Distributivity of composition with a principal reloid over join of reloids

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September 23, 2013

1 Introduction

It is a draft.

I present a proof of the equation $(\bigsqcup T) \circ F = \bigsqcup \{ G \circ F \mid G \in T \}$ for a principal reloid F and a set T of reloids (provided their sources and destination match each other).

First read my book [1].

1.1 Decomposition of composition of binary relations

Remark 1. Sorry for unfortunate choice of terminology: "composition" and "decomposisiton" are unrelated.

The idea of the proof below is that composition of binary relations can be decomposed into two operations: $\otimes$ and dom:

$$g \otimes f = \{ ((x; z); y) \mid x f y \wedge y g z \}.$$

Composition of binary relations can be decomposed: $g \circ f = \operatorname{dom}(g \otimes f)$.

It can be decomposed even further: $g \otimes f = \Theta_0 f \cap \Theta_1 g$ where

$$\Theta_0 f = \{ ((x; z); y) \mid x f y, z \in \mho \} \quad \text{and} \quad \Theta_1 f = \{ ((x; z); y) \mid y f z, x \in \mho \}.$$

(Here $\mho$ is the Grothendieck universe.)

Now we will do a similar trick with reloids.

1.2 Decomposition of composition of reloids

A similar thing for reloids:

$g \circ f = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} (G \circ F) \mid F \in \operatorname{GR} f, G \in \operatorname{GR} g \right\} = \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \operatorname{dom}(G \otimes F) \mid F \in \operatorname{GR} f, G \in \operatorname{GR} g \right\}$.

Lemma 2. $\{ G \otimes F \mid F \in f, G \in g \}$ is a filter base.

Proof. Let $P, Q \in \{ G \otimes F \mid F \in f, G \in g \}$. Then $P = G_0 \otimes F_0$, $Q = G_1 \otimes F_1$ for some $F_0, F_1 \in f$, $G_0, G_1 \in g$. Then $F_0 \cap F_1 \in f$, $G_0 \cap G_1 \in g$ and thus

$$P \cap Q \supseteq (F_0 \cap F_1) \otimes (G_0 \cap G_1) \in \{ G \otimes F \mid F \in f, G \in g \}. \qquad \square$$

Corollary 3. $\left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} (G \otimes F) \mid F \in \operatorname{GR} f, G \in \operatorname{GR} g \right\}$ is a generalized filter base.

Proposition 4. $g \circ f = \operatorname{dom} \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} (G \otimes F) \mid F \in \operatorname{GR} f, G \in \operatorname{GR} g \right\}$.

Proof. $\uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \operatorname{dom}(G \otimes F) \sqsupseteq \operatorname{dom} \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} (G \otimes F) \mid F \in \operatorname{GR} f, G \in \operatorname{GR} g \right\}$. Thus

$$g \circ f \sqsupseteq \operatorname{dom} \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} (G \otimes F) \mid F \in \operatorname{GR} f, G \in \operatorname{GR} g \right\}.$$

Let $X \in \operatorname{dom} \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} (G \otimes F) \mid F \in \operatorname{GR} f, G \in \operatorname{GR} g \right\}$. Then there exist Y such that $X \times Y \in \bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} (G \otimes F) \mid F \in \operatorname{GR} f, G \in \operatorname{GR} g \right\}$. So because it is a generalized filter base $X \times Y \supseteq G \otimes F$ for some $F \in \operatorname{GR} f$, $G \in \operatorname{GR} g$. Thus $X \in \operatorname{dom}(G \otimes F)$, $X \sqsupseteq \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \operatorname{dom}(G \otimes F)$, $X \in \operatorname{GR}(g \circ f)$. □

We can define $g \otimes f$ for reloids f, g:

$$g \otimes f = \{ G \otimes F \mid F \in \operatorname{GR} f, G \in \operatorname{GR} g \}.$$

Then

$$g \circ f = \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \right\rangle \langle \operatorname{dom} \rangle (g \otimes f) = \operatorname{dom} \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle (g \otimes f).$$

2 Lemmas for the main result

Let $F = \uparrow^{\mathsf{RLD}(\operatorname{Src} F; \operatorname{Dst} F)} f$ is a principal reloid.

Lemma 5. $(g \otimes f) \cup (h \otimes f) = (g \cup h) \otimes f$ for binary relations f, g, h.

