(\mathsf{FCD}) \uparrow^{\mathsf{RLD}(A;B)} f = \uparrow^{\mathsf{FCD}(A;B)} f$ *for every small sets* A, B *and binary relation* $f \subseteq A \times B$.

Proof $\mathcal{X} \left[(\mathsf{FCD}) \uparrow^{\mathsf{RLD}(A;B)} f\right] \mathcal{Y} \Leftrightarrow \forall F \in \operatorname{up} \uparrow^{\mathsf{RLD}(A;B)} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(A;B)} F\right] \mathcal{Y} \Leftrightarrow \mathcal{X} \left[\uparrow^{\mathsf{FCD}(A;B)} f\right] \mathcal{Y}$ (for every $\mathcal{X} \in \mathfrak{F}(A)$, $\mathcal{Y} \in \mathfrak{F}(B)$). □

Theorem 61 $\mathcal{X} \left[(\mathsf{FCD}) f\right] \mathcal{Y} \Leftrightarrow (\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}) \not\asymp f$ *for every* $f \in \mathsf{RLD}$ *and* $\mathcal{X} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{Y} \in \mathfrak{F}(\operatorname{Dst} f)$.

Proof

$$\begin{aligned}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}) \not\asymp f \quad &\Leftrightarrow \quad \forall F \in \operatorname{up} f, P \in \operatorname{up}(\mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y}) : P \not\asymp F \\ &\Leftrightarrow \quad \forall F \in \operatorname{up} f, X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : (X \times Y) \not\asymp F \\ &\Leftrightarrow \quad \forall F \in \operatorname{up} f, X \in \operatorname{up} \mathcal{X}, Y \in \operatorname{up} \mathcal{Y} : \uparrow^{\operatorname{Src} f} X \left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F\right] \uparrow^{\operatorname{Dst} f} Y \\ &\Leftrightarrow \quad \forall F \in \operatorname{up} f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F\right] \mathcal{Y} \\ &\Leftrightarrow \quad \mathcal{X} \left[(\mathsf{FCD}) f\right] \mathcal{Y}.\end{aligned}$$

□

Theorem 62 $(\mathsf{FCD}) f = \bigcap \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} \right\rangle \operatorname{up} f$ *for every reloid* f.

Proof Let a be an atomic filter object on $\operatorname{Src} f$.

$\langle(\mathsf{FCD})f\rangle\, a = \bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F\right\rangle a \mid F \in \operatorname{up} f\right\}$ by the definition of (FCD).

$\left\langle\bigcap\left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)}\right\rangle \operatorname{up} f\right\rangle a = \bigcap\left\{\left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)} F\right\rangle a \mid F \in \operatorname{up} f\right\}$ by the theorem 17.

So $\langle(\mathsf{FCD})f\rangle\, a = \left\langle\bigcap\left\langle\uparrow^{\mathsf{FCD}(\operatorname{Src} f;\operatorname{Dst} f)}\right\rangle \operatorname{up} f\right\rangle a$ for every a. □

Lemma 4 *For every two filter bases S and T of binary relations on $U \times V$ for some small sets U, V and every set $A \subseteq U$*

$$\bigcap\left\langle\uparrow^{\mathsf{RLD}(U;V)}\right\rangle S = \bigcap\left\langle\uparrow^{\mathsf{RLD}(U;V)}\right\rangle T \Rightarrow \bigcap\left\{\uparrow^{V} \langle F\rangle A \mid F \in S\right\} = \bigcap\left\{\uparrow^{V} \langle G\rangle A \mid G \in T\right\}.$$

Proof Let $\bigcap\left\langle\uparrow^{\mathsf{RLD}(U;V)}\right\rangle S = \bigcap\left\langle\uparrow^{\mathsf{RLD}(U;V)}\right\rangle T$.

First let prove that $\{\langle F\rangle A \mid F \in S\}$ is a filter base. Let $X, Y \in \{\langle F\rangle A \mid F \in S\}$. Then $X = \langle F_X\rangle A$ and $Y = \langle F_Y\rangle A$ for some $F_X, F_Y \in S$. Because S is a filter base, we have $S \ni F_Z \subseteq F_X \cap F_Y$. So $\langle F_Z\rangle A \subseteq X \cap Y$ and $\langle F_Z\rangle A \in \{\langle F\rangle A \mid F \in S\}$. So $\{\langle F\rangle A \mid F \in S\}$ is a filter base.

Suppose $X \in \operatorname{up}\bigcap\left\{\uparrow^{V} \langle F\rangle A \mid F \in S\right\}$. Then exists $X' \in \{\langle F\rangle A \mid F \in S\}$ where $X \supseteq X'$ because $\{\langle F\rangle A \mid F \in S\}$ is a filter base. That is $X' = \langle F\rangle A$ for some $F \in S$. There exists $G \in T$ such that $G \subseteq F$ because T is a filter base. Let $Y' = \langle G\rangle A$. We have $Y' \subseteq X' \subseteq X$; $Y' \in \{\langle G\rangle A \mid G \in T\}$; $Y' \in \operatorname{up}\bigcap\left\{\uparrow^{V} \langle G\rangle A \mid G \in T\right\}$; $X \in \operatorname{up}\bigcap\left\{\uparrow^{V} \langle G\rangle A \mid G \in T\right\}$. The reverse is symmetric. □

Lemma 5 $\{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ *is a filter base for every reloids f and g.*

Proof Let denote $D = \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$. Let $A \in D \wedge B \in D$. Then $A = G_A \circ F_A \wedge B = G_B \circ F_B$ for some $F_A, F_B \in \operatorname{up} f$ and $G_A, G_B \in \operatorname{up} g$. So $A \cap B \supseteq (G_A \cap G_B) \circ (F_A \cap F_B) \in D$ because $F_A \cap F_B \in \operatorname{up} f$ and $G_A \cap G_B \in \operatorname{up} g$. □

Theorem 63 $(\mathsf{FCD})(g \circ f) = ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f)$ *for every composable reloids f and g.*

Proof

$$\begin{aligned}\langle(\mathsf{FCD})(g \circ f)\rangle^{*} X &= \bigcap\left\{\uparrow^{\operatorname{Dst} g} \langle H\rangle X \mid H \in \operatorname{up}(g \circ f)\right\}\\ &= \bigcap\left\{\uparrow^{\operatorname{Dst} g} \langle H\rangle X \mid H \in \operatorname{up}\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g\right\}\right\}.\end{aligned}$$

Obviously

$$\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g\right\} = \bigcap\left\langle\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g)}\right\rangle \operatorname{up}\bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g\right\};$$

from this by the lemma 4 (taking in account that $\{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}$ and $\operatorname{up} \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g \right\}$ are filter bases)

$$\bigcap \left\{ \uparrow^{\operatorname{Dst} g} \langle H \rangle X \mid H \in \operatorname{up} \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{up} f, G \in \operatorname{up} g \right\} \right\} = \bigcap \left\{ \uparrow^{\operatorname{Dst} g} \langle G \circ F \rangle X \mid F \in \operatorname{up} f, G \in \operatorname{up} g \right\}.$$

On the other side

$$\begin{aligned}\langle ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f) \rangle^* X &= \langle (\mathsf{FCD})g \rangle \langle (\mathsf{FCD})f \rangle^* X \\ &= \langle (\mathsf{FCD})g \rangle \bigcap \left\{ \uparrow^{\operatorname{Dst} f} \langle F \rangle X \mid F \in \operatorname{up} f \right\} \\ &= \bigcap \left\{ \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} g; \operatorname{Dst} g)} G \right\rangle \bigcap \left\{ \uparrow^{\operatorname{Dst} f} \langle F \rangle X \mid F \in \operatorname{up} f \right\} \mid G \in \operatorname{up} g \right\}.\end{aligned}$$

Let's prove that $\{\langle F \rangle X \mid F \in \operatorname{up} f\}$ is a filter base. If $A, B \in \{\langle F \rangle X \mid F \in \operatorname{up} f\}$ then $A = \langle F_1 \rangle X$ and $B = \langle F_2 \rangle X$ where $F_1, F_2 \in \operatorname{up} f$. $A \cap B \supseteq \langle F_1 \cap F_2 \rangle X \in \{\langle F \rangle X \mid F \in \operatorname{up} f\}$. So $\{\langle F \rangle X \mid F \in \operatorname{up} f\}$ is really a filter base.

By the theorem 8 we have

$$\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src} g; \operatorname{Dst} g)} G \right\rangle \bigcap \left\{ \uparrow^{\operatorname{Dst} f} \langle F \rangle X \mid F \in \operatorname{up} f \right\} = \bigcap \left\{ \uparrow^{\operatorname{Dst} g} \langle G \rangle \langle F \rangle X \mid F \in \operatorname{up} f \right\}.$$

So continuing the above equalities,

$$\begin{aligned}\langle ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f) \rangle^* X &= \bigcap \left\{ \bigcap \left\{ \uparrow^{\operatorname{Dst} g} \langle G \rangle \langle F \rangle X \mid F \in \operatorname{up} f \right\} \mid G \in \operatorname{up} g \right\} \\ &= \bigcap \left\{ \uparrow^{\operatorname{Dst} g} \langle G \rangle \langle F \rangle X \mid F \in \operatorname{up} f, G \in \operatorname{up} g \right\} \\ &= \bigcap \left\{ \uparrow^{\operatorname{Dst} g} \langle G \circ F \rangle X \mid F \in \operatorname{up} f, G \in \operatorname{up} g \right\}.\end{aligned}$$

Combining these equalities we get $\langle (\mathsf{FCD})(g \circ f) \rangle^* X = \langle ((\mathsf{FCD})g) \circ ((\mathsf{FCD})f) \rangle^* X$ for every set X. □

Corollary 15

1. $(\mathsf{FCD}) f$ *is a monovalued funcoid if* f *is a monovalued reloid.*

2. $(\mathsf{FCD}) f$ *is an injective funcoid if* f *is an injective reloid.*

Proof We will prove only the first as the second is dual. Let f is a monovalued reloid. Then $f \circ f^{-1} \subseteq I^{\mathsf{RLD}(\operatorname{Dst} f)}$; $(\mathsf{FCD}) \left(f \circ f^{-1} \right) \subseteq I^{\mathsf{FCD}(\operatorname{Dst} f)}$; $(\mathsf{FCD}) f \circ ((\mathsf{FCD}) f)^{-1} \subseteq I^{\mathsf{FCD}(\operatorname{Dst} f)}$ that is $(\mathsf{FCD}) f$ is a monovalued funcoid. □

Proposition 32 $(\mathsf{FCD}) I^{\mathsf{RLD}}_{\mathcal{A}} = I^{\mathsf{FCD}}_{\mathcal{A}}$ *for every f.o.* $\mathcal{A}$.

