In this chapter the term *join-semilattice* means join-semilattice with least element $\bot$.

Definition 1. A *co-frame* is the same as a complete co-brouwerian lattice. [TODO: move it above in the book and use it when appropriate]

Definition 2. It is said that a function f from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$ *preserves finite joins*, when for every finite set $S \in \mathscr{P}\mathfrak{A}$ such that $\bigsqcup^{\mathfrak{A}} S$ exists we have $\bigsqcup^{\mathfrak{B}} \langle f\rangle^* S = f \bigsqcup^{\mathfrak{A}} S$.

Obvious 3. A function between join-semilattices preserves finite joins iff it preserves binary joins ($f(x \sqcup y) = fx \sqcup fy$) and nullary joins ($f(\bot^{\mathfrak{A}}) = \bot^{\mathfrak{B}}$).

Definition 4. A *fixed point* of a function F is such x that $F(x) = x$. We will denote $\operatorname{Fix}(F)$ the set of all fixed points of a function F.

Remark 5. This is based on a Todd Trimble's proof. A shorter but less elementary proof (also by Todd Trimble) is available at http://ncatlab.org/toddtrimble/published/topogeny

Definition 6. Let $\mathfrak{A}$ be a join-semilattice. A *co-nucleus* is a function $F: \mathfrak{A} \to \mathfrak{A}$ such that for every $p, q \in \mathfrak{A}$ we have:

1. $F(p) \sqsubseteq p$;
2. $F(F(p)) = F(p)$;
3. $F(p \sqcup q) = F(p) \sqcup F(q)$.

Proposition 7. Every co-nucleus is a monotone function.

Proof. It follows from $F(p \sqcup q) = F(p) \sqcup F(q)$. □

Lemma 8. $\bigsqcup^{\operatorname{Fix}(F)} S = \bigsqcup S$ for every $S \in \mathscr{P}\operatorname{Fix}(F)$ for every co-nucleus F.

Proof. Obviously $\bigsqcup S \sqsupseteq x$ for every $x \in S$.

Suppose $z \sqsupseteq x$ for every $x \in S$ for a $z \in \operatorname{Fix}(F)$. Then $z \sqsupseteq \bigsqcup S$.

$F(\bigsqcup S) \sqsupseteq F(x)$ for every $x \in S$. Thus $F(\bigsqcup S) \sqsupseteq \bigsqcup_{x \in S} F(x) = \bigsqcup S$. But $F(\bigsqcup S) \sqsubseteq \bigsqcup S$. Thus $F(\bigsqcup S) = \bigsqcup S$ that is $\bigsqcup S \in \operatorname{Fix}(F)$.

So $\bigsqcup^{\operatorname{Fix}(F)} S = \bigsqcup S$ by the definition of join. □

Corollary 9. $\bigsqcup^{\operatorname{Fix}(F)} S$ is defined for every $x, y \in \operatorname{Fix}(F)$.

Lemma 10. $\bigsqcap^{\operatorname{Fix}(F)} S = F(\bigsqcap S)$ for every $S \in \mathscr{P}\operatorname{Fix}(F)$ for every co-nucleus F.

Proof. Obviously $F(\bigsqcap S) \sqsubseteq x$ for every $x \in S$.

Suppose $z \sqsubseteq x$ for every $x \in S$ for a $z \in \operatorname{Fix}(F)$. Then $z \sqsubseteq \bigsqcap S$ and thus $z \sqsubseteq F(\bigsqcap S)$.

So $\bigsqcap^{\operatorname{Fix}(F)} S = F(\bigsqcap S)$ by the definition of meet. □

Corollary 11. $\bigsqcap^{\operatorname{Fix}(F)} S$ is defined for every $x, y \in \operatorname{Fix}(F)$.

Obvious 12. $\operatorname{Fix}(F)$ with induced order is a complete lattice.

