Orderings of filters in terms of reloids*

Extensions of Rudin-Keisler ordering

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Abstract

Orderings of filters which extend Rudin-Keisler preorder of ultrafilters are described in terms of reloids that (roughly speaking) is filters on sets of binary relations between some sets. Also it is defined isomorphism of filters which extends Rudin-Keisler equivalence of ultrafilters.

Keywords: Rudin-Keisler order, Rudin-Keisler preorder, Rudin-Keisler ordering, filters, ultrafilters, reloids, isomorphism, isomorphic

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1 Draft status

It is a draft.

2 Preliminary definitions

Whilist my other works use filters to research funcoids and reloids [3], here it is discussed the opposite thing, the theory of reloids is used to describe properties of filters.

See [4] for the definition of filter objects and [3] for the definition and properties of reloids and funcoids.

I will call *small* sets members of some Grothendieck universe.

Recall that morphisms of the category **Set** (or "**Set**-morphisms" for short) are triples $(F;A;B)$ of a function F and small sets A and B where $\operatorname{dom} F \subseteq A$ and $\operatorname{im} F \subseteq B$.

For $X \in \mathscr{P}A$ we'll denote $\langle (F;A;B) \rangle X = \langle F \rangle X$.

Let $f = (F;A;B)$ is a **Set**-morphism. I will denote in this article

$$\uparrow f = \left(\uparrow^{\mathsf{FCD}(A;B)} F; A; B\right) \quad \text{and} \quad \uparrow^{\mathsf{RLD}} f = \left(\uparrow^{\mathsf{RLD}(A;B)} F; A; B\right).$$

2.1 Equivalent filters

We will restrict to small sets that is members of some Grothendieck universe.

Definition 1. Two filter objects $\mathcal{A}$ and $\mathcal{B}$ (with possibly different base sets) are *equivalent* ($\mathcal{A} \sim \mathcal{B}$) iff there exists a set X such that $X \in \operatorname{up} \mathcal{A}$ and $X \in \operatorname{up} \mathcal{B}$ and $\mathscr{P}X \cap \operatorname{up} \mathcal{A} = \mathscr{P}X \cap \operatorname{up} \mathcal{B}$.

Proposition 2. If two filter objects with the same base are equivalent they are equal.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ are two f.o. and $\mathscr{P}X \cap \operatorname{up} \mathcal{A} = \mathscr{P}X \cap \operatorname{up} \mathcal{B}$ for some set X such that $X \in \operatorname{up} \mathcal{A}$ and $X \in \operatorname{up} \mathcal{B}$, and $\operatorname{Base}(\mathcal{A}) = \operatorname{Base}(\mathcal{B})$. Then $\operatorname{up} \mathcal{A} = (\mathscr{P}X \cap \operatorname{up} \mathcal{A}) \cup \{ Y \in \mathscr{P}\operatorname{Base}(\mathcal{A}) \mid Y \supseteq X \} = (\mathscr{P}X \cap \operatorname{up} \mathcal{B}) \cup \{ Y \in \mathscr{P}\operatorname{Base}(\mathcal{B}) \mid Y \supseteq X \} = \operatorname{up} \mathcal{B}$. □

Proposition 3. $\sim$ restricted to small filter objects is an equivalence relation.

Proof.

Reflexivity. Obvious.

Symmetry. Obvious.

*. This document has been written using the GNU TeXmacs text editor (see `www.texmacs.org`).

Transitivity. Let $\mathcal{A} \sim \mathcal{B}$ and $\mathcal{B} \sim \mathcal{C}$ for some small f.o. $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Then exist a set X such that $X \in \operatorname{up} \mathcal{A}$ and $X \in \operatorname{up} \mathcal{B}$ and $\mathscr{P} X \cap \operatorname{up} \mathcal{A} = \mathscr{P} X \cap \operatorname{up} \mathcal{B}$ and a set Y such that $Y \in \operatorname{up} \mathcal{B}$ and $Y \in \operatorname{up} \mathcal{C}$ and $\mathscr{P} Y \cap \operatorname{up} \mathcal{B} = \mathscr{P} Y \cap \operatorname{up} \mathcal{C}$. So $X \cap Y \in \operatorname{up} \mathcal{A}$ because

$$\mathscr{P} Y \cap \mathscr{P} X \cap \operatorname{up} \mathcal{A} = \mathscr{P} Y \cap \mathscr{P} X \cap \operatorname{up} \mathcal{B} = \mathscr{P}(X \cap Y) \cap \operatorname{up} \mathcal{B} \supseteq \{X \cap Y\} \cap \operatorname{up} \mathcal{B} \ni X \cap Y.$$

Similarly we have $X \cap Y \in \operatorname{up} \mathcal{C}$. Finally $\mathscr{P}(X \cap Y) \cap \operatorname{up} \mathcal{A} = \mathscr{P} Y \cap \mathscr{P} X \cap \operatorname{up} \mathcal{A} = \mathscr{P} Y \cap \mathscr{P} X \cap \operatorname{up} \mathcal{B} = \mathscr{P} X \cap \mathscr{P} Y \cap \operatorname{up} \mathcal{B} = \mathscr{P} X \cap \mathscr{P} Y \cap \operatorname{up} \mathcal{C} = \mathscr{P}(X \cap Y) \cap \operatorname{up} \mathcal{C}$. □

Definition 4. The *rebase* $\mathcal{A} \div A$ for a f.o. $\mathcal{A}$ and a set A (base) such that $\exists X \in \operatorname{up} \mathcal{A}: X \subseteq A$ is defined by the formula

$$\mathcal{A} \div A = \operatorname{up}^{-1}\{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$$

where "up" is taken for the set of f.o. on A.

Proposition 5. If $\exists X \in \operatorname{up} \mathcal{A}: X \subseteq A$ then:

1. $\mathcal{A} \div A$ is a f.o.
2. $\mathcal{A} \div A \sim \mathcal{A}$.

Proof.

1. We need to prove that $\{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$ is a filter on A. That it is an upper set is obvious. It is non-empty because $\exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq A$ and thus $A \in \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$. Let $P, Q \in \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$. Then $P, Q \subseteq A$ and $\exists P' \in \operatorname{up} \mathcal{A}: P' \subseteq P$ and $\exists Q' \in \operatorname{up} \mathcal{A}: Q' \subseteq Q$. So $P \cap Q \subseteq A$ and $P' \cap Q' \subseteq P \cap Q$. Thus $P \cap Q \in \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\}$.
2. $\mathscr{P}(A \cap \operatorname{Base}(\mathcal{A})) \cap \operatorname{up} \mathcal{A} = \operatorname{up}(\mathcal{A} \div (A \cap \operatorname{Base}(\mathcal{A}))) = \{X \in \mathscr{P}(A \cap \operatorname{Base}(\mathcal{A})) \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\} = \mathscr{P}(A \cap \operatorname{Base}(\mathcal{A})) \cap \operatorname{up} \mathcal{A} = \mathscr{P} A \cap \operatorname{up} \mathcal{A} = \{X \in \mathscr{P} A \mid X \in \operatorname{up} \mathcal{A}\} = \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X, X \subseteq \operatorname{Base}(\mathcal{A})\} = \mathscr{P}(A \cap \operatorname{Base}(\mathcal{A})) \cap \operatorname{up}(\mathcal{A} \div A)$.

   Thus $\mathcal{A} \div A \sim \mathcal{A}$ because $A \cap \operatorname{Base}(\mathcal{A}) \supseteq X \in \operatorname{up} \mathcal{A}$ for some $X \in \operatorname{up} \mathcal{A}$ and $A \cap \operatorname{Base}(\mathcal{A}) \supseteq X \cap \operatorname{Base}(\mathcal{A}) \in \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\} = \operatorname{up}(\mathcal{A} \div A)$. □

Proposition 6. $A \in \operatorname{up} \mathcal{A} \Rightarrow \operatorname{up}(\mathcal{A} \div A) = \mathscr{P} A \cap \operatorname{up} \mathcal{A}$.

Proof. Let $A \in \operatorname{up} \mathcal{A}$. Then $\operatorname{up}(\mathcal{A} \div A) = \{X \in \mathscr{P} A \mid \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X\} = \{X \in \mathscr{P} A \mid X \in \operatorname{up} \mathcal{A}\} = \mathscr{P} A \cap \operatorname{up} \mathcal{A}$. □

Lemma 7. If $\mathcal{A} \sim \mathcal{B}$ then $\exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X \Leftrightarrow \exists Y \in \operatorname{up} \mathcal{B}: Y \subseteq X$ for every f.o. $\mathcal{A}$, $\mathcal{B}$, and a set X.

Proof. We will prove $\exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X \Rightarrow \exists Y \in \operatorname{up} \mathcal{B}: Y \subseteq X$ (the other direction is similar).

We have $\mathscr{P} K \cap \operatorname{up} \mathcal{A} = \mathscr{P} K \cap \operatorname{up} \mathcal{B}$ for some set K such that $K \in \operatorname{up} \mathcal{A}$, $K \in \operatorname{up} \mathcal{B}$.

