Orderings of filters in terms of reloids*

Extensions of Rudin-Keisler ordering

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Abstract

Orderings of filters which extend Rudin-Keisler preorder of ultrafilters are described in terms of reloids that (roughly speaking) is filters on sets of binary relations between some sets. Also it is defined isomorphism of filters which extends Rudin-Keisler equivalence of ultrafilters.

Keywords: Rudin-Keisler order, Rudin-Keisler preorder, Rudin-Keisler ordering, filters, ultrafilters, reloids, isomorphism, isomorphic

A.M.S. subject classification: 06F99, 06A99, 54D99

1 Draft status

It is a preliminary draft. Most probably there are little errors in the proofs.

I plan to integrate this article into bigger article [4].

2 Preliminary definitions

See [3] for the definition of filter objects and [4] for the definition and properties of reloids.

I will call *small* sets members of some Grothendieck universe.

Recall that morphisms of the category **Set** (or "**Set**-morphisms" for short) are triples $(f; A; B)$ of a function f and small sets A and B where $\operatorname{dom} f \subseteq A$ and $\operatorname{im} f \subseteq B$.

2.1 Equivalent filters

Earlier I've considered filters on some fixed set $\mho$ (*base set*). But in this section we'll consider relations between filters on different base sets. It's easy to see that the base of a f.o. $\mathcal{A}$ is $\bigcup \operatorname{up} \mathcal{A}$. I will denote filters on set A as $\mathfrak{F}(A)$ and reloids on set A as $\mathsf{RLD}(A)$. In this section saying about filter objects, I will mean filter objects on possibly different sets.

We will restrict to small sets that is members of some Grothendieck universe.

Definition 1. Two filter objects $\mathcal{A}$ and $\mathcal{B}$ (with possibly different base sets) are *equivalent* ($\mathcal{A} \sim \mathcal{B}$) iff there exists a set X such that $X \in \operatorname{up} \mathcal{A}$ and $X \in \operatorname{up} \mathcal{B}$ and $\mathscr{P} X \cap \operatorname{up} \mathcal{A} = \mathscr{P} X \cap \operatorname{up} \mathcal{B}$.

Proposition 2. If two filter objects with the same base are equivalent they are equal.

Proof. Let $\mathcal{A}$ and $\mathcal{B}$ are two f.o. and $\mathscr{P} X \cap \operatorname{up} \mathcal{A} = \mathscr{P} X \cap \operatorname{up} \mathcal{B}$ for some set X and $\bigcup \operatorname{up} \mathcal{A} = \bigcup \operatorname{up} \mathcal{B}$. Then $\operatorname{up} \mathcal{A} = (\mathscr{P} X \cap \operatorname{up} \mathcal{A}) \cup \{Y \in \mathscr{P} \bigcup \operatorname{up} \mathcal{A} \mid Y \supseteq X\} = (\mathscr{P} X \cap \operatorname{up} \mathcal{B}) \cup \{Y \in \mathscr{P} \bigcup \operatorname{up} \mathcal{B} \mid Y \supseteq X\} = \operatorname{up} \mathcal{B}$. □

Proposition 3. $\sim$ restricted to small filter objects is an equivalence relation.

Proof.

Reflexivity. Obvious.

Symmetry. Obvious.

*. This document has been written using the GNU TEXMACS text editor (see www.texmacs.org).

Transitivity. Let $\mathcal{A}\sim\mathcal{B}$ and $\mathcal{B}\sim\mathcal{C}$ for some small f.o. $\mathcal{A}$, $\mathcal{B}$, and $\mathcal{C}$. Then exist a set X such that $X\in\operatorname{up}\mathcal{A}$ and $X\in\operatorname{up}\mathcal{B}$ and $\mathscr{P}X\cap\operatorname{up}\mathcal{A}=\mathscr{P}X\cap\operatorname{up}\mathcal{B}$ and a set Y such that $Y\in\operatorname{up}\mathcal{B}$ and $Y\in\operatorname{up}\mathcal{C}$ and $\mathscr{P}Y\cap\operatorname{up}\mathcal{B}=\mathscr{P}Y\cap\operatorname{up}\mathcal{C}$. So $X\cap Y\in\operatorname{up}\mathcal{A}$ because

$$\mathscr{P}Y\cap\mathscr{P}X\cap\operatorname{up}\mathcal{A}=\mathscr{P}Y\cap\mathscr{P}X\cap\operatorname{up}\mathcal{B}=\mathscr{P}(X\cap Y)\cap\operatorname{up}\mathcal{B}\supseteq\{X\cap Y\}\cap\operatorname{up}\mathcal{B}\ni X\cap Y.$$

Similarly we have $X\cap Y\in\operatorname{up}\mathcal{C}$. Finally $\mathscr{P}(X\cap Y)\cap\operatorname{up}\mathcal{A}=\mathscr{P}Y\cap\mathscr{P}X\cap\operatorname{up}\mathcal{A}=\mathscr{P}Y\cap\mathscr{P}X\cap\operatorname{up}\mathcal{B}=\mathscr{P}X\cap\mathscr{P}Y\cap\operatorname{up}\mathcal{B}=\mathscr{P}X\cap\mathscr{P}Y\cap\operatorname{up}\mathcal{C}=\mathscr{P}(X\cap Y)\cap\operatorname{up}\mathcal{C}$. □

Definition 4. The *rebase* $\mathcal{A}\div A$ for a f.o. $\mathcal{A}$ and a set A (base) such that $\exists X\in\operatorname{up}\mathcal{A}\colon X\subseteq A$ is defined by the formula

$$\mathcal{A}\div A=\operatorname{up}^{-1}\{X\in\mathscr{P}A\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\}$$

where "up" is taken for the set of f.o. on A.

Proposition 5. If $\exists X\in\operatorname{up}\mathcal{A}\colon X\subseteq A$ then:

1. $\mathcal{A}\div A$ is a f.o.
2. $\mathcal{A}\div A\sim\mathcal{A}$.

Proof.

1. We need to prove that $\{X\in\mathscr{P}A\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\}$ is a filter. That it is an upper set is obvious. It is non-empty because $\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq A$ and thus $A\in\{X\in\mathscr{P}A\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\}$. Let $P,Q\in\{X\in\mathscr{P}A\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\}$. Then $P,Q\subseteq A$ and $\exists P'\in\operatorname{up}\mathcal{A}\colon P'\subseteq P$ and $\exists Q'\in\operatorname{up}\mathcal{A}\colon Q'\subseteq P$. So $P\cap Q\subseteq A$ and $P'\cap Q'\subseteq P\cap Q$. Thus $P\cap Q\in\{X\in\mathscr{P}A\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\}$.
2. $\mathscr{P}(A\cap\bigcup\operatorname{up}\mathcal{A})\cap\operatorname{up}\mathcal{A}=\operatorname{up}(\mathcal{A}\div(A\cap\bigcup\operatorname{up}\mathcal{A}))=\{X\in\mathscr{P}(A\cap\bigcup\operatorname{up}\mathcal{A})\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\}=\mathscr{P}(A\cap\bigcup\operatorname{up}\mathcal{A})\cap\operatorname{up}\mathcal{A}=\mathscr{P}A\cap\operatorname{up}\mathcal{A}=\{X\in\mathscr{P}A\mid X\in\operatorname{up}\mathcal{A}\}=\{X\in\mathscr{P}A\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X,\ X\subseteq\bigcup\operatorname{up}\mathcal{A}\}=\{X\in\mathscr{P}A\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X,\ X\subseteq A\cap\bigcup\operatorname{up}\mathcal{A}\}=\mathscr{P}(A\cap\bigcup\operatorname{up}\mathcal{A})\cap\operatorname{up}(\mathcal{A}\div A)$.
   Thus $\mathcal{A}\div A\sim\mathcal{A}$ because $A\cap\bigcup\operatorname{up}\mathcal{A}\supseteq X\cap\bigcup\operatorname{up}\mathcal{A}=X\in\operatorname{up}\mathcal{A}$ for some $X\in\operatorname{up}\mathcal{A}$ and $A\cap\bigcup\operatorname{up}\mathcal{A}\supseteq X\cap\bigcup\operatorname{up}\mathcal{A}\in\{X\in\mathscr{P}A\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\}=\operatorname{up}(\mathcal{A}\div A)$. □

Proposition 6. $A\in\operatorname{up}\mathcal{A}\Rightarrow\operatorname{up}(\mathcal{A}\div A)=\mathscr{P}A\cap\operatorname{up}\mathcal{A}$.

Proof. Let $A\in\operatorname{up}\mathcal{A}$. Then $\operatorname{up}(\mathcal{A}\div A)=\{X\in\mathscr{P}A\mid\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\}=\{X\in\mathscr{P}A\mid X\in\operatorname{up}\mathcal{A}\}=\mathscr{P}A\cap\operatorname{up}\mathcal{A}$. □

Lemma 7. If $\mathcal{A}\sim\mathcal{B}$ then $\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\Leftrightarrow\exists Y\in\operatorname{up}\mathcal{B}\colon Y\subseteq X$ for every f.o. $\mathcal{A}$, $\mathcal{B}$, and a set X.

Proof. We will prove $\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\Rightarrow\exists Y\in\operatorname{up}\mathcal{B}\colon Y\subseteq X$ (the other direction is similar).
We have $\mathscr{P}K\cap\operatorname{up}\mathcal{A}=\mathscr{P}K\cap\operatorname{up}\mathcal{B}$ for some set K.
$\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\Rightarrow\exists Y\in\mathscr{P}K\cap\operatorname{up}\mathcal{A}\colon Y\subseteq X\Rightarrow\exists Y\in\mathscr{P}K\cap\operatorname{up}\mathcal{B}\colon Y\subseteq X\Rightarrow\exists Y\in\operatorname{up}\mathcal{B}\colon Y\subseteq X$. □

Proposition 8. If $\mathcal{A}\sim\mathcal{B}$ then $\mathcal{B}=\mathcal{A}\div\bigcup\operatorname{up}\mathcal{B}$ for every f.o. $\mathcal{A}$, $\mathcal{B}$.