Proof Recall that $I_{\mathcal{A}}^{\mathsf{RLD}} = \bigcap \left\{ \uparrow^{\mathrm{Base}(\mathcal{A})} I_A \quad | \quad A \in \mathrm{up}\,\mathcal{A} \right\}$. For every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Base}(\mathcal{A}))$ we have:

$\mathcal{X} \left[(\mathsf{FCD}) I_{\mathcal{A}}^{\mathsf{RLD}} \right] \mathcal{Y} \Leftrightarrow \mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \not\asymp I_{\mathcal{A}}^{\mathsf{RLD}} \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A} : \mathcal{X} \times^{\mathsf{RLD}} \mathcal{Y} \not\asymp \uparrow^{\mathsf{RLD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} I_A \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A} : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{A}))} I_A \right] \mathcal{Y} \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A} : \mathcal{X} \cap \mathcal{Y} \not\asymp \uparrow^{\mathrm{Base}(\mathcal{A})} A \Leftrightarrow \mathcal{X} \cap \mathcal{Y} \not\asymp \mathcal{A} \Leftrightarrow \mathcal{X} \left[I_{\mathcal{A}}^{\mathsf{FCD}} \right] \mathcal{Y}$ (used properties of generalized filter bases). □

Proposition 33 $(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ *for every f.o.* $\mathcal{A}$ *and* $\mathcal{B}$*.*

Proof $\mathcal{X} \left[(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \right] \mathcal{Y} \Leftrightarrow \forall F \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))} F \right] \mathcal{Y}$ (for every $\mathcal{X}, \mathcal{Y} \in \mathfrak{F}$).

Evidently $\forall F \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))} F \right] \mathcal{Y} \Rightarrow \forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B} : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))} (A \times B) \right] \mathcal{Y}$.

Let $\forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B} : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))} (A \times B) \right] \mathcal{Y}$. Then if $F \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$, there are $A \in \mathrm{up}\,\mathcal{A}$, $B \in \mathrm{up}\,\mathcal{B}$ such that $F \supseteq A \times B$. So $\mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))} F \right] \mathcal{Y}$.

We proved $\forall F \in \mathrm{up}(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))} F \right] \mathcal{Y} \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B} : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))} (A \times B) \right] \mathcal{Y}$.

Further $\forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B} : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Base}(\mathcal{A});\mathrm{Base}(\mathcal{B}))} (A \times B) \right] \mathcal{Y} \Leftrightarrow \forall A \in \mathrm{up}\,\mathcal{A}, B \in \mathrm{up}\,\mathcal{B} : (\mathcal{X} \not\asymp \uparrow^{\mathrm{Base}(\mathcal{A})} A \wedge \mathcal{Y} \not\asymp \uparrow^{\mathrm{Base}(\mathcal{B})} B) \Leftrightarrow \mathcal{X} \not\asymp \mathcal{A} \wedge \mathcal{Y} \not\asymp \mathcal{B} \Leftrightarrow \mathcal{X} \left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right] \mathcal{Y}$.

Thus $\mathcal{X} \left[(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) \right] \mathcal{Y} \Leftrightarrow \mathcal{X} \left[\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \right] \mathcal{Y}$. □

Proposition 34 $\mathrm{dom}(\mathsf{FCD})f = \mathrm{dom}\,f$ *and* $\mathrm{im}(\mathsf{FCD})f = \mathrm{im}\,f$ *for every reloid* f*.*

Proof $\mathrm{im}(\mathsf{FCD})f = \langle (\mathsf{FCD})f \rangle 1^{\mathfrak{F}(\mathrm{Src}\,f)} = \bigcap \left\{ \uparrow^{\mathrm{Dst}\,f} \langle F \rangle (\mathrm{Src}\,f) \quad | \quad F \in \mathrm{up}\,f \right\} = \bigcap \left\{ \uparrow^{\mathrm{Dst}\,f} \mathrm{im}\,F \quad | \quad F \in \mathrm{up}\,f \right\} = \bigcap \left\langle \uparrow^{\mathrm{Dst}\,f} \right\rangle \langle \mathrm{im} \rangle \mathrm{up}\,f = \mathrm{im}\,f$.

$\mathrm{dom}(\mathsf{FCD})f = \mathrm{dom}\,f$ is similar. □

Proposition 35 $(\mathsf{FCD})(f \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})) = (\mathsf{FCD})f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$ *for every reloid* f *and* $\mathcal{A} \in \mathfrak{F}(\mathrm{Src}\,f)$ *and* $\mathcal{B} \in \mathfrak{F}(\mathrm{Dst}\,f)$*.*

Proof $(\mathsf{FCD})(f \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})) = (\mathsf{FCD})(I_{\mathcal{B}}^{\mathsf{RLD}} \circ f \circ I_{\mathcal{A}}^{\mathsf{RLD}}) = (\mathsf{FCD}) I_{\mathcal{B}}^{\mathsf{RLD}} \circ (\mathsf{FCD})f \circ (\mathsf{FCD}) I_{\mathcal{A}}^{\mathsf{RLD}} = I_{\mathcal{B}}^{\mathsf{FCD}} \circ (\mathsf{FCD})f \circ I_{\mathcal{A}}^{\mathsf{FCD}} = (\mathsf{FCD})f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$. □

Corollary 16 $(\mathsf{FCD})(f|_{\mathcal{A}}) = ((\mathsf{FCD})f)|_{\mathcal{A}})$ *for every reloid* f *and f.o.* $\mathcal{A}$*.*

Proposition 36 $\langle (\mathsf{FCD})\,f \rangle\, \mathcal{X} = \mathrm{im}\,(f|_{\mathcal{X}})$ *for every reloid* f *and f.o.* $\mathcal{X}$*.*

Proof $\mathrm{im}\,(f|_{\mathcal{X}}) = \mathrm{im}\,(\mathsf{FCD})\,(f|_{\mathcal{X}}) = \mathrm{im}(((\mathsf{FCD})f)|_{\mathcal{X}}) = \langle (\mathsf{FCD})\,f \rangle\, \mathcal{X}$. □

5.2 Reloids induced by funcoid

Every funcoid $f \in \mathsf{FCD}(A;B)$ induces a reloid from A to B in two ways, intersection of *outward* relations and union of *inward* direct products of filter objects:

$$(\mathsf{RLD})_{\mathrm{out}} f \stackrel{\mathrm{def}}{=} \bigcap \left\langle \uparrow^{\mathsf{RLD}(A;B)} \right\rangle \operatorname{up} f;$$
$$(\mathsf{RLD})_{\mathrm{in}} f \stackrel{\mathrm{def}}{=} \bigcup \left\{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \quad | \quad \mathcal{A} \in \mathfrak{F}(A), \mathcal{B} \in \mathfrak{F}(B), \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\}.$$

Theorem 64 $(\mathsf{RLD})_{\mathrm{in}} f = \bigcup \left\{ a \times^{\mathsf{RLD}} b \quad | \quad a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)}, a \times^{\mathsf{FCD}} b \subseteq f \right\}$.

Proof It follows from the theorem 46. □

Remark 6 It seems that $(\mathsf{RLD})_{\mathrm{in}}$ has smoother properties and is more important than $(\mathsf{RLD})_{\mathrm{out}}$. (However see also the exercize below for $(\mathsf{RLD})_{\mathrm{in}}$ not preserving identities.)

Proposition 37 $(\mathsf{RLD})_{\mathrm{out}} \uparrow^{\mathsf{FCD}(A;B)} f = \uparrow^{\mathsf{RLD}(A;B)} f$ *for every small sets A, B and binary relation $f \subseteq A \times B$.*

Proof $(\mathsf{RLD})_{\mathrm{out}} \uparrow^{\mathsf{FCD}(A;B)} f = \bigcap \left\langle \uparrow^{\mathsf{RLD}(A;B)} \right\rangle \operatorname{up} \uparrow^{\mathsf{FCD}(A;B)} f = \uparrow^{\mathsf{RLD}(A;B)} \min \operatorname{up} \uparrow^{\mathsf{FCD}(A;B)} f = \uparrow^{\mathsf{RLD}(A;B)} f$. □

Surprisingly a funcoid is greater inward than outward:

Theorem 65 $(\mathsf{RLD})_{\mathrm{out}} f \subseteq (\mathsf{RLD})_{\mathrm{in}} f$ *for every funcoid f.*

Proof We need to prove

$$\bigcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} \right\rangle \operatorname{up} f \subseteq \bigcup \left\{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \quad | \quad \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\}.$$

Let

$$K \in \operatorname{up} \bigcup \left\{ \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \quad | \quad \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\}.$$

Then

$$\begin{aligned} K &= \uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} \bigcup \left\{ X_{\mathcal{A}} \times Y_{\mathcal{B}} \quad | \quad \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\} \\ &= \bigcup \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} (X_{\mathcal{A}} \times Y_{\mathcal{B}}) \quad | \quad \mathcal{A}, \mathcal{B} \in \mathfrak{F}, \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f \right\} \\ &\supseteq f \end{aligned}$$

where $X_{\mathcal{A}} \in \operatorname{up} \mathcal{A}$, $Y_{\mathcal{B}} \in \operatorname{up} \mathcal{B}$. So $K \in \operatorname{up} f$; $K \in \operatorname{up} \bigcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} \right\rangle \operatorname{up} f$. □

Theorem 66 $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$ *for every funcoid f.*

Proof For every sets $X \in \mathscr{P}(\operatorname{Src} f)$ and $Y \in \mathscr{P}(\operatorname{Dst} f)$

$$\begin{aligned}
X\ [(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f]^* \ Y &\Leftrightarrow \\
(\uparrow^{\operatorname{Src} f} X \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f} Y) \not\asymp (\mathsf{RLD})_{\mathrm{in}} f &\Leftrightarrow \\
\uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} (X \times Y) \not\asymp \bigcup \left\{ a \times^{\mathsf{RLD}} b \quad \middle| \quad a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)}, a \times^{\mathsf{FCD}} b \subseteq f \right\} &\Leftrightarrow \quad (*) \\
\exists a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} : (a \times^{\mathsf{FCD}} b \subseteq f \wedge \uparrow^{\mathsf{RLD}(\operatorname{Src} f;\operatorname{Dst} f)} (X \times Y) \not\asymp (a \times^{\mathsf{RLD}} b)) &\Leftrightarrow \\
\exists a \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Src} f)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Dst} f)} : (a\ [f]\ b \wedge a \subseteq \uparrow^{\operatorname{Src} f} X \wedge b \subseteq \uparrow^{\operatorname{Dst} f} Y) &\Leftrightarrow \\
X\ [f]^*\ Y.
\end{aligned}$$

* theorem 53 in [15].

Thus $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} f = f$. □

Remark 7 The above theorem allows to represent funcoids as reloids.

Obvious 29. $(\mathsf{RLD})_{\mathrm{in}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for every f.o. $\mathcal{A}$, $\mathcal{B}$.

Conjecture 15 $(\mathsf{RLD})_{\mathrm{out}} I_{\mathcal{A}}^{\mathsf{FCD}} = I_{\mathcal{A}}^{\mathsf{RLD}}$ *for every f.o.* $\mathcal{A}$.

Exercise 1 Prove that generally $(\mathsf{RLD})_{\mathrm{in}} I_{\mathcal{A}}^{\mathsf{FCD}} \neq I_{\mathcal{A}}^{\mathsf{RLD}}$.

Conjecture 16 $\operatorname{dom}(\mathsf{RLD})_{\mathrm{in}} f = \operatorname{dom} f$ *and* $\operatorname{im}(\mathsf{RLD})_{\mathrm{in}} f = \operatorname{im} f$ *for every funcoid* f.