$\exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X \Rightarrow \exists Y \in \mathscr{P} K \cap \operatorname{up} \mathcal{A}: Y \subseteq X \Rightarrow \exists Y \in \mathscr{P} K \cap \operatorname{up} \mathcal{B}: Y \subseteq X \Rightarrow \exists Y \in \operatorname{up} \mathcal{B}: Y \subseteq X$. □

Proposition 8. If $\mathcal{A} \sim \mathcal{B}$ then $\mathcal{B} = \mathcal{A} \div \operatorname{Base}(\mathcal{B})$ for every f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof. $\mathscr{P} Y \cap \operatorname{up} \mathcal{A} = \mathscr{P} Y \cap \operatorname{up} \mathcal{B}$ for some set $Y \in \operatorname{up} \mathcal{A}$, $Y \in \operatorname{up} \mathcal{B}$. There exists a set $X \in \operatorname{up} \mathcal{A}$ such that $X \in \operatorname{up} \mathcal{B}$. Thus $\exists X \in \operatorname{up} \mathcal{A}: X \subseteq \operatorname{Base}(\mathcal{B})$ and so $\mathcal{A} \div \operatorname{Base}(\mathcal{B})$ is a properly defined f.o.

$X \in \operatorname{up}(\mathcal{A} \div \operatorname{Base}(\mathcal{B})) \Leftrightarrow X \in \mathscr{P}\operatorname{Base}(\mathcal{B}) \wedge \exists Y \in \operatorname{up} \mathcal{A}: Y \subseteq X \Leftrightarrow X \in \mathscr{P}\operatorname{Base}(\mathcal{B}) \wedge \exists Y \in \operatorname{up} \mathcal{B}: Y \subseteq X \Leftrightarrow X \in \operatorname{up} \mathcal{B}$ (the lemma used). □

3 Ordering of filters

Below I will define some categories having filter objects (with possibly different bases) as their objects and some relations having two filter objects (with possibly different bases) as arguments induced by these categories (defined as existence of a morphism between these two f.o.).

Theorem 9. $\operatorname{card} a = \operatorname{card} U$ for every ultrafilter a on U if U is infinite.

Proof. Let $f(X)=X$ if $X\in a$ and $f(X)=U\setminus X$ if $X\notin a$. Obviously f is a surjection from U to a.

Every $X\in a$ appears as a value of f exactly twice, as $f(X)$ and $f(U\setminus X)$. So $\operatorname{card} a=U/2=U$. □

Corollary 10. Cardinality of every two ultrafilters on a set U is the same.

Proof. For infinite U it follows from the theorem. For finite case it is obvious. □

Definition 11. $f*\mathcal{A}=\{C\in\mathscr{P}(\operatorname{Dst} f)\mid\langle f^{-1}\rangle C\in\operatorname{up}\mathcal{A}\}$ for every f.o. $\mathcal{A}$ and a **Set**-morphism f.

Below I'll define some directed multigraphs. By an abuse of notation, I will denote these multigraphs the same as (below defined) categories based on some of these these directed multigraphs with added composition of morphisms (of directed multigraphs edges). As such I will call vertices of these multigraphs objects and edges morphisms.

Definition 12. I will denote $\mathbf{GreFunc}_1$ the multigraph whose objects are filter objects and whose morphisms between objects $\mathcal{A}$ and $\mathcal{B}$ are **Set**-morphisms from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that $\operatorname{up}\mathcal{B}\supseteq f*\mathcal{A}$.

Definition 13. I will denote $\mathbf{GreFunc}_2$ the multigraph whose objects are filter objects and whose morphisms between objects $\mathcal{A}$ and $\mathcal{B}$ are **Set**-morphisms from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that $\operatorname{up}\mathcal{B}=f*\mathcal{A}$.

Definition 14. Let $\mathcal{A}$ is a f.o. on a set X and $\mathcal{B}$ is a f.o. on a set Y. $\mathcal{A}\geqslant_1\mathcal{B}$ iff $\operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A};\mathcal{B})$ is not empty.

Definition 15. Let $\mathcal{A}$ is an f.o. on a set X and $\mathcal{B}$ is an f.o. on a set Y. $\mathcal{A}\geqslant_2\mathcal{B}$ iff $\operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A};\mathcal{B})$ is not empty.

Proposition 16.

1. $f\in\operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A};\mathcal{B})$ iff f is a **Set**-morphism from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that
$$C\in\operatorname{up}\mathcal{B}\Leftarrow\langle f^{-1}\rangle C\in\operatorname{up}\mathcal{A}$$
for every $C\in\mathscr{P}\operatorname{Base}(\mathcal{B})$.
2. $f\in\operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A};\mathcal{B})$ iff f is a **Set**-morphism from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$ such that
$$C\in\operatorname{up}\mathcal{B}\Leftrightarrow\langle f^{-1}\rangle C\in\operatorname{up}\mathcal{A}$$
for every $C\in\mathscr{P}\operatorname{Base}(\mathcal{B})$.

Proof.

1. $f\in\operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A};\mathcal{B})\Leftrightarrow\operatorname{up}\mathcal{B}\supseteq f*\mathcal{A}\Leftrightarrow\forall C\in f*\mathcal{A}: C\in\operatorname{up}\mathcal{B}\Leftrightarrow\forall C\in\mathscr{P}\operatorname{Base}(\mathcal{B}): (\langle f^{-1}\rangle C\in\operatorname{up}\mathcal{A}\Rightarrow C\in\operatorname{up}\mathcal{B})$.
2. $f\in\operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A};\mathcal{B})\Leftrightarrow\operatorname{up}\mathcal{B}=f*\mathcal{A}\Leftrightarrow\forall C: (C\in\operatorname{up}\mathcal{B}\Leftrightarrow C\in f*\mathcal{A})\Leftrightarrow\forall C\in\mathscr{P}\operatorname{Base}(\mathcal{B}): (C\in\operatorname{up}\mathcal{B}\Leftrightarrow C\in f*\mathcal{A})\Leftrightarrow\forall C\in\mathscr{P}\operatorname{Base}(\mathcal{B}): (C\in\operatorname{up}\mathcal{B}\Leftrightarrow\langle f^{-1}\rangle C\in\operatorname{up}\mathcal{A})$. □

Definition 17. The multigraph **FuncBij** is the multigraph got from $\mathbf{GreFunc}_2$ by restricting to only bijective morphisms.

Definition 18. A f.o. $\mathcal{A}$ is *directly isomorphic* to a f.o. $\mathcal{B}$ iff there are a morphism $f\in\operatorname{Mor}_{\mathbf{FuncBij}}(\mathcal{A};\mathcal{B})$.

Proposition 19. $f*\mathcal{A}=\operatorname{up}\langle\uparrow f\rangle\mathcal{A}$ for every **Set**-morphism $f:\operatorname{Base}(\mathcal{A})\to\operatorname{Base}(\mathcal{B})$.

Proof. For every set $C\in\mathscr{P}\operatorname{Base}(\mathcal{B})$ we have $C\in f*\mathcal{A}\Leftrightarrow\langle f^{-1}\rangle C\in\operatorname{up}\mathcal{A}\Rightarrow\exists K\in\operatorname{up}\mathcal{A}: \langle f^{-1}\rangle C=K\Rightarrow\exists K\in\operatorname{up}\mathcal{A}: \langle f\rangle\langle f^{-1}\rangle C=\langle f\rangle K\Rightarrow\exists K\in\operatorname{up}\mathcal{A}: C\supseteq\langle f\rangle K\Leftrightarrow\exists K\in\operatorname{up}\mathcal{A}: C\in\operatorname{up}\langle f\rangle K\Leftrightarrow C\in\operatorname{up}\bigcap\langle\uparrow^{\operatorname{Base}(\mathcal{B})}\rangle\langle\langle f\rangle\rangle\operatorname{up}\mathcal{A}\Leftrightarrow C\in\operatorname{up}\langle\uparrow f\rangle\mathcal{A}$.

So $C\in f*\mathcal{A}\Rightarrow C\in\operatorname{up}\langle\uparrow f\rangle\mathcal{A}$.

Let now $C \in \operatorname{up} \langle \uparrow f \rangle \mathcal{A}$. Then $\uparrow^{\operatorname{Base}(\mathcal{A})} \langle f^{-1} \rangle C \supseteq \langle \uparrow f^{-1} \rangle \langle \uparrow f \rangle \mathcal{A} \supseteq \mathcal{A}$ and thus $\langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A}$. □

Corollary 20. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A}; \mathcal{B}) \Leftrightarrow \mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$ for every **Set**-morphism f from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$.

Corollary 21. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B}) \Leftrightarrow \mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$ for every **Set**-morphism f from $\operatorname{Base}(\mathcal{A})$ to $\operatorname{Base}(\mathcal{B})$.

Corollary 22. $\mathcal{A} \geqslant_2 \mathcal{B}$ iff it exists a **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$.

Corollary 23. $\operatorname{up} \mathcal{B} \supseteq f * \mathcal{A} \Leftrightarrow \mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$.

Corollary 24. $\mathcal{A} \geqslant_1 \mathcal{B}$ iff it exists a **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$.

Proposition 25. For a bijective **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ the following are equivalent:

1. $\operatorname{up} \mathcal{B} = f * \mathcal{A}$.
2. $\forall C \in \operatorname{Base}(\mathcal{B}): (C \in \operatorname{up} \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A})$.
3. $\forall C \in \operatorname{Base}(\mathcal{A}): (\langle f \rangle C \in \operatorname{up} \mathcal{B} \Leftrightarrow C \in \operatorname{up} \mathcal{A})$.
4. $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is a bijection from $\operatorname{up} \mathcal{A}$ to $\operatorname{up} \mathcal{B}$.
5. $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is a function onto $\operatorname{up} \mathcal{B}$.
6. $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$.
7. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B})$.
8. $f \in \operatorname{Mor}_{\mathbf{FuncBij}}(\mathcal{A}; \mathcal{B})$.

Proof.

