Connectors and generalized connectedness

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Abstract

I define **connectors** and **generalized connectedness** which generalizes topological connectedness, path connectedness, connectedness of digraphs, proximal connectedness, uniform connectedness, and some other kinds of connectedness. This article also serves as a simple introduction for my future writings where I will consider more difficult topic of filters connected regarding funcoids and reloids.

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1. Related works

In [4] is researched an other way to generalize connectedness. Below is remarked how these two ways are connected.

2. Notation

I will denote $\langle f\rangle X = \{f(x) \mid x \in X\}$ for every function f and set X.

$X\,[f]\,Y \Leftrightarrow \exists x \in X, y \in Y : x\ f\ y \Leftrightarrow (X \times Y) \cap f \neq \emptyset$ for every binary relation f and sets X and Y.

3. Main definition

Let U is a set.

Definition 1 *I will call a* ***connector*** *a binary relation $r \in \mathscr{P}(\mathscr{P}U \times \mathscr{P}U)$ for some set U. The* ***connector space*** *is the pair $(U;r)$. I will call U the* ***base*** *of the connector space $(U;r)$.*

I will denote $A\ (U;r)\ B \Leftrightarrow A\ r\ B$ for every sets A and B.

Definition 2 *Let r is a connector. I call a set A* ***connected*** *(regarding r) when*

$$\forall X, Y \in \mathscr{P}A \setminus \{\emptyset\} : (X \cup Y = A \wedge X \cap Y = \emptyset \Rightarrow X\ r\ Y). \tag{1}$$

I will call ***connectedness*** *the set of connected sets (regarding some connector r). I will denote*

$$\mathrm{CC}(U;r) = \{A \in \mathscr{P}U \mid A \text{ is connected regarding } r\}$$

the connectedness regarding the connector space $(U;r)$. ("CC" is deciphered as "connector connectedness".)

A set is connected regarding a connector space $(U;r)$ iff it is connected regarding the connector r.

Intuitively: A set is connected if for every partition of it into two components these two components are bound with each other ("to be bound" mean to be related by the relation r).

I will call the above formula **generalized connectedness**.

Definition 3 ***Normalized*** *connector space is such a connector space* $(U; r)$ *that*

$\forall X, Y \in \mathscr{P}U : (X = \emptyset \vee Y = \emptyset \Rightarrow \neg(X \mathrel{r} Y))$ and $\forall X, Y \in \mathscr{P}U : (X \cap Y \neq \emptyset \Rightarrow X \mathrel{r} Y)$.

Definition 4 ***Normalization*** *of a connector space* $(U; r)$ *is the connector* $N(U; r) = (U; r')$ *defined by the formula (for every* $X, Y \in \mathscr{P}U$*)*

$$X \mathrel{r'} Y \Leftrightarrow \begin{cases} 0 & \text{if } X = \emptyset \vee Y = \emptyset, \\ 1 & \text{if } X \cap Y \neq \emptyset, \\ X \mathrel{r} Y & \text{otherwise}. \end{cases}$$

Obvious 1 *Normalization of a connector space is a normalized connector space.*

Obvious 2 *A set is connected regarding a connector space iff it is connected regarding its normalization.*

Obvious 3 *For a normalized connector* r *a set* A *is connected iff*

$$\forall X, Y \in \mathscr{P}A \setminus \{\emptyset\} : (X \cup Y = A \Rightarrow X \mathrel{r} Y).$$

Definition 5

- ***Restriction*** $r|_A$ *of a connector* r *to a set* A *is the connector* $r \cap (\mathscr{P}A \times \mathscr{P}A)$.
- ***Restriction*** $(U; r)|_A$ *of a connector space* $(U; r)$ *to a set* $A \in \mathscr{P}U$ *is the connector space* $(A; r \cap (\mathscr{P}A \times \mathscr{P}A))$.

Theorem 1 $\mathrm{CC}((U; r)|_K) = \mathrm{CC}(U; r) \cap \mathscr{P}K$ *for every set* $K \in \mathscr{P}U$.

Proof $A \in \mathrm{CC}((U; r)|_K) \Leftrightarrow A \subseteq K \wedge \forall X, Y \in \mathscr{P}A \setminus \{\emptyset\} : (X \cup Y = A \wedge X \cap Y = \emptyset \Rightarrow X \mathrel{(r \cap (\mathscr{P}K \times \mathscr{P}K))} Y) \Leftrightarrow A \subseteq K \wedge \forall X, Y \in \mathscr{P}A \setminus \{\emptyset\} : (X \cup Y = A \wedge X \cap Y = \emptyset \Rightarrow X \mathrel{r} Y) \Leftrightarrow A \subseteq K \wedge A \in \mathrm{CC}(U; r) \Leftrightarrow A \in \mathrm{CC}(U; r) \cap \mathscr{P}K$ for every set A. □

Corollary 1 $\mathrm{CC}((U; r)|_K) \subseteq \mathrm{CC}(U; r)$.

I will define an order on every set of connectors with the same base by the formula

$$(U; r_0) \subseteq (U; r_1) \Leftrightarrow r_0 \subseteq r_1.$$

4. Examples of connectedness

4.1. Topological connectedness

Let $\mathfrak{A}$ is a topological space. If we take

$$X \; r \; Y \Leftrightarrow \text{(X is not open or Y is not open)}$$

or

$$X \; r \; Y \Leftrightarrow \text{(X is not closed or Y is not closed)}$$

or

$$X \; r \; Y \Leftrightarrow \mathrm{cl}_{X \cup Y}(X) \cap Y \neq \emptyset \vee \mathrm{cl}_{X \cup Y}(Y) \cap X \neq \emptyset \tag{2}$$

where openness and closedness is taken on the space $\mathfrak{A}$ restricted to the set $X \cup Y$ and cl_A means the closure on the subspace A, then we get the classical definition of a set connected regarding a topology.

Observe that there are several connectors which define the same connectedness (because their normalized connectors are identical).

4.2. Path connectedness and similar

Definition 6 *I will call a ternary relation $\tau \in \mathscr{P}(U \times U \times \mathscr{P}U)$* ***link****.*
I will call the pair $(U; \tau)$ a ***link space****.*

I will denote $a \; \tau_A \; b = \tau_A(a, b) = \tau(a, b, A)$.

Remark 1 The expression $\tau(a, b, A)$ generalizes the statement "There exists a path from a to b through A." (where path may be taken in the sense used in topology or the sense used in graph theory).