Proof. $\mathscr{P}Y\cap\operatorname{up}\mathcal{A}=\mathscr{P}Y\cap\operatorname{up}\mathcal{B}$ for some set $Y\in\operatorname{up}\mathcal{A}$, $Y\in\operatorname{up}\mathcal{B}$. Thus exists a set $X\in\operatorname{up}\mathcal{A}$ such that $X\in\operatorname{up}\mathcal{B}\subseteq\bigcup\operatorname{up}\mathcal{B}$. Thus $\exists X\in\operatorname{up}\mathcal{A}\colon X\subseteq\bigcup\operatorname{up}\mathcal{B}$ and so $\mathcal{A}\div\bigcup\operatorname{up}\mathcal{B}$ is a properly defined f.o.
$X\in\operatorname{up}(\mathcal{A}\div\bigcup\operatorname{up}\mathcal{B})\Leftrightarrow X\in\mathscr{P}\bigcup\operatorname{up}\mathcal{B}\wedge\exists Y\in\operatorname{up}\mathcal{A}\colon Y\subseteq X\Leftrightarrow X\in\mathscr{P}\bigcup\operatorname{up}\mathcal{B}\wedge\exists Y\in\operatorname{up}\mathcal{B}\colon Y\subseteq X\Leftrightarrow X\in\operatorname{up}\mathcal{B}$ (the lemma used). □

2.2 Trans-reloids

Definition 9. A *trans-reloid* is a f.o. on the set $A\times B$ for some small sets A and B.
I will denote the set of trans-reloids for sets A, B as $\mathsf{RLD}(A;B)$.

Definition 10. $\operatorname{Src}f=\bigcup\operatorname{Pr}_0\operatorname{up}f$; $\operatorname{Dst}f=\bigcup\operatorname{Pr}_1\operatorname{up}f$ for every trans-reloid f.

Reloids are a special case of trans-reloids (take $A = B$).

In order to not repeat the entire theory of reloids for the more general case of trans-reloids, I will describe properties of trans-reloids through reloids equivalent (in the sense I defined above for filter objects) to them.

Definition 11. I will call trans-reloids f and g *composable* when $\operatorname{Dst} f = \operatorname{Src} g$.

Definition 12. Composition of trans-reloids is defined by the formula

$$g \circ f = \bigcap^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\}.$$

Composition of trans-reloids is a trans-reloid.

Theorem 13. Let f, g, f', g' are trans-reloids. Then $f \sim f' \wedge g \sim g' \Rightarrow g \circ f \sim g' \circ f'$.

Proof. There exist a binary relation K such that $K \in \operatorname{up} f$, $K \in \operatorname{up} f'$, and $\mathscr{P}K \cap \operatorname{up} f = \mathscr{P}K \cap \operatorname{up} f'$ and a binary relation L such that $L \in \operatorname{up} g$, $L \in \operatorname{up} g'$, and $\mathscr{P}L \cap \operatorname{up} g = \mathscr{P}L \cap \operatorname{up} g'$. $g \circ f = \bigcap^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\} = \bigcap^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \{G \circ F \mid F \in \mathscr{P}K \cap \operatorname{up} f, G \in \mathscr{P}K \cap \operatorname{up} g\} = \bigcap^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \{G \circ F \mid F \in \mathscr{P}K \cap \operatorname{up} f', G \in \mathscr{P}K \cap \operatorname{up} g'\} \sim \bigcap^{\mathsf{RLD}(\operatorname{Src} f'; \operatorname{Dst} g')} \{G \circ F \mid F \in \mathscr{P}K \cap \operatorname{up} f', G \in \mathscr{P}K \cap \operatorname{up} g'\} = \bigcap^{\mathsf{RLD}(\operatorname{Src} f'; \operatorname{Dst} g')} \{G \circ F \mid F \in \operatorname{up} f', G \in \operatorname{up} g'\} = g' \circ f'$. □

Theorem 14. The following are equivalent for every three composable trans-reloids f, g, h:

1. $h = g \circ f$.
2. There exist some set and reloids f' and g' on this set such that $f' \sim f$, $g' \sim g$, and $h \sim g' \circ f'$.
3. For every reloids f' and g' such that $f' \sim f$, $g' \sim g$ we have $h \sim g' \circ f'$.

Proof.

(1) ⇒ (2). Let $h = g \circ f$. Get $f' = f \div (A \cup B \cup C)^2$ and $g' = g \div (A \cup B \cup C)^2$. Obviously $f' \sim f$, $g' \sim g$. $g' \circ f' = \bigcap^{\mathsf{RLD}(A \cup B \cup C)} \{G \circ F \mid F \in \operatorname{up} f', G \in \operatorname{up} g'\} = \bigcap^{\mathsf{RLD}(A \cup B \cup C)} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\} \sim \bigcap^{\mathsf{RLD}(A; C)} \{G \circ F \mid F \in \operatorname{up} f, G \in \operatorname{up} g\} = g \circ f$.

(2) ⇒ (3). From the previous theorem.

(3) ⇒ (1). Let $f' = f$ and $g' = g$. We have $h \sim g \circ f$ and thus $h = g \circ f$. □

Theorem 15. Composition of trans-reloids is associative.

Proof. Let $f: A \to B$, $g: B \to C$, $h: C \to D$ are trans-reloids. Consider the reloids

$$f' = f \div (A \cup B \cup C \cup D)^2, \quad g' = g \div (A \cup B \cup C \cup D)^2, \quad h' = h \div (A \cup B \cup C \cup D)^2.$$

Then $g' \circ f' \sim g \circ f$ and $h' \circ g' \sim h \circ g$ thus $h \circ (g \circ f) \sim h' \circ (g' \circ f') = (h' \circ g') \circ f' \sim (h \circ g) \circ f$.

Thus $h \circ (g \circ f) = (h \circ g) \circ f$. □

It is simple to show that trans-reloids with the given composition and identity functions as identities form a category. I'll call it *the category of trans-reloids*.

2.2.1 Some definitions for trans-reloids

Definition 16. The *reverse* trans-reloid of a trans-reloid f is defined by the formula

$$\operatorname{up} f^{-1} = \left\{F^{-1} \mid F \in \operatorname{up} f^{-1}\right\}.$$

Definition 17. *Domain* and *image* of a trans-reloid f are defined as follows:

$$\operatorname{dom} f = \bigcap^{\mathfrak{F}(\operatorname{Dst} f)} \langle \operatorname{dom} \rangle \operatorname{up} f; \quad \operatorname{im} f = \bigcap^{\mathfrak{F}(\operatorname{Dst} f)} \langle \operatorname{im} \rangle \operatorname{up} f.$$

Definition 18.

1. A trans-reloid is *monovalued* when it is a monovalued morhism of the category of trans-reloids.
2. A trans-reloid is *injective* when it is an injective morhism of the category of trans-reloids.

Definition 19. Let f is a trans-reloid and $\mathcal{A}$ is a f.o. on $\operatorname{Src} f$. Then $f|_{\mathcal{A}}^{\mathsf{RLD}} = f \circ I_{\mathcal{A}}^{\mathsf{RLD}(\operatorname{Src} f)}$.

We can define also trans-funcoids in a similar way but we don't need it in this work. Instead I will define applying a **Set**-morhisms to filters:

Definition 20. $\langle (f; A; B) \rangle \mathcal{X} = \bigcap^{\mathfrak{F}(B)} \langle \langle f \rangle \rangle \operatorname{up}(\mathcal{X} \div A)$ for every **Set**-morhism $(f; A; B)$ and sets A and B.

2.2.2 Some theorems about trans-reloids

In this section I do not try to build a complete set of theorems about trans-reloids, just theorems we'll need below.

Theorem 21. If $f' \sim f$ for some trans-reloids f and f' then $\operatorname{im} f \sim \operatorname{im} f'$ and $\operatorname{dom} f \sim \operatorname{dom} f'$.

Proof. There exists a binary relation $K \in \operatorname{up} f, \operatorname{up} f'$ such that $\mathscr{P}K \cap f = \mathscr{P}K \cap f'$. Thus $\operatorname{im} f = \operatorname{im}^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)}(\mathscr{P}K \cap f) = \operatorname{im}^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} f)}(\mathscr{P}K \cap f') \sim \operatorname{im}^{(\operatorname{Src} f'; \operatorname{Dst} f')}(\mathscr{P}K \cap f') = \operatorname{im} f'$. □

Theorem 22. For every trans-reloids f, g:

1. If $\operatorname{im} f \supseteq \operatorname{dom} g$ then $\operatorname{im}(g \circ f) = \operatorname{im} g$.
2. If $\operatorname{im} f \subseteq \operatorname{dom} g$ then $\operatorname{dom}(g \circ f) = \operatorname{dom} f$.

Proof. Let f, g, and $h = g \circ f$ are trans-reloids and $\operatorname{im} f \supseteq \operatorname{dom} g$.

There exist reloids f', g' on some set such that $h \sim g' \circ f'$ and $f' \sim f$, $g' \sim g$.

Obviously $\operatorname{im} f' \supseteq \operatorname{dom} g'$.

$\operatorname{im} h \sim \operatorname{im}(g' \circ f') = \operatorname{im} g' \sim \operatorname{im} g$ (used the theorem 222?? in [4]). □

3 Ordering of filters

Below I will define some categories having filter objects (with possibly different bases) as their objects and some relations having two filter objects (with possibly different bases) as arguments induced by these categories (defined as existence of a morphism between these two f.o.).

Theorem 23. $\operatorname{card} a = \operatorname{card} U$ for every ultrafilter a on U if U is infinite.

Proof. Let $f(X) = X$ if $X \in a$ and $f(X) = U \setminus X$ if $X \notin a$. Obviously f is a surjection.

Every $X \in a$ appears as a value of f exactly twice, as $f(X)$ and $f(U \setminus X)$. So $\operatorname{card} a = U/2 = U$. □

Corollary 24. Cardinality of every two ultrafilters on a set U is the same.

Proof. For infinite U it follows from the theorem. For finite case it is obvious. □

Definition 25. $f * \mathcal{A} = \left\{ C \in \mathscr{P}(\operatorname{Dst} f) \mid \left\langle f^{-1} \right\rangle C \in \operatorname{up} \mathcal{A} \right\}$ for every f.o. $\mathcal{A}$ and a **Set**-morphism f.

