Equalizers and co-Equalizers in Certain Categories

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1 Draft status

It is a rough draft. Errors are possible. Subscribe to my blog for further results.

http://math.stackexchange.com/questions/540220/right-adjoint-of-forgetful-functor-from-top

[TODO: Change notation \( \prod \to \prod^{(L)} \).]

2 Categories with embeddings

Note 1. This section is not used below, it is just to feed your intuition.

The following generalizes the well known concept of embedding function \( A \hookrightarrow B \) for from a set \( A \) to a set \( B \) where \( A \subseteq B \).

I will set that the unique morphism from an object \( A \) to an object \( B \) of a thin category is equal to the pair \( (A; B) \).

Definition 2. A category with embeddings of objects is a dagger category with a preorder of the set of objects together with a functor \( / \) (we will denote applying this functor to the object \( (A; B) \) as \( A / B \)) such that:

- \( \hookrightarrow \) is an identity on objects.
- Every \( A \hookrightarrow B \) is a monomorphism.
- \( (A \hookrightarrow B) \dagger \circ (A \hookrightarrow B) = 1_A \).

Obvious 3. \( A \hookrightarrow B \) is defined when \( (A; B) \) is a morphism of the preorder that is when \( A \subseteq B \).

Obvious 4. \( A \hookrightarrow B; A \to B \) when \( A \supseteq B \).

Proposition 5. \( A \hookrightarrow A = 1_A \).

Proof. Because \( (A; A) \) is an identity morphism and \( \hookrightarrow \) preserves identities. \( \square \)

Proposition 6. \( (B \hookrightarrow C) \circ (A \hookrightarrow B) = A \hookrightarrow C \) whenever \( A \subseteq B \subseteq C \).

Proof. \( (B \hookrightarrow C) \circ (A \hookrightarrow B) = \hookrightarrow (B; C) \circ \hookrightarrow (A; B) = \hookrightarrow ((B; C) \circ (A; B)) = \hookrightarrow (A; C) = A \hookrightarrow C. \) \( \square \)

3 Categories under Rel

Definition 7. The \( \text{Rel} \)-morphism \( A \sqsubseteq B \) (restriction-embedding) is defined by the formula: \( A \sqsubseteq B = (A; B; \text{id}_{A \cap B}) \).

Obvious 8. If \( A \subseteq B \) then \( A \sqsubseteq B \) is an embedding \( A \hookrightarrow B = (A; B; \text{id}_A) \).

Obvious 9. If \( A \supseteq B \) then \( A \sqsubseteq B = (A; B; \text{id}_B) \).
Obvious 10. $A \dagger A = 1^A_{\text{Rel}}$.

Obvious 11. $(A \dagger B)^{-1} = B \dagger A$.

Definition 12. **Dagger functor** between two dagger categories is a functor between these categories, which commutes with the daggers. [TODO: Clearer wording.]

Definition 13. **Category under Rel** is a pair $(C; \uparrow)$ where $C$ is a category whose objects are small sets and $\uparrow$ is an identity-on-objects functor $\text{Rel} \to C$. I call $\uparrow$ **up-arrow functor**. [TODO: $A \subseteq B \to A \subseteq B$ for sets.]

Definition 14. **Dagger category under Rel** is a pair $(C; \uparrow)$ where $C$ is a dagger category whose objects are small sets and $\uparrow$ is a dagger identity-on-objects functor $\text{Rel} \to C$.

Definition 15. $A \dagger^C B = \uparrow (A \dagger B)$. In other words, $\dagger^C = \uparrow \circ \dagger$.

