Equalizers and co-Equalizers in Certain Categories

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1 Draft status

It is a rough draft. Errors are possible. Subscribe to my blog for further results.

http://math.stackexchange.com/questions/540220/right-adjoint-of-forgetful-functor-from-top

[TODO: Change notation $\prod \to \prod^{(L)}$.]

2 Categories with embeddings

Note 1. This section in not used below, it is just to feed your intuition.

The following generalizes the well known concept of embedding function $A \hookrightarrow B$ for from a set A to a set B where $A \subseteq B$.

I will set that the unique morphism from an object A to an object B of a thin category is equal to the pair $(A; B)$.

Definition 2. A *category with embeddings of objects* is a dagger category with a preorder of the set of objects together with a functor $\hookrightarrow$ (we will denote applying this functor to the object $(A; B)$ as $A \hookrightarrow B$.) such that:

- $\hookrightarrow$ is an identity on objects.
- Every $A \hookrightarrow B$ is a monomorphism.
- $(A \hookrightarrow B)^{\dagger} \circ (A \hookrightarrow B) = 1_A$.

Obvious 3. $A \hookrightarrow B$ is defined when $(A; B)$ is a morphism of the preorder that is when $A \sqsubseteq B$.

Obvious 4. $A \hookrightarrow B: A \to B$ when $A \sqsubseteq B$.

Proposition 5. $A \hookrightarrow A = 1_A$.

Proof. Because $(A; A)$ is an identity morphism and $\hookrightarrow$ preserves identities. □

Proposition 6. $(B \hookrightarrow C) \circ (A \hookrightarrow B) = A \hookrightarrow C$ whenever $A \sqsubseteq B \sqsubseteq C$.

Proof. $(B \hookrightarrow C) \circ (A \hookrightarrow B) = \hookrightarrow(B; C) \circ \hookrightarrow(A; B) = \hookrightarrow((B; C) \circ (A; B)) = \hookrightarrow(A; C) = A \hookrightarrow C$. □

3 Categories under Rel

Definition 7. The **Rel**-morphism $A \rightleftarrows B$ (*restriction-embedding*) is defined by the formula: $A \rightleftarrows B = (A; B; \mathrm{id}_{A \cap B})$.

Obvious 8. If $A \subseteq B$ then $A \rightleftarrows B$ is an embedding $A \hookrightarrow B = (A; B; \mathrm{id}_A)$.

Obvious 9. If $A \supseteq B$ then $A \rightleftarrows B = (A; B; \mathrm{id}_B)$.

Obvious 10. $A \rightleftarrows A = 1^A_{\mathbf{Rel}}$.

Obvious 11. $(A \rightleftarrows B)^{-1} = B \rightleftarrows A$.

Definition 12. *Dagger functor* between two dagger categories is a functor between these categories, which commutes with the daggers. [TODO: Clearer wording.]

Definition 13. *Category under* **Rel** is a pair $(C; \uparrow)$ where C is a category whose objects are small sets and $\uparrow$ is an identity-on-objects functor $\mathbf{Rel} \rightarrow C$. I call $\uparrow$ *up-arrow functor*. [TODO: $A \sqsubseteq B \rightarrow A \subseteq B$ for sets.]

Definition 14. *Dagger category under* **Rel** is a pair $(C; \uparrow)$ where C is a dagger category whose objects are small sets and $\uparrow$ is a dagger identity-on-objects functor $\mathbf{Rel} \rightarrow C$.

Definition 15. $A \rightleftarrows^{\mathcal{C}} B = \uparrow (A \rightleftarrows B)$. In other words, $\rightleftarrows^{\mathcal{C}} = \uparrow \circ \rightleftarrows$.

Proposition 16. $A \rightleftarrows^{\mathcal{C}} A = 1^A_{\mathcal{C}}$.

Proof. $A \rightleftarrows^{\mathcal{C}} A = \uparrow(A \rightleftarrows A) = \uparrow 1_{\mathbf{Rel}} = 1^A_{\mathcal{C}}$. □

Proposition 17. If $f: X \rightarrow Y$ is a **Rel**-morphism and $\operatorname{im} f = A \subseteq Y$ then

$$(A \rightleftarrows Y) \circ (Y \rightleftarrows A) \circ f = f.$$

Proof. $(A \rightleftarrows Y) \circ (Y \rightleftarrows A) \circ f = \mathrm{id}_A \circ f = f$. □

Corollary 18. If $f: X \rightarrow Y$ is a morphism of a category under **Rel** and $\operatorname{im} f = A \subseteq Y$, then

$$(A \rightleftarrows^{\mathcal{C}} Y) \circ (Y \rightleftarrows^{\mathcal{C}} A) \circ \uparrow f = \uparrow f.$$

Proposition 19.