(1)⇔(2). $\operatorname{up} \mathcal{B} = f * \mathcal{A} \Leftrightarrow \operatorname{up} \mathcal{B} = \{ C \in \mathscr{P} \operatorname{Base}(\mathcal{B}) \mid \langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A} \} \Leftrightarrow \forall C \in \operatorname{Base}(\mathcal{B}): (C \in \operatorname{up} \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A})$.

(2)⇔(3). Because f is a bijection.

(2)⇒(5). For every $C \in \operatorname{up} \mathcal{B}$ we have $\langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A}$ and thus $\langle f \rangle|_{\operatorname{up} \mathcal{A}} \langle f^{-1} \rangle C = \langle f \rangle \langle f^{-1} \rangle C = C$. Thus $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is onto $\operatorname{up} \mathcal{B}$.

(4)⇒(5). Obvious.

(5)⇒(4). We need to prove only that $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is an injection. But this follows from the fact that f is a bijection.

(4)⇒(3). We have $\forall C \in \operatorname{Base}(\mathcal{A}): ((\langle f \rangle|_{\operatorname{up} \mathcal{A}}) C \in \operatorname{up} \mathcal{B} \Leftrightarrow C \in \operatorname{up} \mathcal{A})$ and consequently $\forall C \in \operatorname{Base}(\mathcal{A}): (\langle f \rangle C \in \operatorname{up} \mathcal{B} \Leftrightarrow C \in \operatorname{up} \mathcal{A})$.

(6)⇔(1). From the last corollary.

(1)⇔(7). Obvious.

(7)⇔(8). Obvious. □

Corollary 26. The following are equivalent for every f.o. $\mathcal{A}$ and $\mathcal{B}$:

1. $\mathcal{A}$ is directly isomorphic to a f.o. $\mathcal{B}$.
2. There are a bijective **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P} \operatorname{Base}(\mathcal{B})$
$$C \in \operatorname{up} \mathcal{B} \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up} \mathcal{A}$$
3. There are a bijective **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P} \operatorname{Base}(\mathcal{B})$
$$\langle f \rangle C \in \operatorname{up} \mathcal{B} \Leftrightarrow C \in \operatorname{up} \mathcal{A}.$$
4. There are a bijective **Set**-morphism $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\langle f \rangle|_{\operatorname{up} \mathcal{A}}$ is a bijection from $\operatorname{up} \mathcal{A}$ to $\operatorname{up} \mathcal{B}$.

5. There are a bijective **Set**-morphism $f\colon \operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{B})$ such that $\langle f \rangle|_{\operatorname{up}\mathcal{A}}$ is a function onto $\operatorname{up}\mathcal{B}$.
6. There are a bijective **Set**-morphism $f\colon \operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$.
7. There are a bijective morphism $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A}; \mathcal{B})$.

Proposition 27. $\mathbf{GreFunc}_1$ and $\mathbf{GreFunc}_2$ with function composition are categories.

Proof. Let $f\colon \mathcal{A} \rightarrow \mathcal{B}$ and $g\colon \mathcal{B} \rightarrow \mathcal{C}$ are morphisms of $\mathbf{GreFunc}_1$. Then $\mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$ and $\mathcal{C} \subseteq \langle \uparrow g \rangle \mathcal{B}$. So $\langle \uparrow (g \circ f) \rangle \mathcal{A} = \langle \uparrow g \rangle \langle \uparrow f \rangle \mathcal{A} \supseteq \langle \uparrow g \rangle \mathcal{B} \supseteq \mathcal{C}$. Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_1$. Associativity law is evident. $\operatorname{Id}_{\operatorname{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_1$ for every f.o. $\mathcal{A}$.

Let $f\colon \mathcal{A} \rightarrow \mathcal{B}$ and $g\colon \mathcal{B} \rightarrow \mathcal{C}$ are morphisms of $\mathbf{GreFunc}_2$. Then $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$ and $\mathcal{C} = \langle \uparrow g \rangle \mathcal{B}$. So $\langle \uparrow (g \circ f) \rangle \mathcal{A} = \langle \uparrow g \rangle \langle \uparrow f \rangle \mathcal{A} = \langle \uparrow g \rangle \mathcal{B} = \mathcal{C}$. Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_2$. Associativity law is evident. $\operatorname{Id}_{\operatorname{Base}(\mathcal{A})}$ is the identity morphism of $\mathbf{GreFunc}_2$ for every f.o. $\mathcal{A}$. □

Corollary 28. $\leqslant_1$ and $\leqslant_2$ are preorders.

Theorem 29. **FuncBij** is a groupoid.

Proof. First let's prove it is a category. Let $f\colon \mathcal{A} \rightarrow \mathcal{B}$ and $g\colon \mathcal{B} \rightarrow \mathcal{C}$ are morphisms of **FuncBij**. Then $f\colon \operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{B})$ and $g\colon \operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{C})$ are bijections and $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$ and $\mathcal{C} = \langle \uparrow g \rangle \mathcal{B}$. Thus $g \circ f\colon \operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{C})$ is a bijection and $\mathcal{C} = \langle \uparrow (g \circ f) \rangle \mathcal{A}$. Thus $g \circ f$ is a morphism of **FuncBij**. $\operatorname{Id}_{\operatorname{Base}(\mathcal{A})}$ is the identity morphism of **FuncBij** for every f.o. $\mathcal{A}$. Thus it is a category.

It remains to prove only that every morphism $f \in \operatorname{Mor}_{\mathbf{FuncBij}}(\mathcal{A}; \mathcal{B})$ has a reverse (for every f.o. $\mathcal{A}$, $\mathcal{B}$). We have f is a bijection $\operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{B})$ such that for every $C \in \mathscr{P}\operatorname{Base}(\mathcal{A})$

$$\langle f \rangle C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in \operatorname{up}\mathcal{A}.$$

Then $f^{-1}\colon \operatorname{Base}(\mathcal{B}) \rightarrow \operatorname{Base}(\mathcal{A})$ is a bijection such that for every $C \in \mathscr{P}\operatorname{Base}(\mathcal{A})$

$$\langle f^{-1} \rangle C \in \operatorname{up}\mathcal{A} \Leftrightarrow C \in \operatorname{up}\mathcal{B}.$$

Thus $f^{-1} \in \operatorname{Mor}_{\mathbf{FuncBij}}(\mathcal{B}; \mathcal{A})$. □

Corollary 30. Being directly isomorphic is an equivalence relation.

Obvious 31. For the case of ultrafilters being directly isomorphic is the same as being Rudin-Keisler equivalent.

Definition 32. A f.o. $\mathcal{A}$ is *isomorphic* to a f.o. $\mathcal{B}$ iff there exist sets $A \in \operatorname{up}\mathcal{A}$ and $B \in \operatorname{up}\mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$.

Obvious 33. Equivalent f.o. are isomorphic.

Theorem 34. Being isomorphic (for filter objects) is an equivalence relation.

Proof.

Reflexivity. Because every f.o. is directly isomorphic to itself.

Symmetry. If f.o. $\mathcal{A}$ is isomorphic to $\mathcal{B}$ then there exist sets $A \in \operatorname{up}\mathcal{A}$ and $B \in \operatorname{up}\mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$ and thus $\mathcal{B} \div B$ is directly isomorphic to $\mathcal{A} \div A$, So $\mathcal{B}$ is isomorphic to $\mathcal{A}$.

Transitivity. Let $\mathcal{A}$ is isomorphic to $\mathcal{B}$ and $\mathcal{B}$ is isomorphic to $\mathcal{C}$. Then exist $A \in \operatorname{up}\mathcal{A}$, $B_1 \in \operatorname{up}\mathcal{B}$, $B_2 \in \operatorname{up}\mathcal{B}$, $C \in \operatorname{up}\mathcal{C}$ such that there are bijections $f\colon A \rightarrow B_1$ and $g \in B_2 \rightarrow C$ such that

$$\forall X \in \mathscr{P}A\colon (X \in \operatorname{up}\mathcal{B} \Leftrightarrow \langle f^{-1} \rangle X \in \operatorname{up}\mathcal{A}) \quad \text{and} \quad \forall X \in \mathscr{P}B_2\colon (X \in \operatorname{up}\mathcal{A} \Leftrightarrow \langle f \rangle X \in \operatorname{up}\mathcal{B}).$$

Also $\forall X \in \mathscr{P}B_2\colon (X \in \operatorname{up}\mathcal{B} \Leftrightarrow \langle g \rangle X \in \operatorname{up}\mathcal{C})$.

So $g \circ f$ is a bijection from $\langle f^{-1} \rangle (B_1 \cap B_2) \in \operatorname{up}\mathcal{A}$ to $\langle g \rangle (B_1 \cap B_2) \in \operatorname{up}\mathcal{C}$ such that

$$X \in \operatorname{up}\mathcal{A} \Leftrightarrow \langle f \rangle X \in \operatorname{up}\mathcal{B} \Leftrightarrow \langle g \rangle \langle f \rangle X \in \operatorname{up}\mathcal{C} \Leftrightarrow \langle g \circ f \rangle X \in \operatorname{up}\mathcal{C}.$$

Thus $g \circ f$ establishes a bijection which proves that $\mathcal{A}$ is isomorphic to $\mathcal{C}$. □

Lemma 35. Let $\operatorname{card} X = \operatorname{card} Y$, u is an atomic f.o. on X and v is an atomic f.o. on Y; let $A \in \operatorname{up} u$ and $B \in \operatorname{up} v$. Let $u \div A$ and $v \div B$ are directly isomorphic. Then if $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ we have u and v directly isomorphic.

Proof. Arbitrary extend the bijection witnessing being directly isomorphic to the sets $X \setminus A$ and $Y \setminus B$. □

Theorem 36. If $\operatorname{card} X = \operatorname{card} Y$ then being isomorphic and being directly isomorphic are the same for atomic f.o. u on X and v on Y.

