

Filters on posets and generalizations

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Abstract

Studied in details properties of filters on lattices, filters on posets, and certain generalizations thereof. Also done some more general lattice theory research. Posed several open problems. Detailed study of filters is required for my ongoing research which will be published as "Algebraic General Topology" series.

Keywords: filters, ideals, lattice of filters, pseudodifference, pseudocomplement

A.M.S. subject classification: 54A20, 06A06, 06B99

Contents

1	Preface	3
2	Notation and basic results	3
2.1	Intersecting and joining elements	3
2.2	Atoms of a poset	4
2.3	Difference and complement	5
2.4	Center of a lattice	7
2.5	Galois connections	8
2.6	Co-Brouwerian Lattices	11
3	Straight maps and separation subsets	14
3.1	Straight maps	14
3.2	Separation subsets and full stars	16
3.3	Atomically separable lattices	17
4	Filtrators	19
4.1	Core part	20
4.2	Filtrators with separable core	21
4.3	Intersecting and joining with an element of the core	22

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5	Filters	23
5.1	Filters on posets	23
5.2	Filters on meet-semilattice	23
5.3	Characterization of finitely meet-closed filtrators	24
6	Filter objects	25
6.1	Definition of filter objects	25
6.2	Order of filter objects	25
7	Lattice of filter objects	26
7.1	Minimal and maximal f.o.	26
7.2	Primary filtrator is filtered	26
7.3	Formulas for meets and joins of filter objects	26
7.4	Distributivity of the lattice of filter objects	28
7.5	Separability of core for primary filtrators	29
7.6	Filters over boolean lattices	30
8	Generalized filter base	30
9	Stars	31
9.1	Free stars	31
9.2	Stars of elements of filtrators	33
9.3	Stars of filters on boolean lattices	33
9.4	More about the lattice of filters	36
10	Atomic filter objects	37
10.1	Prime filtrator elements	38
11	Some criteria	39
12	Quasidifference and quasicomplement	41
13	Complements and core parts	44
13.1	Core part and atomic elements	46
14	Number of filters on a set	46
15	Isomorphic filters	47
16	Partitioning filter objects	50
17	Open problems	50
17.1	Partitioning	51
17.2	Quasidifference	52
17.3	Complementive filter objects	53
17.4	Misc problems	53
17.5	Non-formal problems	53

18 Postface	54
Appendix A Some counter-examples	54
Appendix B Logic of Generalizations	55
Appendix B.1 The formalistic	55

1. Preface

This article is intended to collect in one document the known properties of filters on posets (and some generalizations thereof, namely “filtrators” defined below).

It seems that until now were published no reference on the theory of filters. This text is to fill the gap.

This text will also serve as the reference base for my further articles. It is too hard to find theorems about filters in the literature and even to check if a particular theorem is proved somewhere in the literature. This text provides a definitive place to refer as to the collection of theorems about filters.

Detailed study of filters is required for my ongoing research which will be published as ”Algebraic General Topology” series.

This text contains well known theorems as well as some seemingly new results. Although it is hard to differentiate which of the results are new. Also here are several open problems.

In place of studying filters in this article are instead researched what the author calls “filter objects”. Filter objects are basically the lattice of filters ordered reverse to set inclusion, with principal filters equated with the poset element which generates them. (See below for formal definition of “filter objects”.)

Although our primary interest are properties of filters on a set, in this work are instead researched the more general theory of “filtrators” (see below).

2. Notation and basic results

We denote $\mathcal{P}S$ the set of all subsets of a set S .

$\langle f \rangle X \stackrel{\text{def}}{=} \{fx \mid x \in X\}$ for any set X and function f .

2.1. Intersecting and joining elements

Let \mathfrak{A} is a poset with least element 0.

Definition 1 I will call elements a and b **intersecting** and denote $a \not\asymp b$ when

$$\exists c \in \mathfrak{A} : (c \neq 0 \wedge c \subseteq a \wedge c \subseteq b).$$

Definition 2 $a \asymp b \stackrel{\text{def}}{=} \neg(a \not\asymp b)$.

Obvious 1 If \mathfrak{A} is a meet-semilattice with least element 0 then $a \not\asymp b \Leftrightarrow a \cap b \neq 0$.

Obvious 2 $a_0 \not\subseteq b_0 \wedge a_1 \supseteq a_0 \wedge b_1 \supseteq b_0 \Rightarrow a_1 \not\subseteq b_1$.

Let now \mathfrak{A} is a poset with greatest element 1.

Definition 3 I will call elements a and b **joining** and denote $a \equiv b$ when

$$\nexists c \in \mathfrak{A} : (c \neq 1 \wedge c \supseteq a \wedge c \supseteq b).$$

Definition 4 $a \not\equiv b \stackrel{\text{def}}{=} \neg(a \equiv b)$.

Obvious 3 *Intersecting is the dual of non-joining.*

Obvious 4 *If \mathfrak{A} is a join-semilattice with greatest element 1 then $a \equiv b \Leftrightarrow a \cup b = 1$.*

Obvious 5 $a_0 \equiv b_0 \wedge a_1 \supseteq a_0 \wedge b_1 \supseteq b_0 \Rightarrow a_1 \equiv b_1$.

2.2. Atoms of a poset

I will denote $(\text{atoms}^{\mathfrak{A}} a)$ or just $(\text{atoms } a)$ the set of atoms contained in element a of a poset \mathfrak{A} with least element.

Definition 5 A poset \mathfrak{A} with least element is called **atomic** when $\forall a \in \mathfrak{A} \setminus \{0\} : \text{atoms } a \neq \emptyset$.

Definition 6 **Atomistic lattice** is such lattice that $a = \bigcup \text{atoms } a$ for every element a of this lattice.

Proposition 1 *Let \mathfrak{A} is a poset with least element. If a is an atom of \mathfrak{A} and $B \in \mathfrak{A}$ then $a \subseteq B \Leftrightarrow a \not\subseteq B$.*

Proof

$\Rightarrow a \subseteq B \Rightarrow a \subseteq a \wedge a \subseteq B$, thus $a \not\subseteq B$ because $a \neq 0$.

$\Leftarrow a \not\subseteq B$ implies existence of $x \neq 0$ such that $x \subseteq B$ and $x \subseteq a$. Because a is an atom, we have $x = a$. So $a \subseteq B$.

□

Theorem 1 *If \mathfrak{A} is a distributive lattice with least element then*

$$\text{atoms}(a \cup b) = \text{atoms } a \cup \text{atoms } b$$

for every $a, b \in \mathfrak{A}$.

Proof For any atomic element c

$$\begin{aligned}
c \in \text{atoms}(a \cup b) &\Leftrightarrow \\
c \cap (a \cup b) \neq 0 &\Leftrightarrow \\
(c \cap a) \cup (c \cap b) \neq 0 &\Leftrightarrow \\
c \cap a \neq 0 \vee c \cap b \neq 0 &\Leftrightarrow \\
c \in \text{atoms } a \vee c \in \text{atoms } b. &
\end{aligned}$$

□

Theorem 2 $\text{atoms} \bigcap S = \bigcap \langle \text{atoms} \rangle S$ whenever $\bigcap S$ is defined for every $S \in \mathcal{P}\mathfrak{A}$ where \mathfrak{A} is a poset with least element.

Proof For any atom c

$$\begin{aligned}
c \in \text{atoms} \bigcap S &\Leftrightarrow \\
c \subseteq \bigcap S &\Leftrightarrow \\
\forall a \in S : c \subseteq a &\Leftrightarrow \\
\forall a \in S : c \in \text{atoms } a &\Leftrightarrow \\
c \in \bigcap \langle \text{atoms} \rangle S. &
\end{aligned}$$

□

Corollary 1 $\text{atoms}(a \cap b) = \text{atoms } a \cap \text{atoms } b$ for arbitrary meet-semilattice with least element.

Theorem 3 A complete boolean lattice is atomic iff it is atomistic.

Proof

⇐ Obvious.

⇒ Let \mathfrak{A} is an atomic boolean lattice. Let $a \in \mathfrak{A}$. Suppose $b = \bigcup \text{atoms } a \subset a$.
If $x \in \text{atoms}(a \setminus b)$ then $x \subseteq a \setminus b$ and so $x \subseteq a$ and hence $x \subseteq b$. But we have $x = x \cap b \subseteq (a \setminus b) \cap b = 0$ what contradicts to our supposition.

□

2.3. Difference and complement

Definition 7 Let \mathfrak{A} is a distributive lattice with least element 0. The **difference** (denoted $a \setminus b$) of elements a and b is such $c \in \mathfrak{A}$ that $b \cap c = 0$ and $a \cup b = b \cup c$. I will call b **subtractive** from a when $a \setminus b$ exists.

Theorem 4 *If \mathfrak{A} is a distributive lattice with least element 0, there exists no more than one difference of elements $a, b \in \mathfrak{A}$.*

Proof Let c and d are both differences $a \setminus b$. Then $b \cap c = b \cap d = 0$ and $a \cup b = b \cup c = b \cup d$. So

$$c = c \cap (b \cup c) = c \cap (b \cup d) = (c \cap b) \cup (c \cap d) = 0 \cup (c \cap d) = c \cap d.$$

Analogously, $d = d \cap c$. Consequently $c = c \cap d = d \cap c = d$. □

Definition 8 *I will call b **complementive** to a when there exists $c \in \mathfrak{A}$ such that $b \cap c = 0$ and $b \cup c = a$.*

Proposition 2 *b is complementive to a iff b is subtractive from a and $b \subseteq a$.*

Proof

\Leftarrow Obvious.

\Rightarrow We deduce $b \subseteq a$ from $b \cup c = a$. Thus $a \cup b = a = b \cup c$.

□

Proposition 3 *If b is complementive to a then $(a \setminus b) \cup b = a$.*

Proof Because $b \subseteq a$ by the previous proposition. □

Definition 9 *Let \mathfrak{A} is a bounded distributive lattice. The **complement** (denoted \bar{a}) of element $a \in \mathfrak{A}$ is such $b \in \mathfrak{A}$ that $a \cap b = 0$ and $a \cup b = 1$.*

Proposition 4 *If \mathfrak{A} is a bounded distributive lattice then $\bar{\bar{a}} = 1 \setminus a$.*

Proof $b = \bar{a} \Leftrightarrow b \cap a = 0 \wedge b \cup a = 1 \Leftrightarrow b \cap a = 0 \wedge 1 \cup a = a \cup b \Leftrightarrow b = 1 \setminus a$. □

Corollary 2 *If \mathfrak{A} is a bounded distributive lattice then exists no more than one complement of an element $a \in \mathfrak{A}$.*

Definition 10 *An element of bounded distributive lattice is called **complementmented** when its complement exists.*

Definition 11 *A distributive lattice is a **complementmented lattice** iff every its element is complementmented.*

Proposition 5 *For a distributive lattice $(a \setminus b) \setminus c = a \setminus (b \cup c)$ if $a \setminus b$ and $(a \setminus b) \setminus c$ are defined.*

Proof $((a \setminus b) \setminus c) \cap c = 0$; $((a \setminus b) \setminus c) \cup c = (a \setminus b) \cup c$; $(a \setminus b) \cap b = 0$;
 $(a \setminus b) \cup b = a \cup b$.

We need to prove $((a \setminus b) \setminus c) \cap (b \cup c) = 0$ and $((a \setminus b) \setminus c) \cup (b \cup c) = a \cup (b \cup c)$.
 In fact,

$$\begin{aligned}
 & ((a \setminus b) \setminus c) \cap (b \cup c) = \\
 & (((a \setminus b) \setminus c) \cap b) \cup (((a \setminus b) \setminus c) \cap c) = \\
 & (((a \setminus b) \setminus c) \cap b) \cup 0 = \\
 & ((a \setminus b) \setminus c) \cap b \subseteq \\
 & (a \setminus b) \cap b = 0,
 \end{aligned}$$

so $((a \setminus b) \setminus c) \cap (b \cup c) = 0$;

$$\begin{aligned}
 & ((a \setminus b) \setminus c) \cup (b \cup c) = \\
 & (((a \setminus b) \setminus c) \cup c) \cup b = \\
 & (a \setminus b) \cup c \cup b = \\
 & ((a \setminus b) \cup b) \cup c = \\
 & a \cup b \cup c.
 \end{aligned}$$

□

2.4. Center of a lattice

Definition 12 The **center** $Z(\mathfrak{A})$ of a bounded distributive lattice \mathfrak{A} is the set of its complemented elements.

Remark 1 For definition of center of non-distributive lattices see [3].

Remark 2 In [9] the word center and the notation $Z(\mathfrak{A})$ is used in a different sense.

Definition 13 A complete lattice \mathfrak{A} is **join infinite distributive** when $x \cap \bigcup S = \bigcup \langle x \cap \rangle S$; complete lattice is **meet infinite distributive** when $x \cup \bigcap S = \bigcap \langle x \cup \rangle S$ for all $x \in \mathfrak{A}$ and $S \in \mathcal{P}\mathfrak{A}$.

Definition 14 **Infinitely distributive complete lattice** is a complete lattice which is both join infinite distributive and meet infinite distributive.

Definition 15 A sublattice K of a complete lattice L is a closed sublattice of L if K contains the meet and the join of any its nonempty subset.

Theorem 5 Center of a infinitely distributive lattice is its closed sublattice.

Proof See [6].

□

Remark 3 See [7] for a more strong result.

Theorem 6 *The center of a bounded distributive lattice constitutes its sublattice.*

Proof Let \mathfrak{A} is a bounded distributive lattice and $Z(\mathfrak{A})$ is its center. Let $a, b \in Z(\mathfrak{A})$. Consequently $\bar{a}, \bar{b} \in Z(\mathfrak{A})$. Then $\bar{a} \cup \bar{b}$ is the complement of $a \cap b$ because

$$\begin{aligned}(a \cap b) \cap (\bar{a} \cup \bar{b}) &= (a \cap b \cap \bar{a}) \cup (a \cap b \cap \bar{b}) = 0 \cup 0 = 0 \quad \text{and} \\ (a \cap b) \cup (\bar{a} \cup \bar{b}) &= (a \cup \bar{a} \cup \bar{b}) \cap (b \cup \bar{a} \cup \bar{b}) = 1 \cap 1 = 1.\end{aligned}$$

So $a \cap b$ is complemented, analogously $a \cup b$ is complemented. \square

Theorem 7 *The center of a bounded distributive lattice constitutes a boolean lattice.*

Proof Because it is a distributive complemented lattice. \square

2.5. Galois connections

See [1] and [5] for more detailed treatment of Galois connections.

Definition 16 *Let \mathfrak{A} and \mathfrak{B} are two posets. A **Galois connection** between \mathfrak{A} and \mathfrak{B} is a pair of functions $f = (f^*; f_*)$ with $f^* : \mathfrak{A} \rightarrow \mathfrak{B}$ and $f_* : \mathfrak{B} \rightarrow \mathfrak{A}$ such that:*

$$\forall x \in \mathfrak{A}, y \in \mathfrak{B} : (f^*x \subseteq^{\mathfrak{B}} y \Leftrightarrow x \subseteq^{\mathfrak{A}} f_*y).$$

f_* is called **upper adjoint** of f^* and f^* is called **lower adjoint** of f_* .

Theorem 8 *A pair $(f^*; f_*)$ of functions $f^* : \mathfrak{A} \rightarrow \mathfrak{B}$ and $f_* : \mathfrak{B} \rightarrow \mathfrak{A}$ is a Galois connection iff both of the following:*

1. f^* and f_* are monotone.
2. $x \subseteq^{\mathfrak{A}} f_*f^*x$ and $f^*f_*y \subseteq^{\mathfrak{B}} y$ for every $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$.

Proof

$$\Rightarrow 2. \quad x \subseteq^{\mathfrak{A}} f_*f^*x \text{ since } f^*x \subseteq^{\mathfrak{B}} f^*x; \quad f^*f_*y \subseteq^{\mathfrak{B}} y \text{ since } f_*y \subseteq^{\mathfrak{A}} f_*y.$$

1. Let $a, b \in \mathfrak{A}$ and $a \subseteq^{\mathfrak{A}} b$. Then $a \subseteq^{\mathfrak{A}} b \subseteq^{\mathfrak{A}} f_*f^*b$. So by definition $f^*a \subseteq f^*b$ that is f^* is monotone. Analogously f_* is monotone.

$$\Leftarrow f^*x \subseteq^{\mathfrak{B}} y \Rightarrow f_*f^*x \subseteq^{\mathfrak{A}} f_*y \Rightarrow x \subseteq^{\mathfrak{A}} f_*y. \text{ The other direction is analogous.}$$

\square

Theorem 9

$$1. f^* \circ f_* \circ f^* = f^*.$$

$$2. f_* \circ f^* \circ f_* = f_*.$$

Proof

1. Let $x \in \mathfrak{A}$. We have $x \subseteq^{\mathfrak{A}} f_* f^* x$; consequently $f^* x \subseteq^{\mathfrak{B}} f_* f_* f^* x$. On the other hand, $f_* f_* f^* x \subseteq^{\mathfrak{B}} f^* x$. So $f_* f_* f^* x = f^* x$.

2. Analogously. □

Proposition 6 $f^* \circ f_*$ and $f_* \circ f^*$ are idempotent.

Proof $f^* \circ f_*$ is idempotent because $f^* f_* f^* f_* y = f^* f_* y$. $f_* \circ f^*$ is similar. □

Theorem 10 Each of two adjoints is uniquely determined by the other.

Proof Let p and q are both upper adjoints of f^* . We have for all $x \in \mathfrak{A}$ and $y \in \mathfrak{B}$:

$$x \subseteq p(y) \Leftrightarrow f(x) \subseteq y \Leftrightarrow x \subseteq q(y).$$

For $x = p(y)$ we obtain $p(y) \subseteq q(y)$ and for $x = q(y)$ we obtain $q(y) \subseteq p(y)$. So $p(y) = q(y)$. □

Theorem 11 Let f is a monotone function from \mathfrak{A} to \mathfrak{B} .

1. Both:

1. If $g(b) = \max \{x \in \mathfrak{A} \mid fx \subseteq b\}$ is defined for every $b \in \mathfrak{B}$ then g is the upper adjoint of f .

2. If g is the upper adjoint of f then $g(b) = \max \{x \in \mathfrak{A} \mid fx \subseteq b\}$ for every $b \in \mathfrak{B}$.

2. Both:

1. If $g(b) = \min \{x \in \mathfrak{A} \mid fx \supseteq b\}$ is defined for every $b \in \mathfrak{B}$ then g is the lower adjoint of f .

2. If g is the lower adjoint of f then $g(b) = \min \{x \in \mathfrak{A} \mid fx \supseteq b\}$ for every $b \in \mathfrak{B}$.

Proof We will prove only the first as the second is its dual.

1. Let $g(b) = \max \{x \in \mathfrak{A} \mid fx \subseteq b\}$ for every $b \in \mathfrak{B}$. Then

$$x \subseteq gy \Leftrightarrow x \subseteq \max \{x \in \mathfrak{A} \mid fx \subseteq y\} \Rightarrow fx \subseteq y$$

(because f is monotone) and

$$x \subseteq gy \Leftrightarrow x \subseteq \max \{x \in \mathfrak{A} \mid fx \subseteq y\} \Leftarrow fx \subseteq y.$$

So $fx \subseteq y \Leftrightarrow x \subseteq gy$ that is f is the lower adjoint of g .

2. We have

$$\begin{aligned} g(b) = \max \{x \in \mathfrak{A} \mid fx \subseteq b\} &\Leftrightarrow \\ fgb \subseteq b \wedge \forall x \in \mathfrak{A} : (fx \subseteq b \Rightarrow fgb \subseteq b) &\Leftarrow \\ fgb \subseteq b & \end{aligned}$$

what is true by properties of adjoints.