Proposition 38 $\operatorname{dom}(f|_{\mathcal{A}}) = \mathcal{A} \cap \operatorname{dom} f$ *for every reloid* f *and f.o.* $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$.

Proof $\operatorname{dom}(f|_{\mathcal{A}}) = \operatorname{dom}(\mathsf{FCD}) f|_{\mathcal{A}} = \operatorname{dom}((\mathsf{FCD}) f)|_{\mathcal{A}} = \mathcal{A} \cap \operatorname{dom}(\mathsf{FCD}) f = \mathcal{A} \cap \operatorname{dom} f$. □

Theorem 67 *For every composable reloids* f, g:

1. *If* $\operatorname{im} f \supseteq \operatorname{dom} g$ *then* $\operatorname{im}(g \circ f) = \operatorname{im} g$.

2. *If* $\operatorname{im} f \subseteq \operatorname{dom} g$ *then* $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

Proof

1. $\operatorname{im}(g \circ f) = \operatorname{im}(\mathsf{FCD})(g \circ f) = \operatorname{im}((\mathsf{FCD}) g \circ (\mathsf{FCD}) f) = \operatorname{im}(\mathsf{FCD}) g = \operatorname{im} g$.

2. Similar.

□

Conjecture 17 $(\mathsf{RLD})_{\mathrm{in}}(g \circ f) = (\mathsf{RLD})_{\mathrm{in}} g \circ (\mathsf{RLD})_{\mathrm{in}} f$ *for every composable funcoids* f *and* g.

Theorem 68 *$a \times^{\mathsf{RLD}} b \subseteq (\mathsf{RLD})_{\mathrm{in}} f \Leftrightarrow a \times^{\mathsf{FCD}} b \subseteq f$ for every funcoid f and atomic f.o. a and b on the source and destination of f correspondingly.*

Proof $a \times^{\mathsf{FCD}} b \subseteq f \Rightarrow a \times^{\mathsf{RLD}} b \subseteq (\mathsf{RLD})_{\mathrm{in}} f$ is obvious.

$a \times^{\mathsf{RLD}} b \subseteq (\mathsf{RLD})_{\mathrm{in}} f \Rightarrow a \times^{\mathsf{RLD}} b \not\asymp (\mathsf{RLD})_{\mathrm{in}} f \Rightarrow a \ [(\mathsf{FCD})\,(\mathsf{RLD})_{\mathrm{in}} f] \ b \Rightarrow a \ [f] \ b \Rightarrow a \times^{\mathsf{FCD}} b \subseteq f$. □

A conjecture stronger than the last theorem:

Conjecture 18 *If $\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B} \subseteq (\mathsf{RLD})_{\mathrm{in}} f$ then $\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B} \subseteq f$ for every funcoid f and $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$, $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.*

5.3 Galois connections of funcoids and reloids

Theorem 69 *$(\mathsf{FCD}) : \mathsf{RLD}(A; B) \to \mathsf{FCD}(A; B)$ is the lower adjoint of $(\mathsf{RLD})_{\mathrm{in}} : \mathsf{FCD}(A; B) \to \mathsf{RLD}(A; B)$ for every small sets A, B.*

Proof Because (FCD) and $(\mathsf{RLD})_{\mathrm{in}}$ are trivially monotone, it's enough to prove (for every $f \in \mathsf{RLD}(A; B)$, $g \in \mathsf{FCD}(A; B)$)

$$f \subseteq (\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})f \text{ and} (\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} g \subseteq g.$$

The second formula follows from the fact that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{in}} g = g$.

$$\begin{aligned}
(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD})f &= \\
\bigcup\left\{ a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, a \times^{\mathsf{FCD}} b \subseteq (\mathsf{FCD})f \right\} &= \\
\bigcup\left\{ a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, a \ [(\mathsf{FCD})f] \ b \right\} &= \\
\bigcup\left\{ a \times^{\mathsf{RLD}} b \mid a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, (a \times^{\mathsf{RLD}} b) \not\asymp f \right\} &\supseteq \\
\bigcup\left\{ p \in \operatorname{atoms}(a \times^{\mathsf{RLD}} b) \mid a \in \operatorname{atoms} 1^{\mathfrak{F}(A)}, b \in \operatorname{atoms} 1^{\mathfrak{F}(B)}, p \not\asymp f \right\} &= \\
\bigcup\left\{ p \in \operatorname{atoms} 1^{\mathsf{RLD}(A;B)} \mid p \not\asymp f \right\} &= \\
\bigcup\{ p \mid p \in \operatorname{atoms} f \} = f.
\end{aligned}$$

□

Corollary 17

1. $(\mathsf{FCD}) \bigcup S = \bigcup \langle (\mathsf{FCD}) \rangle S$ *if* $S \in \mathscr{P}\mathsf{RLD}(A; B)$.
2. $(\mathsf{RLD})_{\mathrm{in}} \bigcap S = \bigcap \langle (\mathsf{RLD})_{\mathrm{in}} \rangle S$ *if* $S \in \mathscr{P}\mathsf{FCD}(A; B)$.

Proposition 39 *$(\mathsf{RLD})_{\mathrm{in}}(f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})) = ((\mathsf{RLD})_{\mathrm{in}} f) \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$ for every funcoid f and f.o. $\mathcal{A} \in \mathfrak{F}(\operatorname{Src} f)$ and $\mathcal{B} \in \mathfrak{F}(\operatorname{Dst} f)$.*

Proof $(\mathsf{RLD})_{\mathrm{in}}(f \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})) = (\mathsf{RLD})_{\mathrm{in}} f \cap (\mathsf{RLD})_{\mathrm{in}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = ((\mathsf{RLD})_{\mathrm{in}} f) \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B})$. □

Corollary 18 $(\mathsf{RLD})_{\mathrm{in}}(f|_{\mathcal{A}}) = ((\mathsf{RLD})_{\mathrm{in}} f)|_{\mathcal{A}}$ *for every funcoid f and f.o. $\mathcal{A}$.*

Conjecture 19 $(\mathsf{RLD})_{\mathrm{in}}$ *is not a lower adjoint (in general).*

Conjecture 20 $(\mathsf{RLD})_{\mathrm{out}}$ *is neither a lower adjoint nor an upper adjoint (in general).*

See also the corollary 23 below.

6 Continuous morphisms

This section uses the apparatus from the section "Partially ordered dagger categories".

6.1 Traditional definitions of continuity

In this section we will show that having a funcoid or reloid $\uparrow f$ corresponding to a function f we can express continuity of it by the formula $\uparrow f \circ \mu \subseteq \nu \circ \uparrow f$ (or similar formulas) where μ and ν are some spaces.

6.1.1 Pre-topology

Let μ and ν are funcoids representing some pre-topologies. By definition a function f is continuous map from μ to ν in point a iff

$$\forall \epsilon \in \operatorname{up} \langle \nu \rangle^* \{fa\} \exists \delta \in \operatorname{up} \langle \mu \rangle^* \{a\} : \langle f \rangle \delta \subseteq \epsilon.$$

Equivalently transforming this formula we get:

$$\begin{gathered}
\forall \epsilon \in \operatorname{up} \langle \nu \rangle^* \{fa\} : \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \langle \mu \rangle \uparrow^{\operatorname{Src}\mu} \{a\} \subseteq \epsilon; \\
\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \langle \mu \rangle \uparrow^{\operatorname{Src}\mu} \{a\} \subseteq \langle \nu \rangle^* \{fa\}; \\
\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \langle \mu \rangle \uparrow^{\operatorname{Src}\mu} \{a\} \subseteq \langle \nu \rangle \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \uparrow^{\operatorname{Src}\mu} \{a\}; \\
\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \circ \mu \right\rangle \uparrow^{\operatorname{Src}\mu} \{a\} \subseteq \left\langle \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \right\rangle \uparrow^{\operatorname{Src}\mu} \{a\}.
\end{gathered}$$

So f is a continuous map from μ to ν in every point of its domain iff

$$\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f \circ \mu \subseteq \nu \circ \uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f.$$

6.1.2 Proximity spaces

Let μ and ν are proximity (nearness) spaces (which I consider a special case of funcoids). By definition a function f is a proximity-continuous map (also called equivicontinuous) from μ to ν iff

$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : (X\ [\mu]^{*}\ Y \Rightarrow \langle f\rangle X\ [\nu]^{*}\ \langle f\rangle Y).$$

Equivalently transforming this formula we get (writing $\uparrow$ instead of $\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)}$ for brevity):

$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : (X\ [\mu]^{*}\ Y \Rightarrow \langle f\rangle Y \cap \langle\nu\rangle\langle f\rangle X \neq 0^{\mathfrak{F}(\operatorname{Dst}\nu)});$$
$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : (X\ [\mu]^{*}\ Y \Rightarrow \langle f\rangle Y \cap \langle\nu\circ\uparrow f\rangle^{*} X \neq 0^{\mathfrak{F}(\operatorname{Dst}\nu)});$$
$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : (X\ [\mu]^{*}\ Y \Rightarrow X\ [\nu\circ\uparrow f]^{*}\ \langle f\rangle Y);$$
$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : (X\ [\mu]^{*}\ Y \Rightarrow \langle f\rangle Y\ \left[(\nu\circ\uparrow f)^{-1}\right]^{*}\ X);$$
$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : (X\ [\mu]^{*}\ Y \Rightarrow \langle f\rangle Y\ \left[(\uparrow f)^{-1}\circ\nu^{-1}\right]^{*}\ X);$$
$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : \left(X\ [\mu]^{*}\ Y \Rightarrow \uparrow^{\mathfrak{F}(\operatorname{Src}\mu)} X \cap \left\langle(\uparrow f)^{-1}\circ\nu^{-1}\right\rangle\langle f\rangle Y \neq 0^{\mathfrak{F}(\operatorname{Src}\mu)}\right);$$
$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : \left(X\ [\mu]^{*}\ Y \Rightarrow \uparrow^{\mathfrak{F}(\operatorname{Src}\mu)} X \cap \left\langle(\uparrow f)^{-1}\circ\nu^{-1}\circ\uparrow f\right\rangle^{*} Y \neq 0^{\mathfrak{F}(\operatorname{Src}\mu)}\right);$$
$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : \left(X\ [\mu]^{*}\ Y \Rightarrow Y\ \left[(\uparrow f)^{-1}\circ\nu^{-1}\circ\uparrow f\right]^{*}\ X\right);$$
$$\forall X \in \mathscr{P}(\operatorname{Src}\mu), Y \in \mathscr{P}(\operatorname{Dst}\mu) : \left(X\ [\mu]^{*}\ Y \Rightarrow X\ \left[(\uparrow f)^{-1}\circ\nu\circ\uparrow f\right]^{*}\ Y\right);$$
$$\mu \subseteq (\uparrow f)^{-1}\circ\nu\circ\uparrow f.$$

So a function f is proximity-continuous iff $\mu \subseteq \left(\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f\right)^{-1}\circ\nu\circ\uparrow^{\mathsf{FCD}(\operatorname{Src}\mu;\operatorname{Dst}\nu)} f$.

6.1.3 Uniform spaces

Uniform spaces are a special case of reloids.