Below I'll define some directed multigraphs. By an abuse of notation, I will denote these multigraphs the same as (below defined) categories based on some of these these directed multigraphs with added composition of morphisms (of directed multigraphs edges). As such I will call vertices of these multigraphs objects and edges morphisms.

Definition 26. I will denote $\mathbf{GreFunc}_1$ the multigraph whose objects are filter objects and whose morphisms between objects $\mathcal{A}$ and $\mathcal{B}$ are **Set**-morphisms from $\bigcup \operatorname{up}\mathcal{A}$ to $\bigcup \operatorname{up}\mathcal{B}$ such that $\operatorname{up}\mathcal{B} \supseteq f{*}\mathcal{A}$.

Definition 27. I will denote $\mathbf{GreFunc}_2$ the multigraph whose objects are filter objects and whose morphisms between objects $\mathcal{A}$ and $\mathcal{B}$ are **Set**-morphisms from $\bigcup \operatorname{up}\mathcal{A}$ to $\bigcup \operatorname{up}\mathcal{B}$ such that $\operatorname{up}\mathcal{B} = f{*}\mathcal{A}$.

Definition 28. Let $\mathcal{A}$ is a f.o. on a set X and $\mathcal{B}$ is a f.o. on a set Y. $\mathcal{A} \geqslant_1 \mathcal{B}$ iff $\operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A};\mathcal{B})$ is not empty.

Definition 29. Let $\mathcal{A}$ is an f.o. on a set X and $\mathcal{B}$ is an f.o. on a set Y. $\mathcal{A} \geqslant_2 \mathcal{B}$ iff $\operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A};\mathcal{B})$ is not empty.

Proposition 30.

1. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A};\mathcal{B})$ iff f is a **Set**-morphism from $\bigcup \operatorname{up}\mathcal{A}$ to $\bigcup \operatorname{up}\mathcal{B}$ such that
$$C \in \operatorname{up}\mathcal{B} \Leftarrow \left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A}$$
for every $C \in \mathscr{P}\bigcup \operatorname{up}\mathcal{B}$.
2. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A};\mathcal{B})$ iff f is a **Set**-morphism from $\bigcup \operatorname{up}\mathcal{A}$ to $\bigcup \operatorname{up}\mathcal{B}$ such that
$$C \in \operatorname{up}\mathcal{B} \Leftrightarrow \left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A}$$
for every $C \in \mathscr{P}\bigcup \operatorname{up}\mathcal{B}$.

Proof.

1. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A};\mathcal{B}) \Leftrightarrow \operatorname{up}\mathcal{B} \supseteq f{*}\mathcal{A} \Leftrightarrow \forall C \in f{*}\mathcal{A}\colon C \in \operatorname{up}\mathcal{B} \Leftrightarrow \forall C \in \mathscr{P}\bigcup \operatorname{up}\mathcal{B}\colon \left(\left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A} \Rightarrow C \in \operatorname{up}\mathcal{B}\right)$.
2. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A};\mathcal{B}) \Leftrightarrow \operatorname{up}\mathcal{B} = f{*}\mathcal{A} \Leftrightarrow \forall C\colon (C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in f{*}\mathcal{A}) \Leftrightarrow \forall C \in \mathscr{P}\bigcup \operatorname{up}\mathcal{B}\colon (C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in f{*}\mathcal{A}) \Leftrightarrow \forall C \in \mathscr{P}\bigcup \operatorname{up}\mathcal{B}\colon \left(C \in \operatorname{up}\mathcal{B} \Leftrightarrow \left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A}\right)$. □

Definition 31. The multigraph **FuncBij** is the multigraph got from $\mathbf{GreFunc}_2$ by restricting to only bijective morphisms.

Definition 32. A f.o. $\mathcal{A}$ is directly isomorphic to a f.o. $\mathcal{B}$ iff there are a morphism $f \in \operatorname{Mor}_{\mathbf{FuncBij}}(\mathcal{A};\mathcal{B})$.

Proposition 33. $f{*}\mathcal{A} = \operatorname{up}\langle f\rangle\mathcal{A}$ for every **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$.

Proof. For every set $C \in \mathscr{P}\bigcup \operatorname{up}\mathcal{B}$ we have $C \in f{*}\mathcal{A} \Leftrightarrow \left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A} \Rightarrow \exists K \in \operatorname{up}\mathcal{A}\colon \left\langle f^{-1}\right\rangle C = K \Rightarrow \exists K \in \operatorname{up}\mathcal{A}\colon \langle f\rangle\left\langle f^{-1}\right\rangle C = \langle f\rangle K \Rightarrow \exists K \in \operatorname{up}\mathcal{A}\colon C \supseteq \langle f\rangle K \Leftrightarrow \exists K \in \operatorname{up}\mathcal{A}\colon C \in \operatorname{up}\langle f\rangle K \Leftrightarrow C \in \operatorname{up}\bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle\operatorname{up}\mathcal{A} \Leftrightarrow C \in \operatorname{up}\langle f\rangle\mathcal{A}$.

So $C \in f{*}\mathcal{A} \Rightarrow C \in \operatorname{up}\langle f\rangle\mathcal{A}$.

Let now $C \in \operatorname{up}\langle f\rangle\mathcal{A}$. Then $\left\langle f^{-1}\right\rangle C \supseteq \left\langle f^{-1}\right\rangle\langle f\rangle\mathcal{A} \supseteq \mathcal{A}$ and thus $\left\langle f^{-1}\right\rangle C \in \operatorname{up}\mathcal{A}$. □

Corollary 34. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_1}(\mathcal{A};\mathcal{B}) \Leftrightarrow \mathcal{B} \subseteq \langle f\rangle\mathcal{A}$ for every **Set**-morphism f from $\bigcup \operatorname{up}\mathcal{A}$ to $\bigcup \operatorname{up}\mathcal{B}$.

Corollary 35. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A};\mathcal{B}) \Leftrightarrow \mathcal{B} = \langle f\rangle\mathcal{A}$ for every **Set**-morphism f from $\bigcup \operatorname{up}\mathcal{A}$ to $\bigcup \operatorname{up}\mathcal{B}$.

Corollary 36. $\mathcal{A} \geqslant_2 \mathcal{B}$ iff exists a **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\mathcal{B} = \langle f\rangle\mathcal{A}$.

Corollary 37. $\operatorname{up}\mathcal{B} \supseteq f{*}\mathcal{A} \Leftrightarrow \mathcal{B} \subseteq \langle f\rangle\mathcal{A}$.

Corollary 38. $\mathcal{A} \geqslant_1 \mathcal{B}$ iff exists a **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\mathcal{B} \subseteq \langle f\rangle\mathcal{A}$.

Proposition 39. For a bijective **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ the following are equivalent:

1. $\operatorname{up}\mathcal{B} = f{*}\mathcal{A}$.
2. $\forall C \in \bigcup \operatorname{up}\mathcal{B}\colon \big(C \in \operatorname{up}\mathcal{B} \Leftrightarrow \big\langle f^{-1}\big\rangle C \in \operatorname{up}\mathcal{A}\big)$.
3. $\forall C \in \bigcup \operatorname{up}\mathcal{B}\colon (\langle f\rangle C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in \operatorname{up}\mathcal{A})$.
4. $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is a bijection from $\operatorname{up}\mathcal{A}$ to $\operatorname{up}\mathcal{B}$.
5. $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is a function onto $\operatorname{up}\mathcal{B}$.
6. $\mathcal{B} = \langle f\rangle\mathcal{A}$.
7. $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A};\mathcal{B})$.
8. $f \in \operatorname{Mor}_{\mathbf{FuncBij}}(\mathcal{A};\mathcal{B})$.

Proof.

(1) $\Leftrightarrow$ (2). $\operatorname{up}\mathcal{B} = f{*}\mathcal{A} \Leftrightarrow \operatorname{up}\mathcal{B} = \big\{C \in \mathscr{P}\bigcup \operatorname{up}\mathcal{B} \mid \big\langle f^{-1}\big\rangle C \in \operatorname{up}\mathcal{A}\big\} \Leftrightarrow \forall C \in \bigcup \operatorname{up}\mathcal{B}\colon \big(C \in \operatorname{up}\mathcal{B} \Leftrightarrow \big\langle f^{-1}\big\rangle C \in \operatorname{up}\mathcal{A}\big)$.

(2) $\Leftrightarrow$ (3). Because f is a bijection.

(2) $\Rightarrow$ (5). For every $C \in \operatorname{up}\mathcal{B}$ we have $\big\langle f^{-1}\big\rangle C \in \operatorname{up}\mathcal{A}$ and thus $\langle f\rangle|_{\operatorname{up}\mathcal{A}}\big\langle f^{-1}\big\rangle C = \langle f\rangle\big\langle f^{-1}\big\rangle C = C \in \operatorname{up}\mathcal{B}$. Thus $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is onto $\operatorname{up}\mathcal{B}$.

(4) $\Rightarrow$ (5). Obvious.

(5) $\Rightarrow$ (4). We need to prove only that $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is an injection. But this follows from the fact that f is a bijection.

(4) $\Rightarrow$ (3). We have $\forall C \in \bigcup \operatorname{up}\mathcal{B}\colon ((\langle f\rangle|_{\operatorname{up}\mathcal{A}})C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in \operatorname{up}\mathcal{A})$ and consequently $\forall C \in \bigcup \operatorname{up}\mathcal{B}\colon (\langle f\rangle C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in \operatorname{up}\mathcal{A})$.

(6) $\Leftrightarrow$ (1). From the last corollary.

(1) $\Leftrightarrow$ (7). Obvious.