Definition 7 *I will call a link space $(U; \tau)$* ***increasing*** *iff*

$$\forall A, B \in \mathscr{P}U : (A \supseteq B \Rightarrow \tau_A \supseteq \tau_B).$$

Definition 8 *I will call the* ***restriction*** *of a link space$(U; \tau)$ to a set $A \in \mathscr{P}U$ the link space $(A; \tau \cap (A \times A \times \mathscr{P}A))$.*

Definition 9 *I call a link space $(U; \tau)$* ***symmetric*** *when τ_A is symmetric for every $A \in \mathscr{P}U$,* ***transitive*** *when τ_A is transitive for every $A \in \mathscr{P}U$,* ***reflexive*** *when τ_A is reflexive on A for every $A \in \mathscr{P}U$. I will call a link space* ***equivalence*** *when it is symmetric, transitive, and reflexive.*

Definition 10 *I will call a set A* ***connected regarding a link*** *τ when $\forall x, y \in A : \tau(x, y, A)$. I call* ***connectedness*** *regarding a link space $(U; \tau)$ the collection of all connected (regarding τ) sets on U. I will denote $\mathrm{LC}(U; \tau)$ the connectedness regarding $(U; \tau)$. ("LC" is deciphered as "link connectedness".)*

To get path connectedness we take (for some topology $\mathfrak{A}$)

$$\tau_A(x, y) \Leftrightarrow \exists f \in \mathrm{C}([0; 1]; \mathfrak{A}|_A) : (f(0) = x \wedge f(1) = y). \tag{3}$$

Definition 11 *We can define two connector spaces $T(U; \tau)$ and $Q(U; \tau)$ with the base U corresponding to a link space $(U; \tau)$ by the formulas:*

$$\forall X, Y \in \mathscr{P}U : (X\ T(U; \tau)\ Y \Leftrightarrow \forall x \in X, y \in Y : \tau(x, y, X \cup Y));$$
$$\forall X, Y \in \mathscr{P}U : (X\ Q(U; \tau)\ Y \Leftrightarrow \exists x \in X, y \in Y : \tau(x, y, X \cup Y)).$$

Obvious 4 *If τ is reflexive then $Q(U; \tau)$ is a normalized connector.*

Obvious 5

1. $(T(U; \tau))|_K = T((U; \tau)|_K)$;
2. $(Q(U; \tau))|_K = Q((U; \tau)|_K)$.

Proposition 1 $\mathrm{LC}(U; \tau) = \mathrm{CC}(T(U; \tau))$ *for every reflexive link space* $(U; \tau)$.

Proof Let A is connected regarding $T(U; \tau)$. Then

$$\forall X, Y \in \mathscr{P}A \setminus \{\emptyset\} : (X \cup Y = A \wedge X \cap Y = \emptyset \Rightarrow X\ T(U; \tau)\ Y)$$

that is

$$\forall X, Y \in \mathscr{P}A \setminus \{\emptyset\} : (X \cup Y = A \wedge X \cap Y = \emptyset \Rightarrow \forall x \in X, y \in Y : \tau(x, y, X \cup Y)).$$

Let $a, b \in A$ and $a \neq b$. Then exist $X, Y \in \mathscr{P}A \setminus \{\emptyset\}$ such that $X \cup Y = A \wedge X \cap Y = \emptyset$ and $a \in X$, $b \in Y$. So $\tau(a, b, X \cup Y)$ that is $\tau(a, b, A)$. So taking in account reflexivity of τ we get that A is connected regarding τ.

Let now A is connected regarding $(U; \tau)$. Let $X, Y \in \mathscr{P}A \setminus \{\emptyset\} \wedge X \cup Y = A \wedge X \cap Y = \emptyset$. We have $\tau(a, b, A)$ for every $a \in X$, $b \in Y$. Thus $X\ T(U; \tau)\ Y$. So A is connected regarding $T(U; \tau)$. □

Theorem 2 *For every equivalence link space* $(U; \tau)$

$$\mathrm{LC}(U; \tau) = \mathrm{CC}(T(U; \tau)) = \mathrm{CC}(Q(U; \tau)).$$

Proof Enough to prove $\mathrm{LC}(U; \tau) = \mathrm{CC}(Q(U; \tau))$.

Let A is not connected regarding $(U; \tau)$ that is there are $a, b \in A$ such that $\neg(a\ \tau_A\ b)$. Then $a \in K$ and $b \in A \setminus K$ where K is a equivalence class regarding τ_A. So $\neg(K\ Q(U; \tau)\ A \setminus K)$ and thus A is not connected regarding $Q(U; \tau)$.

Let A is connected regarding $(U; \tau)$. Then for every $X, Y \in \mathscr{P}A \setminus \{\emptyset\}$ we have $\forall x \in X, y \in Y : x\ \tau_A\ y$ and thus $\exists x \in X, y \in Y : x\ \tau_A\ y$ that is $X\ Q(U; \tau)\ Y$. So A is connected regarding $Q(U; \tau)$. □

Remark 2 We may introduce other variants of path-connectedness replacing topology $\mathfrak{A}$ with a proximity or uniformity and continuity with proximal continuity or uniform continuity.

Proposition 2 *The link space is an increasing equivalence for every* $\mathfrak{A}$ *be it a topology, proximity, or uniformity.*

Proof Easy to prove in every of the three cases. □

4.3. Proximal connectedness

The notion of proximal connectedness (also called "equiconnectedness") is defined e.g. in [1], [2], and [3].

To get proximal connectedness we simply take the connector $r = \delta$ for a proximity δ.

Remark 3 Connectedness regarding a proximity can be trivially generalized to connectedness regarding a funcoid [5], but I omit this because the theory of funcoids is not yet officially published.

Proposition 3 *A set* A *is proximally connected iff*

$$\forall X, Y \in \mathscr{P} A \setminus \{\emptyset\} : (X \cup Y = A \Rightarrow X \;\delta\; Y).$$

Proof Because δ is a normalized connector. □

4.4. Connectedness regarding a digraph

The category of binary relations is the category whose objects are sets and whose morphisms from a set A to a set B are triples $(f; A; B)$ where f is a binary relation and $\operatorname{dom} f \subseteq A$ and $\operatorname{im} f \subseteq b$. Composition of morphisms is defined in the natural way.

We will order this category by product order, that is

$$(f; A_0; B_0) \subseteq (g; A_1; B_1) \Leftrightarrow f \subseteq g \wedge A_0 \subseteq A_1 \wedge B_0 \subseteq B_1.$$

For two morphisms $(f; A_0; B_0)$ and $(g; A_1; B_1)$ we have the meet of morphisms by the formula

$$(f; A_0; B_0) \cap (g; A_1; B_1) = (f \cap g; A_0 \cap A_1; B_0 \cap B_1).$$

Easy to see that the right part of this formula is a morphism.

We will define $A \times^{C} B = (A \times B; A; B)$.

I will define a **digraph** as an endomorphism of the category of binary relations. In other words, a digraph is $(U; f)$ where U is a set and f is a binary relation on U.

By definition $a \; (f; A; B) \; b \Leftrightarrow a \; f \; b \Leftrightarrow (a; b) \in f$.

By definition $\langle (f; A; B) \rangle \, X = \langle f \rangle \, X$ and $[(f; A; B)] = [f]$.

Definition 12 ***Connectedness regarding a digraph*** *is the connectedness for the link $(U;\tau)$ where U is the set of vertices of the digraph and $\tau(x,y,A)$ means that there are a path from x to y in the subgraph restricted to A.*

Obvious 6 *The link space $(U;\tau)$ in the above definition is an increasing equivalence.*

Definition 13 $S(U;f) \overset{\text{def}}{=} (U; (=)|_U \cup f \cup f^2 \cup f^3 \cup \ldots)$ *for every digraph $(U;f)$.*

Proposition 4 *There is a path from element a to element b in a set A through a digraph μ iff $a\ S(\mu \cap (A \times^C A))\ b$.*

Proof

$\Rightarrow$ If exists a path from a to b, then $\{b\} \subseteq \left\langle (\mu \cap (A \times^C A))^n \right\rangle \{a\}$ where n is the path length. Consequently $\{b\} \subseteq \left\langle S(\mu \cap (A \times^C A)) \right\rangle \{a\}$; $a\ S(\mu \cap (A \times^C A))\ b$.

