

Generalized Continuity*

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Abstract

It is a part of my Algebraic General Topology research.

The concept of continuity generalized for arbitrary morphisms of a partially ordered category. In particular considered continuity of functors and retracts, what is a generalization of near-continuity and uniform continuity correspondingly.

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1 Intro

1.1 Draft status

This article is a draft.

1.2 Prerequisites

For understanding the concepts of functors and retracts prior reading of [1] is necessary.

For understanding the rest a concept of a *partially ordered dagger category* [2] is necessary.

2 Continuous morphisms

2.1 Traditional definitions of continuity

2.1.1 Pre-topology

Let μ and ν are functors representing some pre-topologies. By definition a function f is continuous map from μ to ν in point a iff

$$\forall \epsilon \in \text{up}\langle \nu \rangle f a \exists \delta \in \text{up}\langle \mu \rangle \{a\} : \langle f \rangle \delta \subseteq \epsilon.$$

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Equivalently transforming this formula we get:

$$\begin{aligned} \forall \epsilon \in \text{up} \langle \nu \rangle f a: \langle f \rangle \langle \mu \rangle \{a\} \subseteq \epsilon; \\ \langle f \rangle \langle \mu \rangle \{a\} \subseteq \langle \nu \rangle f a; \\ \langle f \rangle \langle \mu \rangle \{a\} \subseteq \langle \nu \rangle \langle f \rangle \{a\}. \end{aligned}$$

f is a continuous map from μ to ν in every point of its domain iff $\langle f \rangle \langle \mu \rangle \subseteq \langle \nu \rangle \langle f \rangle$ what is equivalent to $f \circ \mu \subseteq \nu \circ f$.

2.1.2 Proximity spaces

Let μ and ν are proximity (nearness) spaces (which I consider a special case of funcoids). By definition a function f is a nearness-continuous map from μ to ν iff

$$\forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow (\langle f \rangle X)[\nu](\langle f \rangle Y)).$$

Equivalently transforming this formula we get:

$$\begin{aligned} \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow \langle f \rangle Y \cap \langle \nu \rangle \langle f \rangle X \neq \emptyset); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow \langle f \rangle Y \cap \langle \nu \circ f \rangle X \neq \emptyset); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow X[\nu \circ f] \langle f \rangle Y); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow \langle f \rangle Y[(\nu \circ f)^{-1}]X); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow \langle f \rangle Y[f^{-1} \circ \nu^{-1}]X); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow X \cap \langle f^{-1} \circ \nu^{-1} \rangle \langle f \rangle Y \neq \emptyset); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow X \cap \langle f^{-1} \circ \nu^{-1} \circ f \rangle Y \neq \emptyset); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow Y[f^{-1} \circ \nu^{-1} \circ f]X); \\ \forall X, Y \in \mathcal{P}\mathcal{U}: (X[\mu]Y \Rightarrow X[f^{-1} \circ \nu \circ f]Y); \\ \mu \subseteq f^{-1} \circ \nu \circ f. \end{aligned}$$

So a function f is nearness-continuous iff $\mu \subseteq f^{-1} \circ \nu \circ f$.

2.1.3 Uniform spaces

Uniform spaces are a special case of reloids.

Let μ and ν are uniform spaces. By definition a function f is a uniformly continuous map from μ to ν iff

$$\forall \epsilon \in \text{up} \nu \exists \delta \in \text{up} \mu \forall (x; y) \in \delta: (fx; fy) \in \epsilon.$$

Equivalently transforming this formula we get:

$$\begin{aligned} \forall \epsilon \in \text{up} \nu \exists \delta \in \text{up} \mu \forall (x; y) \in \delta: \{(fx; fy)\} \subseteq \epsilon \\ \forall \epsilon \in \text{up} \nu \exists \delta \in \text{up} \mu \forall (x; y) \in \delta: f \circ \{(x; y)\} \circ f^{-1} \subseteq \epsilon \\ \forall \epsilon \in \text{up} \nu \exists \delta \in \text{up} \mu: f \circ \delta \circ f^{-1} \subseteq \epsilon \\ \forall \epsilon \in \text{up} \nu: f \circ \mu \circ f^{-1} \subseteq \epsilon \\ f \circ \mu \circ f^{-1} \subseteq \nu. \end{aligned}$$

So a function f is uniformly continuous iff $f \circ \mu \circ f^{-1} \subseteq \nu$.

