Pseudodifference on atomistic co-brouwerian lattices

by Victor Porton

Email: porton@narod.ru
Web: http://www.mathematics21.org

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Abstract

I prove a partial result on a conjecture about presenting pseudodifference of filters in several equivalent forms from my earlier article, and more generally the same result for arbitrary atomistic co-brouwerian lattices.

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1 The problem

I will call the set of filter objects the set of filters ordered reverse to set theoretic inclusion of filters, with principal filters equated to the corresponding sets. See [1] for the formal definition of filter objects. I will denote \((\text{up } a)\) the filter corresponding to a filter object \(a\). I will denote the set of filter objects (on \(U\)) as \(\mathfrak{F}\). So, \(a \subseteq b \iff \text{up } a \supseteq \text{up } b\) for \(a, b \in \mathfrak{F}\).

\(\mathfrak{F}\) is actually a complete lattice (see [1]).

I will denote \((\text{atoms } A)\) the set of atoms below element \(a\) of a lattice \(A\).

In [1] I’ve formulated the following open problem (problem 1):

Problem 1. Which of the following expressions are pairwise equal for all \(a, b \in \mathfrak{F}\) for each lattice \(\mathfrak{F}\) of filters on a set \(U\)? (If some are not equal, provide counter-examples.)

1. \(\bigcap \mathfrak{F} \{z \in \mathfrak{F} \mid \ a \subseteq b \cup z\}\) (quasidifference of \(a\) and \(b\));
2. \(\bigcup \mathfrak{F} \{z \in \mathfrak{F} \mid z \subseteq a \land z \cap b = \emptyset\}\) (second quasidifference of \(a\) and \(b\));
3. \(\bigcup \mathfrak{F} \{\text{atoms } \mathfrak{F} a \setminus \text{atoms } \mathfrak{F} b\}\);
4. \(\bigcup \mathfrak{F} \{a \cap \mathfrak{F} (U \setminus B) \mid B \in \text{up } b\}\).

2 Solved part of the problem

I have proved equality of the first three expressions. Their equality with the fourth expression remains open.
3 A generalization and the proof

We will prove a more general statement:

**Theorem 2.** For an atomistic co-brouwerian lattice $\mathfrak{A}$ and $a, b \in \mathfrak{A}$ the following expressions are always equal:

1. $\bigcap^\mathfrak{A} \{ z \in \mathfrak{A} \mid a \subseteq b \cup^\mathfrak{A} z \}$ (quasidifference of $a$ and $b$);
2. $\bigcup^\mathfrak{A} \{ z \in \mathfrak{A} \mid z \subseteq a \land z \cap^\mathfrak{A} b = 0 \}$ (second quasidifference of $a$ and $b$);
3. $\bigcup^\mathfrak{A} (\text{atoms}^\mathfrak{A} a \setminus \text{atoms}^\mathfrak{A} b)$.

**Proof.** 

**Proof of (1)=(3):**

$\bigcap^\mathfrak{A} \{ z \in \mathfrak{A} \mid a \subseteq b \cup^\mathfrak{A} z \}$, so it’s enough to prove that

$$\left( \bigcup^\mathfrak{A} \text{atoms}^\mathfrak{A} a \right)^\ast = \bigcup^\mathfrak{A} (\text{atoms}^\mathfrak{A} a \setminus \text{atoms}^\mathfrak{A} b)$$

Really:

$$a \setminus^\ast b = \left( \bigcup^\mathfrak{A} \text{atoms}^\mathfrak{A} a \right)^\ast = (\text{theorem 16})*$$

$$\bigcup^\mathfrak{A} \{ A \setminus^\ast b \mid A \in \text{atoms}^\mathfrak{A} a \} = \bigcup^\mathfrak{A} \{ \left( \begin{cases} A & \text{if } A \not\in \text{atoms}^\mathfrak{A} b \\ 0 & \text{if } A \in \text{atoms}^\mathfrak{A} b \end{cases} \right) \mid A \in \text{atoms}^\mathfrak{A} a \} =$$

$$\bigcup^\mathfrak{A} \{ A \mid A \in \text{atoms}^\mathfrak{A} a, A \not\in \text{atoms}^\mathfrak{A} b \} =$$

$$\bigcup^\mathfrak{A} (\text{atoms}^\mathfrak{A} a \setminus \text{atoms}^\mathfrak{A} b).$$

* The requirement of theorem 16 that our lattice is complete is superfluous and can be removed.

**Proof of (2)=(3):**

$a \setminus^\ast b$ is defined because our lattice is co-brouwerian. Taking the above into account, we have

$$a \setminus^\ast b = \bigcup (\text{atoms} a \setminus \text{atoms} b) = \bigcup \{ z \in \text{atoms} a \mid z \cap^\mathfrak{A} b = 0^\mathfrak{A} \}.$$

So $\bigcup \{ z \in \text{atoms} a \mid z \cap^\mathfrak{A} b = 0^\mathfrak{A} \}$ is defined.

If $z \subseteq a \land z \cap^\mathfrak{A} b = 0^\mathfrak{A}$ then $z’ = \bigcup \{ x \in \text{atoms} z \mid x \cap^\mathfrak{A} b = 0^\mathfrak{A} \}$ is defined. $z’$ is a lower bound for $\{ z \in \text{atoms} a \mid z \cap^\mathfrak{A} b = 0^\mathfrak{A} \}$.

Thus $z’ \subseteq \{ z \in \mathfrak{A} \mid z \subseteq a \land z \cap^\mathfrak{A} b = 0^\mathfrak{A} \}$ and so $\bigcup \{ z \in \text{atoms} a \mid z \cap^\mathfrak{A} b = 0^\mathfrak{A} \}$ is an upper bound of $\{ z \in \mathfrak{A} \mid z \subseteq a \land z \cap^\mathfrak{A} b = 0^\mathfrak{A} \}$.

If $y$ is above every $z’ \in \{ z \in \mathfrak{A} \mid z \subseteq a \land z \cap^\mathfrak{A} b = 0^\mathfrak{A} \}$ then $y$ is above every $z \in \text{atoms} a$ such that $z \cap^\mathfrak{A} b = 0^\mathfrak{A}$ and thus $y$ is above $\bigcup \{ z \in \text{atoms} a \mid z \cap^\mathfrak{A} b = 0^\mathfrak{A} \}$.
Thus \( \bigcup \{ z \in \text{atoms } a \mid z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \} \) is least upper bound of
\[
\{ z \in \mathfrak{A} \mid z \subseteq a \land z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \},
\]
that is
\[
\bigcup \{ z \in \mathfrak{A} \mid z \subseteq a \land z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \} = \bigcup \{ z \in \text{atoms } a \mid z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \} = \bigcup (\text{atoms } a \setminus \text{atoms } b).
\]

Note that \( \mathfrak{F} \) is co-brouwerian by corollary 11 in [1] and atomistic by theorem 48 in [1], so our theorem applies to the lattice \( \mathfrak{F} \).

Bibliography