$\Leftarrow$ If $a\ (S(\mu \cap (A \times^C A)))\ b$ then exists $n \in \mathbb{N}$ such that $a\ (\mu \cap (A \times^C A))^n\ b$. By definition of composition of binary relations this means that there exist finite sequence $x_0 \ldots x_n$ where $x_0 = a$, $x_n = b$ for $n \in \mathbb{N}$ and $x_i\ (\mu \cap (A \times^C A))\ x_{i+1}$ for every $i = 0, \ldots, n-1$. That is there is path from a to b.

□

Lemma 1 *If $X \cap Y = \emptyset$ and $\neg(X\,[f]\,Y)$ then $\neg(X\,[f^n]\,Y)$ for every sets X, Y, digraph f, and natural number n.*

Proof For $n = 0$ it is obvious. Let's prove by induction that it's true for $n \geqslant 1$.

For $n = 1$ it is obvious.

Let it's true for $n = k > 0$. $\neg(X\,[f^{k+1}]\,Y) \Leftrightarrow Y \cap \left\langle f^{k+1} \right\rangle X = \emptyset \Leftrightarrow Y \cap \left\langle f^k \right\rangle \langle f \rangle X = \emptyset \Leftrightarrow \neg(\langle f \rangle X\,[f^k]\,Y)$ what is true by induction because $\langle f \rangle X \cap Y = \emptyset$ is equivalent to $\neg(X\,[f]\,Y)$. □

Theorem 3 *The following statements are equivalent for a digraph μ and a set A:*

1. *A is connected regarding the digraph μ.*
2. *$S(\mu \cap (A \times^C A)) \supseteq A \times^C A$.*
3. *$S(\mu \cap (A \times^C A)) = A \times^C A$.*
4. *A is connected regarding the connector $[\mu]$.*
5. *$\forall X, Y \in A \setminus \{\emptyset\} : (X \cup Y = A \Rightarrow X\,[\mu]\,Y)$.*

Proof

(1)⇒(2) Let for every $a, b \in A$ there is a path between a and b in A through μ. Then $a\ (S(\mu \cap^C (A \times A)))\ b$ for every $a, b \in A$. It is possible only when $S(\mu \cap^C (A \times A)) \supseteq A \times A$.

(3)⇒(1) For every two vertices a and b we have $a\ (S(\mu \cap (A \times^C A)))\ b$. So (by the previous theorem) for every two vertices a and b exist path from a to b.

(3)⇒(4) Suppose that $\neg(X \left[\mu \cap (A \times^C A)\right] Y)$ for some $X, Y \in \mathscr{P}U \setminus \{\emptyset\}$ such that $X \cup Y = A$ and $X \cap Y = \emptyset$. Then by a lemma $\neg(X \left[(\mu \cap (A \times^C A))^n\right] Y)$ for every $n \in \mathbb{N}$. Consequently $\neg(X \left[S(\mu \cap (A \times^C A))\right] Y)$. So $S(\mu \cap (A \times^C A)) \neq A \times A$.

(4)⇒(3) If $\left\langle S(\mu \cap^C (A \times A))\right\rangle \{v\} = A$ for every vertex v then $S(\mu \cap^C (A \times A)) = A \times^C A$. Consider the remaining case when $V \stackrel{\text{def}}{=} \left\langle S(\mu \cap (A \times^C A))\right\rangle \{v\} \subset A$ for some vertex v. Let $W = A \setminus V$. If $\operatorname{card} A = 1$ then $S(\mu \cap^C (A \times A)) \supseteq (=)|_A = A \times^C A$; otherwise $W \neq \emptyset$. Then $V \cup W = A$ and so $V\,[\mu]\,W$ what is equivalent to $V \left[\mu \cap^C (A \times A)\right] W$ that is $\left\langle \mu \cap^C (A \times A)\right\rangle V \cap W \neq \emptyset$. This is impossible because $\left\langle \mu \cap (A \times^C A)\right\rangle V = \left\langle \mu \cap (A \times^C A)\right\rangle \left\langle S(\mu \cap (A \times^C A))\right\rangle V \subseteq \left\langle S(\mu \cap (A \times^C A))\right\rangle V = V$.

(2)⇒(3) Because $S(\mu \cap (A \times^C A)) \subseteq A \times^C A$.

(5)⇒(4) Obvious.

(4)⇒(5) Let (4) holds and let $X \cup Y = A$. If $X = Y = A$ then $X\,[\mu]\,Y$ because $A \neq \emptyset$. Otherwise $X \subset A$ or $Y \subset A$. Let for example $X \subset A$. Then $Y \setminus X \neq \emptyset$. So $X\,[\mu]\,Y \setminus X$ by (4) and consequently $X\,[\mu]\,Y$.

□

Corollary 2 *A set A is connected regarding a digraph μ iff it is connected regarding $\mu \cap (A \times^C A)$.*

Theorem 4 *The following statements are equivalent for each digraph $\mu = (U; f)$ and sets $X, Y \in \mathscr{P}U$:*

1. $X\ T(U; \tau)\ Y$;

2. $X \times^C Y \subseteq S(\mu \cap ((X \cup Y) \times^C (X \cup Y)))$;

3. $X \times^C Y = S(\mu \cap ((X \cup Y) \times^C (X \cup Y)))$.

Proof

$$\begin{aligned} X \times^{C} Y \subseteq S(\mu \cap ((X \cup Y) \times^{C} (X \cup Y))) &\quad\Leftrightarrow \\ \forall x \in X, y \in Y : x\ S(\mu \cap ((X \cup Y) \times^{C} (X \cup Y)))\ y &\quad\Leftrightarrow \\ \forall x \in X, y \in Y : \tau(x, y, X \cup Y) &\quad\Leftrightarrow \quad X\ T(U;\tau)\ Y. \end{aligned}$$

$X \times^{C} Y \subseteq S(\mu \cap ((X \cup Y) \times^{C} (X \cup Y))) \Leftrightarrow X \times^{C} Y = S(\mu \cap ((X \cup Y) \times^{C} (X \cup Y)))$ because $S(\mu \cap ((X \cup Y) \times^{C} (X \cup Y))) \subseteq (X \cup Y) \times^{C} (X \cup Y)$. □

Theorem 5 *$Q(U;\tau)$ and $[\mu]$ have the same normalization (for every digraph $\mu = (U; f)$).*

Proof Let $X, Y \in \mathscr{P}U$, $X, Y \neq \emptyset$, $X \cap Y = \emptyset$. We need to prove $X\ Q(U;\tau)\ Y \Leftrightarrow X\,[\mu]\,Y$.

$X\ Q(U;\tau)\ Y \Leftarrow X\,[\mu]\,Y$ is obvious.

Let $X\ Q(U;\tau)\ Y$. Then there exists a path in $X \cup Y$ from a point of X to a point of Y. Easy to see that there exist consequtive points x, y of this path such that $x\ \mu\ y$. So $X\,[\mu]\,Y$. □

Theorem 6 *Regarding every digraph $(U;\mu)$, connectedness is the same for connector spaces:*

1. *$T(U;\tau)$;*
2. *$Q(U;\tau)$;*
3. *$(U;[\mu])$.*

Proof From the theorems 2 and 3. □

4.5. Weak connectedness

By definition a set A is weakly connected regarding a digraph μ iff it is connected regarding the corresponding graph (that is connected regarding the digraph $\mu \cup \mu^{-1}$). So weak connectedness is also a kind of generalized connectedness.

4.6. Uniform connectedness

4.6.1. Some basic properties of filters

Let $\mathcal{F}$ is the set of filters on some set U.