Proposition 16. $A \dagger^C A = 1^A$.  \hfill $\Box$

Proof. $A \dagger^C A = \uparrow (A \dagger A) = \uparrow 1_{\text{Rel}} = 1^A$. \hfill $\Box$

Proposition 17. If $f: X \to Y$ is a Rel-morphism and $\im f = A \subseteq Y$ then 
$$(A \dagger Y) \circ (Y \dagger A) \circ f = f.$$  \hfill $\Box$

Proof. $(A \dagger Y) \circ (Y \dagger A) \circ f = \id_A \circ f = f$. \hfill $\Box$

Corollary 18. If $f: X \to Y$ is a morphism of a category under Rel and $\im f = A \subseteq Y$, then 
$$(A \dagger^C Y) \circ (Y \dagger^C A) \circ \uparrow f = \uparrow f.$$  \hfill $\Box$

Proposition 19.
1. If $A \subseteq B$ then $A \dagger^C B$ is a monomorphism.
2. If $A \supseteq B$ then $A \dagger^C B$ is an epimorphism.

Proof. We’ll prove only the first as the second is dual.
Let $(A \dagger^C B) \circ f = (A \dagger^C B) \circ g$. Then $(B \dagger^C A) \circ (A \dagger^C B) \circ f = (B \dagger^C A) \circ (A \dagger^C B) \circ g$.

1$^A \circ f = 1^A \circ g$; $f = g$. \hfill $\Box$

Proposition 20. $(B \dagger^C C) \circ (A \dagger B) = A \dagger C$ if $B \supseteq A \cap C$ (for every sets $A$, $B$, $C$).

Proof. $(B \dagger^C C) \circ (A \dagger B) = A \dagger C$ is equivalent to:
$(B; C; \id_{B \cap C}) \circ (A; B; \id_{A \cap B}) = (A; C; \id_{A \cap C});$
$(A; C; \id_{A \cap B \cap C}) = (A; C; \id_{A \cap C});$
$A \cap B \cap C = A \cap C;$
$B \supseteq A \cap C$. \hfill $\Box$

Corollary 21. $(B \dagger^C C) \circ (A \dagger^C C) = (A \dagger^C C)$ if $B \supseteq A \cap C$ (for every sets $A$, $B$, $C$).

Definition 22. **Partially ordered dagger category under Rel** is a category which is both a partially ordered dagger category and a category under Rel such that $\uparrow \circ f^{-1} = (\uparrow \circ f)^\dagger$ and $A \subseteq B \Rightarrow \uparrow A \subseteq \uparrow B$.

Proposition 23. $(A \dagger^C B)^\dagger = B \dagger^C A$ for a dagger category under Rel.

Proof. $(A \dagger^C B)^\dagger = (\uparrow (A \dagger B))^\dagger = \uparrow (A \dagger B)^{-1} = \uparrow (B \dagger A) = B \dagger^C A$. \hfill $\Box$

Proposition 24. For a partially ordered dagger category $\mathcal{C}$ under Rel we have $A \dagger^C B$ is:
1. monovalued;
2. injective;
3. entirely defined if $A \subseteq B$;
4. surjective if $B \subseteq A$.

**Proof.**
1. $(A \sqsupseteq B) \circ (B \sqsupseteq A) \subseteq 1^\text{Rel}_A$; $(A \sqsupseteq B) \circ (A \sqsupseteq B)^{-1} \subseteq 1^\text{Rel}_B$; $(A \sqsupseteq C) \circ (A \sqsupseteq C)^{-1} \subseteq 1^\text{Rel}_B$.
2. $(B \sqsupseteq A) \circ (A \sqsupseteq B) \subseteq 1^\text{Rel}_A$; $(A \sqsupseteq B)^{-1} \circ (A \sqsupseteq B) \subseteq 1^\text{Rel}_A$; $(A \sqsupseteq C)^{-1} \circ (A \sqsupseteq C) \subseteq 1^\text{Rel}_A$.
3. $(B \sqsupseteq A) \circ (A \sqsupseteq B) \supseteq 1^\text{Rel}_A$; $(A \sqsupseteq B)^{-1} \circ (A \sqsupseteq B) \supseteq 1^\text{Rel}_A$; $(A \sqsupseteq C)^{-1} \circ (A \sqsupseteq C) \supseteq 1^\text{Rel}_A$.
4. $(A \sqsupseteq B) \circ (B \sqsupseteq A) \supseteq 1^\text{Rel}_A$; $(A \sqsupseteq B)^{-1} \circ (A \sqsupseteq B) \supseteq 1^\text{Rel}_A$; $(A \sqsupseteq C)^{-1} \circ (A \sqsupseteq C) \supseteq 1^\text{Rel}_A$. \qed

### 4 Rectangular embedding-restriction

**Definition 25.** $\iota_{B_0, B_1} f = (A \sqsupseteq C) \circ f \circ (B_0 \sqsupseteq A)$ for $f \in \text{Mor}_C(A_0; A_1)$.