□

Theorem 12 *Let f is a function from a poset \mathfrak{A} to a poset \mathfrak{B} .*

1. *If f is an upper adjoint, f preserves all existing infima in \mathfrak{A} .*
2. *If \mathfrak{A} is a complete lattice and f preserves all infima, then f is an upper adjoint.*
3. *If f is a lower adjoint, f preserves all existing suprema in \mathfrak{A} .*
4. *If \mathfrak{A} is a complete lattice and f preserves all suprema, then f is a lower adjoint.*

Proof We will prove only first two items because the rest items are similar.

1. Let $S \in \mathcal{P}\mathfrak{A}$ and $\bigcap S$ exists. $f \bigcap S$ is a lower bound for $\langle f \rangle S$ because f is order-preserving. If a is a lower bound for $\langle f \rangle S$ then $\forall x \in S : a \subseteq fx$ that is $\forall x \in S : x \subseteq ga$ where g is the lower adjoint of f . Thus $\bigcap S \subseteq ga$ and hence $f \bigcap S \subseteq a$. So $f \bigcap S$ is the greatest lower bound for $\langle f \rangle S$.
2. Let \mathfrak{A} is a complete lattice and f preserves all infima. Let $g(a) = \bigcap \{x \in \mathfrak{A} \mid fx \supseteq a\}$. Since f preserves infima, we have

$$f(g(a)) = \bigcap \{f(x) \mid x \in \mathfrak{A}, f(x) \supseteq a\} \supseteq a.$$

$$g(f(b)) = \bigcap \{x \in \mathfrak{A} \mid fx \supseteq fb\} \subseteq b.$$

So f is the upper adjoint of g .

□

Corollary 3 *Let f is a function from a complete lattice \mathfrak{A} to a poset \mathfrak{B} . Then:*

1. *f is an upper adjoint iff f preserves all infima in \mathfrak{A} .*
2. *f is a lower adjoint iff f preserves all suprema in \mathfrak{A} .*

2.6. Co-Brouwerian Lattices

Definition 17 Let \mathfrak{A} is a poset with least element. Let $a \in \mathfrak{A}$. **Pseudocomplement** of a is

$$\max \{c \in \mathfrak{A} \mid c \asymp a\}.$$

If z is pseudocomplement of a we will denote $z = a^*$.

Definition 18 Let \mathfrak{A} is a poset with greatest element. Let $a \in \mathfrak{A}$. **Dual pseudocomplement** of a is

$$\min \{c \in \mathfrak{A} \mid c \equiv a\}.$$

If z is dual pseudocomplement of a we will denote $z = a^+$.

Definition 19 Let \mathfrak{A} is a join-semilattice. Let $a, b \in \mathfrak{A}$. **Pseudodifference** of a and b is

$$\min \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}.$$

If z is a pseudodifference of a and b we will denote $z = a \setminus^* b$.

Remark 4 I do not require that a^* is undefined if there are no pseudocomplement of a and likewise for dual pseudocomplement and pseudodifference. In fact below I will define quasicomplement, dual quasicomplment, and quasidifference which will generalize pseudo-* counterparts. I will denote a^* the more general case of quasicomplement than of pseudocomplement, and likewise for other notation.

Obvious 6 Dual pseudocomplement is the dual of pseudocomplement.

Definition 20 **Co-brouwerian lattice** is a lattice for which is defined pseudodifference of any two its elements.

Proposition 7 Every co-brouwerian lattice \mathfrak{A} has least element.

Proof Let a is an arbitrary lattice element. Then $a \setminus^* a = \min \{z \in \mathfrak{A} \mid a \subseteq a \cup z\} = \min \mathfrak{A}$. So $\min \mathfrak{A}$ exists. \square

Definition 21 **Co-Heyting lattice** is co-brouwerian lattice with greatest element.

Theorem 13 For a co-brouwerian lattice $a \cup -$ is an upper adjoint of $- \setminus^* a$ for every $a \in \mathfrak{A}$.

Proof $g(b) = \min \{x \in \mathfrak{A} \mid a \cup x \supseteq b\} = b \setminus^* a$ exists for every $b \in \mathfrak{A}$ and thus is the lower adjoint of $a \cup -$. \square

Corollary 4 $\forall a, x, y \in \mathfrak{A} : (x \setminus^* a \subseteq y \Leftrightarrow x \subseteq a \cup y)$ for a co-brouwerian lattice.

Definition 22 Let $a, b \in \mathfrak{A}$ where \mathfrak{A} is a complete lattice. Quasidifference $a \setminus^* b$ is defined by the formula

$$a \setminus^* b = \bigcap \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}.$$

Remark 5 The more detailed theory of quasidifference (as well as quasicomplement and dual quasicomplement) will be considered below.

Lemma 1 $(a \setminus^* b) \cup b = a \cup b$ for elements a, b of a meet infinite distributive complete lattice.

Proof

$$\begin{aligned} (a \setminus^* b) \cup b &= \\ \bigcap \{z \in \mathfrak{A} \mid a \subseteq b \cup z\} \cup b &= \\ \bigcap \{z \cup b \mid z \in \mathfrak{A}, a \subseteq b \cup z\} &= \\ \bigcap \{t \in \mathfrak{A} \mid t \supseteq b, a \subseteq t\} &= \\ a \cup b. & \end{aligned}$$

□

Theorem 14 The following are equivalent for a complete lattice \mathfrak{A} :

1. \mathfrak{A} is meet infinite distributive.
2. \mathfrak{A} is a co-brouwerian lattice.
3. \mathfrak{A} is a co-Heyting lattice.
4. $a \cup -$ has lower adjoint for every $a \in \mathfrak{A}$.

Proof

(2) \Leftrightarrow (3) Obvious (taking in account completeness of \mathfrak{A}).

(4) \Rightarrow (1) Let $- \setminus^* a$ is the upper adjoint of $a \cup -$. Let $S \in \mathcal{P}\mathfrak{A}$. For every $y \in S$ we have $y \supseteq (a \cup y) \setminus^* a$ by properties of Galois connections; consequently $y \supseteq (\bigcap \langle a \cup \rangle S) \setminus^* a$; $\bigcap S \supseteq (\bigcap \langle a \cup \rangle S) \setminus^* a$. So

$$a \cup \bigcap S \supseteq ((\bigcap \langle a \cup \rangle S) \setminus^* a) \cup a \supseteq \bigcap \langle a \cup \rangle S.$$

But $a \cup \bigcap S \subseteq \bigcap \langle a \cup \rangle S$ is obvious.

(1) \Rightarrow (2) Let $a \setminus^* b = \bigcap \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}$. To prove that \mathfrak{A} is a co-brouwerian lattice is enough to prove that $a \subseteq b \cup (a \setminus^* b)$. But it follows from the lemma.

(2) \Rightarrow (4) $a \setminus^* b = \min \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}$. So $a \cup -$ is an upper adjoint of $-\setminus^* a$.

(1) \Rightarrow (4) Because $a \cup -$ preserves all meets.

□

Corollary 5 *Co-brouwerian lattices are distributive.*

The following theorem is essentially borrowed from [8]:

Theorem 15 *A lattice \mathfrak{A} with least element 0 is co-brouwerian with pseudodifference \setminus^* iff \setminus^* is a binary operation on \mathfrak{A} satisfying the following identities:*

1. $a \setminus^* a = 0$;
2. $a \cup (b \setminus^* a) = a \cup b$;
3. $b \cup (b \setminus^* a) = b$;
4. $(b \cup c) \setminus^* a = (b \setminus^* a) \cup (c \setminus^* a)$.

Proof

\Leftarrow We have

$$c \supseteq b \setminus^* a \Rightarrow c \cup a \supseteq a \cup (b \setminus^* a) = a \cup b \supseteq b;$$

$$c \cup a \supseteq b \Rightarrow c = c \cup (c \setminus^* a) \supseteq (a \setminus^* a) \cup (c \setminus^* a) = (a \cup c) \setminus^* a \supseteq b \setminus^* a.$$

So $c \supseteq b \setminus^* a \Leftrightarrow c \cup a \supseteq b$ that is $a \cup -$ is an upper adjoint of $-\setminus^* a$. By a theorem above our lattice is co-brouwerian. By an other theorem above \setminus^* is a pseudodifference.

\Rightarrow 1. Obvious.

2.

$$\begin{aligned} a \cup (b \setminus^* a) &= \\ a \cup \bigcap \{z \in \mathfrak{A} \mid b \subseteq a \cup z\} &= \\ \bigcap \{a \cup z \mid z \in \mathfrak{A}, b \subseteq a \cup z\} &= \\ a \cup b. & \end{aligned}$$

$$3. \quad b \cup (b \setminus^* a) = b \cup \bigcap \{z \in \mathfrak{A} \mid b \subseteq a \cup z\} = \bigcap \{b \cup z \mid z \in \mathfrak{A}, b \subseteq a \cup z\} = b.$$

4. Obviously $(b \cup c) \setminus^* a \supseteq b \setminus^* a$ and $(b \cup c) \setminus^* a \supseteq c \setminus^* a$, thus $(b \cup c) \setminus^* a \supseteq (b \setminus^* a) \cup (c \setminus^* a)$. We have

$$\begin{aligned}
(b \setminus^* a) \cup (c \setminus^* a) \cup a &= \\
((b \setminus^* a) \cup a) \cup ((c \setminus^* a) \cup a) &= \\
(b \cup a) \cup (c \cup a) &= \\
a \cup b \cup c &\supseteq \\
b \cup c. &
\end{aligned}$$

From this by the definition of adjoints: $(b \setminus^* a) \cup (c \setminus^* a) \supseteq (b \cup c) \setminus^* a$.

□

Theorem 16 $(\bigcup S) \setminus^* a = \bigcup \{x \setminus^* a \mid x \in S\}$ for $a \in \mathfrak{A}$ and $S \in \mathcal{P}\mathfrak{A}$ where \mathfrak{A} is a complete co-brouwerian lattice.

Proof Because lower adjoint preserves all suprema. □

Theorem 17 $(a \setminus^* b) \setminus^* c = a \setminus^* (b \cup c)$ for elements a, b, c of a complete co-brouwerian lattice.

Proof $a \setminus^* b = \bigcap \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}$.

$$(a \setminus^* b) \setminus^* c = \{z \in \mathfrak{A} \mid a \setminus^* b \subseteq c \cup z\}.$$

$$a \setminus^* (b \cup c) = \bigcap \{z \in \mathfrak{A} \mid a \subseteq b \cup c \cup z\}.$$

Left to prove $a \setminus^* b \subseteq c \cup z \Leftrightarrow a \subseteq b \cup c \cup z$.

Let $a \setminus^* b \subseteq c \cup z$. Then $a \cup b \subseteq b \cup c \cup z$ by the lemma and consequently $a \subseteq b \cup c \cup z$.

Let $a \subseteq b \cup c \cup z$. Then $a \setminus^* b \subseteq (b \cup c \cup z) \setminus^* b \subseteq c \cup z$ by a theorem above. □

3. Straight maps and separation subsets

3.1. Straight maps

Definition 23 Let f is a monotone map from a meet-semilattice \mathfrak{A} to some poset \mathfrak{B} . I call f a **straight** map when

$$\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow fa = f(a \cap b)).$$

Proposition 8 The following statements are equivalent for a monotone map f :

1. f is a straight map.
2. $\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow fa \subseteq f(a \cap b))$.
3. $\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow fa \not\supseteq f(a \cap b))$.

4. $\forall a, b \in \mathfrak{A} : (fa \supset f(a \cap b) \Rightarrow fa \not\subseteq fb)$.

Proof

(1) \Leftrightarrow (2) \Leftrightarrow (3) Due $fa \supseteq f(a \cap b)$.

(3) \Leftrightarrow (4) Obvious. □

Remark 6 The definition of straight map can be generalized for any poset \mathfrak{A} by the formula

$$\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow \exists c \in \mathfrak{A} : (c \subseteq a \wedge c \subseteq b \wedge fa = fc)).$$

This generalization is not yet researched however.

Proposition 9 Let f is a monotone map from a meet-semilattice \mathfrak{A} to some poset \mathfrak{B} . If

$$\forall a, b \in \mathfrak{A} : (f(a \cap b) = fa \cap fb)$$

then f is a straight map.

Proof Let $fa \subseteq fb$. Then $f(a \cap b) = fa \cap fb = fa$. □

Proposition 10 Let f is a monotone map from a meet-semilattice \mathfrak{A} to some poset \mathfrak{B} . If

$$\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow a \subseteq b)$$

then f is a straight map.

Proof $fa \subseteq fb \Rightarrow a \subseteq b \Rightarrow a = a \cap b \Rightarrow fa = f(a \cap b)$. □

Theorem 18 If f is a straight monotone map from a meet-semilattice \mathfrak{A} then the following statements are equivalent:

1. f is an injection.
2. $\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow a \subseteq b)$.
3. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow fa \subset fb)$.
4. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow fa \neq fb)$.
5. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow fa \not\supseteq fb)$.
6. $\forall a, b \in \mathfrak{A} : (fa \subseteq fb \Rightarrow a \not\supset b)$.

Proof

- (1) \Rightarrow (3) Let $a, b \in \mathfrak{A}$. Let $fa = fb \Rightarrow a = b$. Let $a \subset b$. $fa \neq fb$ because $a \neq b$. $fa \subseteq fb$ because $a \subseteq b$. So $fa \subset fb$.
- (2) \Rightarrow (1) Let $a, b \in \mathfrak{A}$. Let $fa \subseteq fb \Rightarrow a \subseteq b$. Let $fa = fb$. Then $a \subseteq b \wedge b \subseteq a$ and consequently $a = b$.
- (3) \Rightarrow (2) Let $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow fa \subset fb)$. Let $a \not\subseteq b$. Then $a \supset a \cap b$. So $fa \supset f(a \cap b)$. If $fa \subseteq fb$ then $fa \subseteq f(a \cap b)$ what is a contradiction.
- (3) \Rightarrow (5) \Rightarrow (4) Obvious.
- (4) \Rightarrow (3) Because $a \subset b \Rightarrow a \subseteq b \Rightarrow fa \subseteq fb$.
- (5) \Leftrightarrow (6) Obvious.

□

3.2. Separation subsets and full stars

Definition 24 $\partial_Y a = \{x \in Y \mid x \not\asymp a\}$ for an element a of a poset \mathfrak{A} with least element and $Y \in \mathcal{P}\mathfrak{A}$.

Definition 25 *Full star* of a is $\star a = \partial_{\mathfrak{A}} a$.

Proposition 11 \star is a straight monotone map.

Proof Monotonicity is obvious. Let $\star a \not\subseteq \star(a \cap b)$. Then exists $x \in \star a$ such that $x \notin \star(a \cap b)$. So $x \cap a \notin \star b$ but $x \cap a \in \star a$ and consequently $\star a \not\subseteq \star b$. □

Definition 26 A *separation subset* of a meet-semilattice \mathfrak{A} with least element is such its subset Y that

$$\forall a, b \in \mathfrak{A} : (\partial_Y a = \partial_Y b \Rightarrow a = b).$$

Definition 27 I call *separable* such meet-semilattice with least element that \star is an injection.

Obvious 7 A meet-semilattice with least element is separable iff it has separation subset.

Definition 28 A poset \mathfrak{A} with least element has *disjunction property of Wallman* iff for any $a, b \in \mathfrak{A}$ either $b \subseteq a$ or there exists an element $c \subseteq b$ such that $c \neq 0$ and $a \asymp c$.

Theorem 19 For a meet-semilattice with least element the following statements are equivalent:

1. \mathfrak{A} is separable.

2. $\forall a, b \in \mathfrak{A} : (\star a \subseteq \star b \Rightarrow a \subseteq b)$.
3. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \star a \subset \star b)$.
4. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \star a \neq \star b)$.
5. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \star a \not\subseteq \star b)$.
6. $\forall a, b \in \mathfrak{A} : (\star a \subseteq \star b \Rightarrow a \not\subseteq b)$.
7. \mathfrak{A} conforms to Wallman's disjunction property.
8. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{0\} : (c \succ a \wedge c \subseteq b))$.

Proof

(1) \Leftrightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Leftrightarrow (5) \Leftrightarrow (6) By the above theorem.

(8) \Rightarrow (4) Let disjunction property of Wallman holds. Let $a \subset b$. Then exists element $c \subseteq b$ such that $c \neq 0$ and $c \cap a = 0$. But $c \cap b \neq 0$. So $\star a \neq \star b$.

(2) \Rightarrow (7) Let \mathfrak{A} is separable. Let $a \not\subseteq b$. Then $\star a \not\subseteq \star b$ that is exists $c \in \star a$ such that $c \notin \star b$, in other words $c \cap a \neq 0$ and $c \cap b = 0$. Let $d = c \cap a$. Then $d \subseteq a$ and $d \neq 0$ and $d \cap b = 0$. So disjunction property of Wallman holds.

(7) \Rightarrow (8) Obvious.

(8) \Rightarrow (7) Let $b \not\subseteq a$. Then $a \cap b \subset b$ that is $a' \subset b$ where $a' = a \cap b$. Consequently $\exists c \in \mathfrak{A} \setminus \{0\} : (c \succ a' \wedge c \subseteq b)$. We have $c \cap a = c \cap b \cap a = c \cap a'$. So $c \subseteq b$ and $c \cap a = 0$. Thus Wallman's disjunction property holds.

□

3.3. Atomically separable lattices

Proposition 12 “atoms” is a straight monotone map (for any meet-semilattice with least element).

Proof Monotonicity is obvious. The rest follows from the formula

$$\text{atoms}(a \cap b) = \text{atoms } a \cap \text{atoms } b$$

(the corollary 1).

□

Definition 29 I will call **atomically separable** such a poset with least element that “atoms” is an injection.

Proposition 13 Any atomistic lattice with least element is atomically separable.

Proof We need to prove that $\text{atoms } a = \text{atoms } b \Rightarrow a = b$. But it is obvious because

$$a = \bigcup \text{atoms } a \quad \text{and} \quad b = \bigcup \text{atoms } b.$$

□

Theorem 20 *If a lattice with least element is atomic and separable then it is atomistic.*

Proof Suppose the contrary that is $a \supset \bigcup \text{atoms } a$. Then, because our lattice is separable, exists $c \in \mathfrak{A}$ such that $c \cap a \neq 0$ and $c \cap \bigcup \text{atoms } a = 0$. There exist atom $d \subseteq c$ such that $d \subseteq c \cap a$. $d \cap \bigcup \text{atoms } a \subseteq c \cap \bigcup \text{atoms } a = 0$. But $d \in \text{atoms } a$. Contradiction. □

Theorem 21 *Any atomistic lattice with least element is atomically separable.*

Proof Let \mathfrak{A} is an atomistic lattice. Let $a, b \in \mathfrak{A}$, $a \subset b$. Then $\bigcup \text{atoms } a \subseteq \bigcup \text{atoms } b$ and consequently $\text{atoms } a \subseteq \text{atoms } b$. □

Theorem 22 *Let \mathfrak{A} is an atomic meet-semilattice with least element. Then the following statements are equivalent:*

1. \mathfrak{A} is separable.
2. \mathfrak{A} is atomically separable.
3. \mathfrak{A} conforms to Wallman's disjunction property.
4. $\forall a, b \in \mathfrak{A} : (a \subset b \Rightarrow \exists c \in \mathfrak{A} \setminus \{0\} : (c \asymp a \wedge c \subseteq b))$.

Proof

(1) \Leftrightarrow (3) \Leftrightarrow (4) Proved above.

(2) \Rightarrow (4) Let our semilattice is atomically separable. Let $a \subset b$. Then $\text{atoms } a \subseteq \text{atoms } b$ and so exists $c \in \text{atoms } b$ such that $c \notin \text{atoms } a$. $c \neq 0$ and $c \subseteq b$; $c \not\subseteq a$, from which (taking in account that c is an atom) $c \subseteq b$ and $c \cap a = 0$. So our semilattice conforms to the formula (4).