Let μ and ν are uniform spaces. By definition a function f is a uniformly continuous map from μ to ν iff

$$\forall\epsilon \in \operatorname{up}\nu\exists\delta \in \operatorname{up}\mu\forall(x;y) \in \delta : (fx;fy) \in \epsilon.$$

Equivalently transforming this formula we get:

$$\forall\epsilon \in \operatorname{up}\nu\exists\delta \in \operatorname{up}\mu\forall(x;y) \in \delta : \{(fx;fy)\} \subseteq \epsilon;$$
$$\forall\epsilon \in \operatorname{up}\nu\exists\delta \in \operatorname{up}\mu\forall(x;y) \in \delta : f\circ\{(x;y)\}\circ f^{-1} \subseteq \epsilon;$$
$$\forall\epsilon \in \operatorname{up}\nu\exists\delta \in \operatorname{up}\mu : f\circ\delta\circ f^{-1} \subseteq \epsilon;$$
$$\forall\epsilon \in \operatorname{up}\nu : \uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\circ\mu\circ\left(\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \subseteq \uparrow^{\mathsf{RLD}(\operatorname{Src}\nu;\operatorname{Dst}\nu)} \epsilon;$$
$$\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\circ\mu\circ\left(\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \subseteq \nu.$$

So a function f is uniformly continuous iff $\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\circ\mu\circ\left(\uparrow^{\mathsf{RLD}(\operatorname{Dst}\mu;\operatorname{Dst}\nu)} f\right)^{-1} \subseteq \nu$.

6.2 Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν are endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in \mathrm{C}(\mu; \nu) \Leftrightarrow f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge f \circ \mu \subseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory then continuity also can be defined in two other ways:

$$\begin{array}{rcl} f \in \mathrm{C}'(\mu; \nu) & \Leftrightarrow & f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge \mu \subseteq f^\dagger \circ \nu \circ f; \\ f \in \mathrm{C}''(\mu; \nu) & \Leftrightarrow & f \in \mathrm{Mor}(\mathrm{Ob}\,\mu; \mathrm{Ob}\,\nu) \wedge f \circ \mu \circ f^\dagger \subseteq \nu. \end{array}$$

Remark 8 In the examples (above) about funcoids and reloids the "dagger functor" is the inverse of a funcoid or reloid, that is $f^\dagger = f^{-1}$.

Proposition 40 *Every of these three definitions of continuity forms a subprecategory (subcategory if the original precategory is a category).*

Proof

C Let $f \in \mathrm{C}(\mu; \nu)$, $g \in \mathrm{C}(\nu; \pi)$. Then $f \circ \mu \subseteq \nu \circ f$, $g \circ \nu \subseteq \pi \circ g$; $g \circ f \circ \mu \subseteq g \circ \nu \circ f \subseteq \pi \circ g \circ f$. So $g \circ f \in \mathrm{C}(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}(\mu; \mu)$ is obvious.

C′ Let $f \in \mathrm{C}'(\mu; \nu)$, $g \in \mathrm{C}'(\nu; \pi)$. Then $\mu \subseteq f^\dagger \circ \nu \circ f$, $\nu \subseteq g^\dagger \circ \pi \circ g$;

$$\mu \subseteq f^\dagger \circ g^\dagger \circ \pi \circ g \circ f; \quad \mu \subseteq (g \circ f)^\dagger \circ \pi \circ (g \circ f).$$

So $g \circ f \in \mathrm{C}'(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}'(\mu; \mu)$ is obvious.

C″ Let $f \in \mathrm{C}''(\mu; \nu)$, $g \in \mathrm{C}''(\nu; \pi)$. Then $f \circ \mu \circ f^\dagger \subseteq \nu$, $g \circ \nu \circ g^\dagger \subseteq \pi$;

$$g \circ f \circ \mu \circ f^\dagger \circ g^\dagger \subseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^\dagger \subseteq \pi.$$

So $g \circ f \in \mathrm{C}''(\mu; \pi)$. $1_{\mathrm{Ob}\,\mu} \in \mathrm{C}''(\mu; \mu)$ is obvious.

□

Proposition 41 *For a monovalued morphism f of a partially ordered dagger category and its endomorphisms μ and ν*

$$f \in \mathrm{C}'(\mu; \nu) \Rightarrow f \in \mathrm{C}(\mu; \nu) \Rightarrow f \in \mathrm{C}''(\mu; \nu).$$

Proof Let $f \in \mathrm{C}'(\mu;\nu)$. Then $\mu \subseteq f^{\dagger} \circ \nu \circ f$; $f \circ \mu \subseteq f \circ f^{\dagger} \circ \nu \circ f \subseteq 1_{\mathrm{Dst}\, f} \circ \nu \circ f = \nu \circ f$; $f \in \mathrm{C}(\mu;\nu)$.

Let $f \in \mathrm{C}(\mu;\nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f \circ \mu \circ f^{\dagger} \subseteq \nu \circ f \circ f^{\dagger} \subseteq \nu \circ 1_{\mathrm{Dst}\, f} = \nu$; $f \in \mathrm{C}''(\mu;\nu)$. □

Proposition 42 *For an entirely defined morphism f of a partially ordered dagger category and its endomorphisms μ and ν*

$$f \in \mathrm{C}''(\mu;\nu) \Rightarrow f \in \mathrm{C}(\mu;\nu) \Rightarrow f \in \mathrm{C}'(\mu;\nu).$$

Proof Let $f \in \mathrm{C}''(\mu;\nu)$. Then $f \circ \mu \circ f^{\dagger} \subseteq \nu$; $f \circ \mu \circ f^{\dagger} \circ f \subseteq \nu \circ f$; $f \circ \mu \circ 1_{\mathrm{Src}\, f} \subseteq \nu \circ f$; $f \circ \mu \subseteq \nu \circ f$; $f \in \mathrm{C}(\mu;\nu)$.

Let $f \in \mathrm{C}(\mu;\nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f^{\dagger} \circ f \circ \mu \subseteq f^{\dagger} \circ \nu \circ f$; $1_{\mathrm{Src}\, f} \circ \mu \subseteq f^{\dagger} \circ \nu \circ f$; $\mu \subseteq f^{\dagger} \circ \nu \circ f$; $f \in \mathrm{C}'(\mu;\nu)$. □

For entirely defined monovalued morphisms our three definitions of continuity coincide:

Theorem 70 *If f is a monovalued and entirely defined morphism then*

$$f \in \mathrm{C}'(\mu;\nu) \Leftrightarrow f \in \mathrm{C}(\mu;\nu) \Leftrightarrow f \in \mathrm{C}''(\mu;\nu).$$

Proof From two previous propositions. □

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is near-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

Theorem 71 *If an entirely defined morphism of the category of reloids $f \in \mathrm{C}''(\mu;\nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\mathsf{FCD})f \in \mathrm{C}'((\mathsf{FCD})\mu;(\mathsf{FCD})\nu)$.*

Exercise 2 I leave a simple exercise for the reader to prove the last theorem.

6.3 Continuity of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids regarding the composition.) Consider also some lattice (**lattice of objects**). (For example take the lattice of set theoretic filters.)

We will map every object A to **identity element** I_A of the semigroup (for example identity funcoid or identity reloid). For identity elements we will require

1. $I_A \circ I_B = I_{A \cap B}$;
2. $f \circ I_A \subseteq f$; $I_A \circ f \subseteq f$.

In the case when our semigroup is "dagger" (that is is a dagger precategory) we will require also $(I_A)^\dagger = I_A$.

We can define **restricting** an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define **rectangular restricting** an element μ of our semigroup to objects A and B as $I_B \circ \mu \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for funcoids and for reloids):

$$\mu \cap (A \times B) = I_B \circ \mu \circ I_A.$$

Square restricting of an element μ to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ \mu \circ I_A$ (or by the formula $\mu \cap (A \times A)$).

Theorem 72 *For every elements f, μ, ν of our semigroup and an object A*

1. $f \in \mathrm{C}(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A; \nu)$;
2. $f \in \mathrm{C}'(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A; \nu)$;
3. $f \in \mathrm{C}''(\mu; \nu) \Rightarrow f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A; \nu)$.

(Two last items are true for the case when our semigroup is dagger.)

Proof

1. $f|_A \in \mathrm{C}(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f \circ I_A \Leftarrow f \circ I_A \circ \mu \subseteq \nu \circ f \Leftarrow f \circ \mu \subseteq \nu \circ f \Leftrightarrow f \in \mathrm{C}(\mu; \nu)$.

2. $f|_A \in \mathrm{C}'(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow I_A \circ \mu \circ I_A \subseteq (f|_A)^\dagger \circ \nu \circ f|_A \Leftarrow I_A \circ \mu \circ I_A \subseteq (f \circ I_A)^\dagger \circ \nu \circ f \circ I_A \Leftrightarrow I_A \circ \mu \circ I_A \subseteq I_A \circ f^\dagger \circ \nu \circ f \circ I_A \Leftarrow \mu \subseteq f^\dagger \circ \nu \circ f \Leftrightarrow f \in \mathrm{C}'(\mu; \nu)$.

3. $f|_A \in \mathrm{C}''(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ \mu \circ I_A \circ f^\dagger \subseteq \nu \Leftarrow f \circ \mu \circ f^\dagger \subseteq \nu \Leftrightarrow f \in \mathrm{C}''(\mu; \nu)$.

□

7 Connectedness regarding funcoids and reloids

Definition 55 *I will call **endo-reloids** and **endo-funcoids** reloids and funcoids with the same source and destination.*

7.1 Some lemmas

Lemma 6 *If* $\neg(A\ [f]^*\ B) \wedge A \cup B \in \operatorname{up}(\operatorname{dom} f \cup \operatorname{im} f)$ *then* f *is closed on* $\uparrow^U A$ *for a funcoid* $f \in \mathsf{FCD}(U;U)$ *and sets* $A, B \in \mathscr{P}U$ *(for every small set* U*).*

Proof Let $A \cup B \in \operatorname{up}(\operatorname{dom} f \cup \operatorname{im} f)$. $\neg(A\ [f]^*\ B) \Leftrightarrow \uparrow^U B \cap \langle f \rangle \uparrow^U A = 0^{\mathfrak{F}(U)} \Rightarrow (\operatorname{dom} f \cup \operatorname{im} f) \cap \uparrow^U B \cap \langle f \rangle^* A = 0^{\mathfrak{F}(U)} \Rightarrow ((\operatorname{dom} f \cup \operatorname{im} f) \setminus \uparrow^U A) \cap \langle f \rangle^* A = 0^{\mathfrak{F}(U)} \Leftrightarrow \langle f \rangle^* A \subseteq \uparrow^U A$. □

Corollary 19 *If* $\neg(A\ [f]^*\ B) \wedge A \cup B \in \operatorname{up}(\operatorname{dom} f \cup \operatorname{im} f)$ *then* f *is closed on* $\uparrow^U (A \setminus B)$ *for a funcoid* f *and sets* $A, B \in \mathscr{P}U$ *(for every small set* U*).*

Proof Let $\neg(A\ [f]^*\ B) \wedge A \cup B \in \operatorname{up}(\operatorname{dom} f \cup \operatorname{im} f)$. Then $\neg((A \setminus B)\ [f]^*\ B) \wedge \uparrow^U ((A \setminus B) \cup B) \in \operatorname{up}(\operatorname{dom} f \cup \operatorname{im} f)$. □