For brevity $\iota_B f = \iota_{B, B} f$.

**Proposition 26.** $\iota_{\text{Src}, f, \text{Dst}} f = f$.

**Proof.** $\iota_{\text{Src}, f, \text{Dst}} f = (\text{Dst} f \sqsupseteq C) \circ f \circ (\text{Src} f \sqsupseteq C) = 1^\text{Dst}_f \circ f \circ 1^\text{Src}_f = f$. \qed

**Proposition 27.** The function $\iota_{B_0, B_1} | f \in \text{Mor}_C(A_0; A_1)$ is injective, if $A_0 \subseteq B_0 \land A_1 \subseteq B_1$.

**Proof.** Because $A_1 \sqsupseteq C B_1$ is a monomorphism and $A_0 \sqsupseteq C B_0$ is an epimorphism. \qed

**Proposition 28.** $\iota_{C_0, C_1} \iota_{B_0, B_1} f = \iota_{C_0, C_1} f$ for $B_0 \supseteq A_0 \cap C_0$, $B_1 \supseteq A_1 \cap C_1$ and $f: A_0 \rightarrow A_1$.

**Proof.** $\iota_{C_0, C_1} \iota_{B_0, B_1} f = (B_1 \sqsupseteq C) \circ (A_1 \sqsupseteq C B_1) \circ f \circ (B_0 \sqsupseteq C A_0) \circ (C_0 \sqsupseteq C B_0) = (A_1 \sqsupseteq C) \circ f \circ (C_0 \sqsupseteq C A_0) = \iota_{C_0, C_1} f$. \qed

**Proposition 29.** Let $f: A_0 \rightarrow A_1$ and $g: A_1 \rightarrow A_2$ and $A_1 \subseteq B_1$. Then $\iota_{B_0, B_2} (g \circ f) = \iota_{B_1, B_2} g \circ \iota_{B_0, B_1} f$.

**Proof.** $\iota_{B_0, B_2} (g \circ f) = (A_2 \sqsupseteq C B_2) \circ (g \circ f) \circ (B_0 \sqsupseteq C A_0) = (A_2 \sqsupseteq C B_2) \circ g \circ \iota_{B_1, B_2} f \circ (B_0 \sqsupseteq C A_0) = \iota_{B_1, B_2} g \circ \iota_{B_0, B_1} f$. \qed

### 5 Examples of partially ordered dagger categories under Rel

#### 5.1 Generalized rebase of filters

In [2] I defined rebase $A \div A$ for a set-theoretic filter $A$ and a set $X$ such that $\exists X \in A: X \subseteq A$.

Now define a generalized rebase for every set-theoretic filter $A$ and every set $A$:

**Definition 30.** $A \div A = \bigsqcup \{ \uparrow^A (X \cap A) \mid X \in A \}$.

**Proposition 31.** In the case of $\exists X \in A: X \subseteq A$ these two definitions coincide.

**Proof.** Let $\exists X \in A: X \subseteq A$. Then as it is proved in [2] $\{ X \in \mathcal{P}A \mid \exists Y \in A: Y \subseteq X \}$ is a filter.

If $P \in \{ X \in \mathcal{P}A \mid \exists Y \in A: Y \subseteq X \}$ then $P \in \mathcal{P}A$ and $Y \subseteq P$ for some $Y \in A$. Thus $P \supseteq Y \cap A \in \{ \uparrow^B (Y \cap A) \mid Y \in A \}$.

