Pseudodifference on atomistic co-brouwerian lattices

Victor Porton *†

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Abstract

I prove a conjecture about presenting pseudodifference of filters in several equivalent forms from my earlier article, and more generally a result for arbitrary atomistic co-brouwerian lattices.

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1 The problem

I will call *the set of filter objects* the set of filters ordered reverse to set theoretic inclusion of filters, with principal filters equated to the corresponding sets. See [1] for the formal definition of filter objects. I will denote $(\operatorname{up} a)$ the filter corresponding to a filter object a. I will denote the set of filter objects (on U) as $\mathfrak{F}$. So, $a \subseteq b \Leftrightarrow \operatorname{up} a \supseteq \operatorname{up} b$ for $a, b \in \mathfrak{F}$.

$\mathfrak{F}$ is actually a complete lattice (see [1]).

I will denote $(\operatorname{atoms}^{\mathfrak{A}} a)$ the set of atoms below element a of a lattice $\mathfrak{A}$.

In [1] I've formulated the following open problem (problem 1):

Problem 1 Which of the following expressions are pairwise equal for all $a, b \in \mathfrak{F}$ for each lattice $\mathfrak{F}$ of filters on a set U? (If some are not equal, provide counter-examples.)

1. $\bigcap^{\mathfrak{F}} \left\{ z \in \mathfrak{F} \mid a \subseteq b \cup^{\mathfrak{F}} z \right\}$ (quasidifference of a and b);
2. $\bigcup^{\mathfrak{F}} \left\{ z \in \mathfrak{F} \mid z \subseteq a \wedge z \cap^{\mathfrak{F}} b = \emptyset \right\}$ (second quasidifference of a and b);
3. $\bigcup^{\mathfrak{F}} (\operatorname{atoms}^{\mathfrak{F}} a \setminus \operatorname{atoms}^{\mathfrak{F}} b)$;
4. $\bigcup^{\mathfrak{F}} \left\{ a \cap^{\mathfrak{F}} (U \setminus B) \mid B \in \operatorname{up} b \right\}$.

* *Email:* porton@narod.ru
† *Web:* http://www.mathematics21.org

2 A generalization and the proof

We will prove more general statements:

Theorem 1 *For an atomistic co-brouwerian lattice $\mathfrak{A}$ and $a, b \in \mathfrak{A}$ the following expressions are always equal:*

1. $\bigcap^{\mathfrak{A}} \left\{ z \in \mathfrak{A} \mid a \subseteq b \cup^{\mathfrak{A}} z \right\}$ *(quasidifference of a and b);*

2. $\bigcup^{\mathfrak{A}} \left\{ z \in \mathfrak{A} \mid z \subseteq a \wedge z \cap^{\mathfrak{A}} b = 0 \right\}$ *(second quasidifference of a and b);*

3. $\bigcup^{\mathfrak{A}} (\operatorname{atoms}^{\mathfrak{A}} a \setminus \operatorname{atoms}^{\mathfrak{A}} b)$.

Proof *Proof of (1)=(3):*
$a \setminus^* b = \bigcap^{\mathfrak{A}} \left\{ z \in \mathfrak{A} \mid a \subseteq b \cup^{\mathfrak{A}} z \right\}$, so it's enough to prove that

$$a \setminus^* b = \bigcup^{\mathfrak{A}} (\operatorname{atoms}^{\mathfrak{A}} a \setminus \operatorname{atoms}^{\mathfrak{A}} b).$$

Really:

$$\begin{aligned}
a \setminus^* b &= \\
\left(\bigcup^{\mathfrak{A}} \operatorname{atoms} a \right) \setminus^* b &= \text{(theorem 16 in [1])*} \\
\bigcup^{\mathfrak{A}} \left\{ A \setminus^* b \mid A \in \operatorname{atoms}^{\mathfrak{A}} a \right\} &= \\
\bigcup^{\mathfrak{A}} \left\{ \left(\begin{cases} A & \text{if } A \notin \operatorname{atoms}^{\mathfrak{A}} b \\ 0 & \text{if } A \in \operatorname{atoms}^{\mathfrak{A}} b \end{cases} \right) \mid A \in \operatorname{atoms}^{\mathfrak{A}} a \right\} &= \\
\bigcup^{\mathfrak{A}} \left\{ A \mid A \in \operatorname{atoms}^{\mathfrak{A}} a, A \notin \operatorname{atoms}^{\mathfrak{A}} b \right\} &= \\
\bigcup^{\mathfrak{A}} (\operatorname{atoms}^{\mathfrak{A}} a \setminus \operatorname{atoms}^{\mathfrak{A}} b).
\end{aligned}$$

* The requirement of theorem 16 that our lattice is complete is superfluous and can be removed.

Proof of (2)=(3):

$a \setminus^* b$ is defined because our lattice is co-brouwerian. Taking the above into account, we have

$$\begin{aligned}
a \setminus^* b &= \\
\bigcup (\operatorname{atoms} a \setminus \operatorname{atoms} b) &= \\
\bigcup \left\{ z \in \operatorname{atoms} a \mid z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}.
\end{aligned}$$

So $\bigcup \left\{ z \in \operatorname{atoms} a \mid z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}$ is defined.