Proof. That if two filter objects are isomorphic then they are directly isomorphic is obvious.

Let atomic f.o. u and v are isomorphic that is there is a bijection $f: A \to B$ where $A \in \operatorname{up} u$, $B \in \operatorname{up} v$ witnessing isomorphism of u and v.

If one of the filters u or v is a trivial atomic f.o. then the other is also a trivial atomic f.o. and as it is easy to show they are directly isomorphic. So we can assume u and v are not trivial atomic f.o.

If $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ our statement follows from the last lemma.

Now assume without loss of generality $\operatorname{card}(X \setminus A) < \operatorname{card}(Y \setminus B)$.

$\operatorname{card} B = \operatorname{card} Y$ because $\operatorname{card}(Y \setminus B) < \operatorname{card} Y$.

It is easy to show that thre exists $B' \supset B$ such that $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B')$ and $\operatorname{card} B' = \operatorname{card} B$.

We will find a bijection g from B to B' which witnesses direct isomorphism of v to v itself. Then the composition $g \circ f$ witnesses a direct isomorphism of $u \div A$ and $v \div B'$ and by the lemma u and v are directly isomorphic.

Let $D = B' \setminus B$. We have $D \notin \operatorname{up} v$.

There exists a set $E \subseteq B$ such that $\operatorname{card} E \geqslant \operatorname{card} D$ and $E \notin \operatorname{up} v$.

We have $\operatorname{card} E = \operatorname{card}(D \cup E)$ and thus there exists a bijection $h: E \to D \cup E$.

Let

$$g(x) = \begin{cases} x & \text{if } x \in B \setminus E; \\ h(x) & \text{if } x \in E. \end{cases}$$

$g|_{B \setminus E}$ and $g|_E$ are bijections.

$\operatorname{im}(g|_{B \setminus E}) = B \setminus E$; $\operatorname{im}(g|_E) = \operatorname{im} h = D \cup E$;

$$(D \cup E) \cap (B \setminus E) = (D \cap (B \setminus E)) \cup (E \cap (B \setminus E)) = \emptyset \cup \emptyset = \emptyset.$$

Thus g is a bijection from B to $(B \setminus E) \cup (D \cup E) = B \cup D = B'$.

To finish the proof it's enough to show that $\langle g \rangle v = v$. Indeed it follows from $B \setminus E \in \operatorname{up} v$. □

Proposition 37.

1. For every $A \in \operatorname{up} \mathcal{A}$ and $B \in \operatorname{up} \mathcal{B}$ we have $\mathcal{A} \geqslant_2 \mathcal{B}$ iff $\mathcal{A} \div A \geqslant_2 \mathcal{B} \div B$.
2. For every $A \in \operatorname{up} \mathcal{A}$ and $B \in \operatorname{up} \mathcal{B}$ we have $\mathcal{A} \geqslant_1 \mathcal{B}$ iff $\mathcal{A} \div A \geqslant_1 \mathcal{B} \div B$.

Proof.

1. $\mathcal{A} \geqslant_2 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$.
2. $\mathcal{A} \geqslant_1 \mathcal{B}$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$. □

Rudin-Keisler order of ultrafilters is considered in such a book as [6].

Proposition 38. For ultrafilters $\geqslant_2$ is the same as Rudin-Keisler ordering.

Proof. $x \geqslant_2 y$ iff there exist sets $A \in \operatorname{up} x$ and $B \in \operatorname{up} y$ a bijective **Set**-morphism $f: X \to Y$ such that $\operatorname{up}(y \div B) = \{C \in \mathscr{P} Y \mid \langle f^{-1} \rangle C \in \operatorname{up}(x \div A)\}$ that is when $C \in \operatorname{up}(y \div B) \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up}(x \div A)$ what is equivalent to $C \in \operatorname{up} y \Leftrightarrow \langle f^{-1} \rangle C \in \operatorname{up} x$ what is the definition of Rudin-Keisler ordering. □

Remark 39. The relation of being isomorphic for ultrafilters is traditionally called *Rudin-Keisler equivalence*.

Obvious 40. $(\geqslant_1) \supseteq (\geqslant_2)$.

Definition 41. Let Q and R are binary relations on the set of filter objects. I will denote $\mathbf{MonRld}_{Q,R}$ the directed multigraph with objects being filter objects and morphisms such monovalued reloids f that $(\operatorname{dom} f)\, Q\, \mathcal{A}$ and $(\operatorname{im} f)\, R\, \mathcal{B}$.

I will also denote $\mathbf{CoMonRld}_{Q,R}$ the directed multigraph with objects being filter objects and morphisms such injective reloids f that $(\operatorname{im} f)\, Q\, \mathcal{A}$ and $(\operatorname{dom} f)\, R\, \mathcal{B}$. These are essentially the duals.

Some of these directed multigraphs are categories with reloid composition (see below). By abuse of notation I will denote these categories the same as these directed multigraphs.

Theorem 42. For every f.o. $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \geqslant_1 \mathcal{B}$.
2. $\operatorname{Mor}_{\mathbf{MonRld}_{=,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
3. $\operatorname{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
4. $\operatorname{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
5. $\operatorname{Mor}_{\mathbf{CoMonRld}_{=,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
6. $\operatorname{Mor}_{\mathbf{CoMonRld}_{\subseteq,\supseteq}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
7. $\operatorname{Mor}_{\mathbf{CoMonRld}_{\subseteq,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.

Proof.

(1)⇒(2). There exists a **Set**-morphism $f\colon \operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$. We have

$$\operatorname{dom} (\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \mathcal{A} \cap \operatorname{dom} f = \mathcal{A}$$

and

$$\operatorname{im} (\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im} (\mathsf{FCD})(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im} (\uparrow f)|_{\mathcal{A}} = \langle \uparrow f \rangle \mathcal{A} \supseteq \mathcal{B}.$$

Thus $(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}}$ is a monovalued reloid such that $\operatorname{dom} (\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \mathcal{A}$ and $\operatorname{im} (\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} \supseteq \mathcal{B}$.

(2)⇒(3), (4)⇒(3), (5)⇒(6), (7)⇒(6). Obvious.

(3)⇒(1). We have $\mathcal{B} \subseteq \langle (\mathsf{FCD}) f \rangle \mathcal{A}$ for a monovalued reloid $f \in \mathsf{RLD}(\operatorname{Base}(\mathcal{A}); \operatorname{Base}(\mathcal{B}))$. Then there exists a **Set**-morphism $F\colon \operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \subseteq \langle \uparrow F \rangle \mathcal{A}$ that is $\mathcal{A} \geqslant_1 \mathcal{B}$.

(6)⇒(7). $\operatorname{dom} f|_{\mathcal{B}} = \mathcal{B}$ and $\operatorname{im} f|_{\mathcal{B}} \subseteq \mathcal{A}$.

(2)⇔(5), (3)⇔(6), (4)⇔(7). By duality. □

Theorem 43. For every f.o. $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \geqslant_2 \mathcal{B}$.
2. $\operatorname{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.
3. $\operatorname{Mor}_{\mathbf{CoMonRld}_{=,=}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.

Proof.

(1)⇒(2). Let $\mathcal{A} \geqslant_2 \mathcal{B}$ that is $\mathcal{B} = \langle \uparrow f \rangle \mathcal{A}$ for some **Set**-morphism f: $\operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{B})$. Then $\operatorname{dom} (\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \mathcal{A}$ and $\operatorname{im} (\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im} (\mathsf{FCD})(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}} = \operatorname{im} (\uparrow f)|_{\mathcal{A}} = \langle \uparrow f \rangle \mathcal{A} = \mathcal{B}$. So $(\uparrow^{\mathsf{RLD}} f)|_{\mathcal{A}}$ is a sought for reloid.

(2)⇒(1). There exists a **Set**-morphism $F\colon \operatorname{Base}(\mathcal{A}) \rightarrow \operatorname{Base}(\mathcal{B})$ such that $f = (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}$. Thus $\langle \uparrow F \rangle \mathcal{A} = \operatorname{im}(\uparrow F)|_{\mathcal{A}} = \operatorname{im} (\mathsf{FCD})(\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}} = \operatorname{im} (\mathsf{FCD}) f = \operatorname{im} f = \mathcal{B}$. Thus $\mathcal{A} \geqslant_2 \mathcal{B}$ is testified by the morphism F.

(2)⇔(3). By duality. □

Theorem 44. The following are categories (with reloid composition):

1. $\mathbf{MonRld}_{\subseteq,\supseteq}$;
2. $\mathbf{MonRld}_{\subseteq,=}$;
3. $\mathbf{MonRld}_{=,=}$.
4. $\mathbf{CoMonRld}_{\subseteq,\supseteq}$;
5. $\mathbf{CoMonRld}_{\subseteq,=}$;
6. $\mathbf{CoMonRld}_{=,=}$.