(4) \Rightarrow (2) Let formula (4) holds. Then for any elements $a \subset b$ exists $c \neq 0$ such that $c \subseteq b$ and $c \cap a = 0$. Because \mathfrak{A} is atomic there exists atom $d \subseteq c$. $d \in \text{atoms } b$ and $d \notin \text{atoms } a$. So $\text{atoms } a \neq \text{atoms } b$ and $\text{atoms } a \subseteq \text{atoms } b$. Consequently $\text{atoms } a \subset \text{atoms } b$.

□

4. Filtrators

Definition 30 I will call a **filtrator** a pair $(\mathfrak{A}; \mathfrak{J})$ of a poset \mathfrak{A} and its subset $\mathfrak{J} \subseteq \mathfrak{A}$. I call \mathfrak{A} the **base** of a filtrator and \mathfrak{J} the **core** of a filtrator.

Definition 31 I will call a **lattice filtrator** a pair $(\mathfrak{A}; \mathfrak{J})$ of a lattice \mathfrak{A} and its subset $\mathfrak{J} \subseteq \mathfrak{A}$.

Definition 32 I will call a **complete lattice filtrator** a pair $(\mathfrak{A}; \mathfrak{J})$ of a complete lattice \mathfrak{A} and its subset $\mathfrak{J} \subseteq \mathfrak{A}$.

Definition 33 I will call a **central filtrator** a filtrator $(\mathfrak{A}; Z(\mathfrak{A}))$ where $Z(\mathfrak{A})$ is the center of a bounded lattice \mathfrak{A} .

Remark 7 One use of filtrators is the theory of filters where the base lattice (or the lattice of principal filters) is essentially considered as the core of the lattice of filters. See below for a more exact formulation. Our primary interest is the properties of filters on sets (that is the filtrator of filters on a set), but instead we will research more general theory of filtrators.

Remark 8 An other important example of filtrators is **filtrator of funcoids** whose base is the set of funcoids [12] and whose core is the set of binary relations (or discrete funcoids).

Definition 34 I will call **element** of a filtrator an element of its base.

Definition 35 $\text{up } a = \{c \in \mathfrak{J} \mid c \supseteq a\}$ where $a \in \mathfrak{A}$.

Definition 36 $\text{down } a = \{c \in \mathfrak{J} \mid c \subseteq a\}$ where $a \in \mathfrak{A}$.

Remark 9 I have used “down” in an other sense in an earlier version of this text.

Obvious 8 “up” and “down” are dual.

The main purpose of this text is knowing properties of the core of a filtrator to infer properties of the base of the filtrator, specifically properties of $\text{up } a$ for every element a .

Definition 37 I call a filtrator with **join-closed core** such filtrator $(\mathfrak{A}; \mathfrak{J})$ that $\bigcup^{\mathfrak{J}} S = \bigcup^{\mathfrak{A}} S$ whenever $\bigcup^{\mathfrak{J}} S$ exists for $S \in \mathcal{P}\mathfrak{A}$.

Definition 38 I call a filtrator with **meet-closed core** such filtrator $(\mathfrak{A}; \mathfrak{J})$ that $\bigcap^{\mathfrak{J}} S = \bigcap^{\mathfrak{A}} S$ whenever $\bigcap^{\mathfrak{J}} S$ exists for $S \in \mathcal{P}\mathfrak{A}$.

Definition 39 I call a filtrator with **finitely join-closed core** such filtrator $(\mathfrak{A}; \mathfrak{J})$ that $a \cup^{\mathfrak{J}} b = a \cup^{\mathfrak{A}} b$ whenever $a \cup^{\mathfrak{J}} b$ exists for $a, b \in \mathfrak{J}$.

Definition 40 I call a filtrator with **finitely meet-closed core** such filtrator $(\mathfrak{A}; \mathfrak{B})$ that $a \cap^{\mathfrak{B}} b = a \cap^{\mathfrak{A}} b$ whenever $a \cap^{\mathfrak{B}} b$ exists for $a, b \in \mathfrak{B}$.

In practice we will deal mainly with filtrators with join-closed cores. Proving that a filtrator is with finitely meet-closed core may be a difficult problem. For example as of today it is not known whether the filtrator of functors is with finitely meet-closed core.

Definition 41 *Filtered filtrator* is a filtrator $(\mathfrak{A}; \mathfrak{B})$ such that $\forall a \in \mathfrak{A} : a = \bigcap^{\mathfrak{A}} \text{up } a$.

Definition 42 *Prefiltered filtrator* is a filtrator $(\mathfrak{A}; \mathfrak{B})$ such that “up” is injective.

Definition 43 *Semifiltered filtrator* is a filtrator $(\mathfrak{A}; \mathfrak{B})$ such that

$$\forall a, b \in \mathfrak{A} : (\text{up } a \supseteq \text{up } b \Rightarrow a \subseteq b).$$

Obvious 9 • *Every filtered filtrator is semifiltered.*

• *Every semifiltered filtrator is prefiltered.*

Obvious 10 “up” is a straight map from \mathfrak{A} to the dual of the poset $\mathcal{P}\mathfrak{B}$ if $(\mathfrak{A}; \mathfrak{B})$ is a semifiltered filtrator.

Theorem 23 *Each semifiltered filtrator is a filtrator with join-closed core.*

Proof Let $(\mathfrak{A}; \mathfrak{B})$ is a semifiltered filtrator. Let $S \in \mathcal{P}\mathfrak{B}$ and $\bigcup^{\mathfrak{B}} S$ is defined. We need to prove $\bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} S$. That $\bigcup^{\mathfrak{B}} S$ is an upper bound for S is obvious. Let $a \in \mathfrak{A}$ is an upper bound for S . Enough to prove that $\bigcup^{\mathfrak{B}} S \subseteq a$. Really,

$$c \in \text{up } a \Rightarrow c \supseteq a \Rightarrow \forall x \in S : c \supseteq x \Rightarrow c \supseteq \bigcup^{\mathfrak{B}} S \Rightarrow c \in \text{up } \bigcup^{\mathfrak{B}} S;$$

so $\text{up } a \subseteq \text{up } \bigcup^{\mathfrak{B}} S$ and thus $a \supseteq \bigcup^{\mathfrak{B}} S$ because it is semifiltered. \square

4.1. Core part

Definition 44 The **core part** of an element $a \in \mathfrak{A}$ is $\text{Cor } a = \bigcap^{\mathfrak{B}} \text{up } a$.

Definition 45 The **dual core part** of an element $a \in \mathfrak{A}$ is $\text{Cor}' a = \bigcup^{\mathfrak{B}} \text{down } a$.

Obvious 11 Cor' is dual of Cor .

Theorem 24 $\text{Cor } a \subseteq a$ whenever $\text{Cor } a$ exists for any element a of a filtered filtrator.

Proof $\text{Cor } a = \bigcap^{\mathfrak{B}} \text{up } a \subseteq \bigcap^{\mathfrak{A}} \text{up } a = a$. \square

Corollary 6 $\text{Cor } a \in \text{down } a$ whenever $\text{Cor } a$ exists for any element a of a filtered filtrator.

Theorem 25 $\text{Cor}' a \subseteq a$ whenever $\text{Cor}' a$ exists for any element a of a filtrator with join-closed core.

Proof $\text{Cor}' a = \bigcup^3 \text{down } a = \bigcup^{\mathfrak{A}} \text{down } a \subseteq a$. □

Corollary 7 $\text{Cor}' a \in \text{down } a$ whenever $\text{Cor}' a$ exists for any element a of a filtrator with join-closed core.

Proposition 14 $\text{Cor}' a \supseteq \text{Cor } a$ whenever both $\text{Cor } a$ and $\text{Cor}' a$ exist for any element a of a filtered filtrator.

Proof $\text{Cor } a \in \text{down } a$. So $\text{Cor } a \subseteq \bigcup^3 \text{down } a = \text{Cor}' a$. □

Proposition 15 $\text{Cor}' a \subseteq \text{Cor } a$ whenever both $\text{Cor } a$ and $\text{Cor}' a$ exist for any element a of a filtrator with join-closed core.

Proof $\text{Cor } a = \bigcap^3 \text{up } a \supseteq \text{Cor}' a$ because $\forall A \in \text{up } a : \text{Cor}' a \subseteq A$. □

Exercise 1 Formulate the dual of last proposition.

Theorem 26 $\text{Cor}' a = \text{Cor } a$ whenever both $\text{Cor } a$ and $\text{Cor}' a$ exist for any element a of a filtered filtrator with join-closed core.

Proof From last two propositions. □

Obvious 12 $\text{Cor}' a = \max \text{down } a$ for an element a of a filtrator with join-closed core.

4.2. Filtrators with separable core

Definition 46 Let \mathfrak{A} is a filtrator with least element. \mathfrak{A} is a **filtrator with separable core** when

$$\forall x, y \in \mathfrak{A} : (x \succ^{\mathfrak{A}} y \Rightarrow \exists X \in \text{up } x : X \succ^{\mathfrak{A}} y).$$

Proposition 16 Let \mathfrak{A} is a filtrator with least element. \mathfrak{A} is a filtrator with separable core iff

$$\forall x, y \in \mathfrak{A} : (x \succ^{\mathfrak{A}} y \Rightarrow \exists X \in \text{up } x, Y \in \text{up } y : X \succ^{\mathfrak{A}} Y).$$

Proof

\Rightarrow Apply the definition twice.

\Leftarrow Obvious.

□

Definition 47 Let \mathfrak{A} is a filtrator with least element. \mathfrak{A} is a **filtrator with co-separable core** when

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \text{down } x : X \equiv^{\mathfrak{A}} y).$$

Obvious 13 Co-separability is the dual of separability.

Proposition 17 Let \mathfrak{A} is a filtrator with least element. \mathfrak{A} is a filtrator with co-separable core iff

$$\forall x, y \in \mathfrak{A} : (x \equiv^{\mathfrak{A}} y \Rightarrow \exists X \in \text{down } x, Y \in \text{down } y : X \equiv^{\mathfrak{A}} Y).$$

Proof By duality. □

4.3. *Intersecting and joining with an element of the core*

Definition 48 I call **down-aligned** filtrator such a filtrator $(\mathfrak{A}; \mathfrak{B})$ that \mathfrak{A} and \mathfrak{B} have common least element. (Let's denote it 0.)

Definition 49 I call **up-aligned** filtrator such a filtrator $(\mathfrak{A}; \mathfrak{B})$ that \mathfrak{A} and \mathfrak{B} have common greatest element. (Let's denote it 1.)

Theorem 27 For a filtrator $(\mathfrak{A}; \mathfrak{B})$ where \mathfrak{B} is a boolean lattice, for every $B \in \mathfrak{B}$, $\mathcal{A} \in \mathfrak{A}$:

1. $B \succ^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \supseteq \mathcal{A}$ if it is down-aligned, with finitely meet-closed and separable core;
2. $B \equiv^{\mathfrak{A}} \mathcal{A} \Leftrightarrow \overline{B} \subseteq \mathcal{A}$ if it is up-aligned, with finitely join-closed and co-separable core.

Proof We will prove only the first as the second is dual.

$$\begin{aligned} B \succ^{\mathfrak{A}} \mathcal{A} &\Leftrightarrow \\ \exists A \in \text{up } \mathcal{A} : B \succ^{\mathfrak{A}} A &\Leftrightarrow \\ \exists A \in \text{up } \mathcal{A} : B \cap^{\mathfrak{A}} A = 0 &\Leftrightarrow \\ \exists A \in \text{up } \mathcal{A} : B \cap^{\mathfrak{B}} A = 0 &\Leftrightarrow \\ \exists A \in \text{up } \mathcal{A} : \overline{B} \supseteq A &\Leftrightarrow \\ \overline{B} \in \text{up } \mathcal{A} &\Leftrightarrow \\ \overline{B} \supseteq \mathcal{A}. & \end{aligned}$$

□

5. Filters

5.1. Filters on posets

Let \mathfrak{A} is a poset (partially ordered set) with the partial order \subseteq . I will call it **the base poset**.

Definition 50 *Filter base* is a nonempty subset F of \mathfrak{A} such that

$$\forall X, Y \in F \exists Z \in F : (Z \subseteq X \wedge Z \subseteq Y).$$

Obvious 14 *A nonempty chain is a filter base.*

Definition 51 *Upper set* is a subset F of \mathfrak{A} such that

$$\forall X \in F, Y \in \mathfrak{A} : (Y \supseteq X \Rightarrow Y \in F).$$

Definition 52 *Filter* is a subset of \mathfrak{A} which is both filter base and upper set. I will denote the set of filters \mathfrak{f} .

Proposition 18 *If 1 is the maximal element of \mathfrak{A} then $1 \in F$ for any filter F .*

Proof If $1 \notin F$ then $\forall K \in \mathfrak{A} : K \notin F$ and so F is empty what is impossible. \square

Proposition 19 *Let S is a filter base. If $A_0, \dots, A_n \in S$ ($n \in \mathbb{N}$), then*

$$\exists C \in S : (C \subseteq A_0 \wedge \dots \wedge C \subseteq A_n).$$

Proof Can be easily proved by induction. \square

The dual of filters is called **ideals**. We do not use ideals in this work however.

5.2. Filters on meet-semilattice

Theorem 28 *If \mathfrak{A} is a meet-semilattice and F is a nonempty subset of \mathfrak{A} then the following conditions are equivalent:*

1. F is a filter.
2. $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set.
3. $\forall X, Y \in \mathfrak{A} : (X, Y \in F \Leftrightarrow X \cap Y \in F)$.

Proof

(1) \Rightarrow (2) Let F is a filter. Then F is an upper set. If $X, Y \in F$ then $Z \subseteq X \wedge Z \subseteq Y$ for some $Z \in F$. Because F is an upper set and $Z \subseteq X \cap Y$ then $X \cap Y \in F$.

(2) \Rightarrow (1) Let $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set. We need to prove that F is a filter base. But it is obvious taking $Z = X \cap Y$ (we have also taken in account that $F \neq \emptyset$).

(2) \Rightarrow (3) Let $\forall X, Y \in F : X \cap Y \in F$ and F is an upper set. Then

$$\forall X, Y \in \mathfrak{A} : (X, Y \in F \Rightarrow X \cap Y \in F).$$

Let $X \cap Y \in F$; then $X, Y \in F$ because F is an upper set.

(3) \Rightarrow (2) Let

$$\forall X, Y \in \mathfrak{A} : (X, Y \in F \Leftrightarrow X \cap Y \in F).$$

Then $\forall X, Y \in F : X \cap Y \in F$. Let $X \in F$ and $X \subseteq Y \in \mathfrak{A}$. Then $X \cap Y = X \in F$. Consequently $X, Y \in F$. So F is an upper set.

□

Proposition 20 Let \mathfrak{A} is a meet-semilattice. Let S is a filter base. If $A_0, \dots, A_n \in S$ ($n \in \mathbb{N}$), then

$$\exists C \in S : C \subseteq A_0 \cap \dots \cap A_n.$$

Proof Can be easily proved by induction. □

Proposition 21 If \mathfrak{A} is a meet-semilattice and S is a filter base, $A \in \mathfrak{A}$, then $\langle A \cap \rangle S$ is also a filter base.

Proof $\langle A \cap \rangle S \neq \emptyset$ because $S \neq \emptyset$.

Let $X, Y \in \langle A \cap \rangle S$. Then $X = A \cap X'$ and $Y = A \cap Y'$ where $X', Y' \in S$. Exists $Z' \in S$ such that $Z' \subseteq X' \cap Y'$. So $X \cap Y = A \cap X' \cap Y' \supseteq A \cap Z' \in \langle A \cap \rangle S$.

□

5.3. Characterization of finitely meet-closed filtrators

Theorem 29 The following are equivalent for a filtrator $(\mathfrak{A}; \mathfrak{F})$ whose core is a meet-semilattice such that $\forall a \in \mathfrak{A} : \text{up } a \neq \emptyset$:

1. The filtrator is finitely meet-closed.
2. $\text{up } a$ is a filter on \mathfrak{F} for every $a \in \mathfrak{A}$.

Proof

(1) \Rightarrow (2) Let $X, Y \in \text{up } a$. Then $X \cap^3 Y = X \cap^{\mathfrak{A}} Y \supseteq a$. That $\text{up } a$ is an upper set is obvious. So taking in account that $\text{up } a \neq \emptyset$, $\text{up } a$ is a filter.

(2) \Rightarrow (1) It is enough to prove that $a \subseteq A, B \Rightarrow a \subseteq A \cap^3 B$ for every $A, B \in \mathfrak{A}$. Really:

$$a \subseteq A, B \Rightarrow A, B \in \text{up } a \Rightarrow A \cap^3 B \in \text{up } a \Rightarrow a \subseteq A \cap^3 B.$$

□

6. Filter objects

I want to equate principal filters (see below) with the elements of the base poset. Such thing can be done using the principles described in the appendix Appendix B. The formal definitions follow.

6.1. Definition of filter objects

Let \mathfrak{A} is a poset.

Definition 53 Let $\uparrow a \stackrel{\text{def}}{=} \{x \in \mathfrak{A} \mid x \supseteq a\}$ for every $a \in \mathfrak{A}$. Elements of the set $\langle \uparrow \rangle \mathfrak{A}$ are called **principal filters**.

Obvious 15 \uparrow is an injection from \mathfrak{A} to \mathfrak{f} .

Let M is a bijection defined on \mathfrak{f} such that $M \circ \uparrow = \text{id}_{\mathfrak{A}}$. (See the appendix Appendix B for a proof that such a bijection exists.)

Definition 54 Let $\mathfrak{F} = \text{im } M$. I call elements of \mathfrak{F} as **filter objects** (f.o. for short).

Remark 10 Below we will show that $\text{up } \mathcal{A} = M^{-1}\mathcal{A}$ for each $\mathcal{A} \in \mathfrak{F}$.

Obvious 16 $\uparrow = M^{-1}|_{\mathfrak{A}}$.

Obvious 17 M^{-1} is a bijection $\mathfrak{F} \rightarrow \mathfrak{f}$.

Proposition 22 $\mathfrak{A} \subseteq \mathfrak{F}$.

Proof $x \in \mathfrak{A} \Rightarrow M \uparrow x = x \Rightarrow x \in \text{im } M \Rightarrow x \in \mathfrak{F}$. □

6.2. Order of filter objects

Proposition 23 $a \subseteq b \Leftrightarrow M^{-1}a \supseteq M^{-1}b$.

Proof $a \subseteq b \Leftrightarrow \uparrow a \supseteq \uparrow b \Leftrightarrow M^{-1}a \supseteq M^{-1}b$. □

As a generalization of the last proposition we may define the order on \mathfrak{F} :

Definition 55 $\mathcal{A} \subseteq \mathcal{B} \stackrel{\text{def}}{=} M^{-1}\mathcal{A} \supseteq M^{-1}\mathcal{B}$ for all $\mathcal{A}, \mathcal{B} \in \mathfrak{A}$.

I will call the pair $(\mathfrak{F}; \mathfrak{A})$ the **primary filtrator**.

Theorem 30 For the primary filtrator $(\mathfrak{F}; \mathfrak{A})$ we have $\text{up } \mathcal{A} = M^{-1}\mathcal{A}$ for each $\mathcal{A} \in \mathfrak{F}$.

Proof $x \in \text{up } \mathcal{A} \Leftrightarrow x \supseteq \mathcal{A} \Leftrightarrow M^{-1}x \subseteq M^{-1}\mathcal{A} \Leftrightarrow \uparrow x \subseteq M^{-1}\mathcal{A} \Leftrightarrow x \in M^{-1}\mathcal{A}$ for every $x \in \mathfrak{A}$. □

So we have:

- "up" is a bijection from \mathfrak{F} to \mathfrak{f} .
- $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \text{up } \mathcal{A} \supseteq \text{up } \mathcal{B}$ for each $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.
- $\text{up } a = \uparrow a$ for every $a \in \mathfrak{A}$.