1. If $A \subseteq B$ then $A \rightleftarrows^{\mathcal{C}} B$ is a monomorphism.
2. If $A \supseteq B$ then $A \rightleftarrows^{\mathcal{C}} B$ is a epimorphism.

Proof. We'll prove only the first as the second is dual.

Let $(A \rightleftarrows^{\mathcal{C}} B) \circ f = (A \rightleftarrows^{\mathcal{C}} B) \circ g$. Then $(B \rightleftarrows^{\mathcal{C}} A) \circ (A \rightleftarrows^{\mathcal{C}} B) \circ f = (B \rightleftarrows^{\mathcal{C}} A) \circ (A \rightleftarrows^{\mathcal{C}} B) \circ g$; $1^A \circ f = 1^A \circ g$; $f = g$. □

Proposition 20. $(B \rightleftarrows C) \circ (A \rightleftarrows B) = A \rightleftarrows C$ iff $B \supseteq A \cap C$ (for every sets A, B, C).

Proof. $(B \rightleftarrows C) \circ (A \rightleftarrows B) = A \rightleftarrows C$ is equivalent to:
$(B; C; \mathrm{id}_{B \cap C}) \circ (A; B; \mathrm{id}_{A \cap B}) = (A; C; \mathrm{id}_{A \cap C})$;
$(A; C; \mathrm{id}_{A \cap B \cap C}) = (A; C; \mathrm{id}_{A \cap C})$;
$A \cap B \cap C = A \cap C$;
$B \supseteq A \cap C$. □

Corollary 21. $(B \rightleftarrows^{\mathcal{C}} C) \circ (A \rightleftarrows^{\mathcal{C}} B) = (A \rightleftarrows^{\mathcal{C}} C)$ if $B \supseteq A \cap C$ (for every sets A, B, C).

Definition 22. *Partially ordered dagger category under* **Rel** is a category which is both a partially ordered dagger category and a category under **Rel** such that $\uparrow \circ f^{-1} = (\uparrow \circ f)^{\dagger}$ and $A \sqsubseteq B \Rightarrow \uparrow A \sqsubseteq \uparrow B$.

Proposition 23. $(A \rightleftarrows^{\mathcal{C}} B)^{\dagger} = B \rightleftarrows^{\mathcal{C}} A$ for a dagger category under **Rel**.

Proof. $(A \rightleftarrows^{\mathcal{C}} B)^{\dagger} = (\uparrow(A \rightleftarrows B))^{\dagger} = \uparrow(A \rightleftarrows B)^{-1} = \uparrow(B \rightleftarrows A) = B \rightleftarrows^{\mathcal{C}} A$. □

Proposition 24. For a partially ordered dagger category $\mathcal{C}$ under **Rel** we have $A \rightleftarrows^{\mathcal{C}} B$ is:

1. monovalued;

2. injective;
3. entirely defined if $A \subseteq B$;
4. surjective if $B \subseteq A$.

Proof.

1. $(A \rightleftarrows B) \circ (B \rightleftarrows A) \sqsubseteq 1_B^{\mathbf{Rel}}$; $(A \rightleftarrows B) \circ (A \rightleftarrows B)^{-1} \sqsubseteq 1_B^{\mathbf{Rel}}$; $(A \rightleftarrows^{\mathcal{C}} B) \circ (A \rightleftarrows^{\mathcal{C}} B)^{\dagger} \sqsubseteq 1_B^{\mathcal{C}}$.
2. $(B \rightleftarrows A) \circ (A \rightleftarrows B) \sqsubseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows B)^{-1} \circ (A \rightleftarrows B) \sqsubseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows^{\mathcal{C}} B)^{\dagger} \circ (A \rightleftarrows^{\mathcal{C}} B) \sqsubseteq 1_A^{\mathcal{C}}$.
3. $(B \rightleftarrows A) \circ (A \rightleftarrows B) \sqsupseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows B)^{-1} \circ (A \rightleftarrows B) \sqsupseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows^{\mathcal{C}} B)^{\dagger} \circ (A \rightleftarrows^{\mathcal{C}} B) \sqsupseteq 1_A^{\mathcal{C}}$.
4. $(A \rightleftarrows B) \circ (B \rightleftarrows A) \sqsupseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows B) \circ (A \rightleftarrows B)^{-1} \sqsupseteq 1_A^{\mathbf{Rel}}$; $(A \rightleftarrows^{\mathcal{C}} B) \circ (A \rightleftarrows^{\mathcal{C}} B)^{\dagger} \sqsupseteq 1_A^{\mathcal{C}}$. □