Proof. We will prove only the first three. The rest follow from duality. We need to prove only that composition of morphisms is a morphism, because associativity and existence of identity morphism are evident. We have:

1. Let $f \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{A}; \mathcal{B})$, $g \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{B}; \mathcal{C})$. Then $\operatorname{dom} f \subseteq \mathcal{A}$, $\operatorname{im} f \supseteq \mathcal{B}$, $\operatorname{dom} g \subseteq \mathcal{B}$, $\operatorname{im} g \supseteq \mathcal{C}$. So $\operatorname{dom}(g \circ f) \subseteq \mathcal{A}$, $\operatorname{im}(g \circ f) \supseteq \mathcal{C}$ that is $g \circ f \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{A}; \mathcal{C})$.
2. Let $f \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{A}; \mathcal{B})$, $g \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{B}; \mathcal{C})$. Then $\operatorname{dom} f \subseteq \mathcal{A}$, $\operatorname{im} f = \mathcal{B}$, $\operatorname{dom} g \subseteq \mathcal{B}$, $\operatorname{im} g = \mathcal{C}$. So $\operatorname{dom}(g \circ f) \subseteq \mathcal{A}$, $\operatorname{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{A}; \mathcal{C})$.
3. Let $f \in \mathrm{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A}; \mathcal{B})$, $g \in \mathrm{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{B}; \mathcal{C})$. Then $\operatorname{dom} f = \mathcal{A}$, $\operatorname{im} f = \mathcal{B}$, $\operatorname{dom} g = \mathcal{B}$, $\operatorname{im} g = \mathcal{C}$. So $\operatorname{dom}(g \circ f) = \mathcal{A}$, $\operatorname{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \mathrm{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A}; \mathcal{C})$. □

Definition 45. Let **BijRld** is the groupoid of all bijections of the category of reloid triples. Its objects are filter objects and its morphisms from a f.o. $\mathcal{A}$ to f.o. $\mathcal{B}$ are monovalued injective reloids f such that $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{im} f = \mathcal{B}$.

Theorem 46. Filter objects $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff $\mathrm{Mor}_{\mathbf{BijRld}}(\mathcal{A}; \mathcal{B}) \neq \emptyset$.

Proof.

$\Rightarrow$. Let $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Then there are sets $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$ and a bijective **Set**-morphism $F: A \to B$ such that $\langle F \rangle: \mathscr{P}A \cap \operatorname{up} \mathcal{A} \to \mathscr{P}B \cap \operatorname{up} \mathcal{B}$ is a bijection.

Obviously $f = (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}$ is monovalued and injective.

$\operatorname{im} f = \bigcap \{\uparrow^B \operatorname{im} G \mid G \in \operatorname{up} (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}\} = \bigcap \{\uparrow^B \operatorname{im}(H \cap F|_X) \mid H \in \operatorname{up} (\uparrow^{\mathsf{RLD}} F)|_{\mathcal{A}}, X \in \operatorname{up} \mathcal{A}\} = \bigcap \{\uparrow^B \operatorname{im} F|_P \mid P \in \operatorname{up} \mathcal{A}\} = \bigcap \{\uparrow^B \langle F \rangle P \mid P \in \operatorname{up} \mathcal{A}\} = \bigcap \{\uparrow^B \langle F \rangle P \mid P \in \mathscr{P}A \cap \operatorname{up} \mathcal{A}\} = \bigcap \langle \uparrow^B \rangle (\mathscr{P}B \cap \operatorname{up} \mathcal{B}) = \bigcap \langle \uparrow^B \rangle \operatorname{up} \mathcal{B} = \mathcal{B}$.

Thus $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{im} f = \mathcal{B}$.

$\Leftarrow$. Let f is a monovalued injective reloid such that $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{im} f = \mathcal{B}$. Then exist a function F' and an injective binary relation F'' such that $F', F'' \in \operatorname{up} f$. Thus $F = F' \cap F''$ is an injection such that $F \in \operatorname{up} f$. The function F is a bijection from $A = \operatorname{dom} F$ to $B = \operatorname{im} F$. The function $\langle F \rangle$ is an injection on $\mathscr{P}A \cap \operatorname{up} \mathcal{A}$ (and moreover on $\mathscr{P}A$). It's simple to show that $\forall X \in \mathscr{P}A \cap \operatorname{up} \mathcal{A}: \langle F \rangle X \in \mathscr{P}B \cap \operatorname{up} \mathcal{B}$ and similarly $\forall Y \in \mathscr{P}B \cap \operatorname{up} \mathcal{B}: \langle F \rangle^{-1} Y = \langle F^{-1} \rangle Y \in \mathscr{P}A \cap \operatorname{up} \mathcal{A}$. Thus $\langle F \rangle|_{\mathscr{P}A \cap \operatorname{up} \mathcal{A}}$ is a bijection $\mathscr{P}A \cap \operatorname{up} \mathcal{A} \to \mathscr{P}B \cap \operatorname{up} \mathcal{B}$. So filter objects $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. □

Proposition 47. $(\geqslant_1) = (\subseteq) \circ (\geqslant_2)$ (when we limit to small f.o.).

Proof. $\mathcal{A} \geqslant_1 \mathcal{B}$ iff exists a function $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$. But $\mathcal{B} \subseteq \langle \uparrow f \rangle \mathcal{A}$ is equivalent to $\exists \mathcal{B}' \in \mathfrak{F}: (\mathcal{B} \subseteq \mathcal{B}' \wedge \mathcal{B}' = \langle \uparrow f \rangle \mathcal{A})$. So $\mathcal{A} \geqslant_1 \mathcal{B}$ is equivalent to existence of $\mathcal{B}' \in \mathfrak{F}$ such that $\mathcal{B} \subseteq \mathcal{B}'$ and existence of a function $f: \operatorname{Base}(\mathcal{A}) \to \operatorname{Base}(\mathcal{B})$ such that $\mathcal{B}' = \langle \uparrow f \rangle \mathcal{A}$. That is equivalent to $\mathcal{A}((\subseteq) \circ (\geqslant_2))\mathcal{B}$. □

Proposition 48. If a and b is an atomic f.o. then $b \geqslant_1 a \Leftrightarrow b \geqslant_2 a$.

Proof. We need to prove only $b \geqslant_1 a \Rightarrow b \geqslant_2 a$. If $b \geqslant_1 a$ then there exists a monovalued reloid $f: \operatorname{Base}(b) \to 1^{\mathfrak{F}(\operatorname{Base}(a))}$ such that $\operatorname{dom} f = b$ and $\operatorname{im} f \supseteq a$. Then $\operatorname{im} f = \operatorname{im} (\mathsf{FCD}) f \in \left\{0^{\mathfrak{F}(\operatorname{Base}(a))}\right\} \cup \operatorname{atoms} 1^{\mathfrak{F}(\operatorname{Base}(a))}$ because $(\mathsf{FCD}) f$ is a monovalued funcoid. So $\operatorname{im} f = a$ (taken in account $a \neq 0^{\mathfrak{F}(\operatorname{Base}(a))}$) and thus $b \geqslant_2 a$. □

Corollary 49. For atomic filter objects $\geqslant_1$ is the same as $\geqslant_2$.

Thus I will write simply $\geqslant$ for atomic f.o.

3.1 Existence of no more than one monovalued injective reloid for a given pair of filter objects

3.1.1 The lemmas

The lemmas in this section were provided to me by Robert Martin Solovay in [5]. These are based on Wistar Comfort's work.

In this section we will assume μ is an ultrafilter on a set I and $f: I \rightarrow I$ has the property $X \in \mu \Leftrightarrow \langle f^{-1} \rangle X \in \mu$.

Lemma 50. If $X \in \mu$ then $X \cap \langle f \rangle X \in \mu$.

Proof. If $\langle f \rangle X \notin \mu$ then $X \subseteq \langle f^{-1} \rangle \langle f \rangle X \notin \mu$ and so $X \notin \mu$. Thus $X \in \mu \wedge \langle f \rangle X \in \mu$ and consequently $X \cap \langle f \rangle X \in \mu$. □

We will say that x is *periodic* when $f^n(x) = x$ for some positive integer x. The least such n is called *the period* of x.

Let's define $x \sim y$ iff there exist $i, j \in \mathbb{N}$ such that $f^i(x) = f^j(y)$. Trivially it is an equivalence relation. If x and y are periodic, then $x \sim y$ iff exists $n \in \mathbb{N}$ such that $f^n(y) = x$.

Let $A = \{x \in I \mid x \text{ is periodic with period} > 1\}$.

We will show that $A \notin \mu$. Let's assume $A \in \mu$.

Let a set $D \subseteq A$ contains (by the axiom of choice) exactly one element from each equivalence class of A defined by the relation $\sim$.

Let α is a function $A \rightarrow \mathbb{N}$ defined as follows. Let $x \in A$. Let y be the unique element of D such that $x \sim y$. Let $\alpha(x)$ be the least $n \in \mathbb{N}$ such that $f^n(y) = x$.

Let $B_0 = \{x \in A \mid \alpha(x) \text{ is even}\}$ and $B_1 = \{x \in A \mid \alpha(x) \text{ is odd}\}$.

Let $B_2 = \{x \in A \mid \alpha(x) = 0\}$.

Lemma 51. $B_0 \cap \langle f \rangle B_0 \subseteq B_2$.

Proof. If $x \in B_0 \cap \langle f \rangle B_0$ then $f^n(y) = x$ for a minimal even n and $x = f(x')$ where $f^m(y') = x'$ for a minimal even m. Thus $f^n(y) = f(x')$ thus y and x' laying in the same equivalence class and thus $y = y'$. So we have $f^n(y) = f^{m+1}(y)$. Thus $n \leqslant m + 1$ by minimality.

x' lies on an orbit and thus $x' = f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit; $f^m(y) = f^{-1}(x)$ and thus $x' = f^{n-1}(y)$ thus $n - 1 \geqslant m$ by minimality or $n = 0$.

Thus $n = m + 1$ what is impossible for even n and m. We have a contradiction what proves $B_0 \cap \langle f \rangle B_0 \subseteq \emptyset$.

Remained the case $n = 0$, then $x = f^0(y)$ and thud $\alpha(x) = 0$. □

Lemma 52. $B_1 \cap \langle f \rangle B_1 = \emptyset$.