Proof. $(g \cup h) \otimes f = \Theta_0\, f \cap \Theta_1\, (g \cup h) = \Theta_0\, f \cap (\Theta_1\, g \cup \Theta_1\, h) = (\Theta_0\, f \cap \Theta_1\, g) \cup (\Theta_0\, f \cap \Theta_1\, h) = (g \otimes f) \cup (h \otimes f)$. □

Lemma 6. $\bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} (G \otimes f) \mid G \in \bigsqcup T \right\} = \bigsqcup \left\{ \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle (G \otimes F) \mid G \in T \right\}$.

Proof. $\bigsqcap \left\{ \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} (G \otimes f) \mid G \in \operatorname{GR} \bigsqcup T \right\} \sqsupseteq \bigsqcup \left\{ \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle (G \otimes F) \mid G \in T \right\}$ is obvious.

Let $K \in \bigsqcup \left\{ \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle (G \otimes F) \mid G \in T \right\}$. Then for each $G \in T$

$$K \in \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle (G \otimes F);$$

$K \in \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle \{ \Gamma \otimes f \mid \Gamma \in \operatorname{GR} G \}$.

$K \in \{ (\Gamma_0 \otimes f) \cap \ldots \cap (\Gamma_n \otimes f) \mid n \in \mathbb{N}, \Gamma_i \in \operatorname{GR} G \}$.

$\forall G \in T \colon K \supseteq (\Gamma_{G,0} \otimes f) \cap \ldots \cap (\Gamma_{G,n} \otimes f)$ for some $n \in \mathbb{N}, \Gamma_{G,i} \in G$.

Let $G \in \bigcap T$.

$K \supseteq (\Gamma_0 \otimes f) \cap \ldots \cap (\Gamma_n \otimes f)$ where $\Gamma_i = \bigcup_{g \in G} \Gamma_{g,i} \in \operatorname{GR} \bigsqcup T$.

$K \in \{ (\Gamma_0 \otimes f) \cap \ldots \cap (\Gamma_n \otimes f) \mid n \in \mathbb{N} \}$.

So $K \in \{ (\Gamma'_0 \otimes f) \cap \ldots \cap (\Gamma'_n \otimes f) \mid n \in \mathbb{N}, \Gamma'_i \in \operatorname{GR} \bigsqcup T \} = \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle \{ G \otimes f \mid G \in \operatorname{GR} \bigsqcup T \}$. □

3 Proof of the main result

Theorem 7. $(\bigsqcup T) \circ F = \bigsqcup \{ G \circ F \mid G \in T \}$ for every principal reloid $F = \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} f$.

Proof.

$$\begin{aligned} \left(\bigsqcup T \right) \circ F &= \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \right\rangle \langle \operatorname{dom} \rangle \left(\left(\bigsqcup T \right) \otimes F \right) \\ &= \operatorname{dom} \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle \left(\left(\bigsqcup T \right) \otimes F \right) \\ &= \operatorname{dom} \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle \left\{ G \otimes f \mid G \in \operatorname{GR} \bigsqcup T \right\} \end{aligned}$$

$$\begin{aligned} \bigsqcup \{ G \circ F \mid G \in T \} &= \bigsqcup \left\{ \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f; \operatorname{Dst} g)} \right\rangle \langle \operatorname{dom} \rangle (G \otimes F) \mid G \in T \right\} \\ &= \bigsqcup \left\{ \operatorname{dom} \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle (G \otimes F) \mid G \in T \right\} \\ &= \operatorname{dom} \bigsqcup \left\{ \bigsqcap \left\langle \uparrow^{\mathsf{RLD}(\operatorname{Src} f \times \operatorname{Dst} g; \mho)} \right\rangle (G \otimes F) \mid G \in T \right\}. \end{aligned}$$

It's enough to prove

$$\bigsqcap\left\{\uparrow^{\mathsf{RLD}(\mathrm{Src}\, f\times\mathrm{Dst}\, g;\mho)}(G\otimes f)\mid G\in\mathrm{GR}\bigsqcup T\right\}=\bigsqcup\left\{\bigsqcap\left\langle\uparrow^{\mathsf{RLD}(\mathrm{Src}\, f\times\mathrm{Dst}\, g;\mho)}\right\rangle(G\otimes F)\mid G\in T\right\}$$

but this is the statement of the lemma. □

4 Embedding reloids into funcoids

Definition 8. Let f is a reloid. The funcoid $\rho f\in\mathsf{FCD}(\mathrm{Src}\, f;\mathrm{Dst}\, f)$ is defined by the formulas:

$$\langle\rho f\rangle x=f\circ x\quad\text{and}\quad\langle\rho f^{-1}\rangle y=f^{-1}\circ y$$

where x are endo-reloids on $\mathrm{Src}\, f$ and y are endo-reloids on $\mathrm{Dst}\, f$.