4 Rectangular embedding-restriction

Definition 25. $\iota_{B_0, B_1} f = (A_1 \rightleftarrows^{\mathcal{C}} B_1) \circ f \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0)$ for $f \in \mathrm{Mor}_{\mathcal{C}}(A_0; A_1)$.

For brevity $\iota_B f = \iota_{B,B} f$.

Proposition 26. $\iota_{\mathrm{Src}\, f, \mathrm{Dst}\, f} f = f$.

Proof. $\iota_{\mathrm{Src}\, f, \mathrm{Dst}\, f} f = (\mathrm{Dst}\, f \rightleftarrows^{\mathcal{C}} \mathrm{Dst}\, f) \circ f \circ (\mathrm{Src}\, f \rightleftarrows^{\mathcal{C}} \mathrm{Src}\, f) = 1_{\mathcal{C}}^{\mathrm{Dst}\, f} \circ f \circ 1_{\mathcal{C}}^{\mathrm{Src}\, f} = f$. □

Proposition 27. The function $\iota_{B_0, B_1}|_{f \in \mathrm{Mor}_{\mathcal{C}}(A_0; A_1)}$ is injective, if $A_0 \subseteq B_0 \wedge A_1 \subseteq B_1$.

Proof. Because $A_1 \rightleftarrows^{\mathcal{C}} B_1$ is a monomorphism and $A_0 \rightleftarrows^{\mathcal{C}} B_0$ is an epimorphism. □

Proposition 28. $\iota_{C_0, C_1} \iota_{B_0, B_1} f = \iota_{C_0, C_1} f$ for $B_0 \supseteq A_0 \cap C_0$, $B_1 \supseteq A_1 \cap C_1$ and $f: A_0 \rightarrow A_1$.

Proof. $\iota_{C_0, C_1} \iota_{B_0, B_1} f = (B_1 \rightleftarrows^{\mathcal{C}} C_1) \circ (A_1 \rightleftarrows^{\mathcal{C}} B_1) \circ f \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0) \circ (C_0 \rightleftarrows^{\mathcal{C}} B_0) = (A_1 \rightleftarrows^{\mathcal{C}} C_1) \circ f \circ (C_0 \rightleftarrows^{\mathcal{C}} A_0) = \iota_{C_0, C_1} f$. □

Proposition 29. Let $f: A_0 \rightarrow A_1$ and $g: A_1 \rightarrow A_2$ and $A_1 \subseteq B_1$. Then $\iota_{B_0, B_2}(g \circ f) = \iota_{B_1, B_1} g \circ \iota_{B_0, B_1} f$.

Proof. $\iota_{B_0, B_2}(g \circ f) = (A_2 \rightleftarrows^{\mathcal{C}} B_2) \circ (g \circ f) \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0) = (A_2 \rightleftarrows^{\mathcal{C}} B_2) \circ g \circ \mathrm{id}_{A_1} \circ f \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0) = (A_2 \rightleftarrows^{\mathcal{C}} B_2) \circ g \circ (B_1 \rightleftarrows A_1) \circ (A_1 \rightleftarrows B_1) \circ f \circ (B_0 \rightleftarrows^{\mathcal{C}} A_0) = \iota_{B_1, B_1} g \circ \iota_{B_0, B_1} f$. □

5 Examples of partially ordered dagger categories under Rel

5.1 Generalized rebase of filters

In [2] I defined *rebase* $\mathcal{A} \div A$ for a set-theoretic filter $\mathcal{A}$ and a set X such that $\exists X \in \mathcal{A}: X \subseteq A$.

Now define a generalized rebase for every set-theoretic filter $\mathcal{A}$ and every set A:

Definition 30. $\mathcal{A} \div A = \bigsqcap \{\uparrow^A (X \cap A) \mid X \in \mathcal{A}\}$.

Proposition 31. In the case of $\exists X \in \mathcal{A}: X \subseteq A$ these two definitions coincide.