Lemma 13. If F is a co-nucleus on a co-frame $\mathfrak{A}$, then the poset $\operatorname{Fix}(F)$ of fixed points of F, with order inherited from $\mathfrak{A}$, is also a co-frame.

Proof. Let $b \in \operatorname{Fix}(F)$, $S \in \mathscr{P}\operatorname{Fix}(F)$. Then

$$\begin{aligned} b \sqcup^{\operatorname{Fix}(F)} \bigsqcap^{\operatorname{Fix}(F)} S &= \\ b \sqcup^{\operatorname{Fix}(F)} F\left(\bigsqcap S\right) &= \\ F(b) \sqcup F\left(\bigsqcap S\right) &= \\ F\left(b \sqcup \bigsqcap S\right) &= \\ F\left(\bigsqcap \langle b \sqcup \rangle^* S\right) &= \\ \bigsqcap^{\operatorname{Fix}(F)} \langle b \sqcup \rangle^* S &= \\ \bigsqcap^{\operatorname{Fix}(F)} \left\langle b \sqcup^{\operatorname{Fix}(F)} \right\rangle^* S. & \end{aligned}$$

□

Definition 14. *Upper set* is a set X on a poset $\mathfrak{A}$ such that $x \in X \wedge y \sqsupseteq x \Rightarrow y \in X$ for every $y \in \mathfrak{A}$.

Denote $\operatorname{Up}(\mathfrak{A})$ the set of upper sets on $\mathfrak{A}$ ordered *reverse* to set theoretic inclusion. [TODO: move it above in the book]

Lemma 15. The set $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary meets and joins.

Proof. Let $S \in \mathscr{P}\operatorname{Up}(\mathfrak{A})$.

Let $X \in \bigcup S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then there is $P \in S$ such that $X \in P$ and thus $Y \in P$ and so $Y \in \bigcup S$. So $\bigcup S \in \operatorname{Up}(\mathfrak{A})$.

Let now $X \in \bigcap S$ and $Y \sqsupseteq X$ for an $Y \in \mathfrak{A}$. Then $\forall T \in S: X \in T$ and so $\forall T \in S: Y \in T$, thus $Y \in \bigcap S$. So $\bigcap S \in \operatorname{Up}(\mathfrak{A})$. □

Theorem 16. A poset $\mathfrak{A}$ is a complete lattice iff there is a antitone map $s: \operatorname{Up}(\mathfrak{A}) \rightarrow \mathfrak{A}$ such that [TODO: define *antitone*.]

1. $s(\uparrow p) = p$ for every $p \in \mathfrak{A}$;
2. $D \subseteq \uparrow s(D)$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

Moreover, in this case $s(D) = \bigsqcap D$ for every $D \in \operatorname{Up}(\mathfrak{A})$.

Proof.

⇒. Take $s(D) = \bigsqcap D$.

⇐. $\forall x \in D: x \sqsupseteq s(D)$ from the second formula.

Let $\forall x \in D: y \sqsubseteq x$. Then $x \in \uparrow y$, $D \subseteq \uparrow y$; because s is an antitone map, thus follows $s(D) \sqsupseteq s(\uparrow y) = y$. So $\forall x \in D: y \sqsubseteq s(D)$.

That s is the meet follows from the definition of meets.

It remains to prove that $\mathfrak{A}$ is a complete lattice.

Take any subset S of $\mathfrak{A}$. Let D be the smallest upper set containing S. (It exists because $\operatorname{Up}(\mathfrak{A})$ is closed under arbitrary joins.) This is

$$D = \{x \in \mathfrak{A} \mid \exists s \in S : x \sqsupseteq s\}.$$

Any lower bound of D is clearly an upper bound of S since $D \supseteq S$. Conversely any lower bound of S is a lower bound of D. Thus S and D have the same set of lower bounds, hence have the same greatest lower bound. □

Proposition 17. [TODO: Move it above in the book.] For any poset $\mathfrak{A}$ the following are mutually reverse order isomorphisms between upper sets F (ordered reverse to set-theoretic inclusion) on $\mathfrak{A}$ and order homomorphisms $\varphi : \mathfrak{A}^{\mathrm{op}} \to 2$ (here 2 is the partially ordered set of two elements: 0 and 1 where $0 \sqsubseteq 1$), defined by the formulas

1. $\varphi(a) = \left\{ \begin{smallmatrix} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{smallmatrix} \right.$ for every $a \in \mathfrak{A}$;

2. $F = \varphi^{-1}(1)$.