A filter object \mathcal{A} is represented by the value of $\text{up } \mathcal{A}$. We are not interested in the internal structure of filter objects (which can be inferred from the appendix Appendix B), but only in the value of $\text{up } \mathcal{A}$. Thus the name "filter objects" by analogy with an object in object oriented programming where an object is completely characterized by its methods, likewise a filter object \mathcal{A} is completely characterized by $\text{up } \mathcal{A}$.

7. Lattice of filter objects

7.1. Minimal and maximal f.o.

Obvious 18 *The filter object $0 = \text{up}^{-1} \mathfrak{A}$ (equal to the least element of the poset \mathfrak{A} if this least exists) is the least element of the poset of filter objects.*

Proposition 24 *If there exists greatest element 1 of the poset \mathfrak{A} then it is also the greatest element of the poset of filter objects.*

Proof Take in account that filters are nonempty. □

Obvious 19 1. *If the base poset has least element, then the primary filtrator is down-aligned.*

2. *If the base poset has greatest element, then the primary filtrator is up-aligned.*

7.2. Primary filtrator is filtered

Theorem 31 *Primary filtrator is filtered.*

Proof We need to prove that $\mathcal{A} = \bigcap^{\mathfrak{F}} \text{up } \mathcal{A}$ for every $\mathcal{A} \in \mathfrak{F}$.

\mathcal{A} is obviously a lower bound for $\text{up } \mathcal{A}$.

Let \mathcal{B} is a lower bound for $\text{up } \mathcal{A}$ that is $\forall K \in \text{up } \mathcal{A} : K \supseteq \mathcal{B}$. Then $\text{up } \mathcal{A} \subseteq \text{up } \mathcal{B}$; $\mathcal{A} \supseteq \mathcal{B}$. So \mathcal{A} is the greatest lower bound of $\text{up } \mathcal{A}$. □

7.3. Formulas for meets and joins of filter objects

Lemma 2 *If f is an order embedding from a poset \mathfrak{A} to a complete lattice \mathfrak{B} and $S \in \mathcal{P}\mathfrak{A}$ and exists such $\mathcal{F} \in \mathfrak{A}$ that $f\mathcal{F} = \bigcup^{\mathfrak{B}} \langle f \rangle S$, then $\bigcup^{\mathfrak{A}} S$ exists and $f \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} \langle f \rangle S$.*

Proof f is an order isomorphism from \mathfrak{A} to $\mathfrak{B}|_{\langle f \rangle \mathfrak{A}}$. $f\mathcal{F} \in \mathfrak{B}|_{\langle f \rangle \mathfrak{A}}$.

Consequently, $\bigcup^{\mathfrak{B}} \langle f \rangle S \in \mathfrak{B}|_{\langle f \rangle \mathfrak{A}}$ and $\bigcup^{\mathfrak{B}} \mathfrak{B}|_{\langle f \rangle \mathfrak{A}} \langle f \rangle S = \bigcup^{\mathfrak{B}} \langle f \rangle S$.

$f \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} \mathfrak{B}|_{\langle f \rangle \mathfrak{A}} \langle f \rangle S$ because f is an order isomorphism.

Combining, $f \bigcup^{\mathfrak{A}} S = \bigcup^{\mathfrak{B}} \langle f \rangle S$. □

Theorem 32 *If \mathfrak{A} is a meet-semilattice with greatest element 1 then $\bigcup^{\mathfrak{F}} S$ exists and $\text{up} \bigcup^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \mathcal{P}\mathfrak{A} \langle \text{up} \rangle S$ for every $S \in \mathcal{P}\mathfrak{F}$.*

Proof Taking in account the lemma it is enough to prove that exists $\mathcal{F} \in \mathfrak{F}$ such that $\text{up} \mathcal{F} = \bigcap^{\mathfrak{F}} \mathcal{P}\mathfrak{A} \langle \text{up} \rangle S$, that is that $R = \bigcap^{\mathfrak{F}} \mathcal{P}\mathfrak{A} \langle \text{up} \rangle S$ is a filter.

R is nonempty because $1 \in R$. Let $A, B \in R$; then $\forall \mathcal{F} \in S : A, B \in \text{up} \mathcal{F}$, consequently $\forall \mathcal{F} \in S : A \cap^{\mathfrak{A}} B \in \text{up} \mathcal{F}$. Consequently $A \cap^{\mathfrak{A}} B \in \bigcap^{\mathfrak{F}} \mathcal{P}\mathfrak{A} \langle \text{up} \rangle S = R$. So R is a filter base. Let $X \in R$ and $X \subseteq Y \in \mathfrak{A}$; then $\forall \mathcal{F} \in S : X \in \text{up} \mathcal{F}$; $\forall \mathcal{F} \in S : Y \in \text{up} \mathcal{F}$; $Y \in R$. So R is an upper set. \square

Corollary 8 *If \mathfrak{A} is a meet-semilattice with greatest element 1 then \mathfrak{F} is a complete lattice.*

Corollary 9 *If \mathfrak{A} is a meet-semilattice with greatest element 1 then for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$*

$$\text{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \text{up} \mathcal{A} \cap \text{up} \mathcal{B}.$$

Theorem 33 *If \mathfrak{A} is a join-semilattice then \mathfrak{F} is a join-semilattice then and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$*

$$\text{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \text{up} \mathcal{A} \cap \text{up} \mathcal{B}.$$

Proof Taking in account the lemma it is enough to prove that $R = \text{up} \mathcal{A} \cap \text{up} \mathcal{B}$ is a filter.

R is nonempty because exist $X \in \text{up} \mathcal{A}$ and $Y \in \text{up} \mathcal{B}$ and $R \ni X \cup^{\mathfrak{A}} Y$.

Let $A, B \in R$. Then $A, B \in \text{up} \mathcal{A}$; so exists $C \in \text{up} \mathcal{A}$ such that $C \subseteq A \wedge C \subseteq B$. Analogously exists $D \in \text{up} \mathcal{B}$ such that $D \subseteq A \wedge D \subseteq B$. Let $E = C \cup^{\mathfrak{A}} D$. Then $E \in \text{up} \mathcal{A}$ and $E \in \text{up} \mathcal{B}$; $E \in R$ and $E \subseteq A \wedge E \subseteq B$. So R is a filter base.

That R is an upper set is obvious. \square

Theorem 34 *If \mathfrak{A} is a distributive lattice then for $S \in \mathcal{P}\mathfrak{F} \setminus \{\emptyset\}$*

$$\text{up} \bigcap^{\mathfrak{F}} S = \left\{ K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_i \in \bigcup \langle \text{up} \rangle S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N} \right\}.$$

Proof Let's denote the right part of the equality to be proven as R . First we will prove that R is a filter. R is nonempty because S is nonempty.

Let $A, B \in R$. Then $A = X_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} X_k$, $B = Y_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} Y_l$ where $X_i, Y_j \in \bigcup \langle \text{up} \rangle S$. So

$$A \cap^{\mathfrak{A}} B = X_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} X_k \cap^{\mathfrak{A}} Y_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} Y_l \in R.$$

Let $R \ni C \supseteq A$. Consequently (distributivity used)

$$C = C \cup^{\mathfrak{A}} A = (C \cup^{\mathfrak{A}} X_0) \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} (C \cup^{\mathfrak{A}} X_k).$$

$X_i \in \text{up} P$ for some $P \in S$; $C \cup^{\mathfrak{A}} X_i \in \text{up} P$; consequently $C \in \text{up} P$; $C \in R$.

We have proved that R is a filter base and an upper set. So R is a filter.

Consequently the statement of our theorem is equivalent to $\bigcap^{\mathfrak{F}} S = \text{up}^{-1} R$.
Let $\mathcal{A} \in S$. Then $\text{up } \mathcal{A} \in \langle \text{up} \rangle S$; $\text{up } \mathcal{A} \subseteq \bigcup \langle \text{up} \rangle S$;

$$R \supseteq \{K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_i \in \text{up } \mathcal{A} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N}\} = \text{up } \mathcal{A}.$$

Consequently $\mathcal{A} \supseteq \text{up}^{-1} R$.

Let now $\mathcal{B} \in \mathfrak{F}$ and $\forall \mathcal{A} \in S : \mathcal{A} \supseteq \mathcal{B}$. Then $\forall \mathcal{A} \in S : \text{up } \mathcal{B} \supseteq \text{up } \mathcal{A}$. $\text{up } \mathcal{B} \supseteq \bigcup \langle \text{up} \rangle S$. From this $\text{up } \mathcal{B} \supseteq T$ for any finite set $T \subseteq \bigcup \langle \text{up} \rangle S$. Consequently $\text{up } \mathcal{B} \ni \bigcap^{\mathfrak{A}} T$. Thus $\text{up } \mathcal{B} \supseteq R$; $\mathcal{B} \subseteq \text{up}^{-1} R$.

Comparing we get $\bigcap^{\mathfrak{F}} S = \text{up}^{-1} R$. \square

Theorem 35 *If \mathfrak{A} is a distributive lattice then for any $\mathcal{F}_0, \dots, \mathcal{F}_m$ ($m \in \mathbb{N}$)*

$$\text{up}(\mathcal{F}_0 \cap^{\mathfrak{F}} \dots \cap^{\mathfrak{F}} \mathcal{F}_m) = \{K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_m \mid K_i \in \text{up } \mathcal{F}_i, i = 0, \dots, m\}.$$

Proof Let's denote the right part of the equality to be proven as R . First we will prove that R is a filter. Obviously R is nonempty.

Let $A, B \in R$. Then $A = X_0, \dots, X_m$, $B = Y_0, \dots, Y_m$ where $X_i, Y_i \in \text{up } \mathcal{F}_i$.

$$A \cap^{\mathfrak{A}} B = (X_0 \cap^{\mathfrak{A}} Y_0) \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} (X_m \cap^{\mathfrak{A}} Y_m),$$

consequently $A \cap^{\mathfrak{A}} B \in R$.

Let $R \ni C \supseteq A$.

$$C = A \cup^{\mathfrak{A}} C = (X_0 \cup^{\mathfrak{A}} C) \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} (X_m \cup^{\mathfrak{A}} C) \in R.$$

So R is a filter. Consequently the statement of our theorem is equivalent to

$$\mathcal{F}_0 \cap^{\mathfrak{F}} \dots \cap^{\mathfrak{F}} \mathcal{F}_m = \text{up}^{-1} R.$$

Let $P_i \in \text{up } \mathcal{F}_i$. Then $P_i \in R$ because $P_i = (P_i \cup^{\mathfrak{A}} P_0) \cap^{\mathfrak{F}} \dots \cap^{\mathfrak{F}} (P_i \cup^{\mathfrak{A}} P_m)$. So $\text{up } \mathcal{F}_i \subseteq R$; $\mathcal{F}_i \supseteq \text{up}^{-1} R$.

Let now $\mathcal{B} \in \mathfrak{F}$ and $\forall i \in \{0, \dots, m\} : \mathcal{F}_i \supseteq \mathcal{B}$. Then $\forall i \in \{0, \dots, m\} : \text{up } \mathcal{F}_i \subseteq \text{up } \mathcal{B}$.

$L_i \in \text{up } \mathcal{B}$ for any $L_i \in \text{up } \mathcal{F}_i$. $L_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} L_m \in \text{up } \mathcal{B}$. So $\text{up } \mathcal{B} \supseteq R$; $\mathcal{B} \subseteq \text{up}^{-\infty} R$.

So $\mathcal{F}_0 \cap^{\mathfrak{F}} \dots \cap^{\mathfrak{F}} \mathcal{F}_m = \text{up}^{-1} R$. \square

Definition 56 *I will call a **lattice of filter objects on a set** a set of filter objects on the lattice of all subsets of a set. (From the above it follows that it is actually a complete lattice.)*

7.4. Distributivity of the lattice of filter objects

Theorem 36 *If \mathfrak{A} is a distributive lattice with greatest element, $S \in \mathcal{P}\mathfrak{F}$ and $\mathcal{A} \in \mathfrak{F}$ then $\mathcal{A} \cup^{\mathfrak{F}} \bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} \langle \mathcal{A} \cup^{\mathfrak{F}} \rangle S$.*

Proof Taking in account the previous subsection, we have:

$$\begin{aligned}
& \text{up}(\mathcal{A} \cup^{\mathfrak{F}} \bigcap^{\mathfrak{F}} S) = \\
& \text{up} \mathcal{A} \cap \text{up} \bigcap^{\mathfrak{F}} S = \\
& \text{up} \mathcal{A} \cap \left\{ K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_i \in \bigcup \langle \text{up} \rangle S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N} \right\} = \\
& \left\{ K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \in \text{up} \mathcal{A}, K_i \in \bigcup \langle \text{up} \rangle S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N} \right\} = \\
& \left\{ K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_i \in \text{up} \mathcal{A}, K_i \in \bigcup \langle \text{up} \rangle S \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N} \right\} = \\
& \left\{ K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_i \in \text{up} \mathcal{A}, K_i \in \bigcup \{ \text{up} \mathcal{X} \mid \mathcal{X} \in S \} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N} \right\} = \\
& \left\{ K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_i \in \text{up} \mathcal{A} \cap \bigcup \{ \text{up} \mathcal{X} \mid \mathcal{X} \in S \} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N} \right\} = \\
& \left\{ K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_i \in \bigcup \{ \text{up} \mathcal{A} \cap \text{up} \mathcal{X} \mid \mathcal{X} \in S \} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N} \right\} = \\
& \left\{ K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_i \in \bigcup \{ \text{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{X}) \mid \mathcal{X} \in S \} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N} \right\} = \\
& \left\{ K_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} K_n \mid K_i \in \bigcup \langle \text{up} \rangle \{ \mathcal{A} \cup^{\mathfrak{F}} \mathcal{X} \mid \mathcal{X} \in S \} \text{ where } i = 0, \dots, n \text{ for } n \in \mathbb{N} \right\} = \\
& \text{up} \bigcap^{\mathfrak{F}} \{ \mathcal{A} \cup^{\mathfrak{F}} \mathcal{X} \mid \mathcal{X} \in S \}.
\end{aligned}$$

□

Corollary 10 *If \mathfrak{A} is a distributive lattice with greatest element, then \mathfrak{F} is also a distributive lattice.*

Corollary 11 *If \mathfrak{A} is a distributive lattice with greatest element, then \mathfrak{F} is a co-brouwerian lattice.*

7.5. Separability of core for primary filtrators

Theorem 37 *A primary filtrator with least element, whose base is a distributive lattice, is with separable core.*

Proof Let $\mathcal{A} \succ^{\mathfrak{F}} \mathcal{B}$ where $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.

$$\text{up}(\mathcal{A} \cap^{\mathfrak{F}} \mathcal{B}) = \{ A \cap^{\mathfrak{A}} B \mid A \in \text{up} \mathcal{A}, B \in \text{up} \mathcal{B} \}.$$

So

$$\begin{aligned}
\mathcal{A} \succ^{\mathfrak{F}} \mathcal{B} & \Leftrightarrow \\
0 \in \text{up}(\mathcal{A} \cap^{\mathfrak{F}} \mathcal{B}) & \Leftrightarrow \\
\exists A \in \text{up} \mathcal{A}, B \in \text{up} \mathcal{B} : A \cap^{\mathfrak{A}} B = 0 & \Leftrightarrow \\
\exists A \in \text{up} \mathcal{A}, B \in \text{up} \mathcal{B} : A \cap^{\mathfrak{F}} B = 0 &
\end{aligned}$$

(used the theorem 23).

□

Theorem 38 *Let $(\mathfrak{A}; \mathfrak{J})$ is an up-aligned filtered filtrator with finitely join-closed core which is a meet infinite distributive complete lattice. Then this filtrator is with co-separable core.*

Proof Let $a, b \in \mathfrak{A}$. $\text{Cor } a$ and $\text{Cor } b$ exist since \mathfrak{J} is a complete lattice.

$\text{Cor } a \in \text{down } a$ and $\text{Cor } b \in \text{down } b$ by the corollary 6 since our filtrator is filtered. So we have

$$\begin{aligned}
\exists x \in \text{down } a, y \in \text{down } b : x \cup^{\mathfrak{A}} y = 1 &\Leftrightarrow \\
\text{Cor } a \cup^{\mathfrak{A}} \text{Cor } b = 1 &\Leftrightarrow \text{ (by finite join-closedness of the core)} \\
\text{Cor } a \cup^{\mathfrak{J}} \text{Cor } b = 1 &\Leftrightarrow \\
\bigcap^{\mathfrak{A}} \text{up } a \cup^{\mathfrak{J}} \bigcap^{\mathfrak{A}} \text{up } b = 1 &\Leftrightarrow \text{ (by infinite distributivity)} \\
\bigcap^{\mathfrak{A}} \{x \cup^{\mathfrak{J}} y \mid x \in \text{up } a, y \in \text{up } b\} = 1 &\Leftrightarrow \\
\forall x \in \text{up } a, y \in \text{up } b : x \cup^{\mathfrak{J}} y = 1 &\Leftrightarrow \text{ (by finite join-closedness of the core)} \\
\forall x \in \text{up } a, y \in \text{up } b : x \cup^{\mathfrak{A}} y = 1 &\Leftrightarrow \\
a \cup^{\mathfrak{A}} b = 1. &
\end{aligned}$$

□

7.6. Filters over boolean lattices

Theorem 39 *If \mathfrak{A} is a boolean lattice then $a \setminus^{\mathfrak{F}} B = a \cap^{\mathfrak{F}} \overline{B}$ (where the complement is taken on \mathfrak{A}).*

Proof \mathfrak{F} is distributive by the theorem 10. Our filtrator is with finitely meet-closed core by the theorem 29 and with join-closed core by the theorem 23.

$$(a \cap^{\mathfrak{F}} \overline{B}) \cup^{\mathfrak{F}} B = (a \cup^{\mathfrak{F}} B) \cap^{\mathfrak{F}} (\overline{B} \cup^{\mathfrak{F}} B) = (a \cup^{\mathfrak{F}} B) \cap^{\mathfrak{F}} (\overline{B} \cup^{\mathfrak{A}} B) = (a \cup^{\mathfrak{F}} B) \cap^{\mathfrak{F}} 1 = a \cup^{\mathfrak{F}} B.$$

$$(a \cap^{\mathfrak{F}} \overline{B}) \cap^{\mathfrak{F}} B = a \cap^{\mathfrak{F}} (\overline{B} \cap^{\mathfrak{F}} B) = a \cap^{\mathfrak{F}} (\overline{B} \cap^{\mathfrak{A}} B) = a \cap^{\mathfrak{F}} 0 = 0.$$

So $a \cap^{\mathfrak{F}} \overline{B}$ is the difference of a and B .

□

8. Generalized filter base

Definition 57 *Generalized filter base is a filter base on the set \mathfrak{F} .*

Definition 58 *If S is a generalized filter base and $\mathcal{A} = \bigcap^{\mathfrak{F}} S$, then we will call S a generalized base of filter object \mathcal{A} .*

Theorem 40 *If \mathfrak{A} is a distributive lattice and S is a generalized base of filter object \mathcal{F} then for any element K of the base poset*

$$K \in \text{up } \mathcal{F} \Leftrightarrow \exists \mathcal{L} \in S : \mathcal{L} \subseteq K.$$

Proof

\Leftarrow Because $\mathcal{F} = \bigcap^{\mathfrak{F}} S$.

\Rightarrow Let $K \in \text{up}\mathcal{F}$. Then (taken in account distributivity of \mathfrak{A} and that S is nonempty) exist $X_1, \dots, X_n \in \bigcup \langle \text{up} \rangle S$ such that $X_1 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} X_n = K$. Consequently (by theorem 29) $X_1 \cap^{\mathfrak{F}} \dots \cap^{\mathfrak{F}} X_n = K$. Replacing every X_i with such $\mathcal{X}_i \in S$ that $X_i \in \text{up}\mathcal{X}_i$ (this is obviously possible to do), we get a finite set $T_0 \subseteq S$ such that $\bigcap^{\mathfrak{F}} T_0 \subseteq K$. From this exists $\mathcal{C} \in S$ such that $\mathcal{C} \subseteq \bigcap^{\mathfrak{F}} T_0 \subseteq K$.