Proof. Let $\exists X \in \mathcal{A}: X \subseteq A$. Then as it is proved in [2] $\{X \in \mathscr{P} A \mid \exists Y \in \mathcal{A}: Y \subseteq X\}$ is a filter.

If $P \in \{X \in \mathscr{P} A \mid \exists Y \in \mathcal{A}: Y \subseteq X\}$ then $P \in \mathscr{P} A$ and $Y \subseteq P$ for some $Y \in \mathcal{A}$. Thus $P \supseteq Y \cap A \in \bigsqcap \{\uparrow^B (Y \cap A) \mid Y \in \mathcal{A}\}$.

If $P \in \bigsqcap \{\uparrow^B (X \cap A) \mid X \in \mathcal{A}\}$ then by properties of generalized filter bases, there exists $X \in \mathcal{A}$ such that $P \supseteq X \cap A$. Also $P \in \mathscr{P} A$. Thus $P \supseteq X$. Thus $P \in \{X \in \mathscr{P} A \mid \exists Y \in \mathcal{A}: Y \subseteq X\}$.

[TODO: Clear this proof: wording, consistent use of letters.] □

Proposition 32. $(\mathcal{X} \div A) \div B = \mathcal{X} \div B$ if $B \subseteq A$.

Proof. $(\mathcal{X} \div A) \div B = \bigsqcap \{\uparrow^B (Y \cap B) \mid Y \in \bigsqcap \{\uparrow^A (X \cap A) \mid X \in \mathcal{X}\}\} = \bigsqcap \{\uparrow^B (X \cap A) \mid X \in \mathcal{X}\} \sqcap \uparrow^B B = \bigsqcap \{\uparrow^B (X \cap A \cap B) \mid X \in \mathcal{X}\} = \mathcal{X} \div (A \cap B) = \mathcal{X} \div B$. □

5.2 Category Rel

Category **Rel** with the identity up-arrow functor to itself and "reverse relation" as the dagger is an obvious example of a partially ordered dagger category under **Rel**.

Definition 33. $f \div (A \times B) = (A; B; (\mathrm{GR}\, f) \div (A \times B))$ for every **Rel**-morphism f.

Proposition 34. $\iota_{A,B} f = (A; B; \mathrm{GR}\, f \cap (A \times B))$.

Proof. $\iota_{A,B} f = (\mathrm{Dst}\, f \rightleftarrows B) \circ f \circ (A \rightleftarrows \mathrm{Src}\, f) = (A; B; \mathrm{GR}\, f \cap (A \times B))$. □

5.3 Category FCD

Category FCD with the up-arrow functor $\uparrow^{\mathsf{FCD}}$ and "reverse funcoid" as the dagger is a partially ordered dagger category under **Rel**.

Proposition 35. $A \rightleftarrows^{\mathsf{FCD}} B = (A; B; \lambda \mathcal{X} \in \mathfrak{F}(A): \mathcal{X} \div B; \lambda \mathcal{Y} \in \mathfrak{F}(B): \mathcal{Y} \div A)$ for objects $A \subseteq B$ of FCD.

Proof. $\langle A \rightleftarrows^{\mathsf{FCD}} B \rangle \mathcal{X} = \bigsqcap \{\langle A \rightleftarrows^{\mathsf{FCD}} B \rangle^* X \mid X \in \mathcal{X}\} = \bigsqcap \{\uparrow^B \langle A \rightleftarrows B \rangle X \mid X \in \mathcal{X}\} = \bigsqcap \{\uparrow^B (X \cap A \cap B) \mid X \in \mathcal{X}\} = \bigsqcap \{\uparrow^B (X \cap B) \mid X \in \mathcal{X}\} = \mathcal{X} \div B$.

Rest follows from symmetry. □

Proposition 36.

1. $\langle A \rightleftarrows^{\mathsf{FCD}} B \rangle^* X = \uparrow^B X$ for every $X \in \mathscr{P} A$ if $A \subseteq B$.
2. $\langle (B \rightleftarrows^{\mathsf{FCD}} A) \rangle^* Y = \uparrow^A (Y \cap A)$ for every $Y \in \mathscr{P} B$ if $A \subseteq B$.

Proof. By definition of principal funcoid. □

5.4 Category RLD

Category RLD with the up-arrow functor $\uparrow^{\mathsf{RLD}}$ and "reverse reloid" as the dagger is a partially ordered dagger category under **Rel**.