Proof. Let $X \in \varphi^{-1}(1)$ and $Y \sqsupseteq X$. Then $\varphi(X) = 1$ and thus $\varphi(Y) = 1$. Thus $\varphi^{-1}(1)$ is a upper set.

It is easy to show that φ defined by the formula (1) is an order homomorphism $\mathfrak{A}^{\mathrm{op}} \to 2$ whenever F is a upper set.

Finally we need to prove that they are mutually inverse. Really: Let φ be defined by the formula (1). Then take $F' = \varphi^{-1}(1)$ and define $\varphi'(a)$ by the formula (1). We have

$$\varphi'(a) = \begin{cases} 1 & \text{if } a \in \varphi^{-1}(1) \\ 0 & \text{if } a \notin \varphi^{-1}(1) \end{cases} = \begin{cases} 1 & \text{if } \varphi(a) = 1 \\ 0 & \text{if } \varphi(a) \neq 1 \end{cases} = \varphi(a).$$

Let now F be defined by the formula (2). Then take $\varphi'(a) = \left\{ \begin{smallmatrix} 1 & \text{if } a \in F \\ 0 & \text{if } a \notin F \end{smallmatrix} \right.$ as defined by the formula (1) and define $F' = \varphi'^{-1}(1)$. Then

$$F' = \varphi'^{-1}(1) = F.$$ □

Lemma 18. For a complete lattice $\mathfrak{A}$, the map $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves arbitrary meets.

Proof. Let $S \in \mathscr{P}\operatorname{Up}(\mathfrak{A})$. We have $\bigsqcap S \in \operatorname{Up}(\mathfrak{A})$.

$\bigsqcap \bigsqcap S = \bigsqcap \bigsqcap_{X \in S} X = \bigsqcap_{X \in S} \bigsqcap X$ is what we needed to prove. □

Lemma 19. A complete lattice $\mathfrak{A}$ is a co-frame iff $\bigsqcap : \operatorname{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins.

Proof.

⇒. Let $\mathfrak{A}$ be a co-frame. Let $D, D' \in \operatorname{Up}(\mathfrak{A})$. Obviously $\bigsqcap (D \sqcup D') \sqsupseteq \bigsqcap D$ and $\bigsqcap (D \sqcup D') \sqsupseteq \bigsqcap D'$, so $\bigsqcap (D \sqcup D') \sqsupseteq \bigsqcap D \sqcup \bigsqcap D'$.

Also $\bigsqcap D \sqcup \bigsqcap D' = \bigcup D \sqcup \bigcup D' =$ (because $\mathfrak{A}$ is a co-frame)$= \bigcup \{d \sqcup d' \mid d \in D, d' \in D'\}$. Obviously $d \sqcup d' \in D \cap D'$, thus $\bigsqcap D \sqcup \bigsqcap D' \subseteq \bigcup (D \cap D') = \bigsqcap (D \cap D')$ that is $\bigsqcap D \sqcup \bigsqcap D' \sqsupseteq \bigsqcap (D \cap D')$. So $\bigsqcap (D \sqcup D') = \bigsqcap D \sqcup \bigsqcap D'$ that is $\bigsqcap : \operatorname{Up}(A) \to A$ preserves binary joins.

It preserves nullary joins since $\bigsqcap^{\operatorname{Up}(\mathfrak{A})} \bot^{\operatorname{Up}(\mathfrak{A})} = \bigsqcap^{\operatorname{Up}(\mathfrak{A})} \mathfrak{A} = \bot^{\mathfrak{A}}$.

