This article presents another version of star-composition of funcoids.

Let a be a an anchored relation of the form $\mathfrak{A}$ and $\operatorname{dom}\mathfrak{A} = n$.

Let every f_i (for all $i \in n$) be a pointfree funcoid with $\operatorname{Src} f_i = \mathfrak{A}_i$.

The star-composition of a with f is an anchored relation of the form $\lambda i \in \operatorname{dom}\mathfrak{A}: \operatorname{Dst} f_i$ defined by the formula [TODO: it should have a distinct notation with the other star-composition of staroids]

$$L \in \operatorname{GR}\operatorname{StarComp}(a; f) \Leftrightarrow (\lambda i \in n: \langle f_i{}^{-1}\rangle L_i) \in \operatorname{GR} a.$$

Theorem 1. Let $\operatorname{Src} f_i$ be separable join-semilattice and $\operatorname{Dst} f_i$ be a starrish join-semilattice for every $i \in n$ for a set n. Let $\operatorname{form} a = \prod_{i \in n} (\operatorname{Src} f_i)$

1. If a is a prestaroid then $\operatorname{StarComp}(a; f)$ is a prestaroid.
2. If a is a staroid then $\operatorname{StarComp}(a; f)$ is a staroid.
3. If a is a completary staroid and then $\operatorname{StarComp}(a; f)$ is a completary staroid.

Proof. We have $\langle f_i{}^{-1}\rangle(X \sqcup Y) = \langle f_i{}^{-1}\rangle X \sqcup \langle f_i{}^{-1}\rangle Y$.

1. Let $L \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} (\operatorname{form} f_i)$ for some $k \in n$ and $X, Y \in \operatorname{form} f_k$. Then

$$\begin{aligned}
X \sqcup Y \in \langle \operatorname{StarComp}(a; f)\rangle_k L \quad&\Leftrightarrow\\
\left(\lambda i \in \operatorname{dom} f: \langle f_i{}^{-1}\rangle\left(\left\{\begin{array}{ll} X \sqcup Y & \text{if } i = k\\ L_i & \text{if } i \neq k \end{array}\right)_i\right) \in \operatorname{GR} a \quad&\Leftrightarrow\\
\left(\lambda i \in \operatorname{dom} f: \left(\left\{\begin{array}{ll} \langle f_i{}^{-1}\rangle X \sqcup \langle f_i{}^{-1}\rangle Y & \text{if } i = k\\ \langle f_i{}^{-1}\rangle L_i & \text{if } i \neq k \end{array}\right)_i\right) \in \operatorname{GR} a \quad&\Leftrightarrow\\
\langle f_i{}^{-1}\rangle X \sqcup \langle f_i{}^{-1}\rangle Y \in \langle a\rangle_k(\lambda i \in (\operatorname{dom} f) \setminus \{k\}: \langle f_i{}^{-1}\rangle L_i) \quad&\Leftrightarrow\\
\langle f_i{}^{-1}\rangle X \in \langle a\rangle_k(\lambda i \in n \setminus \{k\}: \langle f_i{}^{-1}\rangle L_i) \vee \langle f_i{}^{-1}\rangle Y \in \langle a\rangle_k(\lambda i \in n \setminus \{k\}: \langle f_i{}^{-1}\rangle L_i) \quad&\Leftrightarrow\\
\left(\lambda i \in \operatorname{dom} f: \left(\left\{\begin{array}{ll} \langle f_i{}^{-1}\rangle X & \text{if } i = k\\ \langle f_i{}^{-1}\rangle L_i & \text{if } i \neq k \end{array}\right)_i\right) \in \operatorname{GR} a \vee \left(\lambda i \in \operatorname{dom} f: \left(\left\{\begin{array}{ll} \langle f_i{}^{-1}\rangle Y & \text{if } i = k\\ \langle f_i{}^{-1}\rangle L_i & \text{if } i \neq k \end{array}\right)_i\right) \in \operatorname{GR} a\right. \quad&\Leftrightarrow\\
\left(\lambda i \in \operatorname{dom} f: \langle f_i{}^{-1}\rangle\left(\left\{\begin{array}{ll} X & \text{if } i = k\\ L_i & \text{if } i \neq k \end{array}\right)_i\right) \in \operatorname{GR} a \vee \left(\lambda i \in \operatorname{dom} f: \langle f_i{}^{-1}\rangle\left(\left\{\begin{array}{ll} Y & \text{if } i = k\\ L_i & \text{if } i \neq k \end{array}\right)_i\right) \in \operatorname{GR} a\right. \quad&\Leftrightarrow\\
X \in \langle \operatorname{StarComp}(a; f)\rangle_k L \vee Y \in \langle \operatorname{StarComp}(a; f)\rangle_k L.&
\end{aligned}$$

   Thus $\operatorname{StarComp}(a; f)$ is a pre-staroid.

2. $\langle f_i\rangle$ are monotone functions by the proposition 15.14. Thus $\langle f_i{}^{-1}\rangle X_i \sqsubseteq \langle f_i{}^{-1}\rangle Y_i$ if X, $Y \in \prod_{i \in (\operatorname{arity} f) \setminus \{k\}} (\operatorname{form} f_i)$ and $X \sqsubseteq Y$. So if a is a staroid and $X \in \operatorname{GR}\operatorname{StarComp}(a; f)$ then $(\lambda i \in \operatorname{dom} f: \langle f_i{}^{-1}\rangle X_i) \in \operatorname{GR} a$ then $(\lambda i \in \operatorname{dom} f: \langle f_i{}^{-1}\rangle Y_i) \in \operatorname{GR} a$ that is $Y \in \operatorname{GR}\operatorname{StarComp}(a; f)$.