2.2 Our three definitions of continuity

I have expressed different kinds of continuity with simple algebraic formulas hiding the complexity of traditional epsilon-delta notation behind a smart algebra. Let's summarize these three algebraic formulas:

Let μ and ν are endomorphisms of some partially ordered precategory. Continuous functions can be defined as these morphisms f of this precategory which conform to the following formula:

$$f \in C(\mu; \nu) \Leftrightarrow f \in \text{Mor}(\text{Ob } \mu; \text{Ob } \nu) \wedge f \circ \mu \subseteq \nu \circ f.$$

If the precategory is a partially ordered dagger precategory (see [2]) then continuity also can be defined in two other ways:

$$\begin{aligned} f \in C'(\mu; \nu) &\Leftrightarrow f \in \text{Mor}(\text{Ob } \mu; \text{Ob } \nu) \wedge \mu \subseteq f^\dagger \circ \nu \circ f; \\ f \in C''(\mu; \nu) &\Leftrightarrow f \in \text{Mor}(\text{Ob } \mu; \text{Ob } \nu) \wedge f \circ \mu \circ f^\dagger \subseteq \nu. \end{aligned}$$

Remark 1. In the examples about funcoids and reloids the “dagger functor” is the inverse of a funcoid or reloid, that is $f^\dagger = f^{-1}$.

Proposition 2. Every of these three definitions of continuity forms a sub-precategory (subcategory if the original precategory is a category).

Proof.

C. Let $f \in C(\mu; \nu)$, $g \in C(\nu; \pi)$. Then $f \circ \mu \subseteq \nu \circ f$, $g \circ \nu \subseteq \pi \circ g$; $g \circ f \circ \mu \subseteq g \circ \nu \circ f \subseteq \pi \circ g \circ f$. So $g \circ f \in C(\mu; \pi)$. $1_{\text{Ob } \mu} \in C(\mu; \mu)$ is obvious.

C'. Let $f \in C'(\mu; \nu)$, $g \in C'(\nu; \pi)$. Then $\mu \subseteq f^\dagger \circ \nu \circ f$, $\nu \subseteq g^\dagger \circ \pi \circ g$;

$$\mu \subseteq f^\dagger \circ g^\dagger \circ \pi \circ g \circ f; \quad \mu \subseteq (g \circ f)^\dagger \circ \pi \circ (g \circ f).$$

So $g \circ f \in C'(\mu; \pi)$. $1_{\text{Ob } \mu} \in C'(\mu; \mu)$ is obvious.

C''. Let $f \in C''(\mu; \nu)$, $g \in C''(\nu; \pi)$. Then $f \circ \mu \circ f^\dagger \subseteq \nu$, $g \circ \nu \circ g^\dagger \subseteq \pi$;

$$g \circ f \circ \mu \circ f^\dagger \circ g^\dagger \subseteq \pi; \quad (g \circ f) \circ \mu \circ (g \circ f)^\dagger \subseteq \pi.$$

So $g \circ f \in C''(\mu; \pi)$. $1_{\text{Ob } \mu} \in C''(\mu; \mu)$ is obvious. \square

Proposition 3. For a monovalued morphism (see [2]) f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in C'(\mu; \nu) \Rightarrow f \in C(\mu; \nu) \Rightarrow f \in C''(\mu; \nu).$$

Proof. Let $f \in C'(\mu; \nu)$. Then $\mu \subseteq f^\dagger \circ \nu \circ f$; $f \circ \mu \subseteq f \circ f^\dagger \circ \nu \circ f \subseteq 1_{\text{Dst } f} \circ \nu \circ f = \nu \circ f$; $f \in C(\mu; \nu)$.

Let $f \in C(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f \circ \mu \circ f^\dagger \subseteq \nu \circ f \circ f^\dagger \subseteq \nu \circ 1_{\text{Dst } f} = \nu$; $f \in C''(\mu; \nu)$. \square

Proposition 4. For an entirely defined morphism (see [2]) f of a partially ordered dagger category and its endomorphisms μ and ν

$$f \in C''(\mu; \nu) \Rightarrow f \in C(\mu; \nu) \Rightarrow f \in C'(\mu; \nu).$$

Proof. Let $f \in C''(\mu; \nu)$. Then $f \circ \mu \circ f^\dagger \subseteq \nu$; $f \circ \mu \circ f^\dagger \circ f \subseteq \nu \circ f$; $f \circ \mu \circ 1_{\text{Src } f} \subseteq \nu \circ f$; $f \circ \mu \subseteq \nu \circ f$; $f \in C(\mu; \nu)$.