If $P \in \bigsqcup \{ \uparrow^B (X \cap A) \mid X \in A \}$ then by properties of generalized filter bases, there exists $X \in A$ such that $P \supseteq X \cap A$. Also $P \in \mathcal{P}A$. Thus $P \supseteq X$. Thus $P \in \{ X \in \mathcal{P}A \mid \exists Y \in A: Y \subseteq X \}$. \[TODO: Clear this proof: wording, consistent use of letters.] \qed
Proposition 32. \((X \times A) \div B = X \div B\) if \(B \subseteq A\).

Proof. \((X \times A) \div B = \bigsqcap \{ \uparrow^B (Y \cap B) \mid Y \in \prod \{ \uparrow^A (X \cap A) \mid X \in \mathcal{X} \} \} = \bigsqcap \{ \uparrow^B (X \cap A) \mid X \in \mathcal{X} \} \cap \uparrow^B B = \bigsqcap \{ \uparrow^B (X \cap A \cap B) \mid X \in \mathcal{X} \} = X \div (A \cap B) = X \div B.\)

\[\square\]

5.2 Category Rel

Category \(\text{Rel}\) with the identity up-arrow functor to itself and “reverse relation” as the dagger is an obvious example of a partially ordered dagger category under \(\text{Rel}\).

Definition 33. \(f \div (A \times B) = (A; B; (\text{GR} f) \div (A \times B))\) for every \(\text{Rel}\)-morphism \(f\).

Proposition 34. \(\iota_{A,B} f = (A; B; \text{GR} f \cap (A \times B)).\)

Proof. \(\iota_{A,B} f = (\text{Dst} f \sqsupseteq B) \circ f \circ (A \sqsubseteq \text{Src} f) = (A; B; \text{GR} f \cap (A \times B)).\)

\[\square\]

5.3 Category FCD

Category \(\text{FCD}\) with the up-arrow functor \(\uparrow^{\text{FCD}}\) and “reverse funcocid” as the dagger is a partially ordered dagger category under \(\text{Rel}\).

Proposition 35. \(A \sqsubseteq^{\text{FCD}} B = (A; B; \lambda X \in \mathfrak{g}(A) \colon X \div B; \lambda Y \in \mathfrak{g}(B) \colon Y \div A)\) for objects \(A \subseteq B\) of \(\text{FCD}\).

Proof. \(\langle A \sqsubseteq^{\text{FCD}} B \rangle X = \bigsqcap \{ \langle A \sqsubseteq^{\text{FCD}} B \rangle X \mid X \in \mathcal{X} \} = \bigsqcap \{ \uparrow^B \langle A \sqsubseteq B \rangle X \mid X \in \mathcal{X} \} = \bigsqcap \{ \uparrow^B (X \cap A \cap B) \mid X \in \mathcal{X} \} = X \div B.\)

Rest follows from symmetry.

\[\square\]

Proposition 36.

1. \(\langle A \sqsubseteq^{\text{FCD}} B \rangle X = \uparrow^B X\) for every \(X \in \mathcal{P} A\) if \(A \subseteq B\).
2. \(\langle (B \sqsubseteq^{\text{FCD}} A) \rangle Y = \uparrow^A (Y \cap A)\) for every \(Y \in \mathcal{P} B\) if \(A \subseteq B\).

Proof. By definition of principal funcocid.

\[\square\]

5.4 Category RLD

Category \(\text{RLD}\) with the up-arrow functor \(\uparrow^{\text{RLD}}\) and “reverse reloid” as the dagger is a partially ordered dagger category under \(\text{Rel}\).

Obvious 37. \(A \sqsubseteq^{\text{RLD}} B = \uparrow^{\text{RLD}}(A; B) \circ \text{id}_{A \cap B}.\)

Definition 38. \(f \div (A \times B) = (A; B; (\text{GR} f) \div (A \times B))\) for every reloid \(f\).