$\Leftarrow$. Suppose $\prod$: $\mathrm{Up}(\mathfrak{A}) \to \mathfrak{A}$ preserves finite joins. Let $b \in \mathfrak{A}$, $S \in \mathscr{P}\mathfrak{A}$. Let D be the smallest upper set containing S (so $D = \bigcup \langle\uparrow\rangle^* S$). We have $\prod D = \bigcup S$. So

$$\begin{aligned}
b \sqcup \prod S &= \\
\prod \uparrow b \sqcup \bigcup \prod \langle\uparrow\rangle^* S &= \\
\prod \uparrow b \sqcup \prod \bigcup \langle\uparrow\rangle^* S &= \text{(since } \textstyle\prod \text{ preserves finite joins)}\\
\prod \left(\uparrow b \sqcup \bigcup \langle\uparrow\rangle^* S\right) &= \\
\bigcup \left(\uparrow b \cap \bigcup \langle\uparrow\rangle^* S\right) &= \\
\prod \bigcup_{a \in S} (\uparrow b \cap \uparrow a) &= \\
\prod \bigcup_{a \in S} \uparrow (b \sqcup a) &= \text{(since } \textstyle\prod \text{ preserves all meets)}\\
\bigcup_{a \in S} \prod \uparrow (b \sqcup a) &= \\
\bigcup_{a \in S} (b \sqcup a) &= \\
\prod_{a \in S} (b \sqcup a).
\end{aligned}$$

□

Corollary 20. If $\mathfrak{A}$ is a co-frame, then the composition $F = \uparrow \circ \prod$: $\mathrm{Up}(\mathfrak{A}) \to \mathrm{Up}(\mathfrak{A})$ is a co-nucleus. The embedding $\uparrow$: $\mathfrak{A} \to \mathrm{Up}(\mathfrak{A})$ is an isomorphism of $\mathfrak{A}$ onto the co-frame $\mathrm{Fix}(F)$.

Proof. $D \sqsupseteq F(D)$ follows from theorem 16.

We have $F(F(D)) = F(D)$ for all $D \in \mathrm{Up}(\mathfrak{A})$ since $F(F(D)) = \uparrow \prod \uparrow \prod D =$ (because $\prod \uparrow s = s$ for any s)$= \uparrow \prod D = F(D)$.

And since both $\prod$: $\mathrm{Up}(\mathfrak{A}) \to \mathfrak{A}$ and $\uparrow$ preserve finite joins, F preserves finite joins. Thus F is a co-nucleus.

Finally, we have $a \sqsupseteq a'$ if and only if $\uparrow a \subseteq \uparrow a'$, so that $\uparrow$: $\mathfrak{A} \to \mathrm{Up}(\mathfrak{A})$ maps $\mathfrak{A}$ isomorphically onto its image $\langle\uparrow\rangle^* \mathfrak{A}$. This image is $\mathrm{Fix}(F)$ because if D is any fixed point (i.e. if $D = \uparrow \prod D$), then D clearly belongs to $\langle\uparrow\rangle^* \mathfrak{A}$; and conversely $\uparrow a$ is always a fixed point of $F = \uparrow \circ \prod$ since $F(\uparrow a) = \uparrow \prod \uparrow a = \uparrow a$. □

Definition 21. If $\mathfrak{A}$, $\mathfrak{B}$ are two join-semilattices, then $\mathrm{Join}(\mathfrak{A}; \mathfrak{B})$ is the (ordered pointwise) set of finite joins preserving maps $\mathfrak{A} \to \mathfrak{B}$.

Obvious 22. $\mathrm{Join}(\mathfrak{A}; \mathfrak{B})$ is a join-semilattice, where $f \sqcup g$ is given by the formula $(f \sqcup g)(p) = f(p) \sqcup g(p)$, $\bot^{\mathrm{Join}(\mathfrak{A};\mathfrak{B})}$ is given by the formula $\bot^{\mathrm{Join}(\mathfrak{A};\mathfrak{B})}(p) = \bot^{\mathfrak{B}}$.