Proposition 9. It is really a funcoid (if we equate reloids x and y with corresponding filters on cartesian products of sets).

Proof. $y\not\asymp\langle\rho f\rangle x\Leftrightarrow y\not\asymp f\circ x\Leftrightarrow f^{-1}\circ y\not\asymp x\Leftrightarrow\langle\rho f^{-1}\rangle y\not\asymp x$. □

Corollary 10. $(\rho f)^{-1}=\rho f^{-1}$.

Definition 11. It can be continued to arbitrary funcoids x having source $\mathrm{Src}\, f$ by the formula $\langle\rho^* f\rangle x=\langle\rho f\rangle\mathrm{id}_{\mathrm{Src}\, f}\circ x$.

Proposition 12. ρ is an injection.

Proof. Consider $x=\mathrm{id}_{\mathrm{Src}\, f}$. □

Proposition 13. $\rho(g\circ f)=(\rho g)\circ(\rho f)$.

Proof. $\langle\rho(g\circ f)\rangle x=g\circ f\circ x=\langle\rho g\rangle\langle\rho f\rangle x=(\langle\rho g\rangle\circ\langle\rho f\rangle)x$. Thus $\langle\rho(g\circ f)\rangle=\langle\rho g\rangle\circ\langle\rho f\rangle=\langle(\rho g)\circ(\rho f)\rangle$ and so $\rho(g\circ f)=(\rho g)\circ(\rho f)$. □

Theorem 14. $\rho\bigsqcup F=\bigsqcup\langle\rho\rangle F$ for a set F of reloids.

Proof. It's enough to prove $\langle\rho\bigsqcup F\rangle^* X=\langle\bigsqcup\langle\rho\rangle F\rangle^* X$ for a set X.

Really, $\langle\rho\bigsqcup F\rangle^* X=\langle\rho\bigsqcup F\rangle\uparrow X=\bigsqcup F\circ\uparrow X=\bigsqcup\{f\circ\uparrow X\mid f\in F\}=\bigsqcup\{\langle\rho f\rangle\uparrow X\mid f\in F\}=\langle\bigsqcup\{\rho f\mid f\in F\}\rangle\uparrow X=\langle\bigsqcup\langle\rho\rangle F\rangle^* X$. □

Conjecture 15. $\rho\bigsqcap F=\bigsqcap\langle\rho\rangle F$ for a set F of reloids.

Proposition 16. $\rho\mathrm{id}^{\mathsf{RLD}(A)}=\mathrm{id}^{\mathsf{FCD}(A)}$.

Proof. $\left\langle\rho\mathrm{id}^{\mathsf{RLD}(A)}\right\rangle x=\mathrm{id}^{\mathsf{RLD}(A)}\circ x=x$. □

We can try to develop further theory by applying embedding of reloids into funcoids for researching of properties of reloids.

Theorem 17. Reloid f is monovalued iff funcoid ρf is monovalued.

Proof. ρf is monovalued$\Leftrightarrow(\rho f)\circ(\rho f)^{-1}\sqsubseteq 1_{\mathrm{Dst}\,\rho f}\Leftrightarrow\rho(f\circ f^{-1})\sqsubseteq 1_{\mathrm{Dst}\, f}\Leftrightarrow\rho(f\circ f^{-1})\sqsubseteq\mathrm{id}^{\mathsf{RLD}(A)}\Leftrightarrow\rho(f\circ f^{-1})\sqsubseteq\rho\mathrm{id}^{\mathsf{FCD}(A)}\Leftrightarrow f\circ f^{-1}\sqsubseteq\mathrm{id}^{\mathsf{FCD}(A)}\Leftrightarrow f$ is monovalued. □

Bibliography

[1] Victor Porton. *Algebraic General Topology. Volume 1*. 2013.