Obvious 37. $A \rightleftarrows^{\mathsf{RLD}} B = \uparrow^{\mathsf{RLD}(A;B)} \mathrm{id}_{A \cap B}$.

Definition 38. $f \div (A \times B) = (A; B; (\mathrm{GR}\, f) \div (A \times B))$ for every reloid f.

Proposition 39. $\iota_{A,B} f = f \div (A \times B)$.

Proof. $\iota_{A,B} f = (\mathrm{Dst}\, f \rightleftarrows^{\mathsf{RLD}} B) \circ f \circ (A \rightleftarrows^{\mathsf{RLD}} \mathrm{Src}\, f) = \bigsqcap \{\uparrow^{\mathsf{RLD}} ((\mathrm{Dst}\, f \rightleftarrows B) \circ F \circ (A \rightleftarrows \mathrm{Src}\, f)) \mid F \in \mathrm{xyGR}\, f\} = \bigsqcap \{\uparrow^{\mathsf{RLD}} (F \cap (A \times B)) \mid F \in \mathrm{xyGR}\, f\} = f \div (A \times B)$. [TODO: Filters on cartesian products vs reloids.] □

6 Equalizers

Categories $\mathbf{cont}(\mathcal{C})$ are defined in [1].

I will denote W the forgetful functor from $\mathbf{cont}(\mathcal{C})$ to $\mathcal{C}$.

In the definition of the category $\mathbf{cont}(\mathcal{C})$ take values of $\uparrow$ as principal morphisms. [TODO: Wording.]

Lemma 40. Let $f: X \rightarrow Y$ be a morphism of the category $\mathbf{cont}(\mathcal{C})$ where $\mathcal{C}$ is a concrete category (so $W f = \uparrow \varphi$ for a **Rel**-morphism φ because f is principal) and $\operatorname{im} \varphi = A \subseteq \operatorname{Ob} Y$. Factor it $\varphi = (A \rightleftarrows \operatorname{Ob} Y) \circ u$ where $u: \operatorname{Ob} X \rightarrow A$ using properties of **Set**. Then u is a morphism of $\mathbf{cont}(\mathcal{C})$ (that is a continuous function $X \rightarrow \iota_A Y$).

Proof. $(A \rightleftarrows \operatorname{Ob} Y)^{-1} \circ \varphi = (A \rightleftarrows \operatorname{Ob} Y)^{-1} \circ (A \rightleftarrows \operatorname{Ob} Y) \circ u$;

$(A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y)^{-1} \circ \uparrow \varphi = (A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y)^{-1} \circ (A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y) \circ \uparrow u$;

$(A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y)^{-1} \circ \uparrow \varphi = \uparrow u$;

$X \sqsubseteq (\uparrow u)^{-1} \circ \pi_A Y \circ \uparrow u \Leftrightarrow X \sqsubseteq (\uparrow \varphi)^{-1} \circ (A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y) \circ \pi_A Y \circ (A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y)^{-1} \circ \uparrow \varphi \Leftrightarrow X \sqsubseteq (\uparrow \varphi)^{-1} \circ (A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y) \circ (A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y)^{-1} \circ Y \circ (A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y) \circ (A \rightleftarrows^{\mathcal{C}} \operatorname{Ob} Y)^{-1} \circ \uparrow \varphi \Leftrightarrow X \sqsubseteq (\uparrow \varphi)^{-1} \circ Y \circ \uparrow \varphi \Leftrightarrow X \sqsubseteq (W f)^{-1} \circ Y \circ W f$ what is true by definition of continuity. □

Equational definition of equalizers:
http://nforum.mathforge.org/comments.php?DiscussionID=5328/

Theorem 41. The following is an equalizer of parallel morphisms $f, g: A \rightarrow B$ of category $\mathbf{cont}(\mathcal{C})$:

- the object $X = \iota_{\{x \in \operatorname{Ob} A \mid f x = g x\}} A$;
- the morphism $\operatorname{Ob} X \rightleftarrows \operatorname{Ob} A$ considered as a morphism $X \rightarrow A$.

Proof. Denote $e = \operatorname{Ob} X \rightleftarrows \operatorname{Ob} A$.

Let $f \circ z = g \circ z$ for some morphism z.

Let's prove $e \circ u = z$ for some $u: \operatorname{Src} z \rightarrow X$. Really, as a morphism of **Set** it exists and is unique.

Consider z as as a generalized element.