Lemma 7 *If* $\neg(A\ [f]^*\ B) \wedge A \cup B \in \operatorname{up}(\operatorname{dom} f \cup \operatorname{im} f)$ *then* $\neg(A\ [f^n]^*\ B)$ *for every whole positive* n*.*

Proof Let $\neg(A\ [f]^*\ B) \wedge A \cup B \in \operatorname{up}(\operatorname{dom} f \cup \operatorname{im} f)$. From the above lemma $\langle f \rangle^* A \subseteq \uparrow^U A$. $\uparrow^U B \cap \langle f \rangle \uparrow^U A = 0^{\mathfrak{F}(U)}$, consequently $\langle f \rangle^* A \subseteq \uparrow^U (A \setminus B)$. Because (by the above corollary) f is closed on $\uparrow^U (A \setminus B)$, then $\langle f \rangle \langle f \rangle \uparrow^U A \subseteq \uparrow^U (A \setminus B)$, $\langle f \rangle \langle f \rangle \langle f \rangle \uparrow^U A \subseteq \uparrow^U (A \setminus B)$, etc. So $\langle f^n \rangle \uparrow^U A \subseteq \uparrow^U (A \setminus B)$, $\uparrow^U B \asymp \langle f^n \rangle \uparrow^U A$, $\neg(A\ [f^n]^*\ B)$. □

7.2 Endomorphism series

Definition 56 $S_1(\mu) \stackrel{\text{def}}{=} \mu \cup \mu^2 \cup \mu^3 \cup \ldots$ *for an endomorphism* μ *of a precategory with countable join of morphisms.*

Definition 57 $S(\mu) \stackrel{\text{def}}{=} \mu^0 \cup S_1(\mu) = \mu^0 \cup \mu \cup \mu^2 \cup \mu^3 \cup \ldots$ *where* $\mu^0 \stackrel{\text{def}}{=} I_{\operatorname{Ob}\mu}$ *(identity morphism for the object* $\operatorname{Ob}\mu$*) where* $\operatorname{Ob}\mu$ *is the object of endomorphism* μ *for an endomorphism* μ *of a category with countable join of morphisms.*

I call S_1 and S **endomorphism series**.

We will consider the collection of all binary relations (on a set $\mho$), as well as the collection of all funcoids and the collection of all reloids on a fixed set, as categories with single object $\mho$ and the identity morphisms $I_\mho$, $I^{\mathsf{FCD}(\mho)}$, $I^{\mathsf{RLD}(\mho)}$.

Proposition 43 *The relation* $S(\mu)$ *is transitive for the category of binary relations.*

Proof

$$\begin{aligned}
S(\mu)\circ S(\mu) &= \mu^0\circ S(\mu)\cup\mu\circ S(\mu)\cup\mu^2\circ S(\mu)\cup\dots\\
&= (\mu^0\cup\mu^1\cup\mu^2\cup\dots)\cup(\mu^1\cup\mu^2\cup\mu^3\cup\dots)\cup(\mu^2\cup\mu^3\cup\mu^4\cup\dots)\\
&= \mu^0\cup\mu^1\cup\mu^2\cup\dots\\
&= S(\mu).
\end{aligned}$$

□

7.3 Connectedness regarding binary relations

Before going to research connectedness for funcoids and reloids we will excurse into the basic special case of connectedness regarding binary relations on a set $\mho$.

Definition 58 *A set A is called* ***(strongly) connected*** *regarding a binary relation μ when*

$$\forall X\in\mathscr{P}(\operatorname{dom}\mu)\setminus\{\emptyset\},Y\in\mathscr{P}(\operatorname{im}\mu)\setminus\{\emptyset\}:(X\cup Y=A\Rightarrow X\,[\mu]\,Y).$$

Let $\mho$ be a set.

Definition 59 ***Path*** *between two elements $a,b\in\mho$ in a set $A\subseteq\mho$ through binary relation μ is the finite sequence $x_0\dots x_n$ where $x_0=a$, $x_n=b$ for $n\in\mathbb{N}$ and $x_i(\mu\cap A\times A)x_{i+1}$ for every $i=0,\dots,n-1$. n is called path* ***length****.*

Proposition 44 *There exists path between every element $a\in\mho$ and that element itself.*

Proof It is the path consisting of one vertex (of length 0). □

Proposition 45 *There is a path from element a to element b in a set A through a binary relation μ iff $a\ (S(\mu\cap A\times A))\ b$ (that is $(a,b)\in S(\mu\cap A\times A)$).*

Proof

$\Rightarrow$ If a path from a to b exists, then $\{b\}\subseteq\langle(\mu\cap A\times A)^n\rangle\{a\}$ where n is the path length. Consequently $\{b\}\subseteq\langle S(\mu\cap A\times A)\rangle\{a\}$; $a\ (S(\mu\cap A\times A))\ b$.

$\Leftarrow$ If $a\ (S(\mu\cap A\times A))\ b$ then exists $n\in\mathbb{N}$ such that $a\ (\mu\cap A\times A)^n\ b$. By definition of composition of binary relations this means that there exist finite sequence $x_0\dots x_n$ where $x_0=a$, $x_n=b$ for $n\in\mathbb{N}$ and $x_i\ (\mu\cap A\times A)\ x_{i+1}$ for every $i=0,\dots,n-1$. That is there is path from a to b.

□

Theorem 73 *The following statements are equivalent for a relation μ and a set A:*

1. *For every $a, b \in A$ there is a path between a and b in A through μ.*
2. *$S(\mu \cap (A \times A)) \supseteq A \times A$.*
3. *$S(\mu \cap (A \times A)) = A \times A$.*
4. *A is connected regarding μ.*

Proof

(1)⇒(2) Let for every $a, b \in A$ there is a path between a and b in A through μ. Then $a\ (S(\mu \cap A \times A))\ b$ for every $a, b \in A$. It is possible only when $S(\mu \cap (A \times A)) \supseteq A \times A$.

(3)⇒(1) For every two vertices a and b we have $a\ (S(\mu \cap A \times A))\ b$. So (by the previous theorem) for every two vertices a and b exist path from a to b.

(3)⇒(4) Suppose that $\neg(X\ [\mu \cap (A \times A)]\ Y)$ for some $X, Y \in \mathscr{P}\mho \setminus \{\emptyset\}$ such that $X \cup Y = A$. Then by a lemma $\neg(X\ [(\mu \cap A \times A)^n]\ Y)$ for every $n \in \mathbb{N}$. Consequently $\neg(X\ [S(\mu \cap A \times A)]\ Y)$. So $S(\mu \cap A \times A) \neq A \times A$.

(4)⇒(3) If $\langle S(\mu \cap (A \times A))\rangle \{v\} = A$ for every vertex v then $S(\mu \cap A \times A) = A \times A$. Consider the remaining case when $V \stackrel{\text{def}}{=} \langle S(\mu \cap A \times A)\rangle \{v\} \subset A$ for some vertex v. Let $W = A \setminus V$. If $\operatorname{card} A = 1$ then $S(\mu \cap A \times A) \supseteq (=) = A \times A$; otherwise $W \neq \emptyset$. Then $V \cup W = A$ and so $V\ [\mu]\ W$ what is equivalent to $V\ [\mu \cap A \times A]\ W$ that is $\langle \mu \cap A \times A\rangle V \cap W \neq \emptyset$. This is impossible because $\langle \mu \cap A \times A\rangle V = \langle \mu \cap A \times A\rangle \langle S(\mu \cap A \times A)\rangle V = \langle S_1(\mu \cap A \times A)\rangle V \subseteq \langle S(\mu \cap A \times A)\rangle V = V$.

(2)⇒(3) Because $S(\mu \cap A \times A) \subseteq A \times A$.

□

Corollary 20 *A set A is connected regarding a binary relation μ iff it is connected regarding $\mu \cap (A \times A)$.*

Definition 60 *A **connected component** of a set A regarding a binary relation F is a maximal connected subset of A.*

Theorem 74 *The set A is partitioned into connected components (regarding every binary relation F).*

Proof Consider the binary relation $a \sim b \Leftrightarrow a\ (S(F))\ b \wedge b\ (S(F))\ a$. $\sim$ is a symmetric, reflexive, and transitive relation. So all points of A are partitioned into a collection of sets Q. Obviously each component is (strongly) connected. If a set $R \subseteq A$ is greater than one of that connected components A then it contains a point $b \in B$ where B is some other connected component. Consequently R is disconnected. □

Proposition 46 *A set is connected (regarding a binary relation) iff it has one connected component.*

Proof Direct implication is obvious. Reverse is proved by contradiction. □

7.4 Connectedness regarding funcoids and reloids

Definition 61 $S_1^*(\mu) = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} S_1(M) \quad | \quad M \in \operatorname{up}\mu \right\}$ *for an endo-reloid* μ.

Definition 62 ***Connectivity reloid*** $S^*(\mu)$ *for an endo-reloid* μ *is defined as follows:*

$$S^*(\mu) = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Ob}\mu;\operatorname{Ob}\mu)} S(M) \quad | \quad M \in \operatorname{up}\mu \right\}.$$

Remark 9 Do not mess the word *connectivity* with the word *connectedness* which means being connected.[1]

Proposition 47 $S^*(\mu) = I^{\mathsf{RLD}(\operatorname{Ob}\mu)} \cup S_1^*(\mu)$ *for every endo-reloid* μ.

Proof It follows from the theorem about distributivity of $\cup$ regarding $\bigcap$ (see [15]). □

Proposition 48 $S^*(\mu) = S(\mu)$ *if* μ *is a principal reloid.*

Proof $S^*(\mu) = \bigcap \{S(\mu)\} = S(\mu)$. □

Definition 63 *A filter object* $\mathcal{A} \in \mathfrak{F}(\operatorname{Ob}\mu)$ *is called* ***connected*** *regarding an endo-reloid* μ *when* $S^*(\mu \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})) \supseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

Obvious 30. A filter object $\mathcal{A} \in \operatorname{Ob}\mu$ is connected regarding a reloid μ iff $S^*(\mu \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})) = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

Definition 64 *A filter object* $\mathcal{A}$ *is called* ***connected*** *regarding an endo-funcoid* μ *when*

$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\operatorname{Ob}\mu) \setminus \left\{ 0^{\mathfrak{F}(\operatorname{Ob}\mu)} \right\} : (\mathcal{X} \cup \mathcal{Y} = \mathcal{A} \Rightarrow \mathcal{X}\ [\mu]\ \mathcal{Y}).$$

[1] In some math literature these two words are used interchangeably.

Proposition 49 *Let A be a set. The f.o. $\uparrow^{\mathrm{Ob}\,\mu} A$ is connected regarding an endo-funcoid μ iff*

$$\forall X, Y \in \mathscr{P}(\mathrm{Ob}\,\mu) \setminus \{\emptyset\} : (X \cup Y = A \Rightarrow X\,[\mu]^*\,Y).$$

Proof

$\Rightarrow$ Obvious.

$\Leftarrow$ It follows from co-separability of filter objects.