Proof. Let $x \in B_1 \cap \langle f \rangle B_1$. Then $f^n(y) = x$ for an odd n and $x = f(x')$ where $f^m(y') = x'$ for an odd m. Thus $f^n(y) = f(x')$ thus y and x' laying in the same equivalence class and thus $y = y'$. So we have $f^n(y) = f^{m+1}(y)$. Thus $n \leqslant m + 1$ by minimality.

x' lies on an orbit and thus $x' = f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit;

$f^m(y) = f^{-1}(x)$ and thus $x' = f^{n-1}(y)$ thus $n - 1 \geqslant m$ by minimality ($n = 0$ is impossible because n is odd.

Thus $n = m + 1$ what is impossible for odd n and m. We have a contradiction what proves $B_0 \cap \langle f \rangle B_0 = \emptyset$. □

Lemma 53. $B_2 \cap \langle f \rangle B_2 = \emptyset$.

Proof. Let $x \in B_2 \cap \langle f \rangle B_2$. Then $x = y$ and $x' = y$ where $x = f(x')$. Thus $x = f(x)$ and so $x \notin A$ what is impossible. □

Lemma 54. $A \notin \mu$.

Proof. Suppose $A \in \mu$.

Since $A \in \mu$ we have $B_0 \in \mu$ or $B_1 \in \mu$.

So either $B_0 \cap \langle f \rangle B_0 \subseteq B_2$ or $B_1 \cap \langle f \rangle B_1 \subseteq B_2$. As such by the lemma 50 we have $B_2 \in \mu$. This is incompatible with $B_2 \cap \langle f \rangle B_2 = \emptyset$. So we got a contradiction. □

Let C be the set of points x which are not periodic but $f^n(x)$ is periodic for some positive n.

Lemma 55. $C \notin \mu$.

Proof. Let β be a function $C \to \mathbb{N}$ such that $\beta(x)$ is the least $n \in \mathbb{N}$ such that $f^n(x)$ is periodic.

Let $C_0 = \{x \in C \mid \beta(x) \text{ is even}\}$ and $C_1 = \{x \in C \mid \beta(x) \text{ is odd}\}$.

Obviously $C_j \cap \langle f \rangle C_j = \emptyset$ for $j = 0, 1$. Hence by the lemma 50 we have $C_0, C_1 \notin \mu$ and thus $C = C_0 \cup C_1 \notin \mu$. □

Let E be the set of $x \in I$ such that for no $n \in \mathbb{N}$ we have $f^n(x)$ periodic.

Lemma 56. Let $x, y \in E$ are such that $f^i(x) = f^j(y)$ and $f^{i'}(x) = f^{j'}(y)$ for some $i, j, i', j' \in \mathbb{N}$. Then $i - j = i' - j'$.

Proof. $i \mapsto f^i(x)$ is a bijection.

So $y = f^{i-j}(y)$ and $y = f^{i'-j'}(y)$. Thus $f^{i-j}(y) = f^{i'-j'}(y)$ and so $i - j = i' - j'$. □

Lemma 57. $E \notin \mu$.

Proof. Let $D' \subseteq E$ be a subset of E with exactly one element from each equivalence class of the relation $\sim$ on E.

Define the function $\gamma: E \to \mathbb{Z}$ as follows. Let $x \in E$. Let y be the unique element of D' such that $x \sim y$. Choose $i, j \in \mathbb{N}$ such that $f^i(y) = f^j(x)$. Let $\gamma(x) = i - j$. By the last lemma, γ is well-defined.

It is clear that if $x \in E$ then $f(x) \in E$ and moreover $\gamma(f(x)) = \gamma(x) + 1$.

Let $E_0 = \{x \in E \mid \gamma(x) \text{ is even}\}$ and $E_1 = \{x \in E \mid \gamma(x) \text{ is odd}\}$.

We have $E_0 \cap \langle f \rangle E_0 = \emptyset \notin \mu$ and hence $E_0 \notin \mu$.

Similarly $E_1 \notin \mu$.

Thus $E = E_0 \cup E_1 \notin \mu$. □

Lemma 58. f is the identity function on a set in μ.

Proof. We have shown $A, C, E \notin \mu$. But the points which lie in none of these sets are exactly points periodic with period 1 that is fixed points of f. Thus the set of fixed points of f belongs to the filter μ. □

3.1.2 The main theorem and its consequences

Theorem 59. For every atomic filter object a the morphism $((=)|_a; a; a)$ is the only

1. monovalued morphism of the category of reloids from a to a;
2. injective morphism of the category of reloids from a to a;
3. bijective morphism of the category of reloids from a to a.

Proof. We will prove only (1) because the rest follow from it.

Let f is a monovalued morphism from a to a. Then it exists a **Set**-morphism $(F; a; a)$ such that $F \in \operatorname{up} f$. Trivially $\langle \uparrow(F; a; a) \rangle a \supseteq a$ and thus $\langle F \rangle A \in \operatorname{up} a$ for every $A \in \operatorname{up} a$. Thus by the lemma we have that F is the identity function on a set in $\operatorname{up} a$ and so obviously f is an identity. □

Corollary 60. For every two atomic filter objects (with possibly different bases) $\mathcal{A}$ and $\mathcal{B}$ there exists at most one bijective reloid from $\mathcal{A}$ to $\mathcal{B}$.

Proof. Suppose that f and g are two different bijective reloids from $\mathcal{A}$ to $\mathcal{B}$. Then $g^{-1}\circ f$ is not the identity reloid (otherwise $g^{-1}\circ f = I^{\mathsf{RLD}}_{\operatorname{dom} f}$ and so $f = g$). But $g^{-1}\circ f$ is a bijective reloid (as a composition of bijective reloids) from $\mathcal{A}$ to $\mathcal{A}$ what is impossible. $\square$

4 Rudin-Keisler equivalence and Rudin-Keisler order

Theorem 61. Atomic filter objects a and b (with possibly different bases) are isomorphic iff $a \geqslant b \wedge b \geqslant a$.

Proof. Let $a \geqslant b \wedge b \geqslant a$. Then there are a monovalued reloids f and g such that $\operatorname{dom} f = a$ and $\operatorname{im} f = b$ and $\operatorname{dom} g = b$ and $\operatorname{im} g = a$. Thus $g \circ f$ is a monovalued morphism from a to a. By the above we have $g \circ f = I^{\mathsf{RLD}}_a$ so $g = f^{-1}$ and $f^{-1}\circ f = I^{\mathsf{RLD}}_a$ so f is monovalued. Thus f is an injective monovalued reloid from a to b and thus a and b are isomorphic. $\square$

The last theorem cannot be generalized from atomic f.o. to arbitrary f.o., as it's shown by the following two examples:

Example 62. $\mathcal{A} \geqslant_1 \mathcal{B} \wedge \mathcal{B} \geqslant_1 \mathcal{A}$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ for some f.o. $\mathcal{A}$ and $\mathcal{B}$.

Proof. Consider $\mathcal{A} = \uparrow^{\mathbb{R}}[0; 1]$ and $\mathcal{B} = \bigcap \{\uparrow^{\mathbb{R}}[0; 1 + \varepsilon) \mid \varepsilon > 0\}$. Then the function $f = \{(x; x/2) \mid x \in \mathbb{R}\}$ witnesses both inequalities $\mathcal{A} \geqslant_1 \mathcal{B}$ and $\mathcal{B} \geqslant_1 \mathcal{A}$. But these filters cannot be isomorphic because only one of them is principal. $\square$

Lemma 63. Let f_0 and f_1 are **Set**-morphisms. Let $f(x; y) = (f_0 x; f_1 y)$ for a function f. Then $\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Dst} f_0; \operatorname{Dst} f_1)} f \right\rangle (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \langle \uparrow f_0 \rangle \mathcal{A} \times^{\mathsf{RLD}} \langle \uparrow f_1 \rangle \mathcal{B}$.

Proof. $\left\langle \uparrow^{\mathsf{FCD}(\operatorname{Dst} f_0; \operatorname{Dst} f_1)} f \right\rangle (\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}) = \left\langle \uparrow^{\mathsf{FCD}(\operatorname{Dst} f_0; \operatorname{Dst} f_1)} f \right\rangle \bigcap \{\uparrow^{\operatorname{Src} f_0 \times \operatorname{Src} f_1}(A \times B) \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\} = \bigcap \{\uparrow^{\operatorname{Src} f_0 \times \operatorname{Src} f_1} \langle f \rangle (A \times B) \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\} = \bigcap \{\uparrow^{\operatorname{Dst} f_0 \times \operatorname{Dst} f_1}(\langle f_0 \rangle A \times \langle f_1 \rangle B) \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\} = \bigcap \{\uparrow^{\operatorname{Dst} f_0} \langle f_0 \rangle A \times^{\mathsf{RLD}} \uparrow^{\operatorname{Dst} f_1} \langle f_1 \rangle B \mid A \in \operatorname{up} \mathcal{A}, B \in \operatorname{up} \mathcal{B}\} =$ (theorem 164?? in [3])$= \bigcap \{\uparrow^{\operatorname{Dst} f_0} \langle f_0 \rangle A \mid A \in \operatorname{up} \mathcal{A}\} \times^{\mathsf{RLD}} \bigcap \{\uparrow^{\operatorname{Dst} f_1} \langle f_1 \rangle B \mid A \in \operatorname{up} \mathcal{B}\} = \langle \uparrow f_0 \rangle \mathcal{A} \times^{\mathsf{RLD}} \langle \uparrow f_1 \rangle \mathcal{B}$. $\square$

Lemma 64. If an f.o. $\mathcal{A}$ is isomorphic to an f.o. $\mathcal{B}$ then if X is a set and $\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}$ is an atomic f.o., then there exists a set Y such that $\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}$ is an atomic f.o. isomorphic to $\uparrow^{\operatorname{Base}(\mathcal{A})} Y \cap \mathcal{B}$. [FIXME: See the book for a corrected proof.]