I will denote $[A)$ the principal filter corresponding to a set A.

Note that I do not require that filters do not contain the empty set, thus $[\emptyset)$ is well defined.

Proposition 5 *$a \cup^{\mathcal{F}} b = \{A \cap B \mid A \in a, B \in b\}$ for every filters a and b.*

Proof First prove that $\{A \cap B \mid A \in a, B \in b\}$ is a filter. Let $X, Y \in \{A \cap B \mid A \in a, B \in b\}$. Then $X = A_1 \cap B_1$ and $Y = A_2 \cap B_2$ where $A_1, A_2 \in a$ and $B_1, B_2 \in b$. Consequently $X \cap Y = (A_1 \cap A_2) \cap (B_1 \cap B_2)$ where $A_1 \cap A_2 \in a$, $B_1 \cap B_2 \in b$; thus $X \cap Y \in \{A \cap B \mid A \in a, B \in b\}$. Let $X \in \{A \cap B \mid A \in a, B \in b\}$ and $C \supseteq X$. We have $X = A \cap B$ where $A \in a$, $B \in b$. We have $C = C \cup X = C \cup (A \cap B) = (C \cup A) \cap (C \cup B)$ where $C \cup A \in a$ and $C \cup B \in b$; thus $C \in \{A \cap B \mid A \in a, B \in b\}$. So $\{A \cap B \mid A \in a, B \in b\}$ is a filter.

We need to prove that $\{A \cap B \mid A \in a, B \in b\}$ is the lowest upper bound of $\{a, b\}$. We have $\{A \cap B \mid A \in a, B \in b\} \supseteq a$ because if $X \in a$ then $X = X \cap U \in \{A \cap B \mid A \in a, B \in b\}$. Similarly $\{A \cap B \mid A \in a, B \in b\} \supseteq b$. Thus it is an upper bound.

Let p is an upper bound of $\{a, b\}$. Then $p \supseteq a$ that is $\forall A \in a : A \in p$ and $\forall B \in b : B \in p$. Thus because p is a filter we have $\forall A \in a, B \in b : A \cap B \in p$ that is $p \supseteq \{A \cap B \mid A \in a, B \in b\}$. □

Proposition 6 $[A) \cup^{\mathcal{F}} [B) = [A \cap B)$ *for every subsets* A *and* B *of* U.

Proof We need to prove that $[A \cap B)$ is the least upper bound of $\{[A), [B)\}$.

That $[A \cap B) \supseteq [A), [B)$ is obvious.

Remained to prove that $\forall a \in \mathcal{F} : (a \supseteq [A), [B) \Rightarrow a \supseteq [A \cap B))$. Really,

$$a \supseteq [A), [B) \Rightarrow A, B \in a \Rightarrow A \cap B \in a \Rightarrow a \supseteq [A \cap B).$$

□

4.6.2. Uniform triples

I will define uniform connectedness. Below I will show that my definition is equivalent to the classical definition of uniform connectedness.

I will call a **uniform triple** on a set U the triple $(f; A; B)$ where f is a filter on $\mathscr{P}(U \times U)$ and A, B are such sets that $A \times B \in f$. Note that uniform spaces can be considered as uniform triples with $A = B$. I will denote $\mathcal{R}$ the set of filters on $\mathscr{P}(U \times U)$ and $\mathcal{U}$ the set of uniform triples.

I will call a **generalized uniform space** a uniform triple with $A = B$.

Remark 4 In fact there can be defined composition of uniform triples and they thus form morphisms of certain category. But in this article I'll not dive into details here. See my draft article [5].

We will introduce order on the set of uniform triples on a set by the formula

$$(f; A_0; B_0) \subseteq (g; A_1; B_1) \Leftrightarrow f \subseteq g \wedge A_0 \supseteq A_1 \wedge B_0 \supseteq B_1.$$

Easy to see that $(f; A_0; B_0) \cup^{\mathcal{U}} (g; A_1; B_1) = (f \cup^{\mathcal{R}} g; A_0 \cap A_1; B_0 \cap B_1)$.

For a morphism $(f; A; B)$ of the category of binary relations, I will denote $[(f; A; B)) = ([f); A; B)$. Easy to see that $[(f; A; B))$ is a uniform triple.

By abuse of notation I will denote

$$(f; A_0; B_0) \in (g; A_1; B_1) \Leftrightarrow f \in g \wedge A_0 = A_1 \wedge B_0 = B_1$$

where f is a binary relation and g is a filter on $\mathscr{P}(U \times U)$.

4.6.3. Uniform connectedness

Let μ is a generalized uniform space.

Definition 14 *I will denote* $S^*(\mu) = \bigcup^{\mathcal{U}} \{[S(f)) \mid f \in \mu\}$.

Obvious 7 S^* *is a monotone function.*

Definition 15 *A set* A *is* ***(uniformly) connected*** *regarding* μ *iff* $S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \subseteq [A \times^C A)$.

Proposition 7 $S^*([f)) = [S(f))$ *for every digraph* f.

Proof $S^*([f)) = \bigcup^{\mathcal{U}} \{[S(g)) \mid g \in [f)\} = \bigcup^{\mathcal{U}} \{[S(f))\} = [S(f))$. □

Obvious 8 *A set* A *is connected regarding a generalized uniform space* μ *iff* $S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) = [A \times^C A)$.

Uniform connectedness is a generalization of digraph connectedness:

Proposition 8 *A set* A *is uniformly connected regarding* $[\mu)$ *iff it is connected regarding* μ *(for every digraph* μ*).*

Proof $S^*([\mu) \cup^{\mathcal{U}} [A \times A)) = S^*([\mu \cap (A \times^C A))) = [S(\mu \cap (A \times^C A)))$.

Thus $S^*([\mu) \cup^{\mathcal{U}} [A \times A)) = [A \times^C A) \Leftrightarrow S(\mu \cap (A \times^C A)) = A \times^C A$. □

Obvious 9 *A set* A *is connected regarding a generalized uniform space* μ *iff* $\forall X \in S^*(\mu \cup^{\mathcal{U}} [A \times A)) : X \supseteq A \times^C A$.

Obvious 10 *A set* A *is connected regarding a generalized uniform space* μ *iff it is connected regarding* $\mu \cup^{\mathcal{U}} [A \times A)$.

Proposition 9 *A set* A *is connected regarding a generalized uniform space* μ *iff* A *is connected regarding every digraph* $f \in \mu$.

Proof

⇒ Let a set A is connected regarding μ and $f \in \mu$. Then $[f) \subseteq \mu$; consequently $[f) \cup^{\mathcal{U}} [A \times^C A) \subseteq \mu \cup^{\mathcal{U}} [A \times^C A)$ and so $S^*([f) \cup^{\mathcal{U}} [A \times^C A)) \subseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \subseteq [A \times^C A)$. Thus $S^*([f \cap (A \times^C A))) \subseteq [A \times^C A)$; $[S(f \cap (A \times^C A))) \subseteq [A \times^C A)$; $S(f \cap (A \times^C A)) \supseteq A \times^C A$ that is A is connected regarding f.

⇐ $S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) = \bigcup^{\mathcal{U}} \left\{[S(f)) \mid f \in \mu \cup^{\mathcal{U}} [A \times^C A)\right\} = \bigcup^{\mathcal{U}} \left\{[S(g \cap h)) \mid g \in \mu, h \in [A \times^C A)\right\} \subseteq \bigcup^{\mathcal{U}} \left\{[S(g \cap (A \times^C A))) \mid g \in \mu\right\} = \bigcup^{\mathcal{U}} \left\{[A \times^C A) \mid g \in \mu\right\} = [A \times^C A)$.