$f(z) = g(z)$. So $z \in X$ (that is $\operatorname{Dst} z \in X$). Thus $z = e \circ u$ for some u (by properties of **Set**). The generalized element u is a $\mathbf{cont}(\mathcal{C})$-morphism because of the lemma above. It is unique by properties of **Set**. □

We can (over)simplify the above theorem by the obvious below:

Obvious 42. $\{x \in \operatorname{Ob} A \mid f x = g x\} = \operatorname{dom}(f \cap g)$.

7 Co-equalizers

http://math.stackexchange.com/questions/539717/how-to-construct-co-equalizers-in-mathbftop

Let $\sim$ be an equivalence relation. Let's denote π its canonical projection.

Definition 43. $f/{\sim} = \uparrow \pi \circ f \circ \uparrow \pi^{-1}$ for every morphism f.

Obvious 44. $\operatorname{Ob}(f/{\sim}) = (\operatorname{Ob} f)/r$.

Obvious 45. $f/{\sim} = \left\langle \uparrow^{\mathsf{FCD}} \pi \times^{(C)} \uparrow^{\mathsf{FCD}} \pi \right\rangle f$ for every morphism f.

To define co-equalizers of morphisms f and g let $\sim$ be is the smallest equivalence relation such that $f x = g x$.

Lemma 46. Let $f: X \rightarrow Y$ be a morphism of the category $\mathbf{cont}(\mathcal{C})$ where $\mathcal{C}$ is a concrete category (so $W f = \uparrow \varphi$ for a **Rel**-morphism φ because f is principal) such that φ respects $\sim$. Factor it $\varphi = u \circ \pi$ where $u: \operatorname{Ob}(X/{\sim}) \rightarrow \operatorname{Ob} Y$ using properties of **Set**. Then u is a morphism of $\mathbf{cont}(\mathcal{C})$ (that is a continuous function $X/{\sim} \rightarrow Y$).

Proof. $f \circ X \circ f^{-1} \sqsubseteq Y$; $\uparrow u \circ \uparrow \pi \circ X \circ \uparrow \pi^{-1} \circ \uparrow u^{-1} \sqsubseteq Y$; $\uparrow u \in \mathrm{C}(\uparrow \pi \circ X \circ \uparrow \pi^{-1}; Y) = \mathrm{C}(X/{\sim}; Y)$. □

Theorem 47. The following is a co-equalizer of parallel morphisms f, g: $A \rightarrow B$ of category $\mathbf{cont}(\mathcal{C})$:

- the object $Y = f/{\sim}$;

- the morphism π considered as a morphism $B \to Y$.

Proof. Let $z \circ f = z \circ g$ for some morphism z.

Let's prove $u \circ \pi = z$ for some $u: Y \to \operatorname{Dst} z$. Really, as a morphism of **Set** it exists and is unique.

$\operatorname{Src} z \in Y$. Thus $z = u \circ \pi$ for some u (by properties of **Set**). The function u is a **cont**$(\mathcal{C})$-morphism because of the lemma above. It is unique by properties of **Set** (π obviously respects equivalence classes). □

8 Rest

[TODO: Specify what is $\mathcal{C}$.]

Theorem 48. The categories **cont**$(\mathcal{C})$ (for example in **Fcd** and **Rld**) are complete.

Proof. They have products [1] and equalizers. □

Theorem 49. The categories **cont**$(\mathcal{C})$ (for example in **Fcd** and **Rld**) are co-complete.

Proof. They have co-products [1] and co-equalizers. □

Definition 50. I call morphisms f and g of a category with embeddings *equivalent* ($f \sim g$) when there exist a morphism p such that $\operatorname{Src} p \sqsubseteq \operatorname{Src} f$, $\operatorname{Src} p \sqsubseteq \operatorname{Src} g$, $\operatorname{Dst} p \sqsubseteq \operatorname{Dst} f$, $\operatorname{Dst} p \sqsubseteq \operatorname{Dst} g$ and $\iota_{\operatorname{Src} f, \operatorname{Dst} f} p = f$ and $\iota_{\operatorname{Src} g, \operatorname{Dst} g} p = g$.

Problem 51. Find under which conditions:

1. Equivalence of morphisms is an equivalence relation.
2. Equivalence of morphisms is a congruence for our category.

Bibliography

[1] Victor Porton. Products in dagger categories with complete ordered mor-sets. At `http://www.mathematics21.org/binaries/product.pdf`.

[2] Victor Porton. *Algebraic General Topology. Volume 1*. 2013.