Proposition 39. \(\iota_{A,B} f = f \div (A \times B).\)

Proof. \(\iota_{A,B} f = (\text{Dst} f \sqsupseteq B) \circ f \circ (A \sqsubseteq \text{Src} f) = \bigsqcap \{ \uparrow^{\text{RLD}} ((\text{Dst} f \sqsupseteq B) \circ f \circ (A \sqsubseteq \text{Src} f)) \mid F \in \text{xyGR} f \} = \bigsqcap \{ \uparrow^{\text{RLD}} (F \cap (A \times B)) \mid F \in \text{xyGR} f \} = f \div (A \times B).\) [TODO: Filters on cartesian products vs reloids.]

\[\square\]

6 Equalizers

Categories \(\text{cont}(\mathcal{C})\) are defined in [1].

I will denote \(W\) the forgetful functor from \(\text{cont}(\mathcal{C})\) to \(\mathcal{C}\).

In the definition of the category \(\text{cont}(\mathcal{C})\) take values of \(\uparrow\) as principal morphisms. [TODO: Wording.]
Lemma 40. Let \( f : X \to Y \) be a morphism of the category \( \text{cont}(C) \) where \( C \) is a concrete category (so \( Wf = \uparrow \varphi \) for a Rel-morphism \( \varphi \) because \( f \) is principal) and \( \text{im} \varphi = A \subseteq \text{Ob} Y \). Factor it \( \varphi = (A \rightrightarrows \text{Ob} Y) \circ u \) where \( u : \text{Ob} X \to A \) using properties of \( \text{Set} \). Then \( u \) is a morphism of \( \text{cont}(C) \) (that is a continuous function \( X \to t_A Y \)).

Proof. \((A \rightrightarrows \text{Ob} Y)^{-1} \circ \varphi = (A \rightrightarrows \text{Ob} Y) \circ (A \rightrightarrows \text{Ob} Y)^{-1} \circ (A \rightrightarrows \text{Ob} Y) \circ u;\)
\((A \rightrightarrows \text{Ob} Y)^{-1} \circ \uparrow \varphi = (A \rightrightarrows \text{Ob} Y)^{-1} \circ (A \rightrightarrows \text{Ob} Y) \circ \uparrow u;\)
\((A \rightrightarrows \text{Ob} Y)^{-1} \circ \uparrow \varphi \equiv \uparrow u;\)
\(X \subseteq (\uparrow u)^{-1} \circ \pi_A Y \circ \uparrow u \Leftrightarrow X \subseteq (\uparrow \varphi)^{-1} \circ (A \rightrightarrows \text{Ob} Y) \circ \pi_A Y \circ (A \rightrightarrows \text{Ob} Y)^{-1} \circ \uparrow \varphi \Leftrightarrow X \subseteq (\uparrow \varphi)^{-1} \circ Y \circ \uparrow \varphi \Leftrightarrow X \subseteq (\uparrow \varphi)^{-1} \circ f \circ \uparrow \varphi \Leftrightarrow X \subseteq (Wf)^{-1} \circ f \circ W \) what is true by definition of continuity. □

Equational definition of equalizers:
http://math.stackexchange.com/questions/539717/how-to-construct-co-equalizers-in-mathbf{top}

Theorem 41. The following is an equalizer of parallel morphisms \( f, g : A \to B \) of category \( \text{cont}(C) \):

- the object \( X = t_{\{x \in \text{Ob} A \mid f(x) = g(x)\}} A; \)
- the morphism \( \text{Ob} X \rightrightarrows \text{Ob} A \) considered as a morphism \( X \to A \).