□

4.6.4. Connectors for uniform connectedness

Let's find a connector which generates the same connectedness as the described above uniform connectedness.

Proposition 10 $\forall x \in U : [\{x\} \times^C \{x\}) \supseteq S^*(\mu)$ *for every generalized uniform space* $\mu = (U; f)$.

Proof $S^*(\mu) = \bigcup^{\mathcal{U}} \{[S(f)) \mid f \in \mu\}$. But $\{x\} \times^C \{x\} \subseteq S(f)$; thus $[\{x\} \times^C \{x\}) \supseteq [S(f))$ and consequently $\bigcup^{\mathcal{U}} \{[S(f)) \mid f \in \mu\} \subseteq [\{x\} \times^C \{x\})$. □

Lemma 2 $[\bigcup S) \supseteq F \Leftrightarrow \forall X \in S : [X) \supseteq F$ *for every collection* S *of sets and every filter* F.

Proof

$\Rightarrow$ Obvious.

$\Leftarrow$ Let $\forall X \in S : [X) \supseteq F$ that is $\forall X \in S, Y \in F : X \subseteq Y$. Then $\forall Y \in F : \bigcup S \subseteq Y$ that is $[\bigcup S) \supseteq F$.

□

From the above lemma follows that

$$[A \times^C A) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \quad \Leftrightarrow$$
$$\forall x \in A : [\{x\} \times^C (A \setminus \{x\})) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \wedge [\{x\} \times^C \{x\}) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)).$$

Because $\forall x \in A : [\{x\} \times^C \{x\}) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A))$, we have

$$[A \times^C A) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \Leftrightarrow \forall x \in A : [\{x\} \times^C (A \setminus \{x\})) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times A))$$

Consequently

$$[A \times^C A) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)) \quad \Leftrightarrow$$
$$\forall X, Y \in \mathscr{P} A : (X \cap Y = \emptyset \wedge X \cup Y = A \Rightarrow [X \times^C Y) \supseteq S^*(\mu \cup^{\mathcal{U}} [A \times^C A)).$$

So, our sought-for connector is defined (for example) by the formula

$$X \; r \; Y \Leftrightarrow [X \times^C Y) \supseteq S^*(\mu \cup^{\mathcal{U}} [(X \cup Y) \times^C (X \cup Y))).$$

A is connected regarding μ iff $\forall f \in \mu, X, Y \in \mathscr{P} U : (X \cup Y = A \Rightarrow X \, [f] \, Y) \Leftrightarrow \forall X, Y \in \mathscr{P} U : (X \cup Y = A \Rightarrow \forall f \in \mu : X \, [f] \, Y)$. Thus

$$X \; r \; Y \Leftrightarrow \forall f \in \mu : X \, [f] \, Y \Leftrightarrow \forall f \in \mu : X \times Y \cap f \neq \emptyset \qquad (4)$$

is also a connector which induces uniform connectedness.

If μ is a uniformity, $X \; r \; Y \Leftrightarrow X \; \delta \; Y$ where δ is the proximity induced by μ. Thus my definition of uniform connectedness is equivalent to traditional definition of uniform connectedness. (See theorem 1 in [3].)

5. Properties

5.1. Extendability

Definition 16 *I will call a connector space $(U;r)$* ***up-directed*** *when*

$$\forall X_0, Y_0, X_1, Y_1 \in \mathscr{P}U : (X_0 \; r \; Y_0 \wedge X_1 \supseteq X_0 \wedge Y_1 \supseteq Y_0 \Rightarrow X_1 \; r \; Y_1).$$

Definition 17 *I will call a connector space $(U;r)$* ***extendable*** *when*

$$\forall X_0, Y_0, X_1, Y_1 \in \mathscr{P}U : (X_1 \cap Y_1 = \emptyset \wedge X_0 \; r \; Y_0 \wedge X_1 \supseteq X_0 \wedge Y_1 \supseteq Y_0 \Rightarrow X_1 \; r \; Y_1).$$

Obvious 11 *Every up-directed connector space is extendable.*

Example 1 The following connector spaces are up-directed (and thus extendable):

1. the connector space defined by the formula (2);
2. $(U;[f])$ for every digraph $(U;f)$;
3. $Q(U;\tau)$ for an increasing link space $(U;\tau)$;
4. the connector space defined by the formula (4);
5. A proximity space $(U;\delta)$.

Proposition 11 *A connector space is extendable iff its normalization is up-directed.*

Proof

- $\Rightarrow$ Let $X \; N(r) \; Y$ and $X' \supseteq X$, $Y' \supseteq Y$. We have $X' \neq \emptyset$, $Y' \neq \emptyset$. If $X' \cap Y' \neq \emptyset$ then $X' \; N(r) \; Y'$. Otherwise by extendabilty $X' \; r \; Y'$ and consequently $X' \; N(r) \; Y'$. Thus $N(r)$ is up-directed.
- $\Leftarrow$ Let $X_1 \cap Y_1 = \emptyset \wedge X_0 \; r \; Y_0 \wedge X_1 \supseteq X_0 \wedge Y_1 \supseteq Y_0$. Then $X_0 \; N(r) \; Y_0$ and consequently $X_1 \; N(r) \; Y_1$. So $X_1 \; r \; Y_1$.

□

5.2. Criteria of connectedness

Obvious 12 *Empty set is connected regarding every connector.*

Obvious 13 *Every singleton is connected regarding every connector.*

5.2.1. Connectedness of unions of sets

Lemma 3 *If $X \cup Y = A \cup B$ and $X, Y \neq \emptyset$ and $X \cap Y = \emptyset$ then either $\{X, Y\} = \{A, B\}$ or A intersects both X and Y or B intersects both X and Y (for every sets A, B, X, Y).*

Proof Let $\{X, Y\} \neq \{A, B\}$. Suppose that "A intersects both X and Y" does not hold (for example suppose that $A \cap X = 0$) and prove "B intersects both X and Y".

We have $X \subseteq B$ and thus $B \cap X \neq 0$. If also $B \cap Y = 0$ then $B \subseteq X$. So $X = B$ and thus either $Y = A$ what contradicts to our supposition or $A \supset Y$ in which case A intersects both X and Y. □

Theorem 7 *If sets $A, B \in \mathscr{P}U$ are connected regarding an extendable connector space $(U; r)$ and $A\ r\ B$ then $A \cup B$ is also connected regarding $(U; r)$.*

Proof We need to prove that

$$\forall X, Y \in \mathscr{P}(A \cup B) \setminus \{\emptyset\} : (X \cup Y = A \cup B \wedge X \cap Y = \emptyset \Rightarrow X\ r\ Y).$$

Let $X, Y \in \mathscr{P}(A \cup B) \setminus \{\emptyset\}$ and $X \cup Y = A \cup B \wedge X \cap Y = \emptyset$. Then by the lemma either $\{X, Y\} = \{A, B\}$ and thus $X\ r\ Y \Leftrightarrow A\ r\ B$ so having $X\ r\ Y$, or A intersects both X and Y or B intersects both X and Y. Consider for example then case $X \cap A \neq \emptyset$ and $Y \cap A \neq \emptyset$.