□

Corollary 12 *If \mathfrak{A} is a distributive lattice with least element 0 and S is a generalized base of filter object \mathcal{F} then $0 \in S \Leftrightarrow \mathcal{F} = 0$.*

Proof Substitute 0 as K .

□

Theorem 41 *Let \mathfrak{A} is a distributive lattice with least element 0 and S is a nonempty set of filter objects on \mathfrak{A} such that $\mathcal{F}_0 \cap^{\mathfrak{F}} \dots \cap^{\mathfrak{F}} \mathcal{F}_n \neq 0$ for every $\mathcal{F}_0, \dots, \mathcal{F}_n \in S$. Then $\bigcap^{\mathfrak{F}} S \neq 0$.*

Proof Consider the set

$$S' = \{ \mathcal{F}_0 \cap^{\mathfrak{F}} \dots \cap^{\mathfrak{F}} \mathcal{F}_n \mid \mathcal{F}_0, \dots, \mathcal{F}_n \in S \}.$$

Obviously S' is nonempty and finitely meet-closed. So S' is a generalized filter base. Obviously $0 \notin S'$. So by properties of generalized filter bases $\bigcap^{\mathfrak{F}} S' \neq 0$. But obviously $\bigcap^{\mathfrak{F}} S = \bigcap^{\mathfrak{F}} S'$. So $\bigcap^{\mathfrak{F}} S \neq 0$.

□

Corollary 13 *Let \mathfrak{A} is a distributive lattice with least element 0 and let $S \in \mathcal{P}\mathfrak{A}$ such that $S \neq \emptyset$ and $A_0 \cap^{\mathfrak{A}} \dots \cap^{\mathfrak{A}} A_n \neq 0$ for every $A_0, \dots, A_n \in S$. Then $\bigcap^{\mathfrak{F}} S \neq 0$.*

Proof Because \mathfrak{A} is finitely meet-closed (by the theorem 29).

□

9. Stars

9.1. Free stars

Definition 59 *Let \mathfrak{A} is a poset with least element 0. **Free stars** on \mathfrak{A} are such $S \in \mathcal{P}\mathfrak{A}$ that $0 \notin S$ and for every $X, Y \in \mathfrak{A}$*

$$\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S) \Leftrightarrow X \in S \vee Y \in S.$$

Proposition 25 *$S \in \mathcal{P}\mathfrak{A}$ where \mathfrak{A} is a poset with least element 0 is a free star iff all of the following:*

1. $0 \notin S$.

2. $\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S) \Rightarrow X \in S \vee Y \in S$ for every $X, Y \in \mathfrak{A}$.

3. S is an upper set.

Proof

\Rightarrow (1) and (2) are obvious. Let prove that S is an upper set. Let $X \in S$ and $X \subseteq Y \in \mathfrak{A}$. Then $X \in S \vee X \in S$ and thus $\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq X \Rightarrow Z \in S)$ that is $\forall Z \in \mathfrak{A} : (Z \supseteq X \Rightarrow Z \in S)$, and so $Y \in S$.

\Leftarrow We need to prove that

$$\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S) \Leftarrow X \in S \vee Y \in S.$$

Let $X \in S \vee Y \in S$. Then $Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S$ for every $Z \in \mathfrak{A}$ because S is an upper set.

□

Proposition 26 Let \mathfrak{A} is a join-semilattice with least element 0. $S \in \mathcal{P}\mathfrak{A}$ is a free star iff all of the following:

1. $0 \notin S$.

2. $X \cup Y \in S \Rightarrow X \in S \vee Y \in S$ for every $X, Y \in \mathfrak{A}$.

3. S is an upper set.

Proof

\Rightarrow We need to prove only $X \cup Y \in S \Rightarrow X \in S \vee Y \in S$. Let $X \cup Y \in S$. Because S is an upper set, we have $\forall Z \in \mathfrak{A} : (Z \supseteq X \cup Y \Rightarrow Z \in S)$ and thus $\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S)$ from which we conclude $X \in S \vee Y \in S$.

\Leftarrow We need to prove $\forall Z \in \mathfrak{A} : (Z \supseteq X \wedge Z \supseteq Y \Rightarrow Z \in S) \Leftarrow X \in S \vee Y \in S$. But this trivially follows from that S is an upper set.

□

Proposition 27 Let \mathfrak{A} is a join-semilattice with least element 0. $S \in \mathcal{P}\mathfrak{A}$ is a free star iff $0 \notin S$ and for every $X, Y \in \mathfrak{A}$

$$X \cup Y \in S \Leftrightarrow X \in S \vee Y \in S.$$

Proof

\Rightarrow We need to prove only that $X \cup Y \in S \Leftarrow X \in S \vee Y \in S$ what follows from that S is an upper set.

\Leftarrow We need to prove only that S is an upper set. Let $X \in S$ and $X \subseteq Y \in \mathfrak{A}$. Then $X \in S \Rightarrow X \in S \vee Y \in S \Leftrightarrow X \cup Y \in S \Rightarrow Y \in S$. So S is an upper set.

□

9.2. Stars of elements of filtrators

Definition 60 Let $(\mathfrak{A}; \mathfrak{F})$ is a filtrator with least element. **Core star** of an element a of this filtrator is

$$\partial a = \{x \in \mathfrak{F} \mid x \not\leq^{\mathfrak{A}} a\}.$$

Proposition 28 $\text{up } a \subseteq \partial a$ for any element $a \neq 0$ of a filtrator with least element.

Proof For any element $X \in \mathfrak{F}$

$$X \in \text{up } a \Rightarrow X \cap^{\mathfrak{A}} a = a \neq 0 \Rightarrow X \in \partial a.$$

□

Theorem 42 Let $(\mathfrak{A}; \mathfrak{F})$ is a distributive lattice filtrator with least element and finitely join-closed core which is a join-semilattice. Then ∂a is a free star for each $a \in \mathfrak{A}$.

Proof For every $A, B \in \mathfrak{F}$

$$\begin{aligned} A \cup^{\mathfrak{F}} B \in \partial a &\Leftrightarrow \\ A \cup^{\mathfrak{A}} B \in \partial a &\Leftrightarrow \\ (A \cup^{\mathfrak{A}} B) \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\ (A \cap^{\mathfrak{A}} a) \cup^{\mathfrak{A}} (B \cap^{\mathfrak{A}} a) \neq 0 &\Leftrightarrow \\ A \cap^{\mathfrak{A}} a \neq 0 \vee B \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\ A \in \partial a \vee B \in \partial a. & \end{aligned}$$

That ∂a doesn't contain 0 is obvious.

□

Definition 61 I call a filtrator **star-separable** when its core is a separation subset of its base.

9.3. Stars of filters on boolean lattices

In this section we will consider the set of filter objects \mathfrak{F} on a boolean lattice \mathfrak{A} .

Theorem 43 If \mathfrak{A} is a boolean lattice and $\mathcal{A} \in \mathfrak{F}$ then

1. $\partial \mathcal{A} = \{\overline{X} \mid X \in \mathfrak{A} \setminus \text{up } \mathcal{A}\};$
2. $\text{up } \mathcal{A} = \{\overline{X} \mid X \in \mathfrak{A} \setminus \partial \mathcal{A}\}.$

Proof 1. For any $K \in \mathfrak{A}$ (taking into account the theorems 29, 37, and 27)

$$\begin{aligned}
K \in \{\overline{X} \mid X \in \mathfrak{A} \setminus \text{up } \mathcal{A}\} &\Leftrightarrow \\
\overline{K} \in \mathfrak{A} \setminus \text{up } \mathcal{A} &\Leftrightarrow \\
\overline{K} \notin \text{up } \mathcal{A} &\Leftrightarrow \\
\overline{K} \not\supseteq \mathcal{A} &\Leftrightarrow \\
K \not\prec^{\mathfrak{F}} \mathcal{A} &\Leftrightarrow \\
K \in \partial \mathcal{A}. &
\end{aligned}$$

2. For any $K \in \mathfrak{A}$ (taking into account the same theorems)

$$\begin{aligned}
K \in \{\overline{X} \mid X \in \mathfrak{A} \setminus \partial \mathcal{A}\} &\Leftrightarrow \\
\overline{K} \in \mathfrak{A} \setminus \partial \mathcal{A} &\Leftrightarrow \\
\overline{K} \notin \partial \mathcal{A} &\Leftrightarrow \\
\overline{K} \succ^{\mathfrak{F}} \mathcal{A} &\Leftrightarrow \\
K \supseteq \mathcal{A} &\Leftrightarrow \\
K \in \text{up } \mathcal{A}. &
\end{aligned}$$

□

Corollary 14 *If \mathfrak{A} is a boolean lattice, $X \in \text{up } \mathcal{A} \Leftrightarrow \overline{X} \notin \partial \mathcal{A}$ for every $X \in \mathfrak{A}$, $\mathcal{A} \in \mathfrak{F}$.*

Corollary 15 *If \mathfrak{A} is a boolean lattice, ∂ is an injection.*

Theorem 44 *If \mathfrak{A} is a boolean lattice, then for any set $S \in \mathcal{P}\mathfrak{A}$ exists filter object \mathcal{A} such that $\partial \mathcal{A} = S$ iff S is a free star.*

Proof

\Rightarrow That $0 \notin S$ is obvious. For every $A, B \in \mathfrak{A}$

$$\begin{aligned}
A \cup^{\mathfrak{A}} B \in S &\Leftrightarrow \\
(A \cup^{\mathfrak{A}} B) \cap^{\mathfrak{F}} \mathcal{A} \neq 0 &\Leftrightarrow \\
(A \cup^{\mathfrak{F}} B) \cap^{\mathfrak{F}} \mathcal{A} \neq 0 &\Leftrightarrow \\
(A \cap^{\mathfrak{F}} \mathcal{A}) \cup^{\mathfrak{F}} (B \cap^{\mathfrak{F}} \mathcal{A}) \neq 0 &\Leftrightarrow \\
A \cap^{\mathfrak{F}} \mathcal{A} \neq 0 \vee B \cap^{\mathfrak{F}} \mathcal{A} \neq 0 &\Leftrightarrow \\
A \in S \vee B \in S. &
\end{aligned}$$

(taken into account the theorems 10 and 29).

\Leftarrow Let $0 \notin S$ and $\forall A, B \in S : (A \cup^{\mathfrak{A}} B \in S \Leftrightarrow A \in S \vee B \in S)$. Let $T = \{\overline{X} \mid X \in \mathfrak{A} \setminus S\}$. We will prove that T is a filter.

$1 \in T$ because $0 \notin S$; so T is nonempty. To prove that T is a filter is enough to show that $\forall X, Y \in \mathfrak{A} : (X, Y \in T \Leftrightarrow X \cap^{\mathfrak{A}} Y \in T)$. In fact,

$$\begin{aligned}
X, Y \in T &\Leftrightarrow \\
\overline{X}, \overline{Y} \notin S &\Leftrightarrow \\
\neg(\overline{X} \in S \vee \overline{Y} \in S) &\Leftrightarrow \\
\overline{X \cup^{\mathfrak{A}} Y} \notin S &\Leftrightarrow \\
\overline{X \cup^{\mathfrak{A}} Y} \in T &\Leftrightarrow \\
X \cap^{\mathfrak{A}} Y \in T. &
\end{aligned}$$

So T is a filter. Let $\text{up } \mathcal{A} = T$ for some filter object \mathcal{A} .

To finish the proof we will show that $\partial \mathcal{A} = S$. In fact, for every $X \in \mathfrak{A}$

$$X \in \partial \mathcal{A} \Leftrightarrow \overline{X} \notin \text{up } \mathcal{A} \Leftrightarrow \overline{X} \notin T \Leftrightarrow X \in S.$$

□

Proposition 29 *If \mathfrak{A} is a boolean lattice then $\mathcal{A} \subseteq \mathcal{B} \Leftrightarrow \partial \mathcal{A} \subseteq \partial \mathcal{B}$ for every $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$.*

Proof

$$\begin{aligned}
\partial \mathcal{A} \subseteq \partial \mathcal{B} &\Leftrightarrow \\
\{\overline{X} \mid X \in \mathfrak{A} \setminus \text{up } \mathcal{A}\} \subseteq \{\overline{X} \mid X \in \mathfrak{A} \setminus \text{up } \mathcal{B}\} &\Leftrightarrow \\
\mathfrak{A} \setminus \text{up } \mathcal{A} \subseteq \mathfrak{A} \setminus \text{up } \mathcal{B} &\Leftrightarrow \\
\text{up } \mathcal{A} \supseteq \text{up } \mathcal{B} &\Leftrightarrow \\
\mathcal{A} \subseteq \mathcal{B}. &
\end{aligned}$$

□

Corollary 16 *∂ is a straight monotone map.*

Theorem 45 *If \mathfrak{A} is a boolean lattice then $\partial \bigcup^{\mathfrak{A}} S = \bigcup \langle \partial \rangle S$.*

Proof For boolean lattices ∂ is an order embedding from the poset \mathfrak{F} to the complete lattice $\mathcal{P}\mathfrak{A}$. So accordingly the lemma 2 it enough to prove that exists $\mathcal{F} \in \mathfrak{F}$ such that $\partial \mathcal{F} = \bigcup \langle \partial \rangle S$. To prove this is enough to show that $0 \notin \bigcup \langle \partial \rangle S$ and

$$\forall A, B \in S : (A \cup^{\mathfrak{A}} B \in \bigcup \langle \partial \rangle S \Leftrightarrow A \in \bigcup \langle \partial \rangle S \vee B \in \bigcup \langle \partial \rangle S).$$

$0 \notin \bigcup \langle \partial \rangle S$ is obvious.

Let $A \cup^{\mathfrak{A}} B \in \bigcup \langle \partial \rangle S$. Then exists $Q \in \langle \partial \rangle S$ such that $A \cup^{\mathfrak{A}} B \in Q$. Then $A \in Q \vee B \in Q$, consequently $A \in \bigcup \langle \partial \rangle S \vee B \in \bigcup \langle \partial \rangle S$. Let now $A \in \bigcup \langle \partial \rangle S$. Then exists $Q \in \langle \partial \rangle S$ such that $A \in Q$, consequently $A \cup^{\mathfrak{A}} B \in Q$ and $A \cup^{\mathfrak{A}} B \in \bigcup \langle \partial \rangle S$. □

9.4. More about the lattice of filters

Theorem 46 *If \mathfrak{A} is a distributive lattice with least element 0 then \mathfrak{F} is an atomic lattice.*

Proof Let $\mathcal{F} \in \mathfrak{F}$. Let choose (by Kuratowski's lemma) a maximal chain S from 0 to \mathcal{F} . Let $S' = S \setminus \{0\}$. $a = \bigcap^{\mathfrak{F}} S' \neq 0$ by properties of generalized filter bases (the corollary 12 which uses the fact that \mathfrak{A} is a distributive lattice with least element). If $a \notin S$ then the chain S can be extended adding there element a because $0 \subset a \subseteq \mathcal{X}$ for any $\mathcal{X} \in S'$ what contradicts to maximality of the chain. So $a \in S$ and consequently $a \in S'$. Obviously a is the minimal element of S' . Consequently (taking in account maximality of the chain) there are no $\mathcal{Y} \in \mathfrak{F}$ such that $0 \subset \mathcal{Y} \subset a$. So a is an atomic filter. Obviously $a \subseteq \mathcal{F}$. \square

Obvious 20 *If \mathfrak{A} is a boolean lattice then \mathfrak{F} is separable.*

Theorem 47 *If \mathfrak{A} is a boolean lattice then \mathfrak{F} is an atomistic lattice.*

Proof Because (used the theorem 20) \mathfrak{F} is atomic (the theorem 46) and separable. \square

Corollary 17 *If \mathfrak{A} is a boolean lattice then \mathfrak{F} is an atomically separable.*

Proof By the theorem 13. \square

Theorem 48 *When the base poset \mathfrak{A} is a boolean lattice, then the filtrator $(\mathfrak{F}; \mathfrak{A})$ is central.*

Proof We can conclude that \mathfrak{F} is atomically separable (the corollary 17) and with separable core (the theorem 37).

We need to prove that $Z(\mathfrak{F}) = \mathfrak{A}$.

Let $\mathcal{X} \in Z(\mathfrak{F})$. Then exists $\mathcal{Y} \in Z(\mathfrak{F})$ such that $\mathcal{X} \cap^{\mathfrak{F}} \mathcal{Y} = 0$ and $\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = 1$. Consequently there are $X \in \text{up } \mathcal{X}$ such that $X \cap^{\mathfrak{F}} \mathcal{Y} = 0$; we have also $X \cup^{\mathfrak{F}} \mathcal{Y} = 1$. Suppose $X \supset \mathcal{X}$. Then exists $a \in \text{atoms}^{\mathfrak{F}} X$ such that $a \notin \text{atoms}^{\mathfrak{F}} \mathcal{X}$. We can conclude also $a \notin \text{atoms}^{\mathfrak{F}} \mathcal{Y}$ (otherwise $X \cap^{\mathfrak{F}} \mathcal{Y} \neq 0$). Thus $a \notin \text{atoms}^{\mathfrak{F}} (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y})$ and consequently $\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} \neq 1$ what is a contradiction. We have $\mathcal{X} = X \in \mathfrak{A}$.

Let now $X \in \mathfrak{A}$. Let $Y = 1 \setminus^{\mathfrak{A}} X$. We have $X \cap^{\mathfrak{A}} Y = 0$ and $X \cup^{\mathfrak{A}} Y = 1$. Thus $X \cap^{\mathfrak{F}} Y = \bigcap^{\mathfrak{A}} \{X \cap^{\mathfrak{A}} Y\} = 0$; $X \cup^{\mathfrak{F}} Y = \bigcap^{\mathfrak{F}} (\text{up } X \cap \text{up } Y) = \bigcap^{\mathfrak{F}} \{1\} = 1$. We have shown that $X \in Z(\mathfrak{F})$. \square

10. Atomic filter objects

See [2] and [4] for more detailed treatment of ultrafilters and prime filters.

Theorem 49 *Let $(\mathfrak{A}; \mathfrak{F})$ is a semifiltered down-aligned filtrator with finitely meet-closed core \mathfrak{F} which is a meet-semilattice. Then a is an atom of \mathfrak{F} iff $a \in \mathfrak{F}$ and a is an atom of \mathfrak{A} .*

Proof

\Leftarrow Obvious.

\Rightarrow We need to prove that if a is an atom of \mathfrak{F} then a is an atom of \mathfrak{A} . Suppose the contrary that a is not an atom of \mathfrak{A} . Then exists $x \in \mathfrak{A}$ such that $0 \neq x \subset a$. Because “up” is a straight monotone map from \mathfrak{A} to the dual of the poset $\mathcal{P}\mathfrak{F}$ (the theorem 10), $\text{up } a \subset \text{up } x$. So exists $K \in \text{up } x$ such that $K \notin \text{up } a$. Also $a \in \text{up } x$. We have $K \cap^{\mathfrak{F}} a = K \cap^{\mathfrak{A}} a \in \text{up } x$; $K \cap^{\mathfrak{F}} a \neq 0$ and $K \cap^{\mathfrak{F}} a \subset a$. So a is not an atom of \mathfrak{F} .

□

Theorem 50 *Let $(\mathfrak{A}; \mathfrak{F})$ is a down-aligned semifiltered filtrator and \mathfrak{A} is a meet-semilattice. Then $a \in \mathfrak{A}$ if an atom of \mathfrak{A} iff $\text{up } a = \partial a$.*

Proof

\Rightarrow Let a is an atom of \mathfrak{A} . $\text{up } a \supseteq \partial a$ because $a \neq 0$. $\text{up } a \subseteq \partial a$ because for any $K \in \mathfrak{A}$

$$K \in \text{up } a \Leftrightarrow K \supseteq a \Rightarrow K \cap^{\mathfrak{A}} a \neq 0 \Leftrightarrow K \in \partial a.$$

\Leftarrow Let $\text{up } a = \partial a$. Then $a \neq 0$. Consequently for every $x \in \mathfrak{A}$ we have

$$\begin{aligned} 0 \subset x \subset a &\Rightarrow \\ x \cap^{\mathfrak{A}} a \neq 0 &\Rightarrow \\ \forall K \in \text{up } x : K \in \partial a &\Rightarrow \\ \forall K \in \text{up } x : K \in \text{up } a &\Rightarrow \\ \text{up } x \subseteq \text{up } a &\Rightarrow \\ x \supseteq a. & \end{aligned}$$

So a is an atom of \mathfrak{A} .

