Dual filters and ideals

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WARNING: There are errors in this draft. See instead my book.

[TODO: Define commutatitve diagrams.]

For a lattice $\mathfrak{Z}$ I denote meets and joins correspondingly as $\sqcap$ and $\sqcup$.

In my earlier work I denoted filters on a poset $\mathfrak{Z}$ as $\mathfrak{F}(\mathfrak{Z})$ (or just $\mathfrak{F}$) and corresponding principal filters as $\mathfrak{P}(\mathfrak{Z})$ (or just $\mathfrak{P}$).

I will denote $\overline{X} = \mathfrak{A} \setminus X$ for a set $X \subseteq \mathfrak{A}$.

Filters and ideals are well known concepts:

Filters $\mathfrak{F}$ are subsets F of $\mathfrak{A}$ such that:

1. F does not contain the least element of $\mathfrak{A}$ (if it exists).

2. $A \sqcap B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$).

Ideals $\mathfrak{I}$ are subsets F of $\mathfrak{A}$ such that:

1. F does not contain the greatest element of $\mathfrak{A}$ (if it exists).

2. $A \sqcup B \in F \Leftrightarrow A \in F \wedge B \in F$ (for every $A, B \in \mathfrak{Z}$).

Free stars $\mathfrak{S}$ are subsets F of $\mathfrak{A}$ such that:

1. F does not contain the least element of $\mathfrak{A}$ (if it exists).

2. $A \sqcup B \in F \Leftrightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$).

Mixers $\mathfrak{M}$ are subsets F of $\mathfrak{A}$ such that:

1. F does not contain the greatest element of $\mathfrak{A}$ (if it exists).

2. $A \sqcap B \in F \Leftrightarrow A \in F \vee B \in F$ (for every $A, B \in \mathfrak{Z}$).

Proposition 1. A set F is a lower set iff $\overline{F}$ is an upper set.

Proof. $X \in \overline{F} \wedge Z \sqsupseteq X \Rightarrow Z \in \overline{F}$ is equivalent to $Z \in F \Rightarrow X \in F \vee Z \not\sqsupseteq X$ is equivalent $Z \in F \Rightarrow (Z \sqsupseteq X \Rightarrow X \in F)$ is equivalent $Z \in F \wedge X \sqsubseteq Z \Rightarrow X \in F$. □

I will denote $\operatorname{dual} A$ where $A \in \mathfrak{Z}$ the corresponding element of the dual poset $\mathfrak{Z}^*$. Also I denote

$$\langle \operatorname{dual} \rangle X \overset{\text{def}}{=} \{ \operatorname{dual} x \mid x \in X \}.$$

Then we have the following bijections between above described four sets:

$A \sqcup B \in \overline{F} \Leftrightarrow A \in \overline{F} \vee B \in \overline{F}$ is equivalent to $\neg(A \sqcup B \in F) \Leftrightarrow \neg(A \in F \wedge B \in F)$ is equivalent to $A \sqcup B \in F \Leftrightarrow A \in F \wedge B \in F$;

$A \sqcap B \in \overline{F} \Leftrightarrow A \in \overline{F} \vee B \in \overline{F}$ is equivalent to $\neg(A \sqcap B \in F) \Leftrightarrow \neg(A \in F \wedge B \in F)$ is equivalent to $A \sqcap B \in F \Leftrightarrow A \in F \wedge B \in F$.

We have the following commutative diagram in category Set, every arrow of this diagram is an isomorphism, every cycle in this diagram is an identity:

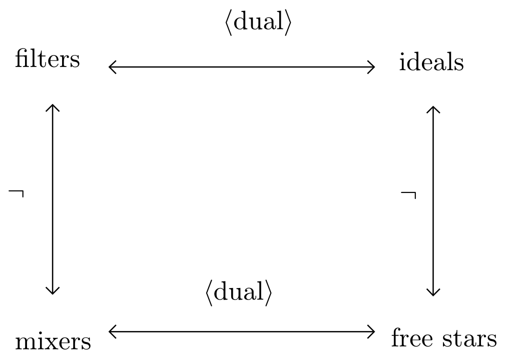

Figure 1. Diagram Υ

(where $\neg$ denotes set-theoretic complement).

These isomorphisms are also order isomorphisms if we define order in the right way.

The above it is defined for lattices only. Generalizing this for arbitrary posets is straigthforward:

Definition 2. Let $\mathfrak{A}$ be a poset.

- *Filters* are sets F without the greatest element of $\mathfrak{A}$ with $A, B \in F \Leftrightarrow \exists Z \in F: (Z \sqsubseteq A \wedge Z \sqsubseteq B)$ (for every $A, B \in \mathfrak{Z}$).

- *Ideals* are sets F without the least element of $\mathfrak{A}$ with $A, B \in F \Leftrightarrow \exists Z \in F: (Z \sqsupseteq A \wedge Z \sqsupseteq B)$ (for every $A, B \in \mathfrak{Z}$).