In this case we have $(X \cap A) \cup (Y \cap A) = (X \cup Y) \cap A = (A \cup B) \cap A = A$ and $(X \cap A) \cap (Y \cap A) \subseteq X \cap Y = \emptyset$. Thus $X \cap A\ r\ Y \cap A$ and consequently $X\ r\ Y$ (taken in account extendability). □

Corollary 3 *If sets $A, B \in \mathscr{P}U$ are connected regarding an extendable connector space $(U; r)$ and $A \cap B \neq \emptyset$ then $A \cup B$ is also connected regarding $(U; r)$.*

Proof Replace r with its normalization $N(r)$. This preserves the same connectedness. $A \cap B \neq \emptyset \Rightarrow A\ N(r)\ B$. Thus we can apply the theorem. □

There holds also infinite version of the previous corollary:

Theorem 8 *If $S \in \mathscr{P}\mathscr{P}U$ is a collection of connected (regarding an extendable connector space $(U; r)$) sets and $\bigcap S \neq \emptyset$ then $\bigcup S$ is connected (regarding this connector space).*

Proof Let $\{X, Y\}$ is a partition of $\bigcup S$. Then exist a point $p \in \bigcap S$ such that $p \in X$ or $p \in Y$. Without lost of generality we may assume $p \in X$. Since $Y \neq \emptyset$, we have $q \in Y$ for some $q \in \bigcup S$ that is $q \in A$ for some $A \in S$. So $A \cap X, A \cap Y \neq \emptyset$ and thus $\{A \cap X, A \cap Y\}$ is a partition of A. Since A is connected, we have $A \cap X\ r\ A \cap Y$ and thus (taken in account extendability) $X\ r\ Y$. So $\bigcup S$ is connected. □

Corollary 4 *Connectedness generated by an extendable connector space is a c-structure in the sense of [4].*

Remark 5 Connectedness generated by an extendable connector space is not necessarily a connective structure in the sense of [4]. A counter-example is proximal connectedness on the set $\mathbb{R} \setminus \{0\}$. (Take $A = (-\infty; 0)$, $B = (0; +\infty)$ to violate the axiom (iii) in the main definition of [4].)

5.3. Links generated by a connector

Definition 18 $a\ \rho(E)\ b \Leftrightarrow \exists K \in E : a, b \in K$ *for every collection* E *of sets.*

Definition 19 $L(E)_A = \rho(\mathscr{P}A \cap E)$ *for every collection* E *of sets and a set* A*.*

Let $(U; r)$ is a connector space.

Definition 20 $\zeta_{(U;r)}(\star)$ *is the link space defined by the formula* $\zeta_{(U;r)}(\star)_A = (U; \star_{(U;r)|_A})$.

Definition 21 *Let* $(\equiv_{(U;r)}) = \rho(\mathrm{CC}(U; r))$.

Proposition 12 $\zeta_{(U;r)}(\equiv)_K = (\equiv_{(U;r)|_K}) = L(\mathrm{CC}(U; r))_K = \rho(\mathrm{CC}((U; r)|_K))$ *for every connector space* $(U; r)$ *and set* $K \in \mathscr{P}U$*.*

Proof $(\equiv_{(U;r)|_K}) = \rho(\mathrm{CC}((U; r)|_K)) = \rho(\mathrm{CC}(U; r) \cap \mathscr{P}K) = L(\mathrm{CC}(U; r))_K$.

$\zeta_{(U;r)}(\equiv)_K = (\equiv_{(U;r)|_K})$ by definition. □

Obvious 14 $\zeta_{(U;r)}(\equiv)$ *is an increasing link space.*

Obvious 15 $(\equiv_{(U;r)})$ *is symmetric for every connector space* $(U; r)$*.*

Proposition 13 $(\equiv_{(U;r)})$ *is reflexive on* U *for every connector space* $(U; r)$*.*

Proof Follows from the fact that singletons are connected. □

Theorem 9 $(\equiv_{(U;r)})$ *is an equivalence relation on* U *for every extendable connector space* $(U; r)$*.*

Proof We need to prove only transitivity. Let $a \equiv_{(U;r)} b$ and $b \equiv_{(U;r)} c$. Then exist $X, Y \in \mathrm{CC}(U; r)$ such that $a, b \in X$ and $b, c \in Y$. Because $X \cap Y \neq \emptyset$ we have $X \cup Y \in \mathrm{CC}(U; r)$. So $a \equiv_{(U;r)} c$. □

Definition 22 *A* ***connected component*** *(regarding a connectedness space* $(U; r)$*) is a non-empty maximal connected set.*

Proposition 14 *A set* $A \in \mathscr{P}U$ *is connected regarding a connector space* $(U; r)$ *iff there are exactly one connected component of the connector space* $(U; r)|_A$*.*

Proof If A is connected regarding $(U;r)$ then A is connected regarding $(U;r)|_A$ and thus is a connected component regarding $(U;r)|_A$.

If A is a connected component regarding $(U;r)|_A$ then A is connected regarding $(U;r)|_A$ and thus is connected regarding $(U;r)$. □

Theorem 10 *Equivalence classes regarding $\equiv_{(U;r)}$ are exactly connected components for every extendable connector space $(U;r)$.*

Proof Let K is a connected component. Then K is connected and thus $a \equiv_{(U;r)} b$ for every $a, b \in K$. If $a \not\equiv_{(U;r)} b$ then there are no connected set X such that $a, b \in X$ and thus $a \notin K \vee b \notin K$. Thus K is an equivalence class of $\equiv_{(U;r)}$.

Let now K is an equivalence class of $\equiv_{(U;r)}$. Let choose arbitrary $k \in K$. For every $x \in K$ exists a connected set X_x such that $k, x \in X$. Having a common point k the union A of all X_x is a connected set. It's impossible $A \supset K$ because otherwise $y \equiv_{(U;r)} k$ for some $y \notin K$. So $A = K$ is the maximal connected set. □

Corollary 5 *For every extendable connector space $(U;r)$ its connectedness is equal to connectedness regarding the link $\zeta_{(U;r)}(\equiv)$.*

Proof $A \in \mathrm{CC}(U;r) \Leftrightarrow A \in \mathrm{CC}((U;r)|_A)$ what is equivalent to A being a connected component regarding $(U;r)|_A$ what is equivalent to A being an equivalence class regarding $\equiv_{(U;r)|_A}$ that is regarding $\zeta_{(U;r)}(\equiv)_A$ that is equivalent to A being connected regarding $\zeta_{(U;r)}(\equiv)$. □

Corollary 6 *The set U is partitioned into connected components for every extendable connector space $(U;r)$.*

Corollary 7 *If a set is connected then it is a subset of a connected component (for extendable connector spaces).*

Theorem 11 *For every extendable connector space exists a link space with the same connectedness.*

Proof Let $(U;r)$ is an extendable connector space. Let $A \in \mathscr{P}U$. Then A is connected regarding $(U;r)$ iff there are one connected component of the connector space $(U;r)|_A$. Thus A is connected regarding $(U;r)$ iff A is connected regarding τ where τ_A is the equivalence relation defined by the partition of the set A into connected components by the connector space $(U;r)|_A$. (Taken in account that connected components of an extendable connector space are a partition.) □

Theorem 12 *Let $(U;\tau)$ is an increasing equivalence link space. Then $L(\mathrm{LC}(U;\tau)) = \tau$.*

Proof K is connected regarding $(U;\tau)$ iff every two points of K are linked by τ_K.