3.

$$\begin{aligned}
L_0 \sqcup L_1 \in \operatorname{GR}\operatorname{StarComp}(a; f) \quad&\Leftrightarrow\\
(\lambda i \in n: \langle f_i{}^{-1}\rangle (L_0 \sqcup L_1) i) \in \operatorname{GR} a \quad&\Leftrightarrow\\
(\lambda i \in n: \langle f_i{}^{-1}\rangle L_0 i \sqcup \langle f_i{}^{-1}\rangle L_1 i) \in \operatorname{GR} a \quad&\Leftrightarrow\\
\exists c \in \{0, 1\}: (\lambda i \in n: \langle f_i{}^{-1}\rangle L_{c(i)} i) \in \operatorname{GR} a \quad&\Leftrightarrow\\
\exists c \in \{0, 1\}: (\lambda i \in n: L_{c(i)} i) \in \operatorname{GR}\operatorname{StarComp}(a; f).&
\end{aligned}$$

□

Conjecture 2. $b \not\asymp^{\mathrm{Anch}(\mathfrak{A})} \mathrm{StarComp}(a; f) \Leftrightarrow \forall A \in \mathrm{GR}\, a, B \in \mathrm{GR}\, b, i \in n: A_i\, [f_i]\, B_i$ for anchored relations a and b on powersets.

It's conequence:

Conjecture 3. $b \not\asymp^{\mathrm{Anch}(\mathfrak{A})} \mathrm{StarComp}(a; f) \Leftrightarrow a \not\asymp^{\mathrm{Anch}(\mathfrak{A})} \mathrm{StarComp}(b; f^{\dagger})$ for anchored relations a and b on powersets.

Conjecture 4. $b \not\asymp^{\mathsf{pStrd}(\mathfrak{A})} \mathrm{StarComp}(a; f) \Leftrightarrow a \not\asymp^{\mathsf{pStrd}(\mathfrak{A})} \mathrm{StarComp}(b; f^{\dagger})$ for pre-staroids a and b on powersets.

Proposition 5. Anchored relations with objects being posets with above defined star-morphisms is a category with star morphisms.

Proof. We need to prove:

1. $\mathrm{StarComp}(\mathrm{StarComp}(m; f); g) = \mathrm{StarComp}(m; \lambda i \in \mathrm{arity}\, m: g_i \circ f_i)$;

2. $\mathrm{StarComp}(m; \lambda i \in \mathrm{arity}\, m: \mathrm{id}_{\mathrm{Obj}_m i}) = m$.

(the rest is obvious).

Really, $L \in \mathrm{GR}\, \mathrm{StarComp}(\mathrm{StarComp}(m; f); g) \Leftrightarrow (\lambda i \in \mathrm{arity}\, m: \langle g_i{}^{-1} \rangle L_i) \in \mathrm{GR}\, \mathrm{StarComp}(m; f) \Leftrightarrow (\lambda i \in n: \langle f_i{}^{-1} \rangle (\lambda i \in n: \langle g_i{}^{-1} \rangle L_i)_i) \in \mathrm{GR}\, m \Leftrightarrow (\lambda i \in \mathrm{arity}\, m: \langle f_i{}^{-1} \rangle \langle g_i{}^{-1} \rangle L_i) \in \mathrm{GR}\, m \Leftrightarrow (\lambda i \in \mathrm{arity}\, m: \langle (g_i \circ f_i)^{-1} \rangle L_i) \in \mathrm{GR}\, m \Leftrightarrow L \in \mathrm{GR}\, \mathrm{StarComp}(m; \lambda i \in \mathrm{arity}\, m: g_i \circ f_i)$;

$L \in \mathrm{GR}\, \mathrm{StarComp}(m; \lambda i \in \mathrm{arity}\, m: \mathrm{id}_{\mathrm{Obj}_m i}) \Leftrightarrow (\lambda i \in n: \langle \mathrm{id}_{\mathrm{Obj}_m i} \rangle L_i) \in \mathrm{GR}\, m \Leftrightarrow (\lambda i \in \mathrm{arity}\, m: \langle \mathrm{id}_{\mathrm{Obj}_m i} \rangle L_i) \in \mathrm{GR}\, m \Leftrightarrow (\lambda i \in \mathrm{arity}\, m: L_i) \in \mathrm{GR}\, m \Leftrightarrow L \in \mathrm{GR}\, m$. □

Conjecture 6. $\mathrm{StarComp}(a \sqcup b; f) = \mathrm{StarComp}(a; f) \sqcup \mathrm{StarComp}(b; f)$ for anchored relations a, b of a form $\mathfrak{A}$, where every $\mathfrak{A}_i$ is a distributive lattice, and an indexed family f of pointfree funcoids with $\mathrm{Src}\, f_i = \mathfrak{A}_i$.

[TODO: Put conjectures from this article to agt-open-problems.pdf]