Pointfree funcoids as a generalization of frames

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Abstract

I define an injection from the set of frames to the set of pointfree endo-funcoids.

1 Preliminaries

This article is a rough partial draft of a future longer writting.

Read my book [2] before this article. The preprint of [2] is not final, so theorem numbers may change.

2 Confession of a non-professional

I am not a professional mathematician. I just recently started my study of pointfree topology by the book "Stone Spaces".

I don't understand how to prove the statement I use below that every frame can be embedded into a boolean lattice. (I take it below as granted, without understanding the proof.)

Despite of all that, this article presents a new branch of mathematics discovered by me: relationships between frames and locales on one side and pointfree funcoids (pointfree funcoids are also my discovery) on an other side. (Frames and locales can be considered as a special kind of pointfree funcoids.) I expect that analyzing pointfree funcoids will turn to be much more easy that customary ways of doing pointfree topology.

3 Definitions

3.1 Pointfree funcoid induced by a co-frame

Let $\mathfrak{L}$ is a co-frame.

We will define pointfree funcoid $\Uparrow\mathfrak{L}$.

Let $\mathcal{B}(\mathfrak{L})$ is a boolean lattice whose co-subframe $\mathfrak{L}$ is. (That this mapping exists follows from [1], page 53.) There may be probably more than one such mapping, but we just choose one $\mathcal{B}$ arbitrarily.

Define $\operatorname{cl}(A) = \bigsqcap \{X \in \mathfrak{L} \mid X \sqsupseteq A\}$.

Here $\bigsqcap$ can be taken on either $\mathfrak{L}$ or $\mathcal{B}(\mathfrak{L})$ as they are the same.

Obvious 1. $\operatorname{cl} \in \mathfrak{L}^{\mathcal{B}(\mathfrak{L})}$.

$\operatorname{cl}(A \sqcup B) = \bigsqcap \{X \in \mathfrak{L} \mid X \sqsupseteq A \sqcup B\} = \bigsqcap \{X \in \mathfrak{L} \mid X \sqsupseteq A, X \sqsupseteq B\} = \bigsqcap \{X_1 \sqcup X_2 \mid X_1 \sqsupseteq A, X_2 \sqsupseteq B\} = \bigsqcap \{X_1 \mid X_1 \sqsupseteq A\} \sqcup \bigsqcap \{X_2 \mid X_2 \sqsupseteq B\} = \operatorname{cl} A \sqcup \operatorname{cl} B$.

$\operatorname{cl} 0 = 0$ is obvious.

Hence we are under conditions of the theorem 14.26 in my book.

So there exists a unique pointfree endo-funcoid $\Uparrow\mathfrak{L} \in \mathsf{FCD}(\mathfrak{F}(\mathcal{B}(\mathfrak{L})); \mathfrak{F}(\mathcal{B}(\mathfrak{L})))$ such that

$$\langle \Uparrow\mathfrak{L} \rangle \mathcal{X} = \bigsqcap^{\mathfrak{F}(\mathcal{B}(\mathfrak{L}))} \langle \operatorname{cl} \rangle \operatorname{up}^{(\mathfrak{F}(\mathcal{B}(\mathfrak{L})); \mathfrak{P}(\mathcal{B}(\mathfrak{L})))} \mathcal{X}$$

for every filter $\mathcal{X} \in \mathfrak{F}(\mathcal{B}(\mathfrak{L}))$.

3.2 Co-frame induced by a pointfree funcoid

The co-frame $\Downarrow f$ for some pointfree endo-funcoids f will be defined to be the reverse of $\Uparrow$. See below for exact meaning of being reverse.

Let restore the co-frame $\mathfrak{L}$ from the pointfree funcoid $\Uparrow\mathfrak{L}$.

Let poset $\Downarrow f$ for every pointfree funcoid f is defined by the formula:

$$\Downarrow f = \{X \in Z(\operatorname{Ob} f) \mid \langle f \rangle X = X\}.$$

Remark 2. It seems that $\Downarrow$ is *not* a monovalued function from $\mathrm{pf}\mathsf{FCD}$ to $\operatorname{Ob}(\mathbf{Frm})$.

3.3 Isomorphism of co-frames through pointfree funcoids

Remark 3. $\mathfrak{P}(\mathcal{B}(\mathfrak{L})) = Z(\mathfrak{F}(\mathcal{B}(\mathfrak{L})))$ (theorem 4.137 in [2]).

Theorem 4. $\mathfrak{L} \mapsto \Downarrow\Uparrow\mathfrak{L}$ *(where $\mathfrak{L}$ ranges all small frames) is an order isomorphism.*

Proof. Let $A' \in \Downarrow\Uparrow\mathfrak{L}$. Then there exists $A \in \mathcal{B}(\mathfrak{L})$ such that $A' = \uparrow^{\mathcal{B}(\mathfrak{L})} A$.

$\langle f \rangle A' = \uparrow^{\mathcal{B}(\mathfrak{L})} \operatorname{cl} A$.

$\langle f \rangle A' = A'$ that is $\uparrow^{\mathcal{B}(\mathfrak{L})} \operatorname{cl} A = A' = \uparrow^{\mathcal{B}(\mathfrak{L})} A$. So $\operatorname{cl} A = A$ and thus $A \in \mathfrak{L}$.

Let now $A \in \mathfrak{L}$. Then take $A' = \uparrow^{\mathcal{B}(\mathfrak{L})} A$. We have $\langle f \rangle A' = \operatorname{cl} A = \uparrow^{\mathcal{B}(\mathfrak{L})} A = A'$. So $A' \in \Downarrow\Uparrow\mathfrak{L}$.

We have proved that it is a bijection.

Because A and A' are related by the equation $A' = \uparrow^{\mathcal{B}(\mathfrak{L})} A$ it is obvious that this is an order embedding. □

4 Postface

Pointfree funcoids are a **massive** generalization of locales and frames: They don't only require the lattice of filters to be boolean but these can be even not lattices of filters at all but just arbitrary posets. I think a new era in pointfree topology starts.

Much work is yet needed to relate different properties of frames and locales with corresponding properties of pointfree funcoids.

Bibliography

[1] Peter T. Johnstone. *Stone Spaces*. Cambridge University Press, 1982.

[2] Victor Porton. *Algebraic General Topology. Volume 1*. 2013.