Proof. Let $\mathcal{A}$ is isomorphic to $\mathcal{B}$. Then there are sets $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$ such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$. So there are a bijection $f: \mathscr{P} A \cap \operatorname{up} \mathcal{A} \to \mathscr{P} B \cap \operatorname{up} \mathcal{B}$ such that $\mathcal{B} = \langle f \rangle \mathcal{A}$.

[FIXME: ?? equality is wrong.]

$\operatorname{up}\left(\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}\right) = \operatorname{up}\left(\uparrow^{\operatorname{Base}(\mathcal{A})} (X \cap A) \cap \mathcal{A}\right) = ?? = \langle X \cap A \cap \rangle \operatorname{up} \mathcal{A} = \langle X \cap \rangle (\mathscr{P} A \cap \operatorname{up} \mathcal{A})$.

Thus $\langle \langle f \rangle \rangle \operatorname{up}\left(\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}\right) = \langle \langle f \rangle \rangle \langle X \cap \rangle (\mathscr{P} A \cap \operatorname{up} \mathcal{A}) = \langle f(X) \cap \rangle \langle \langle f \rangle \rangle (\mathscr{P} A \cap \operatorname{up} \mathcal{A}) = \langle f(X) \cap \rangle (\mathscr{P} B \cap \operatorname{up} \mathcal{B}) = \langle f(X) \cap B \cap \rangle \operatorname{up} \mathcal{B} = \langle f(X) \cap \rangle \operatorname{up} \mathcal{B} = \operatorname{up}\left(\uparrow^{\operatorname{Base}(\mathcal{B})} (f(X)) \cap \mathcal{B}\right)$.

So $\langle f \rangle \left(\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}\right) = \bigcap \left\langle \uparrow^{\operatorname{Base}(\mathcal{B})} \right\rangle \langle \langle f \rangle \rangle \operatorname{up}\left(\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}\right) = \bigcap \left\langle \uparrow^{\operatorname{Base}(\mathcal{B})} \right\rangle \operatorname{up}\left(\uparrow^{\operatorname{Base}(\mathcal{B})} (f(X)) \cap \mathcal{B}\right) = \uparrow^{\operatorname{Base}(\mathcal{B})} (f(X)) \cap \mathcal{B}$.

Finally we have $\uparrow^{\operatorname{Base}(\mathcal{B})} (f(X)) \cap \mathcal{B}$ is isomorphic to $\uparrow^{\operatorname{Base}(\mathcal{A})} X \cap \mathcal{A}$ from the last equality. $\square$

Theorem 65. Let f is a monovalued injective reloid. Then f is isomorphic to the f.o. $\operatorname{dom} f$.

Proof. Let f is a monovalued injective reloid. There exists a bijection $F \in \operatorname{up} f$. Consider the bijective function $p = \{(x; F x) \mid x \in \operatorname{dom} F\}$.

$\langle p \rangle \operatorname{dom} F = F$ and consequently $\langle p \rangle \operatorname{dom} f = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Src} f)} \langle p \rangle \operatorname{dom} K \mid K \in \operatorname{up} f \right\} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Src} f)} \langle p \rangle \operatorname{dom}(K \cap F) \mid K \in \operatorname{up} f \right\} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Src} f)} (K \cap F) \mid K \in \operatorname{up} f \right\} = \bigcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Dst} f; \operatorname{Src} f)} K \mid K \in \operatorname{up} f \right\} = f$. Thus p witnesses that f is isomorphic to the f.o. $\operatorname{dom} f$. $\square$

Corollary 66. A monovalued injective reloid with atomic domain is atomic.

Corollary 67. $I_{\mathcal{A}}^{\mathsf{RLD}}$ is isomorphic to $\mathcal{A}$ for every f.o. $\mathcal{A}$.

Theorem 68. There are atomic f.o. incomparable by Rudin-Keisler order.

Proof. See [2]. □

Theorem 69. $\geqslant_1$ and $\geqslant_2$ are different relations.

Proof. Consider a is an arbitrary non-empty f.o. Then $a \geqslant_1 0^{\mathfrak{F}(\mathrm{Base}(a))}$ but not $a \geqslant_2 0^{\mathfrak{F}(\mathrm{Base}(a))}$. □

Proposition 70. If $a \geqslant_2 b$ where a is an atomic f.o. then b is also an atomic f.o.

Proof. $b = \langle \uparrow f \rangle a$ for some $f\colon \mathrm{Base}(a) \to \mathrm{Base}(b)$. So b is an atomic f.o. since f is monovalued. □

Corollary 71. If $a \geqslant_1 b$ where a is an atomic f.o. then b is also an atomic f.o. or $0^{\mathfrak{F}(\mathrm{Base}(a))}$.

Proof. $b \subseteq \langle \uparrow f \rangle a$ for some $f\colon \mathrm{Base}(a) \to \mathrm{Base}(b)$. Therefore $b' = \langle \uparrow f \rangle a$ is an atomic f.o. From this follows our statement. □

Proposition 72. Principal filters, generated by sets of the same cardinality, are isomorphic.

Proof. Let A and B are sets of the same cardinality. Then there are a bijection f from A to B. We have $\langle f \rangle A = B$ and thus A and B are isomorphic. □

Proposition 73. If a filter object is isomorphic to a principal f.o., then it is also a principal f.o. induced by a set with the same cardinality.

Proof. Let A is a set and B is a f.o. isomorphic to A. Then there are sets $X \in \mathrm{up}\, A$ and $Y \in \mathrm{up}\, B$ such that there are a bijection $f\colon X \to Y$ such that $\langle f \rangle A = B$. Thus A is a set of the same cardinality as B. □

Proposition 74. A filter isomorphic to a non-trivial atomic f.o. is a non-trivial atomic f.o.

Proof. Let a is a non-trivial atomic f.o. and a is isomorphic to b. Then $a \geqslant_2 b$ and thus b is an atomic f.o. The f.o. b cannot be trivial because otherwise a would be also trivial. □

Theorem 75. For an infinite set U there exist $2^{2^{\mathrm{card} U}}$ equivalence classes of isomorphic ultrafilters.

Proof. The number of bijections between any two given subsets of U is no more than $(\mathrm{card}\, U)^{\mathrm{card}\, U} = 2^{\mathrm{card}\, U}$. The number of bijections between all pairs of subsets of U is no more than $2^{\mathrm{card}\, U} \cdot 2^{\mathrm{card}\, U} = 2^{\mathrm{card}\, U}$. Therefore each isomorphism class contains at most $2^{\mathrm{card}\, U}$ ultrafilters. But there are $2^{2^{\mathrm{card}\, U}}$ ultrafilters. So there are $2^{2^{\mathrm{card}\, U}}$ classes. □

Remark 76. One of the above mentioned equivalence classes contains trivial ultrafilters.

Corollary 77. There exist non-isomorphic nontrivial ultrafilters on any infinite set.

5 Consequences

Theorem 78. The reloid $\uparrow^A \{a\} \times^{\mathsf{RLD}} \mathcal{F}$ is isomorphic to the filter object $\mathcal{F}$ for every set A and $a \in A$.

Proof. Consider $B = \{a\} \times \mathrm{Base}(\mathcal{F})$ and $f = \{(x; (a; x)) \mid x \in \mathrm{Base}(\mathcal{F})\}$. Then f is a bijection from $\mathrm{Base}(\mathcal{F})$ to B.

If $X \in \mathrm{up}\, \mathcal{F}$ then $\langle f \rangle X \subseteq B$ and $\langle f \rangle X = \{a\} \times X \in \mathrm{up}(\uparrow^A \{a\} \times^{\mathsf{RLD}} \mathcal{F})$.

For every $Y \in \mathrm{up}(\uparrow^A \{a\} \times^{\mathsf{RLD}} \mathcal{F}) \cap \mathscr{P} B$ we have $Y = \{a\} \times X$ for some $X \in \mathrm{up}\, \mathcal{F}$ and thus $Y = \langle f \rangle X$.

So $\langle f\rangle|_{\mathrm{up}\,\mathcal{F}\cap\mathscr{P}\mathrm{Base}(\mathcal{F})}=\langle f\rangle|_{\mathrm{up}\,\mathcal{F}}$ is a bijection from $\mathrm{up}\,\mathcal{F}\cap\mathscr{P}B$ to $\mathrm{up}(\uparrow^A\{a\}\times^{\mathsf{RLD}}\mathcal{F})\cap\mathscr{P}B$.

We have $\mathrm{up}\,\mathcal{F}\cap\mathscr{P}\mathrm{Base}(\mathcal{F})$ and $\mathrm{up}(\uparrow^A\{a\}\times^{\mathsf{RLD}}\mathcal{F})\cap\mathscr{P}B$ directly isomorphic and thus $\mathrm{up}\,\mathcal{F}$ is isomorphic to $\mathrm{up}(\uparrow^A\{a\}\times^{\mathsf{RLD}}\mathcal{F})$. □

Theorem 79. A monovalued reloid with atomic domain is atomic.

Proof. Let f is a monovalued reloid with atomic domain. There exists a function $F\in\mathrm{up}\,f$.

We have $f\subseteq\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}$. Thus it suffices to prove that $\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}$ is atomic.

Let the function $\tau:\mathrm{dom}\,F\to F$ is defined by the formula $\tau x=(x;Fx)$ (for every $x\in\mathrm{dom}\,F$).