(7) $\Leftrightarrow$ (8). Obvious. □

Corollary 40. The following are equivalent for every f.o. $\mathcal{A}$ and $\mathcal{B}$:

1. $\mathcal{A}$ is directly isomorphic to a f.o. $\mathcal{B}$.
2. There are a bijective **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that for every $C \in \mathscr{P}\bigcup \operatorname{up}\mathcal{B}$
$$C \in \operatorname{up}\mathcal{B} \Leftrightarrow \big\langle f^{-1}\big\rangle C \in \operatorname{up}\mathcal{A}$$
3. There are a bijective **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that for every $C \in \mathscr{P}\bigcup \operatorname{up}\mathcal{B}$
$$\langle f\rangle C \in \operatorname{up}\mathcal{B} \Leftrightarrow C \in \operatorname{up}\mathcal{A}.$$
4. There are a bijective **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is a bijection from $\operatorname{up}\mathcal{A}$ to $\operatorname{up}\mathcal{B}$.
5. There are a bijective **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\langle f\rangle|_{\operatorname{up}\mathcal{A}}$ is a function onto $\operatorname{up}\mathcal{B}$.
6. There are a bijective **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\mathcal{B} = \langle f\rangle\mathcal{A}$.
7. There are a bijective morphism $f \in \operatorname{Mor}_{\mathbf{GreFunc}_2}(\mathcal{A};\mathcal{B})$.

Proposition 41. $\mathbf{GreFunc}_1$ and $\mathbf{GreFunc}_2$ with function composition are categories.

Proof. Let $f\colon \mathcal{A} \to \mathcal{B}$ and $g\colon \mathcal{B} \to \mathcal{C}$ are morphisms of $\mathbf{GreFunc}_1$. Then $\mathcal{B} \subseteq \langle f\rangle\mathcal{A}$ and $\mathcal{C} \subseteq \langle g\rangle\mathcal{B}$. So $\langle g \circ f\rangle\mathcal{A} = \langle g\rangle\langle f\rangle\mathcal{A} \supseteq \langle g\rangle\mathcal{B} \supseteq \mathcal{C}$. Thus $g \circ f$ is a morphism of $\mathbf{GreFunc}_1$. Associativity law is evident. $\operatorname{Id}_{\bigcup \operatorname{up}\mathcal{A}}$ is the identity morphism of $\mathbf{GreFunc}_1$ for every f.o. $\mathcal{A}$.

Let $f\colon \mathcal{A}\to\mathcal{B}$ and $g\colon \mathcal{B}\to\mathcal{C}$ are morphisms of $\mathbf{GreFunc}_2$. Then $\mathcal{B}=\langle f\rangle\mathcal{A}$ and $\mathcal{C}=\langle g\rangle\mathcal{B}$. So $\langle g\circ f\rangle\mathcal{A}=\langle g\rangle\langle f\rangle\mathcal{A}=\langle g\rangle\mathcal{B}=\mathcal{C}$. Thus $g\circ f$ is a morphism of $\mathbf{GreFunc}_2$. Associativity law is evident. $\mathrm{Id}_{\bigcup\operatorname{up}\mathcal{A}}$ is the identity morphism of $\mathbf{GreFunc}_2$ for every f.o. $\mathcal{A}$. □

Corollary 42. $\leqslant_1$ and $\leqslant_2$ are preorders.

Theorem 43. **FuncBij** is a groupoid.

Proof. First let's prove it is a category. Let $f\colon \mathcal{A}\to\mathcal{B}$ and $g\colon \mathcal{B}\to\mathcal{C}$ are morphisms of **FuncBij**. Then $f\colon \bigcup\operatorname{up}\mathcal{A}\to\bigcup\operatorname{up}\mathcal{B}$ and $g\colon \bigcup\operatorname{up}\mathcal{B}\to\bigcup\operatorname{up}\mathcal{C}$ are bijections and $\mathcal{B}=\langle f\rangle\mathcal{A}$ and $\mathcal{C}=\langle g\rangle\mathcal{B}$. Thus $g\circ f\colon \bigcup\operatorname{up}\mathcal{A}\to\bigcup\operatorname{up}\mathcal{C}$ is a bijection and $\mathcal{C}=\langle g\circ f\rangle\mathcal{A}$. Thus $g\circ f$ is a morphism of **FuncBij**. $\mathrm{Id}_{\bigcup\operatorname{up}\mathcal{A}}$ is the identity morphism of **FuncBij** for every f.o. $\mathcal{A}$. Thus it is a category.

It remains to prove only that every morphism $f\in\mathrm{Mor}_{\mathbf{FuncBij}}(\mathcal{A};\mathcal{B})$ has a reverse (for every f.o. $\mathcal{A}$, $\mathcal{B}$). We have f is a bijection $\bigcup\operatorname{up}\mathcal{A}\to\bigcup\operatorname{up}\mathcal{B}$ such that for every $C\in\mathscr{P}\bigcup\operatorname{up}\mathcal{A}$

$$\langle f\rangle C\in\operatorname{up}\mathcal{B}\Leftrightarrow C\in\operatorname{up}\mathcal{A}.$$

Then $f^{-1}\colon \bigcup\operatorname{up}\mathcal{B}\to\bigcup\operatorname{up}\mathcal{A}$ is a bijection such that for every $C\in\mathscr{P}\bigcup\operatorname{up}\mathcal{A}$

$$\left\langle f^{-1}\right\rangle C\in\operatorname{up}\mathcal{A}\Leftrightarrow C\in\operatorname{up}\mathcal{B}.$$

Thus $f^{-1}\in\mathrm{Mor}_{\mathbf{FuncBij}}(\mathcal{B};\mathcal{A})$. □

Corollary 44. Being directly isomorphic is an equivalence relation.

Obvious 45. For the case of ultrafilters being directly isomorphic is the same as being Rudin-Keisler equivalent.

Definition 46. A f.o. $\mathcal{A}$ is *isomorphic* to a f.o. $\mathcal{B}$ iff there exist sets $A\in\operatorname{up}\mathcal{A}$ and $B\in\operatorname{up}\mathcal{B}$ such that $\mathcal{A}\div A$ is directly isomorphic to $\mathcal{B}\div B$.

Obvious 47. Equivalent f.o. are isomorphic.

Theorem 48. Being isomorphic (for filter objects) is an equivalence relation.

Proof.

Reflexivity. Because every f.o. is directly isomorphic to itself.

Symmetry. If f.o. $\mathcal{A}$ is isomorphic to $\mathcal{B}$ then there exist sets $A\in\operatorname{up}\mathcal{A}$ and $B\in\operatorname{up}\mathcal{B}$ such that $\mathcal{A}\div A$ is directly isomorphic to $\mathcal{B}\div B$ and thus $\mathcal{B}\div B$ is directly isomorphic to $\mathcal{A}\div A$, So $\mathcal{B}$ is isomorphic to $\mathcal{A}$.

Transitivity. Let $\mathcal{A}$ is isomorphic to $\mathcal{B}$ and $\mathcal{B}$ is isomorphic to $\mathcal{C}$. Then exist $A\in\operatorname{up}\mathcal{A}$, $B_1\in\operatorname{up}\mathcal{B}$, $B_2\in\operatorname{up}\mathcal{B}$, $C\in\operatorname{up}\mathcal{C}$ such that there are bijections $f\colon A\to B_1$ and $g\in B_2\to C$ such that

$$\forall X\in\mathscr{P}A\colon\left(X\in\operatorname{up}\mathcal{B}\Leftrightarrow\left\langle f^{-1}\right\rangle X\in\operatorname{up}\mathcal{A}\right)\quad\text{and}\quad\forall X\in\mathscr{P}B_2\colon(X\in\operatorname{up}\mathcal{A}\Leftrightarrow\langle f\rangle X\in\operatorname{up}\mathcal{B}).$$

Also $\forall X\in\mathscr{P}B_2\colon(X\in\operatorname{up}\mathcal{B}\Leftrightarrow\langle g\rangle X\in\operatorname{up}\mathcal{C})$.

So $g\circ f$ is a bijection from $\left\langle f^{-1}\right\rangle(B_1\cap B_2)\in\operatorname{up}\mathcal{A}$ to $\langle g\rangle(B_1\cap B_2)\in\operatorname{up}\mathcal{C}$ such that

$$X\in\operatorname{up}\mathcal{A}\Leftrightarrow\langle f\rangle X\in\operatorname{up}\mathcal{B}\Leftrightarrow\langle g\rangle\langle f\rangle X\in\operatorname{up}\mathcal{C}\Leftrightarrow\langle g\circ f\rangle X\in\operatorname{up}\mathcal{C}.$$

Thus $g\circ f$ establishes a bijection which proves that $\mathcal{A}$ is isomorphic to $\mathcal{C}$. □

Lemma 49. Let $\operatorname{card}X=\operatorname{card}Y$, u is an atomic f.o. on X and v is an atomic f.o. on Y; let $A\in\operatorname{up}u$ and $B\in\operatorname{up}v$. Let $u\div A$ and $v\div B$ are directly isomorphic. Then if $\operatorname{card}(X\setminus A)=\operatorname{card}(Y\setminus B)$ we have u and v directly isomorphic.

Proof. Arbitrary extend the bijection witnessing being directly isomorphic to the sets $X\setminus A$ and $Y\setminus B$. [TODO: A little more detailed proof.] □

Theorem 50. If $\operatorname{card} X = \operatorname{card} Y$ then isomorphism and direct isomorphism are the same for atomic f.o. u on X and v on Y.

Proof. That if two filter objects are isomorphic then they are directly isomorphic is obvious.

[TODO: Carefully choose a is isomorphic to b vs. b is isomorphic to a.]

Let atomic f.o. u and v are isomorphic that is there are a bijection $f\colon A \to B$ where $A \in \operatorname{up} u$, $B \in \operatorname{up} v$ witnessing isomorphism of u and v.

If one of the filters u or v is a trivial atomic f.o. then the other is also a trivial atomic f.o. and as it is easy to show they are directly isomorphic. So we can assume u and v are not trivial atomic f.o.

If $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B)$ our statement follows from the last lemma.

Now assume without loss of generality $\operatorname{card}(X \setminus A) < \operatorname{card}(Y \setminus B)$.

$\operatorname{card} B = \operatorname{card} Y$ because $\operatorname{card}(Y \setminus B) < \operatorname{card} Y$.

Let $B' \supset B$ such that $\operatorname{card}(X \setminus A) = \operatorname{card}(Y \setminus B')$ and $\operatorname{card} B' = \operatorname{card} B$. [TODO: Detailed proof that B' exists.]