$a\ L(\mathrm{LC}(U;\tau))_A\ b \Leftrightarrow \exists K \in \mathscr{P}A : (a,b \in K \wedge K \in \mathrm{LC}(U;\tau)) \Leftrightarrow \exists K \in \mathscr{P}A : (a,b \in K \wedge \forall x,y \in K : x\ \tau_K\ y)$.

$a\ L(\mathrm{LC}(U;\tau))_A\ b \Rightarrow \exists K \in \mathscr{P}A : (a,b \in K \wedge a\ \tau_K\ b) \Rightarrow \exists K \in \mathscr{P}A : a\ \tau_K\ b \Rightarrow a\ \tau_A\ b$.

Reversely, if $a\ \tau_A\ b$ then a and b are in the same connected component K and thus $a\ L(\mathrm{LC}(U;\tau))_A\ b$. □

Definition 23 *For a connectedness space* $(U;r)$*:*

$$a \rightarrow_{(U;r)} b \Leftrightarrow \forall X,Y \in \mathscr{P}U : (a \in X \wedge b \in Y \wedge X \cup Y = U \wedge X \cap Y = \emptyset \Rightarrow X\ r\ Y).$$

Obvious 16 $a\ \zeta_{(U;r)}(\rightarrow)_K\ b \Leftrightarrow a \rightarrow_{(U;r)|_K} b \Leftrightarrow \forall X,Y \in \mathscr{P}U : (a \in X \wedge b \in Y \wedge X \cup Y = K \wedge X \cap Y = \emptyset \Rightarrow X\ r\ Y)$ *for every* $K \in \mathscr{P}U$.

Definition 24 $\sim$ *is defined by the formula*

$$a \sim_{(U;r)} b \Leftrightarrow a \rightarrow_{(U;r)} b \wedge b \rightarrow_{(U;r)} a.$$

Obvious 17 $a \sim_{(U;r)} b \Leftrightarrow \forall X,Y \in \mathscr{P}U : (a \in X \wedge b \in Y \wedge X \cup Y = U \wedge X \cap Y = \emptyset \Rightarrow X\ r\ Y \wedge Y\ r\ X)$.

Obvious 18 $a\ \zeta_{(U;r)}(\sim)_K\ b \Leftrightarrow a \sim_{(U;r)|_K} b \Leftrightarrow \forall X,Y \in \mathscr{P}U : (a \in X \wedge b \in Y \wedge X \cup Y = K \wedge X \cap Y = \emptyset \Rightarrow X\ r\ Y \wedge Y\ r\ X)$ *for every* $K \in \mathscr{P}U$.

Remark 6 $\sim$ bears less information about the connector than $\equiv$. For example for the connector $T(U;\tau)$ of a graph consisting of two connected components $\sim|_{T(U;\tau)}$ is just the diagonal relation.

Proposition 15 $x \rightarrow_{(U;r)} x$ *and* $x \sim_{(U;r)} x$ *for every* $x \in U$.

Proof $x \rightarrow_{(U;r)} x$ follows from that $a \in X \wedge b \in Y \wedge X \cup Y = U \wedge X \cap Y = \emptyset$ is always false if $a = b$. $x \sim_{(U;r)} x$ follows from $x \rightarrow_{(U;r)} x$. □

Proposition 16 $\rightarrow_{(U;r)}$ *is transitive.*

Proof Let $a \rightarrow_{(U;r)} b$ and $b \rightarrow_{(U;r)} c$. Let $a \in X$, $c \in Z$, $X \cup Z = U$, $X \cap Z = \emptyset$.

We need to prove $X\ r\ Z$.

Obviously $b \in X \vee b \in Z$. We can assume $b \in X$.

Then $X\ r\ Z$ because $b \rightarrow_{(U;r)} c$. □

Theorem 13 $\sim_{(U;r)}$ *is an equivalence relation.*

Proof

Reflexivity Follows from reflexivity of $\rightarrow_{(U;r)}$.

Symmetry Obvious.

Transitivity Let $a \sim_{(U;r)} b$ and $b \sim_{(U;r)} c$. Then $a \rightarrow_{(U;r)} b$ and $b \rightarrow_{(U;r)} c$. So by transitivity of $\rightarrow_{(U;r)}$ we have $a \rightarrow_{(U;r)} c$. Similarly $c \rightarrow_{(U;r)} a$. So $a \sim_{(U;r)} c$.

□

Theorem 14 *The following statements are equivalent for every connector space $(U;r)$ and set $K \in \mathscr{P}U$:*

1. *The set K is connected regarding $(U;r)$.*
2. *$\forall x, y \in K : x \sim_{(U;r)|_K} y$.*
3. *$\forall x, y \in K : x \rightarrow_{(U;r)|_K} y$.*
4. *$\forall x, y \in K : x \equiv_{(U;r)|_K} y$.*

Proof

(1)⇒(3) Let K is connected. Then we have $X \; r \; Y$ and $Y \; r \; X$ for every $X, Y \in \mathscr{P}K \setminus \{\emptyset\}$ such that $X \cup Y = K \wedge X \cap Y = \emptyset$ and consequently $a \rightarrow_{(U;r)|_K} b$ for every $a, b \in K$.

(3)⇔(2) Obvious.

(3)⇒(1) Let $\forall x, y \in K : x \rightarrow_{(U;r)|_K} y$. Then if $X, Y \in \mathscr{P}K \setminus \{\emptyset\} \wedge X \cup Y = K \wedge X \cap Y = \emptyset$, we have some $x \in X$ and $y \in Y$ thus $X \; r \; Y$ because $x \rightarrow_{(U;r)} y$. So K is connected.

(4)⇒(1) If $\forall x, y \in K : x \equiv_{(U;r)|_K} y$ then K is a subset of a connected component regarding $(U;r)|_K$. This component cannot be greater than K, so K is connected regarding $(U;r)|_K$ and consequently connected regarding $(U;r)$.

(1)⇒(4) If K is connected regarding $(U;r)$ then K is connected regarding $(U;r)|_K$ and thus K is a connected component regarding $(U;r)|_K$ so having $\forall x, y \in K : x \equiv_{(U;r)|_K} y$.

□

Theorem 15 $\zeta_{Q(U;\tau)}(\equiv) = \zeta_{T(U;\tau)}(\equiv) = \zeta_{Q(U;\tau)}(\sim) = \zeta_{Q(U;\tau)}(\rightarrow) = \tau$ *for every equivalence link space $(U;\tau)$.*

Proof $a\ \zeta_{Q(U;\tau)}(\equiv)_K\ b$ iff a and b are in the same connected component regarding $Q((U;\tau)|_K)$.

Let's prove that $a\ \zeta_{Q(U;\tau)}(\rightarrow)_K\ b = a\ \zeta_{Q(U;\tau)}(\equiv)_K\ b$.

We need to prove that $a\ \zeta_{Q(U;\tau)}(\rightarrow)_K\ b$ iff a and b are in the same connected component regarding $Q((U;\tau)|_K)$. (Then also $a\ \zeta_{Q(U;\tau)}(\sim)_K\ b$ iff a and b are in the same connected component regarding $Q((U;\tau)|_K)$.) If a and b are in the same connected component then $x \rightarrow_{(Q(U;\tau))|_K} y$ that is $a\ \zeta_{Q(U;\tau)}(\rightarrow)_K\ b$. Let now $a\ \zeta_{Q(U;\tau)}(\rightarrow)_K\ b$. Suppose $a \in X$ and $b \in Y$ where X and Y are distinct connected components regarding $Q((U;\tau)|_K)$. Then $b \in U \setminus X$, $X \cup (U \setminus X) = U$ and $X \cap (U \setminus X) = \emptyset$. Thus $X\ Q((U;\tau)|_K)\ (U \setminus X)$ that is for some $x \in X$ and $y \in U \setminus X$ we have $x\ \tau_K\ y$ what is impossible because x and y lie in different connected components.