Let $f \in C(\mu; \nu)$. Then $f \circ \mu \subseteq \nu \circ f$; $f^\dagger \circ f \circ \mu \subseteq f^\dagger \circ \nu \circ f$; $1_{\text{Src } f} \circ \mu \subseteq f^\dagger \circ \nu \circ f$; $\mu \subseteq f^\dagger \circ \nu \circ f$; $f \in C'(\mu; \nu)$. \square

For entirely defined monovalued morphisms our three definitions of continuity coincide:

Theorem 5. If f is a monovalued and entirely defined morphism then

$$f \in C'(\mu; \nu) \Leftrightarrow f \in C(\mu; \nu) \Leftrightarrow f \in C''(\mu; \nu).$$

Proof. From two previous propositions. \square

The classical general topology theorem that uniformly continuous function from a uniform space to an other uniform space is near-continuous regarding the proximities generated by the uniformities, generalized for reloids and funcoids takes the following form:

Theorem 6. If an entirely defined morphism of the category of reloids $f \in C''(\mu; \nu)$ for some endomorphisms μ and ν of the category of reloids, then $(\text{FCD})f \in C'((\text{FCD})\mu; (\text{FCD})\nu)$.

Exercise 1. I leave a simple exercise for the reader to prove the last theorem.

3 Continuousness of a restricted morphism

Consider some partially ordered semigroup. (For example it can be the semigroup of funcoids or semigroup of reloids.) Consider also some lattice (*lattice of objects*). (For example take the lattice of set theoretic filters.)

We will map every object A to *identity element* I_A of the semigroup (for example identity funcoid or identity reloid). For identity elements we will require

1. $I_A \circ I_B = I_{A \cap B}$;
2. $f \circ I_A \subseteq f$; $I_A \circ f \subseteq f$.

In the case when our semigroup is “dagger” (that is is a dagger precategory) we will require also $(I_A)^\dagger = I_A$.

We can define *restricting* an element f of our semigroup to an object A by the formula $f|_A = f \circ I_A$.

We can define *rectangular restricting* an element μ of our semigroup to objects A and B as $I_B \circ \mu \circ I_A$. Optionally we can define direct product $A \times B$ of two objects by the formula (true for funcoids and for reloids):

$$\mu \cap (A \times B) = I_B \circ \mu \circ I_A.$$

Square restricting of an element μ to an object A is a special case of rectangular restricting and is defined by the formula $I_A \circ \mu \circ I_A$ (or by the formula $\mu \cap (A \times A)$).

Theorem 7. For any elements f, μ, ν of our semigroup and an object A

1. $f \in C(\mu; \nu) \Rightarrow f|_A \in C(I_A \circ \mu \circ I_A; \nu)$;
2. $f \in C'(\mu; \nu) \Rightarrow f|_A \in C'(I_A \circ \mu \circ I_A; \nu)$;
3. $f \in C''(\mu; \nu) \Rightarrow f|_A \in C''(I_A \circ \mu \circ I_A; \nu)$.

(Two last items are true for the case when our semigroup is with inverses.)

Proof.

1. $f|_A \in C(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f|_A \Leftrightarrow f \circ I_A \circ \mu \circ I_A \subseteq \nu \circ f \circ I_A \Leftrightarrow f \circ I_A \circ \mu \subseteq \nu \circ f \Leftrightarrow f \circ \mu \subseteq \nu \circ f \Leftrightarrow f \in C(\mu; \nu)$.
2. $f|_A \in C'(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow I_A \circ \mu \circ I_A \subseteq (f|_A)^\dagger \circ \nu \circ f|_A \Leftrightarrow I_A \circ \mu \circ I_A \subseteq (f \circ I_A)^\dagger \circ \nu \circ f \circ I_A \Leftrightarrow I_A \circ \mu \circ I_A \subseteq I_A \circ f^\dagger \circ \nu \circ f \circ I_A \Leftrightarrow \mu \subseteq f^\dagger \circ \nu \circ f \Leftrightarrow f \in C'(\mu; \nu)$.
3. $f|_A \in C''(I_A \circ \mu \circ I_A; \nu) \Leftrightarrow f|_A \circ I_A \circ \mu \circ I_A \circ (f|_A)^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ I_A \circ \mu \circ I_A \circ I_A \circ f^\dagger \subseteq \nu \Leftrightarrow f \circ I_A \circ \mu \circ I_A \circ f^\dagger \subseteq \nu \Leftrightarrow f \circ \mu \circ f^\dagger \subseteq \nu \Leftrightarrow f \in C''(\mu; \nu)$. \square

Bibliography

- [1] Victor Porton. Funcoids and reloids. At <http://www.mathematics21.org/binaries/funcoids-reloids.pdf>.
- [2] Victor Porton. Partially ordered dagger categories. At <http://www.mathematics21.org/binaries/ordered-cats-with-inv.pdf>.