We will find a bijection g from B and B' which witnesses direct isomorphism of v to v itself. Then the composition $g \circ f$ witnesses a direct isomorphism of $u \div A$ and $v \div B'$ and by the lemma u and v are directly isomorphic.

Let $D = B' \setminus B$. We have $D \notin v$.

There exists a set $E \subseteq B$ such that $\operatorname{card} E \geqslant \operatorname{card} D$ and $E \notin \operatorname{up} v$.

We have $\operatorname{card} E = \operatorname{card}(D \cup E)$ and thus there exists a bijection $h\colon E \to D \cup E$.

Let

$$g(x) = \begin{cases} x & \text{if } x \in B \setminus E; \\ h(x) & \text{if } x \in E. \end{cases}$$

$g|_{B \setminus E}$ and $g|_E$ are bijections.

$\operatorname{im}\big(g|_{B \setminus E}\big) = B \setminus E$; $\operatorname{im}(g|_E) = \operatorname{im} h = D \cup E$;

$$(D \cup E) \cap (B \setminus E) = (D \cap (B \setminus E)) \cup (E \cap (B \setminus E)) = \emptyset \cup \emptyset = \emptyset.$$

Thus g is a bijection from B to $(B \setminus E) \cup (D \cup E) = B \cup D = B'$.

To finish the proof it's enough to show that $\langle g \rangle v = v$. Indeed it follows from $B \setminus E \in \operatorname{up} v$. □

Proposition 51. [TODO: Can be generalized for $A \supset \bigcup \operatorname{up} \mathcal{A}$ and likewise for B?]

1. For every $A \in \operatorname{up} \mathcal{A}$ and $B \in \operatorname{up} \mathcal{B}$ we have $\mathcal{A} \geqslant_2 \mathcal{B}$ iff $\mathcal{A} \div A \geqslant_2 \mathcal{B} \div B$.
2. For every $A \in \operatorname{up} \mathcal{A}$ and $B \in \operatorname{up} \mathcal{B}$ we have $\mathcal{A} \geqslant_1 \mathcal{B}$ iff $\mathcal{A} \div A \geqslant_1 \mathcal{B} \div B$.

Proof.

1. $\mathcal{A} \div A \geqslant_2 \mathcal{B} \div B$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} = \langle f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$.
2. $\mathcal{A} \div A \geqslant_1 \mathcal{B} \div B$ iff there exist a bijective **Set**-morphism f such that $\mathcal{B} \subseteq \langle f \rangle \mathcal{A}$. The equality is obviously preserved replacing $\mathcal{A}$ with $\mathcal{A} \div A$ and $\mathcal{B}$ with $\mathcal{B} \div B$. □

Rudin-Keisler order of ultrafilters is considered in such a book as [6].

Proposition 52. For ultrafilters $\geqslant_2$ is the same as Rudin-Keisler ordering.

Proof. $x \geqslant_2 y$ iff there exist sets $A \in \operatorname{up} x$ and $B \in \operatorname{up} y$ a bijective **Set**-morphism $f\colon X \to Y$ such that $\operatorname{up}(y \div B) = \big\{C \in \mathscr{P}Y \mid \big\langle f^{-1} \big\rangle C \in \operatorname{up}(x \div A)\big\}$ that is when $C \in \operatorname{up}(y \div B) \Leftrightarrow \big\langle f^{-1} \big\rangle C \in \operatorname{up}(x \div A)$ what is equivalent to $C \in \operatorname{up} y \Leftrightarrow \big\langle f^{-1} \big\rangle C \in \operatorname{up} x$ what is the definition of Rudin-Keisler ordering. □

Remark 53. The relation of being isomorphic for ultrafilters is traditionally called *Rudin-Keisler equivalence*.

Obvious 54. $(\geqslant_1) \supseteq (\geqslant_2)$.

Definition 55. Let Q and R are binary relations on the set of filter objects. I will denote $\mathbf{MonRld}_{Q,R}$ the directed multigraph with objects being filter objects and morphisms such mono-valued trans-reloids f that $(\operatorname{dom} f)\, Q\, \mathcal{A}$ and $(\operatorname{im} f)\, R\, \mathcal{B}$.

I will also denote $\mathbf{CoMonRld}_{Q,R}$ the directed multigraph with objects being filter objects and morphisms such injective trans-reloids f that $(\operatorname{im} f)\, Q\, \mathcal{A}$ and $(\operatorname{dom} f)\, R\, \mathcal{B}$. These are essentially the duals.

Some of these directed multigraphs are categories with reloid composition (see below). By abuse of notation I will denote these categories the same as these directed multigraphs.

Theorem 56. For every f.o. $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \geqslant_1 \mathcal{B}$.
2. $\operatorname{Mor}_{\mathbf{MonRld}_{=,\supseteq}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.
3. $\operatorname{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.
4. $\operatorname{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.
5. $\operatorname{Mor}_{\mathbf{CoMonRld}_{=,\supseteq}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.
6. $\operatorname{Mor}_{\mathbf{CoMonRld}_{\subseteq,\supseteq}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.
7. $\operatorname{Mor}_{\mathbf{CoMonRld}_{\subseteq,=}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.

Proof.

(1) ⇒ (2). There exists a **Set**-morphism $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\mathcal{B} \subseteq \langle f\rangle\mathcal{A}$. We have

$$\operatorname{dom} f|_{\mathcal{A}}^{\mathsf{RLD}} = \mathcal{A} \cap^{\mathfrak{F}} \operatorname{dom} f = \mathcal{A}$$

and

$$\operatorname{im} f|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im}(\mathsf{FCD}) f|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im}((\mathsf{FCD}) f)|_{\mathcal{A}}^{\mathsf{FCD}} = \operatorname{im}\big(f|_{\mathcal{A}}^{\mathsf{FCD}}\big) = \langle f\rangle\mathcal{A} \supseteq \mathcal{B}.$$

Thus $f|_{\mathcal{A}}^{\mathsf{RLD}}$ is a monovalued reloid such that $\operatorname{dom} f|_{\mathcal{A}}^{\mathsf{RLD}} = \mathcal{A}$ and $\operatorname{im} f|_{\mathcal{A}}^{\mathsf{RLD}} \supseteq \mathcal{B}$.

(2) ⇒ (3), (4) ⇒ (3), (5) ⇒ (6), (7) ⇒ (6). Obvious.

(3) ⇒ (1). We have $\mathcal{B} \subseteq \langle f\rangle\mathcal{A}$. Then there exists a **Set**-morphism $F\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\mathcal{B} \subseteq \langle F\rangle\mathcal{A}$ that is $\mathcal{A} \geqslant_1 \mathcal{B}$.

(6) ⇒ (7). $\operatorname{dom} f|_{\mathcal{B}} = \mathcal{B}$ and $\operatorname{im} f|_{\mathcal{B}} \subseteq \mathcal{A}$.

(2) ⇔ (5), (3) ⇔ (6), (4) ⇔ (7). By duality. □

Theorem 57. For every f.o. $\mathcal{A}$ and $\mathcal{B}$ the following are equivalent:

1. $\mathcal{A} \geqslant_2 \mathcal{B}$.
2. $\operatorname{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.
3. $\operatorname{Mor}_{\mathbf{CoMonRld}_{=,=}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.

Proof.

(1) ⇒ (2). Let $\mathcal{A} \geqslant_2 \mathcal{B}$ that is $\mathcal{B} = \langle f\rangle\mathcal{A}$ for some $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$. Then $\operatorname{dom} f|_{\mathcal{A}}^{\mathsf{RLD}} = \mathcal{A}$ (where RLD is taken on the set $\bigcup \operatorname{up}\mathcal{A} \cup \bigcup \operatorname{up}\mathcal{B}$) and $\operatorname{im} f|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im}(\mathsf{FCD}) f|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im}((\mathsf{FCD}) f)|_{\mathcal{A}}^{\mathsf{FCD}} = \operatorname{im}\big(f|_{\mathcal{A}}^{\mathsf{FCD}}\big) = \langle f\rangle\mathcal{A} = \mathcal{B}$. So $f|_{\mathcal{A}}^{\mathsf{RLD}}$ is a sought for trans-reloid.

(2) ⇒ (1). There exists a function $F\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $f = F|_{\mathcal{A}}^{\mathsf{RLD}}$ (where RLD is taken on the set $\bigcup \operatorname{up}\mathcal{A} \cup \bigcup \operatorname{up}\mathcal{B}$). Thus $\langle F\rangle\mathcal{A} = \operatorname{im}\big(F|_{\mathcal{A}}^{\mathsf{FCD}}\big) = \operatorname{im}(\mathsf{FCD}) F|_{\mathcal{A}}^{\mathsf{RLD}} = \operatorname{im}(\mathsf{FCD}) f = \operatorname{im} f = \mathcal{B}$. Thus $\mathcal{A} \geqslant_2 \mathcal{B}$ is testified by the function F.

(2) ⇔ (3). By duality. □

Theorem 58. The following are categories (with trans-reloid composition):

1. $\mathbf{MonRld}_{\subseteq,\supseteq}$;
2. $\mathbf{MonRld}_{\subseteq,=}$;
3. $\mathbf{MonRld}_{=,=}$.

4. $\mathbf{CoMonRld}_{\subseteq,\supseteq}$;
5. $\mathbf{CoMonRld}_{\subseteq,=}$;
6. $\mathbf{CoMonRld}_{=,=}$.