$\zeta_{Q(U;\tau)}(\equiv)_K = (\equiv|_{(U;r)|_K})$;

$a \equiv |_{Q(U;\tau)|_K} b \Leftrightarrow a\ L(\mathrm{CC}(Q(U;\tau))_K\ b \Leftrightarrow a\ L(\mathrm{CC}(T(U;\tau)))_K\ b \Leftrightarrow a \equiv |_{T(U;\tau)|_K} b$ (used the theorem 2).

$L(\mathrm{CC}(Q(U;\tau)))_K = \rho(\mathscr{P}K \cap \mathrm{CC}(Q(U;\tau))) = \rho(\mathrm{CC}((Q(U;\tau))|_K)) = \rho(\mathrm{CC}(Q((U;\tau)|_K))) = \rho(\mathrm{LC}((U;\tau)|_K))$. So if $a \equiv |_{Q(U;\tau)|_K} b$ then a and b lie in the same connected component regarding $(U;\tau)|_K$. Thus $a\ \tau_K\ b$.

Let now $a\ \tau_K\ b$. Suppose that a and b lie in different connected components regarding $(U;\tau)|_K$. Then by equivalence every points of these components are linked and thus they are one connected component. By contradiction a and b lie in the same connected component regarding $(U;\tau)|_K$.

So we proved $a \equiv |_{Q(U;\tau)|_K} b \Leftrightarrow a\ \tau_K\ b$. □

5.4. Relationships of $Q(U;\tau)$ and $T(U;\tau)$

Let find a formula which allows to find $T(U;\tau)$ knowing $Q(U;\tau)$ (to the extent of equal normalization).

Let $(U;r)$ is a connector space.

Definition 25 *I will define the connector space $\beta(U;r) = (U;r')$ by the formula (for every $A, B \in \mathscr{P}U$)*

$$A\ r'\ B \Leftrightarrow A \cup B \in \mathrm{CC}(r).$$

Lemma 4 *Let X, Y, A, B are sets. If $X, Y, A, B \neq \emptyset$ and $X \cup Y = A \cup B$ then $X \cap A \neq \emptyset \wedge Y \cap B \neq \emptyset$ or $X \cap B \neq \emptyset \wedge Y \cap A \neq \emptyset$.*

Proof If $a \in X$ then $a \in A$ or $a \in B$. Let for example $a \in A$. Thus $X \cap A \neq \emptyset$. If $Y \cap B = \emptyset$ then $B \subseteq X$ and $Y \subseteq A$, so having $X \cap B \neq \emptyset \wedge Y \cap A \neq \emptyset$. □

Theorem 16 *$N(T(U;\tau)) = N(\beta(Q(U;\tau)))$ for every increasing equivalence link space $(U;\tau)$.*

Proof Let $A, B \neq \emptyset$ and $A \cap B = \emptyset$. We need to prove that $A\ T(U;\tau)\ B \Leftrightarrow A\ \beta(Q(U;\tau))\ B$.

Let $A\ \beta(Q(U;\tau))\ B$. Then $A \cup B \in \mathrm{CC}(Q(U;\tau))$ that is by the theorem 2 we have $A \cup B \in \mathrm{LC}(U;\tau)$. So $\forall x, y \in A \cup B : x\ \tau_{A\cup B}\ y$ that is $A\ T(U;\tau)\ B$.

Let now $A\ T(U;\tau)\ B$. Then $\forall a \in A, b \in B : \tau(a, b, A \cup B)$.

Let $X \cup Y = A \cup B$ and $X \cap Y = \emptyset$ and $X, Y \neq \emptyset$.

By the lemma there exist $a \in X$, $b \in Y$ such that $a \in A$, $b \in B$ (or $a \in X$, $b \in Y$ such that $a \in B$, $b \in A$ what is analogous). So $\tau(a, b, A \cup B)$ and consequently $X\ Q(U;\tau)\ Y$.

Thus $A \cup B \in \mathrm{CC}(Q(U;\tau))$ that is $A\ \beta(Q(U;\tau))\ B$. □

Proposition 17 *$N(\beta(U;r)) \subseteq N(U;r)$ for every connector space $(U;r)$.*

Proof Let $A\ N(\beta(U;r))\ B$ for some $A, B \neq \emptyset$, $A \cap B = \emptyset$, then $A \cup B \in \mathrm{CC}(U;r)$. Then $A\ r\ B$ and thus $A\ N(U;r)\ B$. □

Theorem 17 *$\mathrm{CC}(\beta(U;r)) \subseteq \mathrm{CC}(U;r)$ for every connector space $(U;r)$.*

Proof From the previous proposition. □

Proposition 18 *$N(\beta(\beta(U;r))) = N(\beta(U;r))$ for every connector space $(U;r)$.*

Proof If $A\ N(\beta(\beta(U;r)))\ B$ then either $A \cap B \neq \emptyset$ and thus $A\ N(\beta(U;r))\ B$ or $A \cap B = \emptyset$ and $A, B \neq \emptyset$ and $A\ \beta(\beta(U;r))\ B$. Then $A \cup B \in \mathrm{CC}(\beta(U;r))$ and thus $A\ \beta(U;r)\ B$ with consequence $A\ N(\beta(U;r))\ B$.

Let now $A\ N(\beta(U;r))\ B$. Then either $A \cap B \neq \emptyset$ and thus $A\ N(\beta(\beta(U;r)))\ B$ or $A \cap B = \emptyset$ and $A, B \neq \emptyset$ and $A\ \beta(U;r)\ B$. So $A \cup B \in \mathrm{CC}(U;r)$.

$\forall X, Y \in \mathscr{P}(A \cup B) \setminus \{\emptyset\} : (X \cup Y = A \cup B \wedge X \cap Y = \emptyset \Rightarrow X \cup Y \in \mathrm{CC}(U;r))$;

$\forall X, Y \in \mathscr{P}(A \cup B) \setminus \{\emptyset\} : (X \cup Y = A \cup B \wedge X \cap Y = \emptyset \Rightarrow X\ \beta(U;r)\ Y)$.

So $A \cup B \in \mathrm{CC}(\beta(U;r))$ that is $A\ \beta(\beta(U;r))\ B$ and thus $A\ N(\beta(\beta(U;r)))\ B$. □

Remark 7 $\mathrm{CC}(\beta(U;r)) = \mathrm{CC}(U;r)$ if $(U;r) = T(U;\tau)$ or $(U;r) = Q(U;\tau)$ for every equivalence link space $(U;\tau)$.

Question 1 *$\beta(\beta(U;r)) = \beta(U;r)$?*

Question 2 *Under which conditions $\mathrm{CC}(\beta(U;r)) = \mathrm{CC}(U;r)$ in general?*

6. Future research

How connectedness is related with continuity?

Research the lattice of connectors and the lattice of links.

To define product of two connectors is not trivial if possible at all.

We also may attempt to define quotient spaces for connectors.

In my further research I am going to study generalized connectedness of filters.

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