□

Theorem 75 *The following are equivalent for every set A and binary relation μ on a set U:*

1. *A is connected regarding binary relation μ.*
2. *$\uparrow^U A$ is connected regarding $\uparrow^{\mathsf{RLD}(U;U)} \mu$.*
3. *$\uparrow^U A$ is connected regarding $\uparrow^{\mathsf{FCD}(U;U)} \mu$.*

Proof

(1)⇔(2) $S^*\left(\uparrow^{\mathsf{RLD}(U;U)} \mu \cap \left(\uparrow^U A \times^{\mathsf{RLD}} \uparrow^U A\right)\right) = S^*\left(\uparrow^{\mathsf{RLD}(U;U)} (\mu \cap (A \times A))\right) = \uparrow^{\mathsf{RLD}(U;U)} S(\mu \cap (A \times A))$. So $S^*\left(\uparrow^{\mathsf{RLD}(U;U)} \mu \cap (\uparrow^U A \times^{\mathsf{RLD}} \uparrow^U A)\right) \supseteq \uparrow^U A \times^{\mathsf{RLD}} \uparrow^U A \Leftrightarrow \uparrow^{\mathsf{RLD}(U;U)} S(\mu \cap (A \times A)) \supseteq \uparrow^{\mathsf{RLD}(U;U)} (A \times A) = \uparrow^U A \times^{\mathsf{RLD}} \uparrow^U A$.

(1)⇔(3) It follows from the previous proposition.

□

Next is conjectured a statement more strong than the above theorem:

Conjecture 21 *Let $\mathcal{A}$ is a f.o. on a set U and F is a binary relation on U.*
$\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{FCD}(U;U)} F$ iff $\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{RLD}(U;U)} F$.

Obvious 31. A filter object $\mathcal{A}$ is connected regarding a reloid μ iff it is connected regarding the reloid $\mu \cap (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A})$.

Obvious 32. A filter object $\mathcal{A}$ is connected regarding a funcoid μ iff it is connected regarding the funcoid $\mu \cap (\mathcal{A} \times^{\mathsf{FCD}} \mathcal{A})$.

Theorem 76 *A filter object $\mathcal{A}$ is connected regarding a reloid f iff $\mathcal{A}$ is connected regarding every $F \in \left\langle \uparrow^{\mathsf{RLD}(\mathrm{Ob}\,f;\mathrm{Ob}\,f)} \right\rangle \mathrm{up}\, f$.*

Proof

$\Rightarrow$ Obvious.

$\Leftarrow$ $\mathcal{A}$ is connected regarding $\uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F$ iff $S(F) = F^0 \cup F^1 \cup F^2 \cup \ldots \in \mathrm{up}\left(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}\right)$.

$S^*(f) = \bigcap\left\{\uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\mu;\mathrm{Ob}\,\mu)} S(F) \mid F \in \mathrm{up}\, f\right\} \supseteq \bigcap\left\{\mathcal{A} \times^{\mathsf{RLD}} \mathcal{A} \mid F \in \mathrm{up}\, f\right\} = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{A}$.

□

Conjecture 22 *A filter object $\mathcal{A}$ is connected regarding a funcoid μ iff $\mathcal{A}$ is connected for every $F \in \left\langle \uparrow^{\mathsf{FCD}(\mathrm{Ob}\,\mu;\mathrm{Ob}\,\mu)}\right\rangle \mathrm{up}\,\mu$.*

The above conjecture is open even for the case when $\mathcal{A}$ is a principal f.o.

Conjecture 23 *A filter object $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding the funcoid $(\mathsf{FCD})f$.*

The above conjecture is true in the special case of principal filters:

Proposition 50 *A f.o. $\uparrow^{\mathrm{Ob}\,\mu} A$ (for a set A) is connected regarding an endo-reloid f iff it is connected regarding the endo-funcoid $(\mathsf{FCD})f$.*

Proof $\uparrow^{\mathrm{Ob}\, f} A$ is connected regarding a reloid f iff A is connected regarding every $F \in \mathrm{up}\, f$ that is when (taken in account that connectedness for $\uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F$ is the same as connectedness of $\uparrow^{\mathsf{FCD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F$)

$$\forall F \in \mathrm{up}\, f \forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \left\{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\right\} : (\mathcal{X} \cup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F\right] \mathcal{Y}) \Leftrightarrow$$
$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \left\{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\right\} \forall F \in \mathrm{up}\, f : (\mathcal{X} \cup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F\right] \mathcal{Y}) \Leftrightarrow$$
$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \left\{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\right\} : (\mathcal{X} \cup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \forall F \in \mathrm{up}\, f : \mathcal{X} \left[\uparrow^{\mathsf{FCD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} F\right] \mathcal{Y}) \Leftrightarrow$$
$$\forall \mathcal{X}, \mathcal{Y} \in \mathfrak{F}(\mathrm{Ob}\, f) \setminus \left\{0^{\mathfrak{F}(\mathrm{Ob}\, f)}\right\} : (\mathcal{X} \cup \mathcal{Y} = \uparrow^{\mathrm{Ob}\, f} A \Rightarrow \mathcal{X} \left[(\mathsf{FCD})f\right] \mathcal{Y})$$

that is when the set $\uparrow^{\mathrm{Ob}\, f} A$ is connected regarding the funcoid $(\mathsf{FCD})f$. □

7.5 Algebraic properties of S and S^*

Theorem 77 *$S^*(S^*(f)) = S^*(f)$ for every endo-reloid f.*

Proof $S^*(S^*(f)) = \bigcap\left\{\uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} S(R) \mid R \in \mathrm{up}\, S^*(f)\right\} \subseteq \bigcap\left\{\uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} S(R) \mid R \in \{S(F) \mid F \in \mathrm{up}\, f\}\right\} = \bigcap\left\{\uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} S(S(F)) \mid F \in \mathrm{up}\, f\right\} = \bigcap\left\{\uparrow^{\mathsf{RLD}(\mathrm{Ob}\, f;\mathrm{Ob}\, f)} S(F) \mid F \in \mathrm{up}\, f\right\} = S^*(f)$.

So $S^*(S^*(f)) \subseteq S^*(f)$. That $S^*(S^*(f)) \supseteq S^*(f)$ is obvious. □

Corollary 21 *$S^*(S(f)) = S(S^*(f)) = S^*(f)$ for any endo-reloid f.*

Proof Obviously $S^*(S(f)) \supseteq S^*(f)$ and $S(S^*(f)) \supseteq S^*(f)$.

But $S^*(S(f)) \subseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \subseteq S^*(S^*(f)) = S^*(f)$. $\square$

Conjecture 24 *$S(S(f)) = S(f)$ for*

1. *every endo-reloid f;*

2. *every endo-funcoid f.*

Conjecture 25 *For every endo-reloid f*

1. *$S(f) \circ S(f) = S(f)$;*

2. *$S^*(f) \circ S^*(f) = S^*(f)$;*

3. *$S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$.*

Conjecture 26 *$S(f) \circ S(f) = S(f)$ for every endo-funcoid f.*

8 Postface

8.1 Misc

See this Web page for my research plans: http://www.mathematics21.org/agt-plans.html

I deem that now the most important research topics in Algebraic General Topology are:

- to solve the open problems mentioned in this work;
- define and research compactness of funcoids;
- research are n-ary (where n is an ordinal, or more generally an index set) funcoids and reloids (plain funcoids and reloids are binary by analogy with binary relations).

We should also research relationships between complete funcoids and complete reloids.

All my research of funcoids and reloids is presented at http://www.mathematics21.org/algebraic-general-topology.html

A Some counter-examples

For further examples we will use the filter object Δ defined by the formula

$$\Delta = \bigcap \left\{ \uparrow^{\mathfrak{F}(\mathbb{R})} (-\varepsilon; \varepsilon) \quad | \quad \varepsilon \in \mathbb{R}, \varepsilon > 0 \right\}.$$

I also will denote $\Omega(A)$ the Fréchet f.o. on the set A.

Example 1 There exist a funcoid f and a set S of funcoids such that $f \cap \bigcup S \neq \bigcup \langle f\cap \rangle S$.

Proof Let $f = \Delta \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}(\mathbb{R})} \{0\}$ and $S = \left\{ \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} ((\varepsilon; +\infty) \times \{0\}) \quad | \quad \varepsilon > 0 \right\}$. Then $f \cap \bigcup S = (\Delta \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}(\mathbb{R})} \{0\}) \cap \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} ((0; +\infty) \times \{0\}) = (\Delta \cap \uparrow^{\mathfrak{F}(\mathbb{R})} (0; +\infty)) \times^{\mathsf{FCD}} \uparrow^{\mathfrak{F}(\mathbb{R})} \{0\} \neq 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$ while $\bigcup \langle f\cap \rangle S = \bigcup \left\{ 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} \right\} = 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$. □

Example 2 There exist a set R of funcoids and a funcoid f such that $f \circ \bigcup R \neq \bigcup \langle f\circ \rangle R$.

Proof Let $f = \Delta \times^{\mathsf{FCD}} \{0\}$, $R = \left\{ \{0\} \times^{\mathsf{FCD}} (\varepsilon; +\infty) \quad | \quad \varepsilon \in \mathbb{R} \right\}$.

We have $\bigcup R = \{0\} \times^{\mathsf{FCD}} (0; +\infty)$; $f \circ \bigcup R = \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} (\{0\} \times \{0\}) \neq 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$ and $\bigcup \langle f\circ \rangle R = \bigcup \left\{ 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} \right\} = 0^{\mathsf{FCD}(\mathbb{R};\mathbb{R})}$. □

Example 3 There exist a set R of funcoids and f.o. $\mathcal{X}$ and $\mathcal{Y}$ such that

1. $\mathcal{X} \left[\bigcup R \right] \mathcal{Y} \wedge \nexists f \in R : \mathcal{X} [f] \mathcal{Y}$;
2. $\left\langle \bigcup R \right\rangle \mathcal{X} \supset \bigcup \{ \langle f \rangle \mathcal{X} \quad | \quad f \in R \}$.

Proof

1. Let $\mathcal{X} = \Delta$ and $\mathcal{Y} = 1^{\mathfrak{F}(\mathbb{R})}$. Let $R = \left\{ \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} ((\varepsilon; +\infty) \times \mathbb{R}) \quad | \quad \varepsilon \in \mathbb{R}, \varepsilon > 0 \right\}$. Then $\bigcup R = \uparrow^{\mathsf{FCD}(\mathbb{R};\mathbb{R})} ((0; +\infty) \times \mathbb{R})$. So $\mathcal{X} \left[\bigcup R \right] \mathcal{Y}$ and $\forall f \in R : \neg(\mathcal{X} [f] \mathcal{Y})$.
2. With the same $\mathcal{X}$ and R we have $\left\langle \bigcup R \right\rangle \mathcal{X} = \mathbb{R}$ and $\langle f \rangle \mathcal{X} = 0^{\mathfrak{F}(\mathbb{R})}$ for every $f \in R$, thus $\bigcup \{ \langle f \rangle \mathcal{X} \quad | \quad f \in R \} = 0^{\mathfrak{F}(\mathbb{R})}$.