Proof. We will prove only the first three. The rest follow from duality. We need to prove only that composition of morphisms is a morphism, because associativity and existence of identity morphism is evident. Using the theorem 22 we have:

1. Let $f \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{A};\, \mathcal{B})$, $g \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{B};\, \mathcal{C})$. Then $\operatorname{dom} f \subseteq \mathcal{A}$, $\operatorname{im} f \supseteq \mathcal{B}$, $\operatorname{dom} g \subseteq \mathcal{B}$, $\operatorname{im} g \supseteq \mathcal{C}$. So $\operatorname{dom}(g \circ f) \subseteq \mathcal{A}$, $\operatorname{im}(g \circ f) \supseteq \mathcal{C}$ that is $g \circ f \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,\supseteq}}(\mathcal{A};\mathcal{C})$.
2. Let $f \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{A};\, \mathcal{B})$, $g \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{B};\, \mathcal{C})$. Then $\operatorname{dom} f \subseteq \mathcal{A}$, $\operatorname{im} f = \mathcal{B}$, $\operatorname{dom} g \subseteq \mathcal{B}$, $\operatorname{im} g = \mathcal{C}$. So $\operatorname{dom}(g \circ f) \subseteq \mathcal{A}$, $\operatorname{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \mathrm{Mor}_{\mathbf{MonRld}_{\subseteq,=}}(\mathcal{A};\mathcal{C})$.
3. Let $f \in \mathrm{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A};\, \mathcal{B})$, $g \in \mathrm{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{B};\, \mathcal{C})$. Then $\operatorname{dom} f = \mathcal{A}$, $\operatorname{im} f = \mathcal{B}$, $\operatorname{dom} g = \mathcal{B}$, $\operatorname{im} g = \mathcal{C}$. So $\operatorname{dom}(g \circ f) = \mathcal{A}, \operatorname{im}(g \circ f) = \mathcal{C}$ that is $g \circ f \in \mathrm{Mor}_{\mathbf{MonRld}_{=,=}}(\mathcal{A};\mathcal{C})$. □

Definition 59. Let **BijRld** is the groupoid of all bijections of the category of reloids. Its objects are filter objects and its morphisms from a f.o. $\mathcal{A}$ to f.o. $\mathcal{B}$ are monovalued injective trans-reloid f such that $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{im} f = \mathcal{B}$.

Theorem 60. Filter objects $\mathcal{A}$ and $\mathcal{B}$ are isomorphic iff $\mathrm{Mor}_{\mathbf{BijRld}}(\mathcal{A};\mathcal{B}) \neq \emptyset$.

Proof.

$\Rightarrow$**.** Let $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. Then there are sets $A \in \operatorname{up} \mathcal{A}$, $B \in \operatorname{up} \mathcal{B}$ and a bijective function $F\colon A \to B$ such that $\langle F \rangle\colon \mathscr{P}A \cap \operatorname{up}\mathcal{A} \to \mathscr{P}B \cap \operatorname{up}\mathcal{B}$ is a bijection.

Obviously $f = F|_{\mathcal{A}}^{\mathsf{RLD}(A;B)}$ is monovalued and injective.

$\operatorname{im} f = \bigcap^{\mathfrak{F}} \left\{ \operatorname{im} G \mid G \in F|_{\mathcal{A}}^{\mathsf{RLD}(A;B)} \right\} = \bigcap^{\mathfrak{F}} \left\{ \operatorname{im}(H \cap F|_X) \mid H \in F|_{\mathcal{A}}^{\mathsf{RLD}(A;B)}, X \in \operatorname{up}\mathcal{A} \right\} = \bigcap^{\mathfrak{F}} \{ \operatorname{im} F|_P \mid P \in \operatorname{up} \mathcal{A} \} = \bigcap^{\mathfrak{F}} \{ \langle F \rangle P \mid P \in \operatorname{up} \mathcal{A} \} = \bigcap^{\mathfrak{F}} \{ \langle F \rangle P \mid P \in \mathscr{P}A \cap \operatorname{up} \mathcal{A} \} = \bigcap^{\mathfrak{F}} (\mathscr{P}B \cap \operatorname{up}\mathcal{B}) = \bigcap^{\mathfrak{F}} \operatorname{up}\mathcal{B} = \mathcal{B}$.

Thus $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{im} f = \mathcal{B}$.

$\Leftarrow$**.** Let f is a monovalued injective trans-reloid such that $\operatorname{dom} f = \mathcal{A}$ and $\operatorname{im} f = \mathcal{B}$. Then exist a function F' and an injective binary relation F'' such that $F', F'' \in \operatorname{up} f$. Thus $F = F' \cap F''$ is an injection such that $F \in \operatorname{up} f$. The function F is a bijection from $A = \operatorname{dom} F$ to $B = \operatorname{im} F$. The function $\langle F \rangle$ is an injection on $\mathscr{P}A \cap \operatorname{up} \mathcal{A}$ (and moreover on $\mathscr{P}A$). Simple to show that $\forall X \in \mathscr{P}A \cap \operatorname{up}\mathcal{A}\colon \langle F \rangle X \in \mathscr{P}B \cap \operatorname{up}\mathcal{B}$ and similarly $\forall Y \in \mathscr{P}B \cap \operatorname{up}\mathcal{B}\colon \langle F \rangle^{-1} Y = \left\langle F^{-1} \right\rangle Y \in \mathscr{P}A \cap \operatorname{up}\mathcal{A}$. Thus $\langle F \rangle|_{\mathscr{P}A \cap \operatorname{up}\mathcal{A}}$ is a bijection $\mathscr{P}A \cap \operatorname{up}\mathcal{A} \to \mathscr{P}B \cap \operatorname{up}\mathcal{B}$. So filter objects $\mathcal{A}$ and $\mathcal{B}$ are isomorphic. □

Proposition 61. $(\geqslant_1) = (\subseteq) \circ (\geqslant_2)$ (when we limit to small f.o.).

Proof. $\mathcal{A} \geqslant_1 \mathcal{B}$ iff exists a function $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $B \subseteq \langle f \rangle \mathcal{A}$. But $\mathcal{B} \subseteq \langle f \rangle \mathcal{A}$ is equivalent to $\exists \mathcal{B}' \in \mathfrak{F}\colon (\mathcal{B} \subseteq \mathcal{B}' \wedge \mathcal{B}' = \langle f \rangle \mathcal{A})$. So $\mathcal{A} \geqslant_1 \mathcal{B}$ is equivalent to existence of $\mathcal{B}' \in \mathfrak{F}$ such that $\mathcal{B} \subseteq \mathcal{B}'$ and existence of a function $f\colon \bigcup \operatorname{up}\mathcal{A} \to \bigcup \operatorname{up}\mathcal{B}$ such that $\mathcal{B}' = \langle f \rangle \mathcal{A}$. That is equivalent to $\mathcal{A}((\subseteq) \circ (\geqslant_2))\mathcal{B}$. □

Proposition 62. If b is an atomic f.o. and $A \neq \emptyset$ then $b \geqslant_1 A \Leftrightarrow b \geqslant_2 A$. [TODO: Generalize it for f.o. on different base sets.]

Proof. We need to prove only $b \geqslant_1 A \Rightarrow b \geqslant_2 A$. If $b \geqslant_1 A$ then there exists a monovalued reloid f such that $\operatorname{dom} f = b$ and $\operatorname{im} f \supseteq A$. Then $\operatorname{im} f = \operatorname{im}(\mathsf{FCD}) f \in \{\emptyset\} \cup \operatorname{atoms}^{\mathfrak{F}}$ because $(\mathsf{FCD}) f$ is a monovalued funcoid. So $\operatorname{im} f = A$ (taken in account $A \neq \emptyset$) and thus $b \geqslant_2 A$. □

Corollary 63. For atomic filter objects $\geqslant_1$ is the same as $\geqslant_2$.

Thus I will write simply $\geqslant$ for atomic f.o.

3.1 Existence of no more than one monovalued injective reloid for a given pair of filter objects

3.1.1 The lemmas

The lemmas in this section were provided to me by Robert Martin Solovay in [5]. These are based on Wistar Comfort's work.

In this section we will assume μ is an ultrafilter on a set I and $f\colon I \to I$ has the property $X \in \mu \Leftrightarrow \left\langle f^{-1} \right\rangle X \in \mu$.

Lemma 64. If $X \in \mu$ then $X \cap \langle f \rangle X \in \mu$.

Proof. If $\langle f \rangle X \notin \mu$ then $X \subseteq \left\langle f^{-1} \right\rangle \langle f \rangle X \notin \mu$ and so $X \notin \mu$. Thus $X \in \mu \wedge \langle f \rangle X \in \mu$ and consequently $X \cap \langle f \rangle X \in \mu$. □

We will say that x is *periodic* when $f^n(x) = x$ for some positive integer x. The least such n is called *the period* of x.

Let's define $x \sim y$ iff there exist $i, j \in \mathbb{N}$ such that $f^i(x) = f^j(y)$. Trivially it is an equivalence relation. If x and y are periodic, then $x \sim y$ iff exists $n \in \mathbb{N}$ such that $f^n(y) = x$.

Let $A = \{x \in I \mid x \text{ is periodic with period} > 1\}$.

We will show that $A \notin \mu$. Let's assume $A \in \mu$.

Let a set $D \subseteq A$ contains (by the axiom of choice) exactly one element from each equivalence class of A defined by the relation $\sim$.

Let α is a function $A \to \mathbb{N}$ defined as follows. Let $x \in A$. Let y be the unique element of D such that $x \sim y$. Let $\alpha(x)$ be the least $n \in \mathbb{N}$ such that $f^n(y) = x$.

Let $B_0 = \{x \in A \mid \alpha(x) \text{ is even}\}$ and $B_1 = \{x \in A \mid \alpha(x) \text{ is odd}\}$.

Let $B_2 = \{x \in A \mid \alpha(x) = 0\}$.

[TODO: Two below lemmas have very similar proofs. Can we refactor it?]

Lemma 65. $B_0 \cap \langle f \rangle B_0 \subseteq B_2$.

Proof. If $x \in B_0 \cap \langle f \rangle B_0$ then $f^n(y) = x$ for a minimal even n and $x = f(x')$ where $f^m(y') = x'$ for a minimal even m. Thus $f^n(y) = f(x')$ thus y and x' laying in the same equivalence class and thus $y = y'$. So we have $f^n(y) = f^{m+1}(y)$. Thus $n \leqslant m + 1$ by minimality.

x' lies on an orbit and thus $x' = f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit; $f^m(y) = f^{-1}(x)$ and thus $x' = f^{n-1}(y)$ thus $n - 1 \geqslant m$ by minimality or $n = 0$.

Thus $n = m + 1$ what is impossible for even n and m. We have a contradiction what proves $B_0 \cap \langle f \rangle B_0 \subseteq \emptyset$.

