Connected funcoids and reloids*

BY VICTOR PORTON

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Abstract

Defined connectedness and connectivity for funcoids and reloids. Researched relationships of different kinds of connectedness. It is a part of my Algebraic General Topology research.

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1 Draft status

This article is a draft.

2 Prerequisites

For understanding this article prior reading of [1] is necessary.

3 Some lemmas

Lemma 1. If $\neg(A[f]B)\wedge A\cup B\supseteq \mathrm{dom}\, f\cup \mathrm{im}\, f$ then f is closed on A for a funcoid f and sets A and B.

Proof. $\neg(A[f]B)\Leftrightarrow B\cap\langle f\rangle A=\emptyset\Leftrightarrow(\mathrm{dom}\, f\cup\mathrm{im}\, f)\cap B\cap\langle f\rangle A=\emptyset\Rightarrow(\mathrm{dom}\, f\cup\mathrm{im}\, f\setminus A)\cap\langle f\rangle A=\emptyset\Leftrightarrow\langle f\rangle A\subseteq A$. □

Corollary 2. If $\neg(A[f]B)\wedge A\cup B\supseteq\mathrm{dom}\, f\cup\mathrm{im}\, f$ then f is closed on $A\setminus B$ for a funcoid f and sets A and B.

Proof. Let $\neg(A[f]B)\wedge A\cup B\supseteq\mathrm{dom}\, f\cup\mathrm{im}\, f$. Then $\neg((A\setminus B)[f]B)\wedge(A\setminus B)\cup B\supseteq\mathrm{dom}\, f\cup\mathrm{im}\, f$. □

Lemma 3. If $\neg(A[f]B)\wedge A\cup B\supseteq\mathrm{dom}\, f\cup\mathrm{im}\, f$ then $\neg(A[f^n]B)$ for any whole positive n.

Proof. Let $\neg(A[f]B)\wedge A\cup B\supseteq\mathrm{dom}\, f\cup\mathrm{im}\, f$. From the above proposition $\langle f\rangle A\subseteq A$. $B\cap\langle f\rangle A=\emptyset$, consequently $\langle f\rangle A\subseteq A\setminus B$. Because (by the above corollary) f is closed on $A\setminus B$, then $\langle f\rangle\langle f\rangle A\subseteq A\setminus B$, $\langle f\rangle\langle f\rangle\langle f\rangle A\subseteq A\setminus B$, etc. So $\langle f^n\rangle A\subseteq A\setminus B$, $B\cap\langle f^n\rangle A=\emptyset$, $\neg(A[f^n]B)$. □

4 Endomorphism series

Definition 4. $S_1(\mu)\overset{\mathrm{def}}{=}\mu\cup\mu^2\cup\mu^3\cup\ldots$ for an endomorphism μ of a precategory with countable union of morphisms.

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*. This document has been written using the GNU $\mathrm{T_EX_{MACS}}$ text editor (see `www.texmacs.org`).

Definition 5. $S(\mu)\overset{\mathrm{def}}{=}\mu^0\cup S_1(\mu)$ where $\mu^0\overset{\mathrm{def}}{=}I_{\mathrm{Ob}\,\mu}$ (identity morphism for the object $\mathrm{Ob}\ \mu$) where $\mathrm{Ob}\ \mu$ is the object of endomorphism μ for an endomorphism μ of a category with countable union of morphisms.

I call S_1 and S *endomorphism series*.

We will consider the collection of all binary relations (on a set $\mho$), as well as the collection of all funcoids and the collection of all reloids, as categories with single object $\mho$ and the identity morphism $(=)$ or $(=)|_{\mho}$.

So if μ is a binary relation or a funcoid or a reloid we have

$S_1(\mu)=\mu\cup\mu^2\cup\mu^3\cup\ldots$ and $S(\mu)=(=)\cup\mu\cup\mu^2\cup\mu^3\cup\ldots$

Proposition 6. $S(\mu)$ is transitive for the category of binary relations.

Proof.

$$\begin{aligned} S(\mu)\circ S(\mu) &= \mu^0\circ S(\mu)\cup\mu\circ S(\mu)\cup\mu^2\circ S(\mu)\cup\ldots\\ &= (\mu^0\cup\mu^1\cup\mu^2\cup\ldots)\cup(\mu^1\cup\mu^2\cup\mu^3\cup\ldots)\cup(\mu^2\cup\mu^3\cup\mu^4\cup\ldots)\\ &= \mu^0\cup\mu^1\cup\mu^2\cup\ldots\\ &= S(\mu). \end{aligned}$$

□

5 Connectedness regarding binary relations

Before going to research connectedness for funcoids and reloids we will excurse into the basic special case of connectedness regarding binary relations.