- *Free stars* are sets F without the greatest element of $\mathfrak{A}$ with

  $A, B \in \overline{F} \Leftrightarrow \exists Z \in \overline{F}: (Z \sqsupseteq A \wedge Z \sqsupseteq B)$

  $A \notin F \wedge B \notin F \Leftrightarrow \exists Z \in \overline{F}: (Z \sqsupseteq A \wedge Z \sqsupseteq B)$

  $A \in F \vee B \in F \Leftrightarrow \neg \exists Z \in \overline{F}: (Z \sqsupseteq A \wedge Z \sqsupseteq B)$

- *Mixers* are lower sets F without the least element of $\mathfrak{A}$ with $\neg \exists Z \in \overline{F}: (Z \sqsubseteq A \wedge Z \sqsubseteq B) \Leftrightarrow A \in F \vee B \in F$ or equivalently $\exists Z \in \overline{F}: (Z \sqsubseteq A \wedge Z \sqsubseteq B) \Leftrightarrow A \notin F \wedge B \notin F$ (for every $A, B \in \mathfrak{Z}$).

Proposition 3. The following are equivelent: [TODO: With one side implications and requirement to be upper/lower set.]

1. F is a free star.

2. $\forall Z \in \mathfrak{A}: (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{A}$ and $F \neq \mathscr{P}\mathfrak{A}$.

Proof. The following is a chain of equivalencies:

$$\exists Z \in \overline{F}: (Z \sqsupseteq A \wedge Z \sqsupseteq B) \Leftrightarrow A \notin F \wedge B \notin F;$$
$$\forall Z \in \overline{F}: \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B) \Leftrightarrow A \in F \vee B \in F;$$
$$\forall Z \in \mathfrak{A}: (Z \notin F \Rightarrow \neg(Z \sqsupseteq A \wedge Z \sqsupseteq B)) \Leftrightarrow A \in F \vee B \in F;$$
$$\forall Z \in \mathfrak{A}: (Z \sqsupseteq A \wedge Z \sqsupseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F.$$

□

Corollary 4. The following are equivelent: [TODO: With one side implications and requirement to be upper/lower set.]

1. F is a mixer.

2. $\forall Z \in \mathfrak{A}: (Z \sqsubseteq A \wedge Z \sqsubseteq B \Rightarrow Z \in F) \Leftrightarrow A \in F \vee B \in F$ for every $A, B \in \mathfrak{A}$ and F does not contain the least element of $\mathfrak{A}$.

1 General isomorphisms

Let θ be an self-inverse order reversing isomorphism of some set P of posets.

We have the following commutative diagram in category Set, every arrow of this diagram is an isomorphism, every cycle in this diagram is an identity:

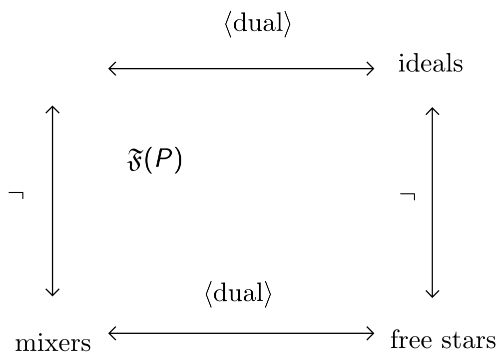

Figure 2. Diagram Υ

2 Boolean lattices

In the case if $\mathfrak{Z}$ is a boolean lattice, there is also an alternative commutative diagram in category Set, every arrow of this diagram is an isomorphism, every loop in this diagram is an identity:

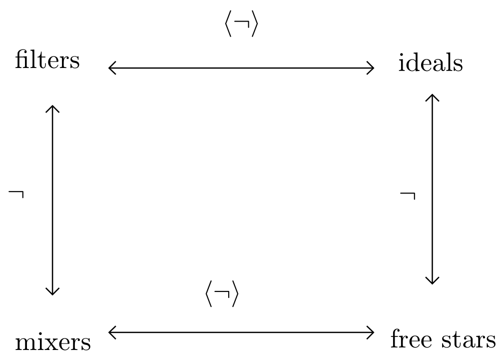

Figure 3. Diagram Ψ

(here $\langle\neg\rangle X \overset{\text{def}}{=} \{\bar{x} \mid x \in X\}$).

[TODO: Other definition where upper/lower set is explicitly said.]

[TODO: Combine two rectangular diagrams into one "cubic".]

Above we have two diagrams D with ??(every isomorphism). I will denote $A \overset{D}{\rightarrow} B$ the unique bijection from A to B which is a composition of arrows of this diagram.

[TODO: Define isomorphism of filtrators.]

Define principal ideals, free stars, mixers as objects isomorphic to principal filters. [FIXME: There are two diagrams which provide different isomorphisms!] These isomorphisms f have in common that $fa \sqsubseteq fb \Leftrightarrow a \sqsupseteq b$. We can *define* the isomorphism $\langle\text{dual}\rangle^*$ as a function such that $fa \sqsubseteq fb \Leftrightarrow a \sqsupseteq b$. It is called *order reversing isomorphism*. So first consider the general case of subsets of filters, ideals, ... with an arbitrary antitone isomorphism. Need it to be an involution?