Remained the case $n = 0$, then $x = f^0(y)$ and thud $\alpha(x) = 0$. □

Lemma 66. $B_1 \cap \langle f \rangle B_1 = \emptyset$.

Proof. Let $x \in B_1 \cap \langle f \rangle B_1$. Then $f^n(y) = x$ for an odd n and $x = f(x')$ where $f^m(y') = x'$ for an odd m. Thus $f^n(y) = f(x')$ thus y and x' laying in the same equivalence class and thus $y = y'$. So we have $f^n(y) = f^{m+1}(y)$. Thus $n \leqslant m + 1$ by minimality.

x' lies on an orbit and thus $x' = f^{-1}(x)$ where by f^{-1} I mean step backward on our orbit;

$f^m(y) = f^{-1}(x)$ and thus $x' = f^{n-1}(y)$ thus $n - 1 \geqslant m$ by minimality ($n = 0$ is impossible because n is odd.

Thus $n = m + 1$ what is impossible for odd n and m. We have a contradiction what proves $B_0 \cap \langle f \rangle B_0 = \emptyset$. □

Lemma 67. $B_2 \cap \langle f \rangle B_2 = \emptyset$.

Proof. Let $x \in B_2 \cap \langle f \rangle B_2$. Then $x = y$ and $x' = y$ where $x = f(x')$. Thus $x = f(x)$ and so $x \notin A$ what is impossible. □

Lemma 68. $A \notin \mu$.

Proof. Suppose $A \in \mu$.

Since $A \in \mu$ we have $B_0 \in \mu$ or $B_1 \in \mu$.

So either $B_0 \cap \langle f \rangle B_0 \subseteq B_2$ or $B_1 \cap \langle f \rangle B_1 \subseteq B_2$. As such by the lemma 64 we have $B_2 \in \mu$. [TODO: Clarify.] This is incompatible with $B_2 \cap \langle f \rangle B_2 = \emptyset$. So we got a contradiction. □

Let C be the set of points x which are not periodic but $f^n(x)$ is periodic for some positive n.

Lemma 69. $C \notin \mu$.

Proof. Let β be a function $C \to \mathbb{N}$ such that $\beta(x)$ is the least $n \in \mathbb{N}$ such that $f^n(x)$ is periodic.

Let $C_0 = \{x \in C \mid \beta(x) \text{ is even}\}$ and $C_1 = \{x \in C \mid \beta(x) \text{ is odd}\}$.

Obviously $C_j \cap \langle f \rangle C_j = \emptyset$ for $j = 0, 1$. Hence by the lemma 64 we have $C_0, C_1 \notin \mu$ and thus $C = C_0 \cup C_1 \notin \mu$. □

Let E be the set of $x \in I$ such that for no $n \in \mathbb{N}$ we have $f^n(x)$ periodic.

Lemma 70. Let $x, y \in E$ are such that $f^i(x) = f^j(y)$ and $f^{i'}(x) = f^{j'}(y)$ for some $i, j, i', j' \in \mathbb{N}$. Then $i - j = i' - j'$.

Proof. $i \mapsto f^i(x)$ is a bijection.

So $y = f^{i-j}(y)$ and $y = f^{i'-j'}(y)$. Thus $f^{i-j}(y) = f^{i'-j'}(y)$ and so $i - j = i' - j'$. □

Lemma 71. $E \notin \mu$.

Proof. Let $D' \subseteq E$ be a subset of E with exactly one element from each equivalence class of the relation $\sim$ on E.

Define the function $\gamma: E \to \mathbb{Z}$ as follows. Let $x \in E$. Let y be the unique element of D' such that $x \sim y$. Choose $i, j \in \mathbb{N}$ such that $f^i(y) = f^j(x)$. Let $\gamma(x) = i - j$. By the last lemma, γ is well-defined.

It is clear that if $x \in E$ then $f(x) \in E$ and moreover $\gamma(f(x)) = \gamma(x) + 1$.

Let $E_0 = \{x \in E \mid \gamma(x) \text{ is even}\}$ and $E_1 = \{x \in E \mid \gamma(x) \text{ is odd}\}$.

We have $E_0 \cap \langle f \rangle E_0 = \emptyset \notin \mu$ and hence $E_0 \notin \mu$.

Similarly $E_1 \notin \mu$.

Thus $E = E_0 \cup E_1 \notin \mu$. □

Lemma 72. f is the identity function on a set in μ.

Proof. We have shown $A, C, E \notin \mu$. But the points which lie in none of these sets are exactly points periodic with period 1 that is fixed points of f. Thus the set of fixed points of f belongs to the filter μ. □

3.1.2 The main theorem and its consequences

Theorem 73. For every atomic filter object a the morphism $((=)|_a; a; a)$ is the only

1. monovalued morphism of the category of reloids from a to a;
2. injective morphism of the category of reloids from a to a;
3. bijective morphism of the category of reloids from a to a.

Proof. We will prove only (1) because the rest follow from it.

Let f is a monovalued morphism from a to a. Then it exists a function F such that $F \in \operatorname{up} f$. Trivially $\langle F \rangle a \supseteq a$ and thus $\langle F \rangle A \in \operatorname{up} a$ for every $A \in \operatorname{up} a$. Thus by the lemma we have that F is the identity function on a set in $\operatorname{up} a$ and so obviously f is an identity. □

Corollary 74. For every two atomic filter objects (with possibly different bases) $\mathcal{A}$ and $\mathcal{B}$ there exists at most one bijective trans-reloid from $\mathcal{A}$ to $\mathcal{B}$.

Proof. Suppose that f and g are two different bijective trans-reloids from $\mathcal{A}$ to $\mathcal{B}$. Then $g^{-1} \circ f$ is not the identity reloid (otherwise $g^{-1} \circ f = I^{\mathsf{RLD}}_{\operatorname{dom} f}$ and so $f = g$). But $g^{-1} \circ f$ is a bijective trans-reloid (as a composition of bijective reloids) from $\mathcal{A}$ to $\mathcal{A}$ what is impossible. □

4 Rudin-Keisler equivalence and Rudin-Keisler order

Theorem 75. Atomic filter objects a and b (with possibly different bases) are isomorphic iff $a \geqslant b \wedge b \geqslant a$.

Proof. Let $a \geqslant b \wedge b \geqslant a$. Then there are a monovalued trans-reloids f and g such that $\operatorname{dom} f = a$ and $\operatorname{im} f = b$ and $\operatorname{dom} g = b$ and $\operatorname{im} g = a$. Thus $g \circ f$ is a monovalued morphism from a to b. By the above we have $g \circ f = I_a^{\mathsf{RLD}}$ so $g = f^{-1}$ and $f^{-1} \circ f = I_a^{\mathsf{RLD}}$ so g is monovalued. Thus f is an injective monovalued trans-reloid from a to b and thus a and b are isomorphic. □

The last theorem cannot be generalized from atomic f.o. to arbitrary f.o., as it's shown by the following two examples:

Example 76. $\mathcal{A} \geqslant_1 \mathcal{B} \wedge \mathcal{B} \geqslant_1 \mathcal{A}$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ for some f.o. $\mathcal{A}$ and $\mathcal{B}$.

Proof. Consider $\mathcal{A} = [0;1]$ and $\mathcal{B} = \bigcap^{\mathfrak{F}}\{[0;1+\varepsilon) \mid \varepsilon > 0\}$. Then the function $f = \{(x; x/2) \mid x \in \mathbb{R}\}$ witnesses both inequalities $\mathcal{A} \geqslant_1 \mathcal{B}$ and $\mathcal{B} \geqslant_1 \mathcal{A}$. But these filters cannot be isomorphic because only one of them is principal. □

Lemma 77. Let a function is defined as $f(x;\ y) = (f_0 x;\ f_1 y)$. Then $\langle f\rangle\big(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}\big) = (f_0\mathcal{A}) \times^{\mathsf{RLD}} (f_1\mathcal{B})$.

Proof. $\langle f\rangle\big(\mathcal{A} \times^{\mathsf{RLD}} \mathcal{B}\big) = \langle f\rangle \bigcap^{\mathsf{RLD}}\{A \times B \mid A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B}\} = \bigcap^{\mathsf{RLD}}\{\langle f\rangle(A \times B) \mid A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B}\} = \bigcap^{\mathsf{RLD}}\{\langle f_0\rangle A \times \langle f_1\rangle B \mid A \in \operatorname{up}\mathcal{A}, B \in \operatorname{up}\mathcal{B}\} =$ (theorem 148 in [4]) $= \bigcap^{\mathfrak{F}}\{\langle f_0\rangle A \mid A \in \operatorname{up}\mathcal{A}\} \times^{\mathsf{RLD}} \bigcap^{\mathfrak{F}}\{\langle f_1\rangle B \mid A \in \operatorname{up}\mathcal{B}\} = (f_0\mathcal{A}) \times^{\mathsf{RLD}} (f_1\mathcal{B})$. □

Lemma 78. If an f.o. $\mathcal{A}$ is isomorphic to an f.o. $\mathcal{B}$ then if X is a set and $X \cap^{\mathfrak{F}} \mathcal{A}$ is an atomic f.o., then there exists a set Y such that $X \cap^{\mathfrak{F}} \mathcal{A}$ is an atomic f.o. isomorphic to $Y \cap^{\mathfrak{F}} \mathcal{B}$.

Proof. Let $\mathcal{A}$ is isomorphic to $\mathcal{B}$. Then there are sets A, B such that $\mathcal{A} \div A$ is directly isomorphic to $\mathcal{B} \div B$. So there are a bijection $f\colon \mathscr{P}A \cap \operatorname{up}\mathcal{A} \to \mathscr{P}B \cap \operatorname{up}\mathcal{B}$ such that $\mathcal{B} = \langle f\rangle\mathcal{A}$.