Definition 23. Let h: $Q \to R$ be a finite joins preserving map. Then by definition $\mathrm{Join}(P, h)$: $\mathrm{Join}(P; Q) \to \mathrm{Join}(P; R)$ takes $f \in \mathrm{Join}(P; Q)$ into the composition $h \circ f \in \mathrm{Join}(P; R)$.

Lemma 24. Above defined $\mathrm{Join}(P, h)$ is a finite joins preserving map.

Proof. $(h \circ (f \sqcup f'))x = h(f \sqcup f')x = h(fx \sqcup f'\,x) = hfx \sqcup hf'\,x = (h \circ f)x \sqcup (h \circ f')x = ((h \circ f) \sqcup (h \circ f'))x$. Thus $h \circ (f \sqcup f') = (h \circ f) \sqcup (h \circ f')$.

$\left(h \circ \bot^{\mathrm{Join}(\mathfrak{A};\mathfrak{B})}\right)x = h \bot^{\mathrm{Join}(\mathfrak{A};\mathfrak{B})} x = h \bot^{\mathfrak{B}} = \bot^{\mathfrak{A}}$. □

Proposition 25. If h, h': $Q \to R$ are finite join preserving maps and $h \sqsupseteq h'$, then $\mathrm{Join}(P, h) \sqsupseteq \mathrm{Join}(P, h')$.

Proof. $\mathrm{Join}(P, h)(f)(x) = (h \circ f)(x) = h f x \sqsupseteq h' f x = (h' \circ f)(x) = \mathrm{Join}(P, h')(f)(x)$. □

Lemma 26. If $g: Q \rightarrow R$ and $h: R \rightarrow S$ are finite joins preserving, then the composition $\mathrm{Join}(P; h) \circ \mathrm{Join}(P; g)$ is equal to $\mathrm{Join}(P; h \circ g)$. Also $\mathrm{Join}(P; \mathrm{id}_Q)$ for identity map id_Q on Q is the identity map $\mathrm{id}_{\mathrm{Join}(P; Q)}$ on $\mathrm{Join}(P; Q)$.

Proof. $\mathrm{Join}(P; h)\, \mathrm{Join}(P; g)\, f = \mathrm{Join}(P; h)(g \circ f) = h \circ g \circ f = \mathrm{Join}(P; h \circ g) f$.

$\mathrm{Join}(P; \mathrm{id}_Q)\, f = \mathrm{id}_Q \circ f = f$. □

Corollary 27. If Q is a join-semilattice and $F: Q \rightarrow Q$ is a co-nucleus, then for any join-semilattice P we have that $\mathrm{Join}(P; F): \mathrm{Join}(P; Q) \rightarrow \mathrm{Join}(P; Q)$ is also a co-nucleus.

Proof. From $\mathrm{id}_Q \sqsupseteq F$ (co-nucleus axiom 1) we have $\mathrm{Join}(P; \mathrm{id}_Q) \sqsupseteq \mathrm{Join}(P; F)$ and since by the last lemma the left side is the identity on $\mathrm{Join}(P; Q)$, we see that $\mathrm{Join}(P; F)$ also satisfies co-nucleus axiom 1.

$\mathrm{Join}(P; F) \circ \mathrm{Join}(P; F) = \mathrm{Join}(P; F \circ F)$ by the same lemma and thus $\mathrm{Join}(P; F) \circ \mathrm{Join}(P; F) = \mathrm{Join}(P; F)$ by the second co-nucleus axiom for F, showing that $\mathrm{Join}(P; F)$ satisfies the second co-nucleus axiom.

By an other lemma, we have that $\mathrm{Join}(P; F)$ preserves finite joins, given that F preserves finite joins, which is the third co-nucleus axiom. □

Lemma 28. $\mathrm{Fix}(\mathrm{Join}(P; F)) = \mathrm{Join}(P; \mathrm{Fix}(F))$ for every join-semilattices P, Q and a join preserving function $F: Q \rightarrow Q$.