□

Theorem 78 *For a f.o. a we have $a \times^{\mathsf{RLD}} a \subseteq I^{\mathsf{RLD}(\mathrm{Base}(a))}$ only in the case if $a = 0^{\mathfrak{F}(\mathrm{Base}(a))}$ or a is a trivial atomic f.o. (that is corresponds to an one-element set).*

Proof If $a \times^{\mathsf{RLD}} a \subseteq I^{\mathsf{RLD}(\mathrm{Base}(a))}$ then exists $m \in \mathrm{up}(a \times^{\mathsf{RLD}} a)$ such that $m \subseteq I_{\mathrm{Base}(a)}$. Consequently exist $A, B \in \mathrm{up}\, a$ such that $A \times B \subseteq I_{\mathrm{Base}(a)}$ what is possible only in the case when $\uparrow^{\mathrm{Base}(a)} A = \uparrow^{\mathrm{Base}(a)} B = a$ and $A = B$ is an one-element set or empty set. □

Corollary 22 *Reloidal product of a non-trivial atomic filter object with itself is non-atomic.*

Proof Obviously $(a \times^{\mathsf{RLD}} a) \cap I^{\mathsf{RLD}(\mathrm{Base}(a))} \neq 0^{\mathfrak{F}(\mathrm{Base}(a))}$ and $(a \times^{\mathsf{RLD}} a) \cap I^{\mathsf{RLD}(\mathrm{Base}(a))} \subset a \times^{\mathsf{RLD}} a$. □

Example 4 Non-convex reloids exist.

Proof Let a is a non-trivial atomic f.o. Then $\mathrm{id}_a^{\mathsf{RLD}}$ is non-convex. This follows from the fact that only direct products which are below $\mathrm{id}^{\mathrm{RLD}(\mathrm{Base}(a))}$are direct products of atomic f.o. and $\mathrm{id}_a^{\mathsf{RLD}}$is not their join. $\square$

Example 5 $(\mathsf{RLD})_{\mathrm{in}} f \neq (\mathsf{RLD})_{\mathrm{out}} f$ for a funcoid f.

Proof Let $f = I^{\mathsf{FCD}(\mathbb{N})}$. Then $(\mathsf{RLD})_{\mathrm{in}} f = \bigcup \left\{ a \times^{\mathsf{RLD}} a \quad | \quad a \in \operatorname{atoms} 1^{\mathfrak{F}(\mathbb{N})} \right\}$ and $(\mathsf{RLD})_{\mathrm{out}} f = I^{\mathsf{RLD}(\mathbb{N})}$. But as we have shown above $a \times^{\mathsf{RLD}} a \nsubseteq I^{\mathsf{RLD}(\mathbb{N})}$ for non-trivial f.o. a, and so $(\mathsf{RLD})_{\mathrm{in}} f \nsubseteq (\mathsf{RLD})_{\mathrm{out}} f$. $\square$

Proposition 51 $I^{\mathsf{FCD}(\mathbb{N})} \cap \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})} ((\mathbb{N} \times \mathbb{N}) \setminus I_{\mathbb{N}}) = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \neq 0^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}$.

Proof Note that $\left\langle I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \right\rangle \mathcal{X} = \mathcal{X} \cap \Omega(\mathbb{N})$.

Let $f = I^{\mathsf{FCD}(\mathbb{N})}$, $g = \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})} ((\mathbb{N} \times \mathbb{N}) \setminus I_{\mathbb{N}})$.

Let x is a non-trivial atomic f.o. If $X \in \operatorname{up} x$ then $\operatorname{card} X \geqslant 2$ (In fact, X is infinite but we don't need this.) and consequently $\langle g \rangle^* X = 1^{\mathfrak{F}(\mathbb{N})}$. Thus $\langle g \rangle x = 1^{\mathfrak{F}(\mathbb{N})}$. Consequently

$$\langle f \cap g \rangle x = \langle f \rangle x \cap \langle g \rangle x = x \cap 1^{\mathfrak{F}(\mathbb{N})} = x.$$

Also $\left\langle I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \right\rangle x = x \cap \Omega(\mathbb{N}) = x$.

Let now x is a trivial f.o. Then $\langle f \rangle x = x$ and $\langle g \rangle x = 1^{\mathfrak{F}(\mathbb{N})} \setminus x$. So

$$\langle f \cap g \rangle x = \langle f \rangle x \cap \langle g \rangle x = x \cap \left(1^{\mathfrak{F}(\mathbb{N})} \setminus x \right) = 0^{\mathfrak{F}(\mathbb{N})}.$$

Also $\left\langle I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \right\rangle x = x \cap \Omega(\mathbb{N}) = 0^{\mathfrak{F}(\mathbb{N})}$.

So $\langle f \cap g \rangle x = \left\langle I^{\mathsf{FCD}}_{\Omega(\mathbb{N})} \right\rangle x$ for every atomic f.o. x. Thus $f \cap g = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. $\square$

Example 6 There exist binary relations f and g such that $\uparrow^{\mathsf{FCD}(A;B)} f \cap \uparrow^{\mathsf{FCD}(A;B)} g \neq \uparrow^{\mathsf{FCD}(A;B)} (f \cap g)$ for some sets A, B such that $f, g \subseteq A \times B$.

Proof From the proposition above. $\square$

Example 7 There exists a principal funcoid which is not a complemented element of the lattice of funcoids.

Proof I will prove that quasi-complement (see [15] for the definition of quasi-complement) of the funcoid $I^{\mathsf{FCD}(\mathbb{N})}$ is not its complement. We have:

$$\begin{aligned}\left(I^{\mathsf{FCD}(\mathbb{N})}\right)^* &= \bigcup\left\{c \in \mathsf{FCD}(\mathbb{N};\mathbb{N}) \quad\middle|\quad c \asymp I^{\mathsf{FCD}(\mathbb{N})}\right\}\\ &\supseteq \bigcup\left\{\uparrow^{\mathbb{N}}\{\alpha\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}}\{\beta\} \quad\middle|\quad \alpha,\beta\in\mathbb{N}, \uparrow^{\mathbb{N}}\{\alpha\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}}\{\beta\} \asymp I^{\mathsf{FCD}(\mathbb{N})}\right\}\\ &= \bigcup\left\{\uparrow^{\mathbb{N}}\{\alpha\} \times^{\mathsf{FCD}} \uparrow^{\mathbb{N}}\{\beta\} \quad|\quad \alpha,\beta\in\mathbb{N}, \alpha\neq\beta\right\}\\ &= \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}\bigcup\left\{\{\alpha\}\times\{\beta\} \quad|\quad \alpha,\beta\in\mathbb{N}, \alpha\neq\beta\right\}\\ &= \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}(\mathbb{N}\times\mathbb{N}\setminus I_{\mathbb{N}})\end{aligned}$$

(used the corollary 10). But by proved above

$$\left(I^{\mathsf{FCD}(\mathbb{N})}\right)^* \cap I^{\mathsf{FCD}(\mathbb{N})} \neq 0^{\mathfrak{F}(\mathbb{N})}.$$

□

Example 8 There exists funcoid h such that $\operatorname{up} h$ is not a filter.

Proof Consider the funcoid $h = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. We have (from the proof of proposition 51) that $f \in \operatorname{up} h$ and $g \in \operatorname{up} h$, but $f \cap g = \emptyset \notin \operatorname{up} h$. □

Example 9 There exists a funcoid $h \neq 0^{\mathsf{FCD}(A;B)}$ such that $(\mathsf{RLD})_{\mathrm{out}} h = 0^{\mathsf{RLD}(A;B)}$.

Proof Consider $h = I^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$. By proved above $h = f \cap g$ where $f = I^{\mathsf{FCD}(\mathbb{N})}$, $g = \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}((\mathbb{N}\times\mathbb{N})\setminus I_{\mathbb{N}})$.

We have $\mathrm{id}_{\mathbb{N}}, (\mathbb{N}\times\mathbb{N})\setminus\mathrm{id}_{\mathbb{N}} \in \operatorname{up} h$.

So $(\mathsf{RLD})_{\mathrm{out}} h = \bigcap\left\langle\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}\right\rangle \operatorname{up} h \subseteq \uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(\mathrm{id}_{\mathbb{N}} \cap ((\mathbb{N}\times\mathbb{N})\setminus\mathrm{id}_{\mathbb{N}})) = 0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$; and thus $(\mathsf{RLD})_{\mathrm{out}} h = 0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$. □

Example 10 There exists a funcoid h such that $(\mathsf{FCD})(\mathsf{RLD})_{\mathrm{out}} h \neq h$.

Proof It follows from the previous example. □

Example 11 $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \neq f$ for some convex reloid f.

Proof Let $f = I^{\mathsf{RLD}(\mathbb{N})}$. Then $(\mathsf{FCD}) f = I^{\mathsf{FCD}(\mathbb{N})}$. Let a be some non-trivial atomic f.o. Then $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \supseteq a \times^{\mathsf{RLD}} a \nsubseteq I^{\mathsf{RLD}(\mathbb{N})}$ and thus $(\mathsf{RLD})_{\mathrm{in}}(\mathsf{FCD}) f \nsubseteq f$. □

Remark 10 Before I found the last counter-example, I thought that $(\mathsf{RLD})_{\mathrm{in}}$ is an isomorphism from the set of of funcoids to the set of convex reloids. As this conjecture failed, we need an other way to characterize the set of reloids isomorphic to funcoids.

Example 12 There exist funcoids f and g such that

$$(\mathsf{RLD})_{\mathrm{out}}(g\circ f)\neq(\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{RLD})_{\mathrm{out}}f.$$

Proof Take $f=I^{\mathsf{FCD}}_{\Omega(\mathbb{N})}$ and $g=1^{\mathfrak{F}(\mathbb{N})}\times^{\mathsf{FCD}}\uparrow^{\mathbb{N}}\{\alpha\}$ for some $\alpha\in\mathbb{N}$. Then $(\mathsf{RLD})_{\mathrm{out}}f=0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$ and thus $(\mathsf{RLD})_{\mathrm{out}}g\circ(\mathsf{RLD})_{\mathrm{out}}f=0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$.

We have $g\circ f=\Omega(\mathbb{N})\times^{\mathsf{FCD}}\uparrow^{\mathbb{N}}\{\alpha\}$.

Let's prove $(\mathsf{RLD})_{\mathrm{out}}(\Omega(\mathbb{N})\times^{\mathsf{FCD}}\uparrow^{\mathbb{N}}\{\alpha\})=\Omega(\mathbb{N})\times^{\mathsf{RLD}}\uparrow^{\mathbb{N}}\{\alpha\}$.

Really: $(\mathsf{RLD})_{\mathrm{out}}(\Omega(\mathbb{N})\times^{\mathsf{FCD}}\uparrow^{\mathbb{N}}\{\alpha\})=\bigcap\left\langle\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}\right\rangle\mathrm{up}(\Omega(\mathbb{N})\times^{\mathsf{FCD}}\uparrow^{\mathbb{N}}\{\alpha\})=\bigcap\left\{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(K\times\{\alpha\})\quad|\quad K\in\mathrm{up}\,\Omega(\mathbb{N})\right\}$.