□

10.1. Prime filtrator elements

Definition 62 Let $(\mathfrak{A}; \mathfrak{F})$ is a down-aligned filtrator with least element 0. **Prime filtrator elements** are such $a \in \mathfrak{A}$ that $\text{up } a$ is a free star.

Proposition 30 Let $(\mathfrak{A}; \mathfrak{F})$ is a down-aligned filtrator with finitely join-closed core, where \mathfrak{A} is a distributive lattice and \mathfrak{F} is a join-semilattice. Then atomic elements of this filtrator are prime.

Proof Let a is an atom of the lattice \mathfrak{A} . We have for every $X, Y \in \mathfrak{F}$

$$\begin{aligned}
X \cup^{\mathfrak{F}} Y \in \text{up } a &\Leftrightarrow \\
X \cup^{\mathfrak{A}} Y \in \text{up } a &\Leftrightarrow \\
X \cup^{\mathfrak{A}} Y \supseteq a &\Leftrightarrow \\
(X \cup^{\mathfrak{A}} Y) \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\
(X \cap^{\mathfrak{A}} a) \cup^{\mathfrak{A}} (Y \cap^{\mathfrak{A}} a) \neq 0 &\Leftrightarrow \\
X \cap^{\mathfrak{A}} a \neq 0 \vee Y \cap^{\mathfrak{A}} a \neq 0 &\Leftrightarrow \\
X \supseteq a \vee Y \supseteq a &\Leftrightarrow \\
X \in \text{up } a \vee Y \in \text{up } a. &
\end{aligned}$$

□

The following theorem is essentially borrowed from [8]:

Theorem 51 Let \mathfrak{A} is a boolean lattice. Let a is a f.o. Then the following are equivalent:

1. a is prime.
2. For every $A \in \mathfrak{A}$ exactly one of $\{A, \bar{A}\}$ is in $\text{up } a$.
3. a is an atom of \mathfrak{F} .

Proof

- (1) \Rightarrow (2) Let a is prime. Then $A \cup^{\mathfrak{A}} \bar{A} = 1 \in \text{up } a$. Therefore $A \in \text{up } a \vee \bar{A} \in \text{up } a$. But since $A \cap^{\mathfrak{A}} \bar{A} = 0 \notin \text{up } a$ it is impossible $A \in \text{up } a \wedge \bar{A} \in \text{up } a$.
- (2) \Rightarrow (3) Obviously $a \neq 0$. Let f.o. $b \subset a$. So $\text{up } b \supset \text{up } a$. Let $X \in \text{up } b \setminus \text{up } a$. Then $X \notin \text{up } a$ and thus $\bar{X} \in \text{up } a$ and consequently $\bar{X} \in \text{up } b$. So $0 = X \cap^{\mathfrak{A}} \bar{X} \in \text{up } b$ and thus $b = 0$. So a is atomic.
- (3) \Rightarrow (1) By the previous proposition (taking into account the corollary 10 and the theorem 23).

□

11. Some criteria

Theorem 52 *For a semifiltered, star-separable, down-aligned filtrator $(\mathfrak{A}; \mathfrak{Z})$ with finitely meet closed, join-closed and separable core where \mathfrak{Z} is a boolean lattice and both \mathfrak{Z} and \mathfrak{A} are atomistic lattices the following conditions are equivalent for any $\mathcal{F} \in \mathfrak{A}$:*

1. $\mathcal{F} \in \mathfrak{Z}$;
2. $\forall S \in \mathcal{P}\mathfrak{A} : (\mathcal{F} \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} S \neq 0 \Rightarrow \exists K \in S : \mathcal{F} \cap^{\mathfrak{A}} K \neq 0)$;
3. $\forall S \in \mathcal{P}\mathfrak{Z} : (\mathcal{F} \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} S \neq 0 \Rightarrow \exists K \in S : \mathcal{F} \cap^{\mathfrak{A}} K \neq 0)$.

Proof

(1) \Rightarrow (2) Let $\mathcal{F} \in \mathfrak{Z}$. Then (taking in account the proposition 27)

$$\mathcal{F} \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} S \neq 0 \Leftrightarrow \overline{\mathcal{F}} \not\subseteq \bigcup^{\mathfrak{A}} S \Rightarrow \exists K \in S : \overline{\mathcal{F}} \not\subseteq K \Leftrightarrow \exists K \in S : \mathcal{F} \cap^{\mathfrak{A}} K \neq 0.$$

(2) \Rightarrow (3) Obvious.

(3) \Rightarrow (1) Let the formula (3) is true. Then for $L \in \mathfrak{Z}$ and $S = \text{atoms}^{\mathfrak{Z}} L$ it takes the form $\mathcal{F} \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} \text{atoms}^{\mathfrak{Z}} L \neq 0 \Rightarrow \exists K \in S : \mathcal{F} \cap^{\mathfrak{A}} K \neq 0$ that is $\mathcal{F} \cap^{\mathfrak{A}} L \neq 0 \Rightarrow \exists K \in S : \mathcal{F} \cap^{\mathfrak{A}} K \neq 0$ because $\bigcup^{\mathfrak{A}} \text{atoms}^{\mathfrak{Z}} L = \bigcup^{\mathfrak{Z}} \text{atoms}^{\mathfrak{Z}} L = L$. That is $\mathcal{F} \cap^{\mathfrak{A}} L \neq 0 \Rightarrow \mathcal{F} \cap^{\mathfrak{A}} K_L \neq 0$ where $K_L \in S$. Thus K_L is an atom of both \mathfrak{A} and \mathfrak{Z} (see the theorem 49), so having $\mathcal{F} \cap^{\mathfrak{A}} L \neq 0 \Rightarrow \mathcal{F} \supseteq K_L$.
Let

$$F = \bigcup^{\mathfrak{A}} \{K_L \mid L \in \mathfrak{Z}, \mathcal{F} \cap^{\mathfrak{A}} L \neq 0\}.$$

Obviously $F \subseteq \mathcal{F}$. We have $L \cap^{\mathfrak{Z}} \mathcal{F} \neq 0 \Rightarrow L \cap^{\mathfrak{A}} \mathcal{F} \neq 0 \Rightarrow K_L \cap^{\mathfrak{Z}} F \neq 0 \Rightarrow L \cap^{\mathfrak{Z}} F \neq 0$, thus by star separability of our filtrator $\mathcal{F} \subseteq F$ and so $\mathcal{F} = F \in \mathfrak{Z}$.

□

Theorem 53 *If \mathfrak{A} is a complete boolean lattice then for each $\mathcal{F} \in \mathfrak{F}$*

$$\mathcal{F} \in \mathfrak{A} \Leftrightarrow \forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcup^{\mathfrak{A}} S \in \partial\mathcal{F} \Rightarrow S \cap \partial\mathcal{F} \neq 0 \right).$$

Proof

$$\begin{aligned} \forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcup^{\mathfrak{A}} S \in \partial\mathcal{F} \Rightarrow S \cap \partial\mathcal{F} \neq 0 \right) &\Leftrightarrow \\ \forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcup^{\mathfrak{A}} S \notin \partial\mathcal{F} \Leftarrow S \cap \partial\mathcal{F} = 0 \right) &\Leftrightarrow \\ \forall S \in \mathcal{P}\mathfrak{A} : \left(\overline{\bigcup^{\mathfrak{A}} S} \in \text{up}\mathcal{F} \Leftarrow (\neg) S \subseteq \text{up}\mathcal{F} \right) &\Leftrightarrow \\ \forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcap^{\mathfrak{A}} S \in \text{up}\mathcal{F} \Leftarrow S \subseteq \text{up}\mathcal{F} \right), & \end{aligned}$$

but

$$\begin{aligned}
& \mathcal{F} \in \mathfrak{A} \Rightarrow \\
\forall S \in \mathcal{P}\mathfrak{A} : \left(\bigcap^{\mathfrak{A}} S \in \text{up } \mathcal{F} \Leftarrow S \subseteq \text{up } \mathcal{F} \right) & \Rightarrow \\
\bigcap^{\mathfrak{A}} \text{up } \mathcal{F} \in \text{up } \mathcal{F} & \Rightarrow \\
\mathcal{F} \in \mathfrak{A}. &
\end{aligned}$$

□

Definition 63 Let \mathfrak{A} is a complete lattice. A set $S \in \mathcal{P}\mathfrak{A}$ is **filter-closed** when for every filter base $T \in \mathcal{P}S$ we have $\bigcap T \in S$.

Theorem 54 Let \mathfrak{A} is a boolean lattice. For any $S \in \mathcal{P}\mathfrak{F}$ the condition $\exists \mathcal{F} \in \mathfrak{F} : S = \star \mathcal{F}$ is equivalent to conjunction of the following items:

1. S is a free star on \mathfrak{F} ;
2. S is filter-closed.

Proof

\Rightarrow 1. That $0 \notin \star \mathcal{F}$ is obvious. For every $a, b \in \mathfrak{F}$

$$\begin{aligned}
a \cup^{\mathfrak{F}} b \in \star S & \Leftrightarrow \\
(a \cup^{\mathfrak{F}} b) \cap^{\mathfrak{F}} \mathcal{F} \neq 0 & \Leftrightarrow \\
(a \cap^{\mathfrak{F}} \mathcal{F}) \cup (b \cap^{\mathfrak{F}} \mathcal{F}) \neq 0 & \Leftrightarrow \\
a \cap^{\mathfrak{F}} \mathcal{F} \neq 0 \vee b \cap^{\mathfrak{F}} \mathcal{F} \neq 0 & \Leftrightarrow \\
a \in \star S \vee b \in \star S. &
\end{aligned}$$

(taken into account the corollary 10). So $\star \mathcal{F}$ is a free star on \mathfrak{F} .

2. We have $T \subseteq S$ and need to prove that $\bigcap^{\mathfrak{F}} T \cap \mathcal{F} \neq 0$. Because $\langle \mathcal{F} \cap^{\mathfrak{F}} \rangle T$ is a generalized filter base, $0 \in \langle \mathcal{F} \cap^{\mathfrak{F}} \rangle T \Leftrightarrow \bigcap^{\mathfrak{F}} \langle \mathcal{F} \cap^{\mathfrak{F}} \rangle T = 0 \Leftrightarrow \bigcap^{\mathfrak{F}} T \cap^{\mathfrak{F}} \mathcal{F} = 0$. So left to prove $0 \notin \langle \mathcal{F} \cap^{\mathfrak{F}} \rangle T$ what follows from $T \subseteq S$.

\Leftarrow Let S is a free star on \mathfrak{F} . Then for every $A, B \in \mathfrak{A}$

$$\begin{aligned}
A, B \in S \cap \mathfrak{A} & \Leftrightarrow \\
A, B \in S & \Leftrightarrow \\
A \cup^{\mathfrak{F}} B \in S & \Leftrightarrow \\
A \cup^{\mathfrak{A}} B \in S & \Leftrightarrow \\
A \cup^{\mathfrak{A}} B \in S \cap \mathfrak{A} &
\end{aligned}$$

(taken into account the theorem 23). So $S \cap \mathfrak{A}$ is a free star on \mathfrak{A} .

Thus there exists $\mathcal{F} \in \mathfrak{F}$ such that $\partial\mathcal{F} = S \cap \mathfrak{A}$. We have $\text{up } \mathcal{X} \subseteq S \Leftrightarrow \mathcal{X} \in S$ (because S is filter-closed) for every $\mathcal{X} \in \mathfrak{F}$; then (taking in account properties of generalized filter bases)

$$\begin{aligned}
\mathcal{X} \in S &\Leftrightarrow \\
\forall X \in \text{up } \mathcal{X} : X \cap^{\mathfrak{F}} \mathcal{F} \neq 0 &\Leftrightarrow \\
0 \notin \langle \mathcal{F} \cap^{\mathfrak{F}} \rangle \text{up } \mathcal{X} &\Leftrightarrow \\
\bigcap^{\mathfrak{F}} \langle \mathcal{F} \cap^{\mathfrak{F}} \rangle \text{up } \mathcal{X} \neq 0 &\Leftrightarrow \\
\mathcal{F} \cap^{\mathfrak{F}} \bigcap^{\mathfrak{F}} \text{up } \mathcal{X} \neq 0 &\Leftrightarrow \\
\mathcal{F} \cap^{\mathfrak{F}} \mathcal{X} \neq 0 &\Leftrightarrow \\
\mathcal{X} \in \partial\mathcal{F}. &
\end{aligned}$$

□

12. Quasidifference and quasicomplement

I've got quasidifference and quasicomplement (and dual quasicomplement) replacing max and min in the definition of pseudodifference and pseudocomplement (and dual pseudocomplement) with \bigcup and \bigcap . Thus quasidifference and (dual) quasicomplement are generalizations of their pseudo- counterparts.

Remark 11 Pseudocomplements and pseudodifferences is standard terminology. **Quasi-** counterparts are my neologisms.

Definition 64 Let \mathfrak{A} is a poset with least element, $a \in \mathfrak{A}$. **Quasicomplement** of a is

$$a^* = \bigcup \{c \in \mathfrak{A} \mid c \succ a\}.$$

Definition 65 Let \mathfrak{A} is a poset with greatest element, $a \in \mathfrak{A}$. **Dual quasicomplement** of a is

$$a^+ = \bigcap \{c \in \mathfrak{A} \mid c \equiv a\}.$$

I will denote quasicomplement and dual quasicomplement for a specific poset \mathfrak{A} as $a^{*(\mathfrak{A})}$ and $a^{+(\mathfrak{A})}$.

Definition 66 Let $a, b \in \mathfrak{A}$ where \mathfrak{A} is a distributive lattice. **Quasidifference** of a and b is

$$a \setminus^* b = \bigcap \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}.$$

Definition 67 Let $a, b \in \mathfrak{A}$ where \mathfrak{A} is a distributive lattice with least element. **Second quasidifference** of a and b is

$$a \# b \stackrel{\text{def}}{=} \bigcup \{z \in \mathfrak{A} \mid z \subseteq a \wedge z \succ b\}.$$

Theorem 55 $a \setminus^* b = \bigcap \{z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z\}$ where \mathfrak{A} is a distributive lattice and $a, b \in \mathfrak{A}$.

Proof Obviously $\{z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z\} \subseteq \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}$. Thus $\bigcap \{z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z\} \supseteq a \setminus^* b$.

Let $z \in \mathfrak{A}$ and $z' = z \cap a$.

$a \subseteq b \cup z \Rightarrow a \subseteq (b \cup z) \cap a \Leftrightarrow a \subseteq (b \cap a) \cup (z \cap a) \Leftrightarrow a \subseteq (b \cap a) \cup z' \Rightarrow a \subseteq b \cup z'$
and $a \subseteq b \cup z \Leftarrow a \subseteq b \cup z'$. Thus $a \subseteq b \cup z \Leftrightarrow a \subseteq b \cup z'$.

If $z \in \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}$ then $a \subseteq b \cup z'$ and thus $z' \in \{z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z\}$.
But $z' \subseteq z$ thus having $\bigcap \{z \in \mathfrak{A} \mid z \subseteq a \wedge a \subseteq b \cup z\} \subseteq \bigcap \{z \in \mathfrak{A} \mid a \subseteq b \cup z\}$. \square

Remark 12 If we drop the requirement that \mathfrak{A} is distributive, two formulas for quasidifference (the definition and the last theorem) fork.

Obvious 21 *Dual quasicomplement is the dual of quasicomplement.*

Obvious 22 • *Every pseudocomplement is quasicomplement.*

- *Every dual pseudocomplement is dual quasicomplement.*
- *Every pseudodifference is quasidifference.*

Below we will stick to the more general quasies than pseudos. If needed, one can check that a quasicomplement a^* is a pseudocomplement by the equation $a^* \asymp a$ (and analogously with other quasies).

Next we will express quasidifference through quasicomplement.

Proposition 31

1. $a \setminus^* b = a \setminus^* (a \cap b)$ for any distributive lattice;
2. $a \# b = a \# (a \cap b)$ for any distributive lattice with least element.

Proof

1. $a \subseteq (a \cap b) \cup z \Leftrightarrow a \subseteq (a \cup z) \cap (b \cup z) \Leftrightarrow a \subseteq a \cup z \wedge a \subseteq b \cup z \Leftrightarrow a \subseteq b \cup z$. Thus $a \setminus^* (a \cap b) = \bigcap \{z \in \mathfrak{A} \mid a \subseteq (a \cap b) \cup z\} = \bigcap \{z \in \mathfrak{A} \mid a \subseteq b \cup z\} = a \setminus^* b$.
2. $a \# (a \cap b) = \bigcup \{z \in \mathfrak{A} \mid z \subseteq a \wedge z \cap a \cap b = 0\} = \bigcup \{z \in \mathfrak{A} \mid z \subseteq a \wedge (z \cap a) \cap a \cap b = 0\} = \bigcup \{z \cap a \mid z \in \mathfrak{A}, z \cap a \cap b = 0\} = \bigcup \{z \in \mathfrak{A} \mid z \subseteq a, z \cap b = 0\} = a \# b$.

\square

I will denote Da the lattice $\{x \in \mathfrak{A} \mid x \subseteq a\}$.

Theorem 56 For $a, b \in \mathfrak{A}$ where \mathfrak{A} is a bounded distributive lattice

1. $a \setminus^* b = (a \cap b)^{+(Da)}$;

$$2. a\#b = (a \cap b)^{*(Da)}.$$

Proof

1.

$$\begin{aligned} (a \cap b)^{+(Da)} &= \\ \bigcap \{c \in Da \mid c \cup (a \cap b) = a\} &= \\ \bigcap \{c \in Da \mid c \cup (a \cap b) \supseteq a\} &= \\ \bigcap \{c \in Da \mid (c \cup a) \cap (c \cup b) \supseteq a\} &= \\ \bigcap \{c \in \mathfrak{A} \mid c \subseteq a \wedge c \cup b \supseteq a\} &= \\ a \setminus^* b. & \end{aligned}$$

2.

$$\begin{aligned} (a \cap b)^{*(Da)} &= \\ \bigcup \{c \in Da \mid c \cap a \cap b = 0\} &= \\ \bigcup \{c \in \mathfrak{A} \mid c \subseteq a \wedge c \cap a \cap b = 0\} &= \\ \bigcup \{c \in \mathfrak{A} \mid c \subseteq a \wedge c \cap b = 0\} &= \\ a\#b. & \end{aligned}$$

□

Theorem 57 Let $(\mathfrak{F}; \mathfrak{A})$ is a primary filtrator where \mathfrak{A} is a boolean lattice. Let $\mathcal{A} \in \mathfrak{F}$. Then for each $\mathcal{X} \in \mathfrak{F}$

$$\mathcal{X} \in Z(D\mathcal{A}) \Leftrightarrow \exists X \in \mathfrak{A} : \mathcal{X} = X \cap^{\mathfrak{F}} \mathcal{A}.$$

Proof

\Leftarrow Let $\mathcal{X} = X \cap^{\mathfrak{F}} \mathcal{A}$ where $X \in \mathfrak{A}$. Let also $\mathcal{Y} = \overline{X} \cap^{\mathfrak{F}} \mathcal{A}$. Then $\mathcal{X} \cap^{\mathfrak{F}} \mathcal{Y} = X \cap^{\mathfrak{F}} \overline{X} \cap^{\mathfrak{F}} \mathcal{A} = (X \cap^{\mathfrak{A}} \overline{X}) \cap^{\mathfrak{F}} \mathcal{A} = 0$ (used the theorems 23 and 29) and $\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = (X \cup^{\mathfrak{F}} \overline{X}) \cap^{\mathfrak{F}} \mathcal{A} = (X \cup^{\mathfrak{A}} \overline{X}) \cap^{\mathfrak{F}} \mathcal{A} = 1 \cap^{\mathfrak{F}} \mathcal{A} = \mathcal{A}$ (used the theorems 23 and 10). So $\mathcal{X} \in Z(D\mathcal{A})$.