Proof. $a \in \mathrm{Fix}(\mathrm{Join}(P; F)) \Leftrightarrow a \in F^P \wedge F \circ a = a \Leftrightarrow a \in F^P \wedge \forall x \in P: F(a(x)) = a(x)$.

$a \in \mathrm{Join}(P; \mathrm{Fix}(F)) \Leftrightarrow a \in \mathrm{Fix}(F)^P \Leftrightarrow a \in F^P \wedge \forall x \in P: F(a(x)) = a(x)$.

Thus $\mathrm{Fix}(\mathrm{Join}(P; F)) = \mathrm{Join}(P; \mathrm{Fix}(F))$. That the order of the left and right sides of the equality agrees is obvious. □

Definition 29. $\mathbf{Pos}(\mathfrak{A}; \mathfrak{B})$ is the pointwise ordered poset of monotone maps from a poset $\mathfrak{A}$ to a poset $\mathfrak{B}$.

Lemma 30. If Q, R are join-semilattices and P is a poset, then $\mathbf{Pos}(P; R)$ is a join-semilattice and $\mathbf{Pos}(P; \mathrm{Join}(Q; R))$ is isomorphic to $\mathrm{Join}(Q; \mathbf{Pos}(P; R))$. If R is a co-frame, then also $\mathbf{Pos}(P; R)$ is a co-frame.

Proof. Let $f, g \in \mathbf{Pos}(P; R)$. Then $\lambda x \in P: (f x \sqcup g x)$ is obviously monotone and then it is evident that $f \sqcup^{\mathbf{Pos}(P; R)} g = \lambda x \in P: (f x \sqcup g x)$. $\lambda x \in P: \bot^R$ is also obviously monotone and it is evident that $\bot^{\mathbf{Pos}(P; R)} = \lambda x \in P: \bot^R$.

Obviously both $\mathbf{Pos}(P; \mathrm{Join}(Q; R))$ and $\mathrm{Join}(Q; \mathbf{Pos}(P; R))$ are sets of order preserving maps.

Let f be a monotone map.

$f \in \mathbf{Pos}(P; \mathrm{Join}(Q; R))$ iff $f \in \mathrm{Join}(Q; R)^P$ iff $f \in \{g \in R^Q \mid g \text{ preserves finite joins}\}^P$ iff $f \in (R^Q)^P$ and every $g = f(x)$ (for $x \in P$) preserving finite joins. This is bijectively equivalent ($f \mapsto f'$) to $f' \in (R^P)^Q$ preserving finite joins.

$f' \in \mathrm{Join}(Q; \mathbf{Pos}(P; R))$ iff f' preserves finite joins and $f' \in \mathbf{Pos}(P; R)^Q$ iff f' preserves finite joins and $f' \in \{g \in (R^P)^Q \mid g(x) \text{ is monotone}\}$ iff f' preserves finite joins and $f' \in (R^P)^Q$.

So we have proved that $f \mapsto f'$ is a bijection between $\mathbf{Pos}(P; \mathrm{Join}(Q; R))$ and $\mathrm{Join}(Q; \mathbf{Pos}(P; R))$. That it preserves order is obvious.

It remains to prove that if R is a co-frame, then also $\mathbf{Pos}(P;R)$ is a co-frame.

First, we need to prove that $\mathbf{Pos}(P;R)$ is a complete lattice. But it is easy to prove that for every set $S\in\mathscr{P}\,\mathbf{Pos}(P;R)$ we have $\lambda x\in P\colon\bigsqcup_{f\in S}f(x)$ and $\lambda x\in P\colon\bigsqcap_{f\in S}f(x)$ are monotone and thus are the joins and meets on $\mathbf{Pos}(P;R)$.

