A note on starrish posets

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Abstract

In this note some theorems from my previous article are strengthened. Starrish posets, a generalization of distributive lattices, are considered.

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In this short note I strengthen some results about distributive lattices in [1] as distributive lattices are a special case of *starrish* posets introduced in this note.

Definition 1. I will call a poset *starrish* when the full star $\star a$ is a free star for every element a of this poset.

Proposition 2. Every distributive lattice is starrish.

Proof. Let $\mathfrak{A}$ is a distributive lattice, $a \in \mathfrak{A}$. Obviously $0 \notin \star a$; obviously $\star a$ is an upper set. If $x \cup y \in \star a$, then $(x \cup y) \cap a$ is non-least that is $(x \cap a) \cup (y \cap a)$ is non-least what is equivalent to $x \cap a$ or $y \cap a$ being non-least that is $x \in \star a \vee y \in \star a$. □

A generalization of theorem 1 in [1]:

Theorem 3. If $\mathfrak{A}$ is a starrish join-semilattice then

$$\operatorname{atoms}(a \cup b) = \operatorname{atoms} a \cup \operatorname{atoms} b$$

Proof. For every atom c we have: $c \in \operatorname{atoms}(a \cup b) \Leftrightarrow c \not\asymp a \cup b \Leftrightarrow a \cup b \in \star c \Leftrightarrow a \in \star c \vee b \in \star c \Leftrightarrow c \not\asymp a \vee c \not\asymp b \Leftrightarrow c \in \operatorname{atoms} a \vee c \in \operatorname{atoms} b$. □

A generalization of proposition 30 in [1]:

Proposition 4. Let $(\mathfrak{A}; \mathfrak{Z})$ be a down-aligned filtrator with finitely join-closed core, where $\mathfrak{A}$ is a starrish join-semilattice and $\mathfrak{Z}$ is a join-semilattice. Then atomic elements of this filtrator are prime.

Proof. Let a be an atom of the lattice $\mathfrak{A}$. We have for every $X, Y \in \mathfrak{Z}$

$$\begin{aligned}
X \cup^{\mathfrak{Z}} Y \in \operatorname{up} a &\Leftrightarrow \\
X \cup^{\mathfrak{A}} Y \in \operatorname{up} a &\Leftrightarrow \\
X \cup^{\mathfrak{A}} Y \supseteq a &\Leftrightarrow \\
X \cup^{\mathfrak{A}} Y \not\asymp^{\mathfrak{A}} a &\Leftrightarrow \\
X \not\asymp^{\mathfrak{A}} a \vee Y \not\asymp^{\mathfrak{A}} a &\Leftrightarrow \\
X \supseteq a \vee Y \supseteq a &\Leftrightarrow \\
X \in \operatorname{up} a \vee Y \in \operatorname{up} a.&
\end{aligned}$$

□

A generalization of theorem 43 in [1]:

Theorem 5. Let $(\mathfrak{A};\mathfrak{Z})$ be a starrish join-semilattice filtrator with finitely join-closed core which is a join-semilattice. Then ∂a is a free star for each $a\in\mathfrak{A}$.

Proof. For every $A,B\in\mathfrak{Z}$

$$\begin{aligned} X\cup^{\mathfrak{Z}} Y\in\partial a &\Leftrightarrow\\ X\cup^{\mathfrak{A}} Y\in\partial a &\Leftrightarrow\\ X\cup^{\mathfrak{A}} Y\not\asymp^{\mathfrak{A}} a &\Leftrightarrow\\ X\not\asymp^{\mathfrak{A}} a\vee Y\not\asymp^{\mathfrak{A}} a &\Leftrightarrow\\ X\in\partial a\vee Y\in\partial a. \end{aligned}$$

□

A generalization of theorem 65 in [1]:

Theorem 6. Let $(\mathfrak{A};\mathfrak{Z})$ be a semifiltered down-aligned filtrator with finitely meet-closed core $\mathfrak{Z}$ which is an atomistic lattice and $\mathfrak{A}$ is a starrish join-semilattice, then $\operatorname{Cor}'(a\cup^{\mathfrak{A}} b)=\operatorname{Cor}' a\cup^{\mathfrak{Z}}\operatorname{Cor}' b$ for every $a,b\in\mathfrak{A}$.

Proof. $\operatorname{Cor}'(a\cup^{\mathfrak{A}} b)=\bigcup^{\mathfrak{Z}}\{x\mid x\text{ is an atom of }\mathfrak{Z}, x\subseteq a\cup^{\mathfrak{A}} b\}$ (use proposition 34 from [1]),

By the theorem 50 from [1] we have $\operatorname{Cor}'(a\cup^{\mathfrak{A}} b)=\bigcup^{\mathfrak{Z}}(\operatorname{atoms}^{\mathfrak{A}}(a\cup^{\mathfrak{A}} b)\cap\mathfrak{Z})=\bigcup^{\mathfrak{Z}}((\operatorname{atoms}^{\mathfrak{A}} a\cup\operatorname{atoms} b)\cap\mathfrak{Z})=\bigcup^{\mathfrak{Z}}((\operatorname{atoms}^{\mathfrak{A}} a\cap\mathfrak{Z})\cup(\operatorname{atoms}^{\mathfrak{A}} b\cap\mathfrak{Z}))=\bigcup^{\mathfrak{Z}}(\operatorname{atoms}^{\mathfrak{A}} a\cap\mathfrak{Z})\cup^{\mathfrak{Z}}\bigcup^{\mathfrak{Z}}(\operatorname{atoms}^{\mathfrak{A}} b\cap\mathfrak{Z})$ (used the theorem 3). Again using the theorem 50 from [1], we get

$\operatorname{Cor}'(a\cup^{\mathfrak{A}} b)=\bigcup^{\mathfrak{Z}}\{x\mid x\text{ is an atom of }\mathfrak{Z}, x\subseteq a\}\cup^{\mathfrak{Z}}\bigcup^{\mathfrak{Z}}\{x\mid x\text{ is an atom of }\mathfrak{Z}, x\subseteq b\}=\operatorname{Cor}' a\cup^{\mathfrak{Z}}\operatorname{Cor}' b$ (again used the proposition 34 from [1]). □

Bibliography

[1] Victor Porton. Filters on posets and generalizations. *International Journal of Pure and Applied Mathematics*, 74(1):55–119, 2012.