\Rightarrow Let $\mathcal{X} \in Z(D\mathcal{A})$. Then exists $\mathcal{Y} \in Z(D\mathcal{A})$ such that $\mathcal{X} \cap^{\mathfrak{F}} \mathcal{Y} = 0$ and $\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y} = \mathcal{A}$. Then (used the theorem 37) exists $X \in \text{up } \mathcal{X}$ such that $X \cap^{\mathfrak{F}} \mathcal{Y} = 0$. We have

$$\mathcal{X} = \mathcal{X} \cup^{\mathfrak{F}} (X \cap^{\mathfrak{F}} \mathcal{Y}) = X \cap^{\mathfrak{F}} (\mathcal{X} \cup^{\mathfrak{F}} \mathcal{Y}) = X \cap^{\mathfrak{F}} \mathcal{A}.$$

□

Proposition 32 $(a \cup b) \setminus^* b \subseteq a$ for an arbitrary complete lattice.

Proof $(a \cup b) \setminus^* b = \bigcap \{z \in \mathfrak{A} \mid a \cup b \subseteq b \cup z\}$.

But $a \subseteq z \Rightarrow a \cup b \subseteq b \cup z$. So $\{z \in \mathfrak{A} \mid a \cup b \subseteq b \cup z\} \supseteq \{z \in \mathfrak{A} \mid a \subseteq z\}$.

Consequently, $(a \cup b) \setminus^* b \subseteq \bigcap \{z \in \mathfrak{A} \mid a \subseteq z\} = a$. \square

13. Complements and core parts

Lemma 3 If $(\mathfrak{A}; \mathfrak{F})$ is a filtered, up-aligned filtrator with co-separable core which is a complete lattice, then for any $a, c \in \mathfrak{A}$

$$c \equiv^{\mathfrak{A}} a \Leftrightarrow c \equiv^{\mathfrak{A}} \text{Cor } a.$$

Proof

\Rightarrow If $c \equiv^{\mathfrak{A}} a$ then by co-separability of the core exists $K \in \text{down } a$ such that $c \equiv^{\mathfrak{A}} K$. To finish the proof we will show that $K \subseteq \text{Cor } a$. To show this is enough to show that $\forall X \in \text{up } a : K \subseteq X$ what is obvious.

\Leftarrow Because $\text{Cor } a \subseteq a$ (by the theorem 24 using that our filtrator is filtered).

\square

Theorem 58 If $(\mathfrak{A}; \mathfrak{F})$ is a filtered up-aligned complete lattice filtrator with finitely join-closed co-separable core which is a complete boolean lattice, then $a^+ = \overline{\text{Cor } a}$ for every $a \in \mathfrak{A}$.

Proof

$$\begin{aligned} a^+ &= \\ \bigcap^{\mathfrak{A}} \{c \in \mathfrak{A} \mid c \cup^{\mathfrak{A}} a = 1\} &= \\ \bigcap^{\mathfrak{A}} \{c \in \mathfrak{A} \mid c \cup^{\mathfrak{A}} \text{Cor } a = 1\} &= \\ \bigcap^{\mathfrak{A}} \{c \in \mathfrak{A} \mid c \supseteq \overline{\text{Cor } a}\} &= \\ \overline{\text{Cor } a}. & \end{aligned}$$

(used the lemma and the theorem 27). \square

Corollary 18 If $(\mathfrak{A}; \mathfrak{F})$ is a filtered up-aligned complete lattice filtrator with finitely join-closed co-separable core which is a complete boolean lattice, then $a^+ \in \mathfrak{F}$ for every $a \in \mathfrak{A}$.

Theorem 59 If $(\mathfrak{A}; \mathfrak{F})$ is a filtered complete lattice filtrator with down-aligned, finitely meet-closed, join-closed, separable core which is a complete boolean lattice, then $a^* = \overline{\text{Cor } a} = \text{Cor}' a$.

Proof $a^* = \bigcup^{\mathfrak{A}} \{c \in \mathfrak{A} \mid c \cap^{\mathfrak{A}} a = 0\}$. But $c \cap^{\mathfrak{A}} a = 0 \Rightarrow \exists C \in \text{up } c : C \cap^{\mathfrak{A}} a = 0$. So

$$\begin{aligned}
a^* &= \\
\bigcup^{\mathfrak{A}} \{C \in \mathfrak{Z} \mid C \cap^{\mathfrak{A}} a = 0\} &= \\
\bigcup^{\mathfrak{A}} \{C \in \mathfrak{Z} \mid a \subseteq \overline{C}\} &= \\
\bigcup^{\mathfrak{A}} \{\overline{C} \mid C \in \mathfrak{Z}, a \subseteq C\} &= \\
\bigcup^{\mathfrak{A}} \{\overline{C} \mid C \in \text{up } a\} &= \\
\bigcup^{\mathfrak{Z}} \{\overline{C} \mid C \in \text{up } a\} &= \\
\overline{\bigcap^{\mathfrak{Z}} \{C \mid C \in \text{up } a\}} &= \\
\overline{\bigcap^{\mathfrak{Z}} \text{up } a} &= \\
\overline{\text{Cor } a}. &
\end{aligned}$$

(used the theorem 27).

$\text{Cor } a = \text{Cor}' a$ by the theorem 26. \square

Corollary 19 *If $(\mathfrak{A}; \mathfrak{Z})$ is filtered down-aligned and up-aligned complete lattice filtrator with join closed, finitely meet-closed, separable and co-separable core which is a complete boolean lattice, then $a^* = a^+$ for every $a \in \mathfrak{A}$.*

Proof Comparing two last theorems. \square

Theorem 60 *If $(\mathfrak{A}; \mathfrak{Z})$ is a complete lattice filtrator with join-closed separable core which is a complete lattice, then $a^* \in \mathfrak{Z}$ for every $a \in \mathfrak{A}$.*

Proof $\{c \in \mathfrak{A} \mid c \cap^{\mathfrak{A}} a = 0\} \supseteq \{A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0\}$; consequently $a^* \supseteq \bigcup^{\mathfrak{A}} \{A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0\}$.

But if $c \in \{c \in \mathfrak{A} \mid c \cap^{\mathfrak{A}} a = 0\}$ then exists $A \in \mathfrak{Z}$ such that $A \supseteq c$ and $A \cap^{\mathfrak{A}} a = 0$ that is $A \in \{A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0\}$. Consequently $a^* \subseteq \bigcup^{\mathfrak{A}} \{A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0\}$.

We have $a^* = \bigcup^{\mathfrak{A}} \{A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0\} = \bigcup^{\mathfrak{Z}} \{A \in \mathfrak{Z} \mid A \cap^{\mathfrak{A}} a = 0\} \in \mathfrak{Z}$. \square

Theorem 61 *If $(\mathfrak{A}; \mathfrak{Z})$ is an up-aligned filtered complete lattice filtrator with finitely join closed co-separable core which is a complete boolean lattice, then a^+ is dual pseudocomplement of a , that is $a^+ = \min \{c \in \mathfrak{A} \mid c \cup^{\mathfrak{A}} a = 1\}$ for every $a \in \mathfrak{A}$.*

Proof It's enough to prove that $a^+ \cup^{\mathfrak{A}} a = 1$. But $a^+ \cup^{\mathfrak{A}} a = \overline{\text{Cor } a} \cup^{\mathfrak{A}} a \supseteq \overline{\text{Cor } a} \cup^{\mathfrak{A}} \text{Cor } a = \overline{\text{Cor } a} \cup^{\mathfrak{Z}} \text{Cor } a = 1$ (used the theorem 24 and the fact that our filtrator is filtered). \square

Definition 68 The *edge part* of an element $a \in \mathfrak{A}$ is $\text{Edg } a = a \setminus \text{Cor } a$, the *dual edge part* is $\text{Edg}' a = a \setminus \text{Cor}' a$.

Proposition 33 For a primary filtrator over a complete boolean lattice both edge part and dual edge part are always defined.

Proof Using the theorem 39. □

Knowing core part and edge part or dual core part and dual edge part of a filter object, the filter object can be restored by the formulas:

$$a = \text{Cor } a \cup^{\mathfrak{A}} \text{Edg } a \text{ and } a = \text{Cor}' a \cup^{\mathfrak{A}} \text{Edg}' a.$$

13.1. Core part and atomic elements

Proposition 34 Let $(\mathfrak{A}; \mathfrak{F})$ is a filtrator with join-closed core and \mathfrak{F} is an atomistic lattice. Then for every $a \in \mathfrak{A}$ such that $\text{Cor}' a$ exists we have

$$\text{Cor}' a = \bigcup^{\mathfrak{F}} \{x \mid x \text{ is an atom of } \mathfrak{F}, x \subseteq a\}.$$

Proof

$$\begin{aligned} \text{Cor}' a &= \\ \bigcup^{\mathfrak{F}} \{A \in \mathfrak{F} \mid A \subseteq a\} &= \\ \bigcup^{\mathfrak{F}} \left\{ \bigcup^{\mathfrak{F}} \text{atoms}^{\mathfrak{F}} A \mid A \in \mathfrak{F}, A \subseteq a \right\} &= \\ \bigcup^{\mathfrak{F}} \bigcup \{ \text{atoms}^{\mathfrak{F}} A \mid A \in \mathfrak{F}, A \subseteq a \} &= \\ \bigcup^{\mathfrak{F}} \{x \mid x \text{ is an atom of } \mathfrak{F}, x \subseteq a\}. & \end{aligned}$$

□

14. Number of filters on a set

Theorem 62 Let U is a set. The number of atomic f.o. on U is $2^{2^{\text{card}U}}$ if U is infinite and $\text{card}U$ if U is finite.

Proof See [11]. □

Corollary 20 The number of filters on U is $2^{2^{\text{card}U}}$ if U is infinite and $2^{\text{card}U}$ if U is finite.

Proof The finite case is obvious. The infinite case follows from the theorem and the fact that filters are collections of sets and there cannot be more than $2^{2^{\text{card}U}}$ collections of sets on U . □

15. Isomorphic filters

Below is defined equivalence relation “being isomorphic” for filters on sets.

Definition 69 I call filters on sets a and b **directly isomorphic** when there are a bijection $f : \bigcup a \rightarrow \bigcup b$ such that $\langle f \rangle|_a$ is a bijection from a to b .

Definition 70 Let U, W are sets. Let $\mathcal{P}U, \mathcal{P}W$ are the lattices of all subsets of these sets. Let a and b are filters on the lattices $\mathcal{P}U$ and $\mathcal{P}W$ correspondingly. I will call filters a and b **isomorphic** when there exist sets $A \in a$ and $B \in b$ such that filters $a \cap \mathcal{P}A$ and $b \cap \mathcal{P}B$ are directly isomorphic.

Obvious 23 Filters a and b are isomorphic iff there exist sets $A \in a$ and $B \in b$ such that there are a bijection $f : A \rightarrow B$ such that $\langle f \rangle|_{a \cap \mathcal{P}A}$ is a bijection from $a \cap \mathcal{P}A$ to $b \cap \mathcal{P}B$.

Remark 13 It seems that there exist a simpler definition of isomorphic filters in terms of reloids [12], but that stumbles on some open problems about reloids.

Proposition 35 If two filters are directly isomorphic then they are isomorphic.

Proof Take $A = \bigcup a, B = \bigcup b$. Then $a \cap \mathcal{P}A = a$ and $b \cap \mathcal{P}B = b$. □

Example 1 There exists a set U such that there are isomorphic filters on U which are not directly isomorphic.

Proof Consider two filters on \mathbb{N} : $a = \{\mathbb{N}\}$ and $b = \{\mathbb{N} \setminus \{0\}, \mathbb{N}\}$. There are no bijection from a to b because $\text{card } a \neq \text{card } b$. So a and b are not directly isomorphic.

Now let $A = \mathbb{N}$ and $B = \mathbb{N} \setminus \{0\}$ and $f : A \rightarrow B$ is defined by the formula $fx = x + 1$. Then $a \cap \mathcal{P}A = \{\mathbb{N}\}$ and $b \cap \mathcal{P}B = \{\mathbb{N} \setminus \{0\}\}$; f is a bijection from A to B and $\langle f \rangle|_{a \cap \mathcal{P}A} = \langle f \rangle|_{\{\mathbb{N}\}}$ is a bijection from $a \cap \mathcal{P}A$ to $b \cap \mathcal{P}B$. So $a \cap \mathcal{P}A$ and $b \cap \mathcal{P}B$ are directly isomorphic, that is a and b are isomorphic filters. □

Theorem 63 “Being directly isomorphic” for filters on sets is an equivalence relation.

Proof

Reflexivity Let a is a filter on some set A . Then the identity function Id_A is a bijection from $\bigcup a$ to $\bigcup a$. Evidently $\langle \text{Id}_A \rangle|_a$ is a bijection from a to a . So a and a are directly isomorphic.

Symmetry Let filters a and b be directly isomorphic. Then exists a bijection $f : \bigcup a \rightarrow \bigcup b$ such that $\langle f \rangle|_a$ is a bijection from a to b . $f^{-1} : \bigcup b \rightarrow \bigcup a$ is a bijection. To finish the proof it is enough to show that $\langle f^{-1} \rangle|_b$ is a bijection from b to a . First $\langle f^{-1} \rangle|_b$ is a function with domain b . Let $X, Y \in b$; if $X \neq Y$ then $\langle f^{-1} \rangle X \neq \langle f^{-1} \rangle Y$ because f^{-1} is a bijection, consequently $\langle f^{-1} \rangle|_b X \neq \langle f^{-1} \rangle|_b Y$. So $\langle f^{-1} \rangle|_b$ is an injection. Let $X \in a$; then $\langle f \rangle X \in b$ that is $\langle f \rangle X = Y$ where $Y \in b$; $X = \langle f^{-1} \rangle \langle f \rangle X = \langle f^{-1} \rangle Y = \langle f^{-1} \rangle|_b Y$; that is $\langle f^{-1} \rangle|_b$ is a function onto a . So $\langle f^{-1} \rangle|_b$ is a bijection from b to a . Hence b and a are directly isomorphic.

Transitivity Let filters a and b be directly isomorphic and filters b and c are directly isomorphic. Then exist bijections $f : \bigcup a \rightarrow \bigcup b$ and $g : \bigcup b \rightarrow \bigcup c$ such that $\langle f \rangle|_a$ is a bijection from a to b and $\langle g \rangle|_b$ is a bijection from b to c . The function $g \circ f$ is a bijection from $\bigcup a$ to $\bigcup c$. We have $\langle g \circ f \rangle|_a = \langle g \rangle|_b \circ \langle f \rangle|_a$ is a bijection as a composition of two bijections. So filters a and c are directly isomorphic.

□

Lemma 4 *If $a \cap \mathcal{P}A$ and $b \cap \mathcal{P}B$ are directly isomorphic, then for every $A' \in a \cap \mathcal{P}A$ there exists $B' \in b \cap \mathcal{P}B$ such that $a \cap \mathcal{P}A'$ and $b \cap \mathcal{P}B'$ are directly isomorphic.*

Proof Let $a \cap \mathcal{P}A$ and $b \cap \mathcal{P}B$ are directly isomorphic; let $A' \in a \cap \mathcal{P}A$. Then there are a bijection $f : A \rightarrow B$ such that $\langle f \rangle|_{a \cap \mathcal{P}A}$ is a bijection from $a \cap \mathcal{P}A$ to $b \cap \mathcal{P}B$. $a \cap \mathcal{P}A' \subseteq a \cap \mathcal{P}A$, hence $\langle f \rangle|_{a \cap \mathcal{P}A'}$ is an injection defined on the set $a \cap \mathcal{P}A'$. Let $B' = \langle f \rangle A'$. We have $B' \subseteq B$. Let $Y \in b \cap \mathcal{P}B'$; then $Y \in b \cap \mathcal{P}B$; by definition of “directly isomorphic” exists $X \in a \cap \mathcal{P}A$ such that $\langle f \rangle X = Y$. We have $X \subseteq A'$ because $Y \subseteq B'$ and because f is a bijection. So $X \in a \cap \mathcal{P}A'$. Thus $\langle f \rangle|_{a \cap \mathcal{P}A'}$ is a function onto $b \cap \mathcal{P}B'$. So $\langle f \rangle|_{a \cap \mathcal{P}A'}$ is a bijection. □

Theorem 64 *“Being isomorphic” for filters on sets is an equivalence relation.*

Proof

Reflexivity Let a is a filter. Let $A = \bigcup a$. Then $a = a \cap \mathcal{P}A$ is directly isomorphic to itself. Consequently a is isomorphic to itself.

Symmetry Let filters a and b are isomorphic. Then there exist sets $A \in a$ and $B \in b$ such that filters $a \cap \mathcal{P}A$ and $b \cap \mathcal{P}B$ are directly isomorphic. By proved above $b \cap \mathcal{P}B$ and $a \cap \mathcal{P}A$ are directly isomorphic. But this means that b and a are isomorphic.

Transitivity Let filters a and b are isomorphic and filters b and c are isomorphic. Then $a \cap \mathcal{P}A$ and $b \cap \mathcal{P}B_1$ are directly isomorphic and $b \cap \mathcal{P}B_2$ are $c \cap \mathcal{P}C$ are directly isomorphic where $A \in a$, $B_1, B_2 \in b$, $C \in c$. Let

$B_0 = B_1 \cap B_2$. We have $B_0 \in b$. By the lemma (taking in account also symmetry proved above) there exist sets $A_0 \in a \cap \mathcal{P}A$ and $C_0 \in c \cap \mathcal{P}C$ such that $a \cap \mathcal{P}A_0$ and $b \cap \mathcal{P}B_0$ are directly isomorphic and $b \cap \mathcal{P}B_0$ and $c \cap \mathcal{P}C_0$ are directly isomorphic. By transitivity of being direct isomorphic we have that $a \cap \mathcal{P}A_0$ and $c \cap \mathcal{P}C_0$ are directly isomorphic. Hence a and c are isomorphic.

□

Proposition 36

1. *Principal filters, generated by sets of the same cardinality, are isomorphic.*
2. *If a filter is isomorphic to a principal filter, then it is also a principal filter with the same cardinality of the sets which generate these principal filters.*

Proof

1. Let a and b are principal filters. Let they are generated by the sets A and B correspondingly where $\text{card } A = \text{card } B$. Enough to prove that $a \cap \mathcal{P}A$ and $b \cap \mathcal{P}B$ are directly isomorphic. But this follows from $a \cap \mathcal{P}A = \{A\}$ and $b \cap \mathcal{P}B = \{B\}$ and $\text{card } A = \text{card } B$, because any bijection from A to B sends $\{A\}$ to $\{B\}$.
2. Let a is a principal filter generated by a set A , let b is a filter isomorphic to a . We shall prove that b is principal and is generated by a set B such that $\text{card } B = \text{card } A$. Let $A \in a$ and $B \in b$ and f is a bijection from A to B such that $\langle f \rangle|_{a \cap \mathcal{P}A}$ is a bijection from $a \cap \mathcal{P}A$ to $b \cap \mathcal{P}B$. We have $b \cap \mathcal{P}B = \{\langle f \rangle|_{a \cap \mathcal{P}A} X \mid X \in a \cap \mathcal{P}A\}$. So $b \cap \mathcal{P}B$ has the smallest element $\langle f \rangle A$. Consequently b has the smallest element $\langle f \rangle A$ and so b is a principal filter generated by the set $B = \langle f \rangle A$ whose cardinality is $\text{card } A$.