Proof. Denote \( e = \text{Ob} X \rightrightarrows \text{Ob} A. \)
Let \( f \circ z = g \circ z \) for some morphism \( z. \)
Let’s prove \( e \circ u = z \) for some \( u : \text{Src} z \to X. \) Really, as a morphism of \( \text{Set} \) it exists and is unique. Consider \( z \) as as a generalized element.
\( f(z) = g(z). \) So \( z \in X \) (that is \( \text{Dst} z \in X \)). Thus \( z = e \circ u \) for some \( u \) (by properties of \( \text{Set} \)).
The generalized element \( u \) is a \( \text{cont}(C) \)-morphism because of the lemma above. It is unique by properties of \( \text{Set}. \) □

We can (over)simplify the above theorem by the obvious below:

Obvious 42. \( \{x \in \text{Ob} A \mid f(x) = g(x)\} = \text{dom}(f \cap g). \)

7 Co-equalizers

http://math.stackexchange.com/questions/539717/how-to-construct-co-equalizers-in-mathbf{top}
Let \( \sim \) be an equivalence relation. Let’s denote \( \pi \) its canonical projection.

Definition 43. \( f/\sim = \uparrow \pi \circ f \circ \uparrow \pi^{-1} \) for every morphism \( f. \)

Obvious 44. \( \text{Ob}(f/\sim) = (\text{Ob} f)/\sim. \)

Obvious 45. \( f/\sim = (\uparrow \text{FCD}_\pi \times (C) \uparrow \text{FCD}_\pi) f \) for every morphism \( f. \)

To define co-equalizers of morphisms \( f \) and \( g \) let \( \sim \) be is the smallest equivalence relation such that \( f x = g x. \)

Lemma 46. Let \( f : X \to Y \) be a morphism of the category \( \text{cont}(C) \) where \( C \) is a concrete category (so \( Wf = \uparrow \varphi \) for a Rel-morphism \( \varphi \) because \( f \) is principal) such that \( \varphi \) respects \( \sim \). Factor it \( \varphi = u \circ \pi \) where \( u : \text{Ob}(X/\sim) \to \text{Ob} Y \) using properties of \( \text{Set} \). Then \( u \) is a morphism of \( \text{cont}(C) \) (that is a continuous function \( X/\sim \to Y \)).

Proof. \( f \circ X \circ f^{-1} \subseteq Y; \uparrow u \circ \uparrow \pi \circ X \circ \uparrow \pi^{-1} \circ \uparrow u^{-1} \subseteq Y; \uparrow u \in C(\uparrow \pi \circ X \circ \uparrow \pi^{-1}; Y) = C(X/\sim; Y). \) □

Theorem 47. The following is a co-equalizer of parallel morphisms \( f, g : A \to B \) of category \( \text{cont}(C) \):

- the object \( Y = f/\sim; \)
the morphism $\pi$ considered as a morphism $B \to Y$.

Proof. Let $z \circ f = z \circ g$ for some morphism $z$.

Let’s prove $u \circ \pi = z$ for some $u: Y \to \text{Dst} z$. Really, as a morphism of Set it exists and is unique. $\text{Src} z \in Y$. Thus $z = u \circ \pi$ for some $u$ (by properties of Set). The function $u$ is a cont$(C)$-morphism because of the lemma above. It is unique by properties of Set ($\pi$ obviously respects equivalence classes). □

8 Rest

[TODO: Specify what is $C$.]

Theorem 48. The categories cont$(C)$ (for example in Fcd and Rld) are complete.

Proof. They have products [1] and equalizers. □

Theorem 49. The categories cont$(C)$ (for example in Fcd and Rld) are co-complete.

Proof. They have co-products [1] and co-equalizers. □

Definition 50. I call morphisms $f$ and $g$ of a category with embeddings equivalent ($f \sim g$) when there exist a morphism $p$ such that $\text{Src} p \sqsubseteq \text{Src} f$, $\text{Src} p \sqsubseteq \text{Src} g$, $\text{Dst} p \sqsubseteq \text{Dst} f$, $\text{Dst} p \sqsubseteq \text{Dst} g$ and $\iota_{\text{Src}} f, \text{Dst} p f = f$ and $\iota_{\text{Src}} g, \text{Dst} g p g = g$.

Problem 51. Find under which conditions:

1. Equivalence of morphisms is an equivalence relation.
2. Equivalence of morphisms is a congruence for our category.

Bibliography