Next we need to prove that $b\sqcup^{\mathbf{Pos}(P;R)}\bigsqcap^{\mathbf{Pos}(P;R)}S=\bigsqcap^{\mathbf{Pos}(P;R)}\left\langle b\sqcup^{\mathbf{Pos}(P;R)}\right\rangle^{*}S$. Really (for every $x\in P$),

$\left(b\sqcup^{\mathbf{Pos}(P;R)}\bigsqcap^{\mathbf{Pos}(P;R)}S\right)x=b(x)\sqcup\left(\bigsqcap^{\mathbf{Pos}(P;R)}S\right)x=b(x)\sqcup\bigsqcap_{f\in S}f(x)=\bigsqcap_{f\in S}(b(x)\sqcup f(x))=\bigsqcap_{f\in S}\left(b\sqcup^{\mathbf{Pos}(P;R)}f\right)x=\left(\bigsqcap_{f\in S}^{\mathbf{Pos}(P;R)}\left(b\sqcup^{\mathbf{Pos}(P;R)}f\right)\right)x$.

Thus $b\sqcup^{\mathbf{Pos}(P;R)}\bigsqcap^{\mathbf{Pos}(P;R)}S=\bigsqcap_{f\in S}^{\mathbf{Pos}(P;R)}\left(b\sqcup^{\mathbf{Pos}(P;R)}f\right)=\bigsqcap^{\mathbf{Pos}(P;R)}\left\langle b\sqcup^{\mathbf{Pos}(P;R)}\right\rangle^{*}S$. □

Definition 31. $P\cong Q$ means that posets P and Q are isomorphic.

Theorem 32. If $\mathfrak{A}$ is a co-frame and L is a distributive lattice with greatest element, then $\operatorname{Join}(L;\mathfrak{A})$ is also a co-frame.

Proof. Let $F=\uparrow\circ\bigsqcap\colon\operatorname{Up}(\mathfrak{A})\rightarrow\operatorname{Up}(\mathfrak{A})$; F is a co-nucleus by above.

Since $\operatorname{Up}(\mathfrak{A})\cong\mathbf{Pos}(\mathfrak{A};2)$ by proposition 17, we may regard F as a co-nucleus on $\mathbf{Pos}(\mathfrak{A};2)$.

$\operatorname{Join}(L;\mathfrak{A})\cong\operatorname{Join}(L;\operatorname{Fix}(F))$ by corollary 20.

$\operatorname{Join}(L;\operatorname{Fix}(F))\cong\operatorname{Fix}(\operatorname{Join}(L;F))$ by lemma 28.

By corollary 27 the function $\operatorname{Join}(L;F)$ is a co-nucleus on $\operatorname{Join}(L;\mathbf{Pos}(\mathfrak{A};2))$.

$$\begin{aligned}\operatorname{Join}(L;\mathbf{Pos}(\mathfrak{A};2)) &\cong \text{(by lemma 30)}\\ \mathbf{Pos}(\mathfrak{A};\operatorname{Join}(L;2)) &\cong\\ \mathbf{Pos}(\mathfrak{A};\mathfrak{F}(X)). &\end{aligned}$$

But $\mathbf{Pos}(\mathfrak{A};\mathfrak{F}(X))$ is a co-frame by lemma 30.

Thus $\operatorname{Join}(L;\mathfrak{A})$ is isomorphic to a poset of fixed points of a co-nucleus on the co-frame $\mathbf{Pos}(\mathfrak{A};\mathfrak{F}(X))$. By lemma 13 $\operatorname{Join}(L;\mathfrak{A})$ is also a co-frame. □

Theorem 33. The set of funcoids from a set A to a set B is a co-frame. [TODO: Generalize for pointfree funcoids.]

Proof. Take $\mathfrak{A}=\mathfrak{F}(B)$ in the previous theorem and use the fact that $\mathsf{FCD}(A;B)$ is isomorphic to finite join preserving maps $\mathscr{P}A\rightarrow\mathfrak{F}(B)$. □

Remark 34. The last theorem was proved without using axiom of choice.