□

Proposition 37 *A filter isomorphic to a nontrivial ultrafilter is a nontrivial ultrafilter.*

Proof Let a is a nontrivial ultrafilter and let b is isomorphic to a . Then there are sets $A \in a$ and $B \in b$ and a bijection $f : A \rightarrow B$ such that $\langle f \rangle|_{a \cap \mathcal{P}A}$ is a bijection from $a \cap \mathcal{P}A$ to $b \cap \mathcal{P}B$.

The filter b cannot be a trivial ultrafilter because otherwise a would be also a principal filter. So it's enough to prove that b is an ultrafilter. We will prove that for any set $Y \in \mathcal{P} \cup b$ either Y or $(\cup b) \setminus Y$ is an element of b .

Let $Y \in \mathcal{P} \cup b$. Then $Y' = Y \cap B \in \mathcal{P}B$; $Y' = \langle f \rangle X'$ for some $X' \in \mathcal{P}A$. Because a is an ultrafilter, either $X' \in a$ or $(\cup a) \setminus X' \in a$. If $X' \in a$ then $X' \in a \cap \mathcal{P}A$ and so $Y' = \langle f \rangle X' = \langle f \rangle|_{a \cap \mathcal{P}A} X' \in b \cap \mathcal{P}B$ and consequently $Y \in b$. If $(\cup a) \setminus X' \in a$ then $A \setminus X' \in a \cap \mathcal{P}A$; consequently $\langle f \rangle|_{a \cap \mathcal{P}A}(A \setminus X') \in b \cap \mathcal{P}B$; $\langle f \rangle|_{a \cap \mathcal{P}A}(A) \setminus \langle f \rangle|_{a \cap \mathcal{P}A}(X') \in b \cap \mathcal{P}B$; $B \setminus Y' \in b$; $(\cup b) \setminus Y \in b$. □

Theorem 65 For an infinite set U there exist $2^{2^{\text{card } U}}$ equivalence classes of isomorphic ultrafilters.

Proof The number of bijections between any two given subsets of U is no more than $(\text{card } U)^{\text{card } U} = 2^{\text{card } U}$. The number of bijections between all pairs of subsets of U is no more than $2^{\text{card } U} \cdot 2^{\text{card } U} = 2^{2^{\text{card } U}}$. Therefore each isomorphism class contains at most $2^{\text{card } U}$ ultrafilters. But there are $2^{2^{\text{card } U}}$ ultrafilters. So there are $2^{2^{\text{card } U}}$ classes. \square

Remark 14 One of the above mentioned equivalence classes contains trivial ultrafilters.

Corollary 21 There exist non-isomorphic nontrivial ultrafilters on any infinite set.

16. Partitioning filter objects

Definition 71 Let \mathfrak{A} is a complete lattice. **Thorning** of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{0\}$ such that

$$\bigcup^{\mathfrak{A}} S = a \quad \text{and} \quad \forall x, y \in S : x \asymp y.$$

Definition 72 Let \mathfrak{A} is a complete lattice. **Weak partitioning** of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{0\}$ such that

$$\bigcup^{\mathfrak{A}} S = a \quad \text{and} \quad \forall x \in S : x \asymp \bigcup^{\mathfrak{A}} (S \setminus \{x\}).$$

Definition 73 Let \mathfrak{A} is a complete lattice. **Strong partitioning** of an element $a \in \mathfrak{A}$ is a set $S \in \mathcal{P}\mathfrak{A} \setminus \{0\}$ such that

$$\bigcup^{\mathfrak{A}} S = a \quad \text{and} \quad \forall A, B \in \mathcal{P}S : (A \asymp B \Rightarrow \bigcup^{\mathfrak{A}} A \asymp \bigcup^{\mathfrak{A}} B).$$

Obvious 24 1. Every strong partitioning is a weak partitioning.

2. Every weak partitioning is a thorning.

See the section “Open problems” for supposed properties of partitionings.

17. Open problems

In this section I will formulate some conjectures about lattices of filter objects on a set. If a conjecture comes true, it may be generalized for more general lattices (such as, for example, lattices of filters on arbitrary lattices). I deem that the main challenge is to prove the special case about lattices of filter objects on a set, and generalizing the conjectures is expected to be a simple task.

17.1. Partitioning

Conjecture 1 *Weak partitioning of an element of a lattice of filter objects on a set is the same as its strong partitioning.*

The above conjecture may be strengthened to:

Conjecture 2 *Weak partitioning of an element of a complete lattice is the same as its strong partitioning.*

Now consider the complete lattice $[S]$ generated by the set S where S is a strong partitioning of some element a .

Conjecture 3 $[S] = \{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S\}$, where $[S]$ is the complete lattice generated by a strong partitioning S of an element of a lattice \mathfrak{F} of filter objects on a set.

Proposition 38 *Provided that the last conjecture is true, we have that $[S]$ is a complete atomic boolean lattice with the set of its atoms being S .*

Remark 15 Consequently S is atomistic, completely distributive and isomorphic to a power set algebra (see [13]).

Proof Completeness of $[S]$ is obvious. Let $A \in [S]$. Then exists $X \in \mathcal{P}S$ such that $A = \bigcup^{\mathfrak{F}} X$. Let $B = \bigcup^{\mathfrak{F}}(S \setminus X)$. Then $B \in [S]$ and $A \cap B = 0$. $A \cup B = \bigcup^{\mathfrak{F}} S$ is the biggest element of S . So we have proved that $[S]$ is a boolean lattice.

Now let prove that $[S]$ is atomic with the set of atoms being S . Let $z \in S$ and $A \in [S]$. If $A \neq z$ then either $A = 0$ or $x \in X$ where $A = \bigcup^{\mathfrak{F}} X$, $X \in \mathcal{P}S$ and $x \neq z$. Because S is a partitioning, $\bigcup^{\mathfrak{F}}(X \setminus \{z\}) \cap^{\mathfrak{F}} z = 0$ and $\bigcup^{\mathfrak{F}}(X \setminus \{z\}) \neq 0$. So $A = \bigcup^{\mathfrak{F}} X = \bigcup^{\mathfrak{F}}(X \setminus \{z\}) \cup^{\mathfrak{F}} z \not\subseteq z$.

Finally we will prove that elements of $[S] \setminus S$ are not atoms. Let $A \in [S] \setminus S$ and $A \neq 0$. Then $A \supseteq x \cup^{\mathfrak{F}} y$ where $x, y \in S$ and $x \neq y$. If A is an atom then $A = x = y$ what is impossible. \square

Proposition 39 *The conjecture about the value of $[S]$ is equivalent to closedness of $\{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S\}$ under arbitrary meets and joins.*

Proof If $\{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S\} = [S]$ then trivially $\{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S\}$ is closed under arbitrary meets and joins.

If $\{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S\}$ is closed under arbitrary meets and joins, then it is the complete lattice generated by the set S because it cannot be smaller than the set of all suprema of subsets of S . \square

That $\{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{P}S\}$ is closed under arbitrary joins is trivial. I have not succeeded to prove that it is closed under arbitrary meets, but have proved a weaker statement that is is closed under finite meets:

Proposition 40 $\{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{PS}\}$ is closed under finite meets.

Proof Let $R = \{\bigcup^{\mathfrak{F}} X \mid X \in \mathcal{PS}\}$. Then

$$\begin{aligned}
& \bigcup^{\mathfrak{A}} X \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y = \\
& \bigcup^{\mathfrak{A}} ((X \cap Y) \cup (X \setminus Y)) \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y = \\
& \left(\bigcup^{\mathfrak{A}} (X \cap Y) \cup^{\mathfrak{A}} \bigcup^{\mathfrak{A}} (X \setminus Y) \right) \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y = \\
& \left(\bigcup^{\mathfrak{A}} (X \cap Y) \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y \right) \cup^{\mathfrak{A}} \left(\bigcup^{\mathfrak{A}} (X \setminus Y) \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y \right) = \\
& \left(\bigcup^{\mathfrak{A}} (X \cap Y) \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y \right) \cup^{\mathfrak{A}} 0 = \\
& \bigcup^{\mathfrak{A}} (X \cap Y) \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y.
\end{aligned}$$

Applying the formula $\bigcup^{\mathfrak{A}} X \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y = \bigcup^{\mathfrak{A}} (X \cap Y) \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y$ twice we get

$$\begin{aligned}
& \bigcup^{\mathfrak{A}} X \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y = \\
& \bigcup^{\mathfrak{A}} (X \cap Y) \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} (Y \cap (X \cap Y)) = \\
& \bigcup^{\mathfrak{A}} (X \cap Y) \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} (X \cap Y) = \\
& \bigcup^{\mathfrak{A}} (X \cap Y).
\end{aligned}$$

But for any $A, B \in R$ exist $X, Y \in \mathcal{PS}$ such that $A = \bigcup^{\mathfrak{A}} X$ and $B = \bigcup^{\mathfrak{A}} Y$.
So $A \cap^{\mathfrak{A}} B = \bigcup^{\mathfrak{A}} X \cap^{\mathfrak{A}} \bigcup^{\mathfrak{A}} Y = \bigcup^{\mathfrak{A}} (X \cap Y) \in R$. \square

Conjecture 4

1. Every filter object on a set can be partitioned into atomic filter objects.
2. This partitioning is unique.

17.2. Quasidifference

Problem 1 Which of the following expressions are pairwise equal for all $a, b \in \mathfrak{F}$ for each lattice \mathfrak{F} of filters on a set? (If some are not equal, provide counter-examples.)

1. $\bigcap^{\mathfrak{F}} \{z \in \mathfrak{F} \mid a \subseteq b \cup^{\mathfrak{F}} z\}$ (quasidifference of a and b);
2. $\bigcup^{\mathfrak{F}} \{z \in \mathfrak{F} \mid z \subseteq a \wedge z \cap^{\mathfrak{F}} b = \emptyset\}$ (second quasidifference of a and b);
3. $\bigcup^{\mathfrak{F}} (\text{atoms}^{\mathfrak{F}} a \setminus \text{atoms}^{\mathfrak{F}} b)$;
4. $\bigcup^{\mathfrak{F}} \{a \cap^{\mathfrak{F}} (U \setminus B) \mid B \in \text{up } b\}$.

17.3. Complementary filter objects

Conjecture 5 $Z(D\mathcal{A})$ is a complete lattice when $\mathcal{A} \in \mathfrak{F}$ where \mathfrak{F} is the set of f.o. on some set U . (If true, this conjecture may be generalized.)

Conjecture 6 $Z(D\mathcal{A})$ is join-closed for any f.o. \mathcal{A} .

Proposition 41 The last conjecture follows from the previous.

Proof Let $S \in \mathcal{P}Z(D\mathcal{A})$. We need to prove that $\bigcup^3 Z(D\mathcal{A})S = \bigcup^{\mathfrak{F}} S$. It is enough to prove that for any $\mathcal{F} \in D\mathcal{A}$

$$\forall K \in S : K \subseteq \mathcal{F} \Rightarrow \mathcal{F} \supseteq \bigcup^3 Z(D\mathcal{A})S.$$

Really: Let $C \in \text{up}^{D\mathcal{A}} \mathcal{F}$ and $\forall K \in S : K \subseteq \mathcal{F}$. Then $C \supseteq \mathcal{F}$; $C \supseteq \bigcup^3 Z(D\mathcal{A})S$; $C \in \text{up} \bigcup^3 Z(D\mathcal{A})S$; consequently $\mathcal{F} \supseteq \bigcup^3 Z(D\mathcal{A})S$. \square

In the light of the theorem 57, it is simple to prove that the above conjecture follows from each of two items of the following conjecture:

Conjecture 7 Consider the filtrator $(\mathfrak{F}; \mathcal{P}U)$ where U is some set and \mathfrak{F} is the set of filter objects on $\mathcal{P}U$. Let $\mathcal{A} \in \mathfrak{F}$ and $T \in \mathcal{P}\mathcal{P}U$. Then:

1. $\bigcup^3 Z(D\mathcal{A}) \{X \cap^{\mathfrak{F}} \mathcal{A} \mid X \in T\} = \mathcal{A} \cap^{\mathfrak{F}} \bigcup T$;
2. $\bigcap^3 Z(D\mathcal{A}) \{X \cap^{\mathfrak{F}} \mathcal{A} \mid X \in T\} = \mathcal{A} \cap^{\mathfrak{F}} \bigcap T$.

We can prove that $a \setminus^* b = a \# b$ if we can prove that Da is a filtered filtrator with both separable and co-separable and join-closed center because of formulas expressing equality of $a \setminus^* b$ and $a \# b$ and the theorem providing the equality $a^* = a^+$.

17.4. Misc problems

Conjecture 8 For a filtrator $(\mathfrak{F}; \mathfrak{A})$ of filters on a set we have $A \cap^{\mathfrak{F}} \bigcup^{\mathfrak{F}} S = \bigcup^{\mathfrak{F}} \langle A \cap^{\mathfrak{F}} \rangle S$ for every $A \in \mathfrak{A}$, $S \in \mathcal{P}\mathfrak{F}$.

Conjecture 9 Let \mathfrak{A} is a complete lattice. A set $S \in \mathcal{P}\mathfrak{A}$ is filter-closed iff for every nonempty chain $T \in \mathcal{P}S$ we have $\bigcap T \in S$.

17.5. Non-formal problems

Develop the theory of atoms on posets not having least element. See [10] for the general definition of atoms.

Should we introduce the concept of star objects, analogous to filter objects, and research the lattice of star objects?

Find a common generalization of two theorems:

1. If \mathfrak{A} is a meet-semilattice with greatest element 1 then for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$

$$\text{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \text{up} \mathcal{A} \cap \text{up} \mathcal{B}.$$

2. If \mathfrak{A} is a join-semilattice then \mathfrak{F} is a join-semilattice then and for any $\mathcal{A}, \mathcal{B} \in \mathfrak{F}$

$$\text{up}(\mathcal{A} \cup^{\mathfrak{F}} \mathcal{B}) = \text{up } \mathcal{A} \cap \text{up } \mathcal{B}.$$

Under which conditions $a \setminus * b$ and $a \# b$ are complementive to a ?

Generalize straight maps for arbitrary posets.

18. Postface

I have set a wiki site to collaboratively write a book which will extend and amend this article and will emerge into the exhaustive reference text about filters on posets, filters on lattices, and generalizations thereof. I call you to collaborate with me writing this book.

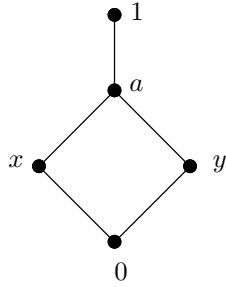
The wiki site URL is: <http://filters.wikidot.com>.

I also call you to solve open problems which I formulated in this article.

Appendix A. Some counter-examples

Example 2 There exist a bounded distributive lattice which is not lattice with separable center.

Proof The lattice with the following Hasse diagram is bounded and distributive because it does not contain “diamond lattice” nor “pentagon lattice” as a sublattice [14].



It's center is $\{0, 1\}$. $x \cap y = 0$ indeed $\text{up } x = \{1\}$ but $1 \cap y \neq 0$ consequently the lattice is not with separable center. \square

For further examples we will use the filter object Δ defined by the formula

$$\Delta = \bigcap^{\mathfrak{F}} \{(-\varepsilon; \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\}$$

and more general

$$\Delta + a = \bigcap^{\mathfrak{F}} \{(a - \varepsilon; a + \varepsilon) \mid \varepsilon \in \mathbb{R}, \varepsilon > 0\}.$$

Example 3 There exists $A \in \mathcal{P}U$ such that $\bigcap^{\mathfrak{S}} A \neq \bigcap^3 \mathcal{P}UA$ for some set U .

Proof $\bigcap^3 \mathcal{P}\mathbb{R} \{(-a; a) \mid a \in \mathbb{R}, a > 0\} = \{0\} \neq \Delta$. \square

Example 4 There exists a set U and there are a f.o. a and a set S of f.o. on the lattice $\mathcal{P}U$ such that $a \cap^{\mathfrak{S}} \bigcup^{\mathfrak{S}} S \neq \bigcup^{\mathfrak{S}} \langle a \cap^{\mathfrak{S}} \rangle S$.

Proof Let $a = \Delta$ and $S = \{(\varepsilon; +\infty) \mid \varepsilon > 0\}$. Then $a \cap^{\mathfrak{S}} \bigcup^{\mathfrak{S}} S = \Delta \cap^{\mathfrak{S}} (0; +\infty) \neq \emptyset$ while $\bigcup^{\mathfrak{S}} \langle a \cap^{\mathfrak{S}} \rangle S = \bigcup^{\mathfrak{S}} \{\emptyset\} = \emptyset$. \square

Example 5 There are thornings which are not weak partitionings.

Proof $\{\Delta + a \mid a \in \mathbb{R}\}$ is a thorning but not weak partitioning of the real line. \square

Appendix B. Logic of Generalizations

In mathematics it is often encountered that a smaller set S naturally bijectively corresponds to a subset R of a larger set B . (In other words, there is specified an injection from S to B .) It is a widespread practice to equate S with R .

Remark 16 I denote the first set S from the first letter of the word “small” and the second set B from the first letter of the word “big”, because S is intuitively considered as smaller than B . (However we do not require $\text{card } S < \text{card } B$.)

The set B is considered as a generalization of the set S , for example: whole numbers generalizing natural numbers, rational numbers generalizing whole numbers, real numbers generalizing rational numbers, complex numbers generalizing real numbers, etc.

But strictly speaking this equating may contradict to the axioms of ZF/ZFC because we are not insured against $S \cap B \neq \emptyset$ incidents. Not wonderful, as it is often labeled as “without proof”.

To work around of this (and formulate things exactly what could benefit computer proof assistants) we will replace the set B with a new set B' having a bijection $M : B \rightarrow B'$ such that $S \subseteq B'$. (I call this bijection M from the first letter of the word “move” which signifies the move from the old set B to a new set B').

Appendix B.1. The formalistic

Let S and B are sets. Let E is an injection from S to B . Let $R = \text{im } E$.

Let $t = \mathcal{P} \bigcup \bigcup S$.

Let $M(x) = \begin{cases} E^{-1}x & \text{if } x \in R; \\ (t; x) & \text{if } x \notin R. \end{cases}$

Recall that in standard ZF $(t; x) = \{\{t\}, \{t, x\}\}$ by definition.

Theorem 66 $(t; x) \notin S$.

Proof Suppose $(t; x) \in S$. Then $\{\{t\}, \{t, x\}\} \in S$. Consequently $\{t\} \in \bigcup S$; $t \subseteq \bigcup \bigcup S$; $t \in \mathcal{P} \bigcup \bigcup S$; $t \in t$ what contradicts to the axiom of foundation (aka axiom of regularity). \square

Definition 74 Let $B' = \text{im } M$.

Theorem 67 $S \subseteq B'$.

Proof Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$; $x \in \text{im } M = B'$. \square

Obvious 25 E is a bijection from S to R .

Theorem 68 M is a bijection from B to B' .

Proof Surjectivity of M is obvious. Let's prove injectivity. Let $a, b \in B$ and $M(a) = M(b)$. Consider all cases:

$a, b \in R$ $M(a) = E^{-1}a$; $M(b) = E^{-1}b$; $E^{-1}a = E^{-1}b$; thus $a = b$ because E^{-1} is a bijection.

$a \in R, b \notin R$ $M(a) = E^{-1}a$; $M(b) = (t; b)$; $M(a) \in S$; $M(b) \notin S$. Thus $M(a) \neq M(b)$.

$a \notin R, b \in R$ Analogous,

$a, b \notin R$ $M(a) = (t; a)$; $M(b) = (t; b)$. Thus $M(a) = M(b)$ implies $a = b$.

\square

Theorem 69 $M \circ E = \text{id}_S$.

Proof Let $x \in S$. Then $Ex \in R$; $M(Ex) = E^{-1}Ex = x$. \square

Obvious 26 $E = M^{-1}|_S$.

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