$F\in\mathrm{up}\bigcap\left\{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(K\times\{\alpha\})\quad|\quad K\in\mathrm{up}\,\Omega(\mathbb{N})\right\}\Leftrightarrow$
$F\in\mathrm{up}\left(\bigcap\left\{\uparrow^{\mathbb{N}}K\quad|\quad K\in\mathrm{up}\,\Omega(\mathbb{N})\right\}\times^{\mathsf{RLD}}\uparrow^{\mathbb{N}}\{\alpha\}\right)$ for every $F\in\mathscr{P}(\mathbb{N}\times\mathbb{N})$. Thus
$\bigcap\left\{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(K\times\{\alpha\})\quad|\quad K\in\mathrm{up}\,\Omega(\mathbb{N})\right\}=$
$\bigcap\left\{\uparrow^{\mathbb{N}}K\quad|\quad K\in\mathrm{up}\,\Omega(\mathbb{N})\right\}\times^{\mathsf{RLD}}\uparrow^{\mathbb{N}}\{\alpha\}=\Omega(\mathbb{N})\times^{\mathsf{RLD}}\uparrow^{\mathbb{N}}\{\alpha\}$.

So $(\mathsf{RLD})_{\mathrm{out}}(\Omega(\mathbb{N})\times^{\mathsf{FCD}}\uparrow^{\mathbb{N}}\{\alpha\})=\Omega(\mathbb{N})\times^{\mathsf{RLD}}\uparrow^{\mathbb{N}}\{\alpha\}$.

Thus $(\mathsf{RLD})_{\mathrm{out}}(g\circ f)=\Omega(\mathbb{N})\times^{\mathsf{RLD}}\uparrow^{\mathbb{N}}\{\alpha\}\neq 0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}$. □

Example 13 (FCD) does not preserve finite meets.

Proof $(\mathsf{FCD})(I^{\mathsf{RLD}(\mathbb{N})}\cap(1^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}\setminus I^{\mathsf{RLD}(\mathbb{N})}))=(\mathsf{FCD})0^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}=0^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}$.

On the other hand

$$(\mathsf{FCD})I^{\mathsf{RLD}(\mathbb{N})}\cap(\mathsf{FCD})(1^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}\setminus I^{\mathsf{RLD}(\mathbb{N})}))=I^{\mathsf{FCD}(\mathbb{N})}\cap\uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}(\mathbb{N}\times\mathbb{N}\setminus I_{\mathbb{N}})=I^{\mathsf{FCD}}_{\Omega(\mathbb{N})}\neq 0^{\mathsf{FCD}(\mathbb{N};\mathbb{N})}$$

(used the proposition 31). □

Corollary 23 (FCD) *is not an upper adjoint (in general).*

Considering restricting polynomials (considered as reloids) to atomic filter objects, it is simple to prove that each that restriction is injective if not restricting a constant polynomial. Does this hold in general? No, see the following example:

Example 14 There exists a monovalued reloid with atomic domain which is neither injective nor constant (that is not a restriction of a constant function).

Proof (based on [16]) Consider the function $F\in\mathbb{N}^{\mathbb{N}\times\mathbb{N}}$ defined by the formula $(x;y)\mapsto x$.

Let ω_x is a non-principal atomic filter object on the vertical line $\{x\}\times\mathbb{N}$ for every $x\in\mathbb{N}$.

Let T is the collection of such sets Y that $Y\cap(\{x\}\times\mathbb{N})\in\mathrm{up}\,\omega_x$ for all but finitely many vertical lines. Obviously T is a filter.

Let $\omega\in\mathrm{atoms}\,\mathrm{up}^{-1}T$.

For every $x\in\mathbb{N}$ we have some $Y\in T$ for which $(\{x\}\times\mathbb{N})\cap Y=\emptyset$ and thus $(\{x\}\times\mathbb{N})\cap\mathrm{up}\,\omega=\emptyset$.

Let $g = \left(\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})} F\right)|_{\omega}$. If g is constant, then there exist a constant function $G \in \operatorname{up} g$ and $F \cap G$ is also constant. Obviously $\operatorname{dom} \uparrow^{\mathsf{RLD}(\mathbb{N}\times\mathbb{N};\mathbb{N})} (F \cap G) \supseteq \omega$. The function $F \cap G$ cannot be constant because otherwise $\omega \subseteq \operatorname{dom} \uparrow^{\mathsf{RLD}(\mathbb{N}\times\mathbb{N};\mathbb{N})} (F \cap G) \subseteq \uparrow^{\mathbb{N}\times\mathbb{N}} (\{x\} \times \mathbb{N})$ for some $x \in \mathbb{N}$ what is impossible by proved above. So g is not constant.

Suppose that g is injective. Then there exists an injection $G \in \operatorname{up} g$. So $\operatorname{dom} G$ intersects each vertical line by atmost one element that is $\overline{\operatorname{dom} G}$ intersects every vertical line by the whole line or the line without one element. Thus $\overline{\operatorname{dom} G} \in T \subseteq \operatorname{up} \omega$ and consequently $\operatorname{dom} G \notin \operatorname{up} \omega$ what is impossible.

Thus g is neither injective nor constant. $\square$

A.1 Second product. Oblique product

Definition 65 $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \stackrel{\text{def}}{=} (\mathsf{RLD})_{\text{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B})$ *for every f.o.* $\mathcal{A}$ *and* $\mathcal{B}$*. I will call it **second direct product** of filter objects* $\mathcal{A}$ *and* $\mathcal{B}$*.*

Remark 11 The letter F is the above definition is from the word "funcoid". It signifies that it seems to be impossible to define $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ directly without referring to funcoidal product.

Definition 66 ***Oblique products*** *of f.o.* $\mathcal{A}$ *and* $\mathcal{B}$ *are defined as*

$\mathcal{A} \ltimes \mathcal{B} = \bigcap\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \forall B \in \operatorname{up} \mathcal{B} : \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \supseteq \mathcal{A} \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B\}$;

$\mathcal{A} \rtimes \mathcal{B} = \bigcap\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \forall A \in \operatorname{up} \mathcal{A} : \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \supseteq \uparrow^{\operatorname{Base}(\mathcal{A})} A \times^{\mathsf{FCD}} \mathcal{B}\}$.

Proposition 52 $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \subseteq \mathcal{A} \ltimes \mathcal{B} \subseteq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ *for every f.o.* $\mathcal{A}$, $\mathcal{B}$*.*

Proof $\mathcal{A} \ltimes \mathcal{B} \subseteq \bigcap\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \forall A \in \operatorname{up} \mathcal{A}, \forall B \in \operatorname{up} \mathcal{B} : \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \supseteq \uparrow^{\operatorname{Base}(\mathcal{A})} A \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B\} \subseteq \bigcap\left\{\uparrow^{\operatorname{Base}(\mathcal{A})} A \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\right\} = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.

$\mathcal{A} \ltimes \mathcal{B} \supseteq \bigcap\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \supseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\} = \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \operatorname{up}\left(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\right)\right\} = (\mathsf{RLD})_{\text{out}}\left(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\right) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$. $\square$

Conjecture 27 $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \subset \mathcal{A} \ltimes \mathcal{B}$ *for some f.o.* $\mathcal{A}$, $\mathcal{B}$*.*

A stronger conjecture:

Conjecture 28 $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \subset \mathcal{A} \ltimes \mathcal{B} \subset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ *for some f.o.* $\mathcal{A}$, $\mathcal{B}$*. Particularly, is this formula true for* $\mathcal{A} = \mathcal{B} = \Delta \cap \uparrow^{\mathbb{R}} (0; +\infty)$*?*

The above conjecture is similar to Fermat Last Theorem as having no value by itself but being somehow challenging to prove it.

Example 15 $\mathcal{A} \ltimes \mathcal{B} \subset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof It's enough to prove $\mathcal{A} \ltimes \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.

Let $\Delta_+ = \Delta \cap \uparrow^{\mathbb{R}} (0; +\infty)$. Let $\mathcal{A} = \mathcal{B} = \Delta_+$.

Let $K = (\leqslant)|_{\mathbb{R}\times\mathbb{R}}$.

Obviously $K \notin \operatorname{up}\left(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}\right)$.

$\mathcal{A} \ltimes \mathcal{B} \subseteq \uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} K$ and thus $K \in \operatorname{up}(\mathcal{A} \ltimes \mathcal{B})$ because $\uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} K \supseteq \Delta_+ \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B = \mathcal{A} \times^{\mathsf{FCD}} \uparrow^{\operatorname{Base}(\mathcal{B})} B$ for $B = (0; +\infty)$.

Thus $\mathcal{A} \ltimes \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

Example 16 $\mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \subset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof This follows from the above example. □

Proposition 53 *$(\mathcal{A} \ltimes \mathcal{B}) \cap (\mathcal{A} \rtimes \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ for every f.o. $\mathcal{A}$, $\mathcal{B}$.*

Proof $(\mathcal{A} \ltimes \mathcal{B}) \cap (\mathcal{A} \rtimes \mathcal{B}) \subseteq \bigcap\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \mathscr{P}(\operatorname{Base}(\mathcal{A}) \times \operatorname{Base}(\mathcal{B})), \uparrow^{\mathsf{FCD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \supseteq \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\} = \bigcap\left\{\uparrow^{\mathsf{RLD}(\operatorname{Base}(\mathcal{A});\operatorname{Base}(\mathcal{B}))} f \mid f \in \operatorname{up}\left(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\right)\right\} = (\mathsf{RLD})_{\mathrm{out}}\left(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}\right) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$.

To finish the proof we need to show $\mathcal{A} \ltimes \mathcal{B} \supseteq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ and $\mathcal{A} \rtimes \mathcal{B} \supseteq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$. By symmetry it's enough to show $\mathcal{A} \ltimes \mathcal{B} \supseteq \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B}$ what is proved above. □

Example 17 $(\mathcal{A} \ltimes \mathcal{B}) \cup (\mathcal{A} \rtimes \mathcal{B}) \subset \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ for some f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof (based on [3]) Let $\mathcal{A} = \mathcal{B} = \Omega(\mathbb{N})$. It's enough to prove $(\mathcal{A} \ltimes \mathcal{B}) \cup (\mathcal{A} \rtimes \mathcal{B}) \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$.

Let $X \in \operatorname{up}\mathcal{A}$, $Y \in \operatorname{up}\mathcal{B}$ that is $X \in \Omega(\mathbb{N})$, $Y \in \Omega(\mathbb{N})$.

Removing one element x from X produces a set P. Removing one element y from Y produces a set Q. Obviously $P \in \Omega(\mathbb{N})$, $Q \in \Omega(\mathbb{N})$.

Obviously $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \in \operatorname{up}((\mathcal{A} \ltimes \mathcal{B}) \cup (\mathcal{A} \rtimes \mathcal{B}))$.

$(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \not\supseteq X \times Y$ because $(x; y) \in X \times Y$ but $(x; y) \notin (P \times \mathbb{N}) \cup (\mathbb{N} \times Q)$.

Thus $(P \times \mathbb{N}) \cup (\mathbb{N} \times Q) \notin \operatorname{up}\left(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}\right)$ by properties of filter bases. □

Example 18 $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD})f \neq f$ for some convex reloid f.

Proof Let $f = \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$ where $\mathcal{A}$ and $\mathcal{B}$ are from the previous example.

$(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}$ by the proposition 33.

So $(\mathsf{RLD})_{\mathrm{out}}(\mathsf{FCD})(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = (\mathsf{RLD})_{\mathrm{out}}(\mathcal{A} \times^{\mathsf{FCD}} \mathcal{B}) = \mathcal{A} \times^{\mathsf{RLD}}_F \mathcal{B} \neq \mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}$. □

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