$\operatorname{up}\big(X \cap^{\mathfrak{F}} \mathcal{A}\big) = \operatorname{up}\big((X \cap A) \cap^{\mathfrak{F}} \mathcal{A}\big) = \langle X \cap A \cap\rangle \operatorname{up}\mathcal{A} = \langle X \cap\rangle(\mathscr{P}A \cap \operatorname{up}\mathcal{A})$

Thus $\langle\langle f\rangle\rangle \operatorname{up}\big(X \cap^{\mathfrak{F}} \mathcal{A}\big) = \langle f(X) \cap\rangle\langle\langle f\rangle\rangle(\mathscr{P}A \cap \operatorname{up}\mathcal{A}) = \langle f(X) \cap\rangle(\mathscr{P}B \cap \operatorname{up}\mathcal{B}) = \langle f(X) \cap B \cap\rangle \operatorname{up}\mathcal{B} = \langle f(X) \cap\rangle \operatorname{up}\mathcal{B} = \operatorname{up}\big(f(X) \cap^{\mathfrak{F}} \mathcal{B}\big)$.

So $\langle f\rangle\big(X \cap^{\mathfrak{F}} \mathcal{A}\big) = \bigcap^{\mathfrak{F}}\langle\langle f\rangle\rangle \operatorname{up}\big(X \cap^{\mathfrak{F}} \mathcal{A}\big) = \bigcap^{\mathfrak{F}} \operatorname{up}\big(f(X) \cap^{\mathfrak{F}} \mathcal{B}\big) = f(X) \cap^{\mathfrak{F}} \mathcal{B}$.

Finally we have $f(X) \cap^{\mathfrak{F}} \mathcal{B}$ is isomorphic to $X \cap^{\mathfrak{F}} \mathcal{A}$ from the last equality. □

Example 79. $\mathcal{A} \geqslant_2 \mathcal{B} \wedge \mathcal{B} \geqslant_2 \mathcal{A}$ but $\mathcal{A}$ is not isomorphic to $\mathcal{B}$ for some f.o. $\mathcal{A}$ and $\mathcal{B}$.

Proof. (proof idea by Andreas Blass, rewritten by me)

Let u_n, h_n with n ranging over the set $\mathbb{Z}$ are sequences of atomic f.o. and functions such that $\langle h_n\rangle u_{n+1} = u_n$ and u_n are pairwise non-isomorphic. (See [1] for a proof that such ultrafilters and functions exist.)

$\mathcal{A} \overset{\text{def}}{=} \bigcup^{\mathsf{RLD}}\big\{\{n\} \times^{\mathsf{RLD}} u_{2n+1} \mid n \in \mathbb{Z}\big\}$; $\mathcal{B} \overset{\text{def}}{=} \bigcup^{\mathsf{RLD}}\big\{\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\big\}$.

Let the functions f, $g\colon \mathbb{Z} \times \mathbb{N} \to \mathbb{Z} \times \mathbb{N}$ are defined by the formulas $f(n;x) = (n; h_{2n}x)$ and $g(n;x) = (n-1; h_{2n-1}x)$.

Using the fact that every function is a complete funcoid and a lemma above we get:

$\langle f\rangle\mathcal{A} = \bigcup^{\mathsf{RLD}}\langle f\rangle\big\{\{n\} \times^{\mathsf{RLD}} u_{2n+1} \mid n \in \mathbb{Z}\big\} = \bigcup^{\mathsf{RLD}}\big\{\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\big\} = \mathcal{B}$.

$\langle g\rangle\mathcal{B} = \bigcup^{\mathsf{RLD}}\langle g\rangle\big\{\{n\} \times^{\mathsf{RLD}} u_{2n} \mid n \in \mathbb{Z}\big\} = \bigcup^{\mathsf{RLD}}\big\{\{n-1\} \times^{\mathsf{RLD}} u_{2n-1} \mid n \in \mathbb{Z}\big\} = \bigcup^{\mathsf{RLD}}\big\{\{n\} \times^{\mathsf{RLD}} u_{2n+1} \mid n \in \mathbb{Z}\big\} = \mathcal{A}$.

It remains to show that $\mathcal{A}$ and $\mathcal{B}$ are not isomorphic.

Let $X \in \operatorname{up}\big(\{n\} \times^{\mathsf{RLD}} u_{2n+1}\big)$ for some $n \in \mathbb{Z}$. Then if $X \cap^{\mathfrak{F}} \mathcal{A}$ is an atomic f.o. we have $X \cap^{\mathfrak{F}} \mathcal{A} = \{n\} \times^{\mathsf{RLD}} u_{2n+1}$ and thus is isomorphic to u_{2n+1}.

If $X \notin \operatorname{up}\big(\{n\} \times^{\mathsf{RLD}} u_{2n+1}\big)$ for every $n \in \mathbb{Z}$ then $(\mathbb{Z} \times \mathbb{N}) \setminus X \in \operatorname{up}\big(\{n\} \times^{\mathsf{RLD}} u_{2n+1}\big)$ and thus $(\mathbb{Z} \times \mathbb{N}) \setminus X \in \operatorname{up}\mathcal{A}$ and thus $X \cap^{\mathfrak{F}} \mathcal{A} = \emptyset$.

We have also $(\{0\}\times\mathbb{N})\cap^{\mathsf{RLD}}\mathcal{B}=(\{0\}\times\mathbb{N})\cap^{\mathsf{RLD}}\bigcup^{\mathsf{RLD}}\{\{n\}\times^{\mathsf{RLD}}u_{2n}\mid n\in\mathbb{Z}\}=\bigcup^{\mathsf{RLD}}\{(\{0\}\times\mathbb{N})\cap^{\mathsf{RLD}}(\{n\}\times^{\mathsf{RLD}}u_{2n})\mid n\in\mathbb{Z}\}=\{0\}\times^{\mathsf{RLD}}u_0$ (an atomic f.o.).

Thus every atomic f.o. generated as intersecting $\mathcal{A}$ with a set X is isomorphic to some u_{2n+1} and thus is not isomorphic to u_0. By the lemma it follows that $\mathcal{A}$ and $\mathcal{B}$ are non-isomorphic. □

Theorem 80. Let f is a monovalued injective trans-reloid. Then f is isomorphic to the f.o. $\operatorname{dom} f$. [TODO: Move this theorem and its corollary above.]

Proof. Let f is a monovalued injective trans-reloid. There exists a bijection $F\in\operatorname{up} f$. Consider the bijective function $p=\{(x;Fx)\mid x\in\operatorname{dom} F\}$.

$\langle p\rangle\operatorname{dom} F=F$ and consequently $\langle p\rangle\operatorname{dom} f=\bigcap^{\mathsf{RLD}(\operatorname{Dst} f;\operatorname{Src} f)}\{\langle p\rangle\operatorname{dom} K\mid K\in\operatorname{up} f\}=\bigcap^{\mathsf{RLD}(\operatorname{Dst} f;\operatorname{Src} f)}\{\langle p\rangle\operatorname{dom}(K\cap F)\mid K\in\operatorname{up} f\}=\bigcap^{\mathsf{RLD}(\operatorname{Dst} f;\operatorname{Src} f)}\{K\cap F\mid K\in\operatorname{up} f\}=\bigcap^{\mathsf{RLD}(\operatorname{Dst} f;\operatorname{Src} f)}\{K\mid K\in\operatorname{up} f\}=f$. Thus p witnesses that f is isomorphic to the f.o. $\operatorname{dom} f$. □

Corollary 81. $I_{\mathcal{A}}^{\mathsf{RLD}}$ is isomorphic to $\mathcal{A}$ for every f.o. $\mathcal{A}$.

Theorem 82. There are atomic f.o. incomparable by Rudin-Keisler order.

Proof. See [2]. □

Theorem 83. $\geqslant_1$ and $\geqslant_2$ are different relations.

Proof. Consider a is an arbitrary non-empty f.o. Then $a\geqslant_1\emptyset$ but not $a\geqslant_2\emptyset$. □

Proposition 84. If $a\geqslant_2 b$ where a is an atomic f.o. then b is also an atomic f.o.

Proof. $b=\langle f\rangle a$ for some $f\colon\bigcup\operatorname{up} a\to\bigcup\operatorname{up} b$. So b is an atomic f.o. since f is monovalued. □

Corollary 85. If $a\geqslant_1 b$ where a is an atomic f.o. then b is also an atomic f.o. or $\emptyset$.

Proof. $b\subseteq\langle f\rangle a$ for some $f\colon\bigcup\operatorname{up} a\to\bigcup\operatorname{up} b$. Therefore $b'=\langle f\rangle a$ is an atomic f.o. From this follows our statement. □

Proposition 86. Principal filters, generated by sets of the same cardinality, are isomorphic.

Proof. Let A and B are sets of the same cardinality. Then there are a bijection f from A to B. We have $\langle f\rangle A=B$ and thus A and B are isomorphic. □

Proposition 87. If a filter object is isomorphic to a set, then it is also a set with the same cardinality.

Proof. Let A is a set and B is a f.o. isomorphic to A. Then there are sets $X\in\operatorname{up} A$ and $Y\in\operatorname{up} B$ such that there are a bijection $f\colon X\to Y$ such that $\langle f\rangle A=B$. Thus A is a set of the same cardinality as A. □

Proposition 88. A filter isomorphic to a non-trivial atomic f.o. is a non-trivial atomic f.o.

Proof. Let a is a non-trivial atomic f.o. and a is isomorphic to b. Then $a\geqslant_2 b$ and thus b is an atomic f.o. The f.o. b cannot be trivial because otherwise a would be also trivial. □

Theorem 89. For an infinite set U there exist $2^{2^{\operatorname{card} U}}$ equivalence classes of isomorphic ultrafilters.

Proof. The number of bijections between any two given subsets of U is no more than $(\operatorname{card} U)^{\operatorname{card} U}=2^{\operatorname{card} U}$. The number of bijections between all pairs of subsets of U is no more than $2^{\operatorname{card} U}\cdot 2^{\operatorname{card} U}=2^{\operatorname{card} U}$. Therefore each isomorphism class contains at most $2^{\operatorname{card} U}$ ultrafilters. But there are $2^{2^{\operatorname{card} U}}$ ultrafilters. So there are $2^{2^{\operatorname{card} U}}$ classes. □

Remark 90. One of the above mentioned equivalence classes contains trivial ultrafilters.

Corollary 91. There exist non-isomorphic nontrivial ultrafilters on any infinite set.

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