If $z \subseteq a \wedge z \cap^{\mathfrak{F}} b = 0^{\mathfrak{A}}$ then $z' = \bigcup \left\{ x \in \operatorname{atoms} z \mid x \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}$ is defined. z' is a lower bound for $\left\{ z \in \operatorname{atoms} a \mid z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}$.

Thus $z' \in \left\{ z \in \mathfrak{A} \mid z \subseteq a \wedge z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}$ and so $\bigcup \left\{ z \in \operatorname{atoms} a \mid z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}$ is an upper bound of $\left\{ z \in \mathfrak{A} \mid z \subseteq a \wedge z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}$.

If y is above every $z' \in \left\{ z \in \mathfrak{A} \mid z \subseteq a \wedge z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}$ then y is above every $z \in \operatorname{atoms} a$ such that $z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}}$ and thus y is above $\bigcup \left\{ z \in \operatorname{atoms} a \mid z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}$.

Thus $\bigcup \left\{ z \in \operatorname{atoms} a \mid z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\}$ is least upper bound of

$$\left\{ z \in \mathfrak{A} \mid z \subseteq a \wedge z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\},$$

that is

$$\bigcup \left\{ z \in \mathfrak{A} \mid z \subseteq a \wedge z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\} = \bigcup \left\{ z \in \operatorname{atoms} a \mid z \cap^{\mathfrak{A}} b = 0^{\mathfrak{A}} \right\} = \bigcup (\operatorname{atoms} a \setminus \operatorname{atoms} b).$$

□

Note that $\mathfrak{F}$ is co-brouwerian by corollary 11 in [1] and atomistic by theorem 48 in [1], so our theorem applies to the lattice $\mathfrak{F}$, and more generally to any filters on a boolean lattice.

Proposition 1 *For filters on boolean lattices the three above ways to express quasidifference of a and b are also equal to $\bigcup^{\mathfrak{F}} \{ a \cap^{\mathfrak{F}} \uparrow \overline{B} \mid B \in \operatorname{up} b \}$ ($\uparrow X$ denotes the principal filter induced by X).*

Remark 1 *By corollary 8 in [1] the set of filters on a boolean lattice is complete. So our formula is well-defined.*

Proof Using results from [1]:

$\bigcup^{\mathfrak{F}} \{ z \in \mathfrak{F} \mid z \subseteq a \wedge z \cap b = 0^{\mathfrak{F}} \} \subseteq \bigcup^{\mathfrak{F}} \{ a \cap^{\mathfrak{F}} \uparrow \overline{B} \mid B \in \operatorname{up} b \}$ because

$$\begin{gathered} z \in \{ z \in \mathfrak{F} \mid z \subseteq a \wedge z \cap b = 0^{\mathfrak{F}} \} \Leftrightarrow z \subseteq a \wedge z \cap b = 0^{\mathfrak{F}} \Leftrightarrow \\ z \subseteq a \wedge \exists B \in \operatorname{up} b : z \cap \uparrow B = 0^{\mathfrak{F}} \Leftrightarrow z \subseteq a \wedge \exists B \in \operatorname{up} b : z \subseteq \uparrow \overline{B} \Leftrightarrow \\ \exists B \in \operatorname{up} b : (z \subseteq a \wedge z \subseteq \uparrow \overline{B}) \Leftrightarrow \exists B \in \operatorname{up} b : z \subseteq a \cap^{\mathfrak{F}} \uparrow \overline{B} \Rightarrow \\ z \subseteq \bigcup^{\mathfrak{F}} \{ a \cap^{\mathfrak{F}} \uparrow \overline{B} \mid B \in \operatorname{up} b \}. \end{gathered}$$

But obviously $a \cap^{\mathfrak{F}} \uparrow \overline{B} \in \{ z \in \mathfrak{F} \mid z \subseteq a \wedge z \cap b = 0^{\mathfrak{F}} \}$ and thus

$$a \cap^{\mathfrak{F}} \uparrow \overline{B} \subseteq \bigcup^{\mathfrak{F}} \{ z \in \mathfrak{F} \mid z \subseteq a \wedge z \cap b = 0^{\mathfrak{F}} \}$$

and so $\bigcup^{\mathfrak{F}} \{ z \in \mathfrak{F} \mid z \subseteq a \wedge z \cap b = 0^{\mathfrak{F}} \} \supseteq \bigcup^{\mathfrak{F}} \{ a \cap^{\mathfrak{F}} \uparrow \overline{B} \mid B \in \operatorname{up} b \}$. □

The above proposition completes the proof of problem 1 in [1].

There is a little more general theorem in my unpublished book "Algebraic General Topology. Volume 1" (available on the Web), currently at the end of the section "Filtrators over Boolean Lattices".

I present a part of this theorem here without a proof, as its (fairly technical, not long however) proof is available in this my free e-book:

The below theorem uses terminology from [1].

Theorem 2 *If* $(\mathfrak{A}; \mathfrak{Z})$ *is a complete co-brouwerian atomistic down-aligned lattice filtrator with binarily meet-closed and separable boolean core, then the three expressions of pseudodifference of a and b in the above theorem are also equal to* $\bigcup^{\mathfrak{F}}\{a \cap^{\mathfrak{F}} \uparrow \overline{B} \mid B \in \operatorname{up} b\}$.

References

[1] Victor Porton. Filters on posets and generalizations. *International Journal of Pure and Applied Mathematics*, 74(1):55–119, 2012. `http://www.mathematics21.org/binaries/filters.pdf`.