That τ is an injection is obvious. That τ is a surjection is also obvious. Thus τ is a bijection.

Let $T=\langle\tau\rangle|_{\mathrm{up}\,\mathrm{dom}\,f\cap\mathscr{P}\mathrm{dom}\,F}$.

If $X\in\mathrm{up}\,\mathrm{dom}\,f\cap\mathscr{P}\mathrm{dom}\,F$ then

$$TX=\{\tau x\mid x\in X\}=\{(x;Fx)\mid x\in X\}=F|_X.$$

Thus $TX\subseteq F$ and $\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}TX\supseteq\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}$. So

$$TX\in\mathrm{up}\left(\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}\right)\cap\mathscr{P}F.$$

So $T:\mathrm{up}\,\mathrm{dom}\,f\cap\mathscr{P}\mathrm{dom}\,F\to\mathrm{up}\left(\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}\right)\cap\mathscr{P}F$.

Let $X,Y\in\mathrm{up}\,\mathrm{dom}\,f\cap\mathscr{P}\mathrm{dom}\,F$ and $X\neq Y$. Then $TX=\langle\tau\rangle X\neq\langle\tau\rangle Y=TY$ because τ is a bijection. So T is an injection.

Let $Y\in\mathrm{up}\left(\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}\right)\cap\mathscr{P}F$. Then $Y\subseteq F$ and thus $Y=F|_{\mathrm{dom}\,Y}$. We have $\mathrm{dom}\,Y\in\mathrm{up}\,\mathrm{dom}\,f\cap\mathscr{P}\mathrm{dom}\,F$ and

$$T\,\mathrm{dom}\,Y=\{\tau x\mid x\in\mathrm{dom}\,Y\}=\{(x;Fx)\mid x\in\mathrm{dom}\,Y\}=F|_{\mathrm{dom}\,Y}=Y.$$

Thus T is a surjection.

Thus T is a bijection and so $\mathrm{up}\,\mathrm{dom}\,f\cap\mathscr{P}\mathrm{dom}\,F$ is directly isomorphic to $\mathrm{up}\left(\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}\right)\cap\mathscr{P}F$. Consequently $\mathrm{up}\,\mathrm{dom}\,f$ is isomorphic to $\mathrm{up}\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}$, and so $\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}$ is an atomic filter object because $\mathrm{dom}\,f$ is atomic by the assumption. □

Theorem 80. If f, g are reloids, $f\subseteq g$ and g is monovalued then $g|_{\mathrm{dom}\,f}=f$.

Proof. It's simple to show that $f=\bigcup\left\{f|_a\mid a\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}\right\}$ (use the fact that every atomic reloid $k\subseteq f|_a$ for some $a\in\mathrm{atoms}\,1^{\mathfrak{F}(\mathrm{Src}\,f)}$ and the fact that $\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)$ is atomistic).

Suppose that $g|_{\mathrm{dom}\,f}\neq f$. Then there exists $a\in\mathrm{atoms}\,\mathrm{dom}\,f$ such that $g|_a\neq f|_a$.

Obviously $g|_a\supseteq f|_a$.

If $g|_a\supset f|_a$ then $g|_a$ is not atomic (because $f|_a\neq 0^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}$) what contradicts to a theorem above. So $g|_a=f|_a$ what is a contradiction and thus $g|_{\mathrm{dom}\,f}=f$. □

Corollary 81. Every monovalued reloid is a restricted discrete monovalued reloid.

Proof. Let f is a monovalued reloid. Then exists a function $F\in\mathrm{up}\,f$. So we have

$$\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}=f.$$ □

Corollary 82. Every monovalued injective reloid is a restricted injective monovalued discrete reloid.

Proof. Let f is a monovalued injective reloid. There exists a function F such that $f=\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}$. Also there exists an injection $G\in\mathrm{up}\,f$.

Thus $f=f\cap\left(\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}G\right)|_{\mathrm{dom}\,f}\right)=\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}F\right)|_{\mathrm{dom}\,f}\cap\left(\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}G\right)|_{\mathrm{dom}\,f}\right)=\left(\uparrow^{\mathsf{RLD}(\mathrm{Src}\,f;\mathrm{Dst}\,f)}(F\cap G)\right)|_{\mathrm{dom}\,f}$. Obviously $F\cap G$ is an injection. □

Theorem 83. If a reloid f is monovalued and $\mathrm{dom}\,f$ is a principal f.o. then f is discrete.

Proof. f is a discrete monovalued reloid. Thus $f = F|_{\operatorname{dom} f}$ where F is a discrete monovalued reloids. Thus f is discrete. □

Example 84. There exist two atomic reloids whose composition is non-atomic and non-empty.

Proof. Let a is a non-trivial atomic filter object on $\mathbb{N}$ and $x \in \mathbb{N}$. Then

$(a \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}}\{x\}) \circ (\uparrow^{\mathbb{N}}\{x\} \times^{\mathsf{RLD}} a) = \bigcap \{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}((A \times \{x\}) \circ (\{x\} \times A)) \mid A \in \operatorname{up} a\} = \bigcap \{\uparrow^{\mathsf{RLD}(\mathbb{N};\mathbb{N})}(A \times A) \mid A \in \operatorname{up} a\} = a \times^{\mathsf{RLD}} a$

is non-atomic despite of $a \times^{\mathsf{RLD}} \uparrow^{\mathbb{N}}\{x\}$ and $\uparrow^{\mathbb{N}}\{x\} \times^{\mathsf{RLD}} a$ are atomic. □

Example 85. There exists non-monovalued atomic reloid.

Proof. From the previous example follows that the atomic reloid $\uparrow^{\mathbb{N}}\{x\} \times^{\mathsf{RLD}} a$ is not monovalued. □

Example 86. $\mathcal{A} \geqslant_2 \mathcal{B} \wedge \mathcal{B} \geqslant_2 \mathcal{A}$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ for some f.o. $\mathcal{A}$ and $\mathcal{B}$.

Proof. (proof idea by Andreas Blass, rewritten using reloids by me)

Let u_n, h_n with n ranging over the set $\mathbb{Z}$ are sequences of atomic f.o. on $\mathbb{N}$ and functions $\mathbb{N} \rightarrow \mathbb{N}$ such that $\langle \uparrow^{\mathsf{FCD}(\mathbb{N};\mathbb{N})} h_n \rangle u_{n+1} = u_n$ and u_n are pairwise non-isomorphic. (See [1] for a proof that such ultrafilters and functions exist.)

$\mathcal{A} \stackrel{\text{def}}{=} \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1} \mid n \in \mathbb{Z}\}$; $\mathcal{B} \stackrel{\text{def}}{=} \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\}$.

Let the **Set**-morphisms $f, g: \mathbb{Z} \times \mathbb{N} \rightarrow \mathbb{Z} \times \mathbb{N}$ are defined by the formulas $f(n; x) = (n; h_{2n} x)$ and $g(n; x) = (n - 1; h_{2n-1} x)$.

Using the fact that every function induces a complete funcoid and a lemma above we get:

$\langle f \rangle \mathcal{A} = \bigcup \langle \langle \uparrow f \rangle \rangle \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1} \mid n \in \mathbb{Z}\} = \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\} = \mathcal{B}$.

$\langle g \rangle \mathcal{B} = \bigcup \langle \langle \uparrow g \rangle \rangle \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\} = \bigcup \{\uparrow^{\mathbb{Z}}\{n - 1\} \times^{\mathsf{RLD}} u_{2n-1} \mid n \in \mathbb{Z}\} = \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1} \mid n \in \mathbb{Z}\} = \mathcal{A}$.

It remains to show that $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic.

Let $X \in \operatorname{up}(\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1})$ for some $n \in \mathbb{Z}$. Then if $\uparrow^{\mathbb{Z} \times \mathbb{N}} X \cap \mathcal{A}$ is an atomic f.o. we have $\uparrow^{\mathbb{Z} \times \mathbb{N}} X \cap \mathcal{A} = \uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1}$ and thus by the theorem 78 is isomorphic to u_{2n+1}.

If $X \notin \operatorname{up}(\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1})$ for every $n \in \mathbb{Z}$ then $(\mathbb{Z} \times \mathbb{N}) \setminus X \in \operatorname{up}(\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n+1})$ and thus $(\mathbb{Z} \times \mathbb{N}) \setminus X \in \operatorname{up} \mathcal{A}$ and thus $\uparrow^{\mathbb{Z} \times \mathbb{N}} X \cap \mathcal{A} = \emptyset$.

We have also $(\uparrow^{\mathbb{Z}}\{0\} \times^{\mathsf{RLD}} \mathbb{N}) \cap \mathcal{B} = (\uparrow^{\mathbb{Z}}\{0\} \times^{\mathsf{RLD}} \mathbb{N}) \cap \bigcup \{\uparrow^{\mathbb{Z}}\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\} = \bigcup \{(\uparrow^{\mathbb{Z}}\{0\} \times^{\mathsf{RLD}} \mathbb{N}) \cap (\{n\} \times^{\mathsf{RLD}} u_{2n}) \mid n \in \mathbb{Z}\} = \uparrow^{\mathbb{Z}}\{0\} \times^{\mathsf{RLD}} u_0$ (an atomic f.o.).

Thus every atomic f.o. generated as intersecting $\mathcal{A}$ with a principal f.o. $\uparrow^{\mathbb{Z} \times \mathbb{N}} X$ is isomorphic to some u_{2n+1} and thus is not isomorphic to u_0. By the lemma it follows that $\mathcal{A}$ and $\mathcal{B}$ are non-isomorphic. □

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