Definition 7. A set A is called *(strongly) connected* regarding a binary relation μ when

$$\forall X,Y\in\mathscr{P}\mho\setminus\{\emptyset\}:(X\cup Y\supseteq A\Rightarrow X[\mu]Y).$$

Definition 8. *Path* between two elements $a,b\in\mho$ in a set A through binary relation μ is the finite sequence $x_0\ldots x_n$ where $x_0=a$, $x_n=b$ for $n\in\mathbb{N}$ and $x_i(\mu\cap A\times A)x_{i+1}$ for any $i=0,\ldots,n-1$. n is called path *length*.

Proposition 9. There exists path between any element $a\in\mho$ and that element itself.

Proof. It is the path consisting of one vertex (of length 0). □

Proposition 10. There is a path from element a to element b in a set A through a binary relation μ iff $a\,(S(\mu\cap A\times A))\,b$ (that is $(a,b)\in S(\mu\cap A\times A)$).

Proof.

Direct implication. If exists a path from a to b, then $\{b\}\subseteq\langle(\mu\cap A\times A)^n\rangle\{a\}$ where n is the path length. Consequently $\{b\}\subseteq\langle S(\mu\cap A\times A)\rangle\{a\}$; $a\,(S(\mu\cap A\times A))\,b$.

Reverse implication. If $a\,(S(\mu\cap A\times A))\,b$ then exists $n\in\mathbb{N}$ such that $a\,(\mu\cap A\times A)^n\,b$. By definition of composition of binary relations this means that there exist finite sequence $x_0\ldots x_n$ where $x_0=a$, $x_n=b$ for $n\in\mathbb{N}$ and $x_i\,(\mu\cap A\times A)\,x_{i+1}$ for any $i=0,\ldots,n-1$. That is there is path from a to b. □

Theorem 11. The following statements are equivalent for a relation μ and a set A:

1. For any $a,b\in A$ there is a path between a and b in A through μ.
2. $S(\mu\cap A\times A)\supseteq A\times A$. [TODO: $\supseteq$ can be replaced with $=$.]
3. A is connected regarding μ.

Proof.

- **(1) ⇒ (2).** Let for any a, $b \in A$ there is a path between a and b in A through μ. Then $a\,(S(\mu\cap A\times A))\,b$ for any $a, b\in A$. It is possible only when $S(\mu\cap A\times A)\supseteq A\times A$.
- **(2) ⇒ (1).** For any two vertices a and b we have $a\,(S(\mu\cap A\times A))\,b$. So (by the previous theorem) for any two vertices a and b exist path from a to b.
- **(2) ⇒ (3).** Suppose that $\neg(X[\mu\cap A\times A]Y)$ for some $X, Y\in\mathscr{P}\mho\setminus\{\emptyset\}$ such that $X\cup Y\supseteq A$. Then by a lemma $\neg(X[(\mu\cap A\times A)^n]Y)$ for any $n\in\mathbb{N}$. Consequently $\neg(X[S(\mu\cap A\times A)]Y)$. So $S(\mu\cap A\times A)\neq A\times A$.
- **(3) ⇒ (2).** If $\langle S(\mu\cap A\times A)\rangle\{v\}=A$ for every vertex v then $S(\mu\cap A\times A)=A\times A$. Consider the remaining case when $V\overset{\text{def}}{=}\langle S(\mu\cap A\times A)\rangle\{v\}\subset A$ for some vertex v. Let $W=A\setminus V$. Then $V\cup W\supseteq A$ and consequently $V\cap A\cup W\cap A=(V\cup W)\cap A=A$ consequently $V\cap A[\mu]W\cap A$ that is $V[\mu\cap A\times A]W$ that is $\langle\mu\cap A\times A\rangle V\cap W\neq\emptyset$. This is impossible because $\langle\mu\cap A\times A\rangle V=\langle\mu\cap A\times A\rangle\langle S(\mu\cap A\times A)\rangle V=\langle S_1(\mu\cap A\times A)\rangle V\subseteq\langle S(\mu\cap A\times A)\rangle V=V$. □

Corollary 12. A set A is connected regarding a relation μ iff it is connected regarding $\mu\cap A\times A$. [TODO: Generalize this for funcoids and reloids.]

Definition 13. A *connected component* of a set A regarding a binary relation F is a maximal connected subset of A.

Theorem 14. The set A is partitioned into connected components (regarding any binary relation F).

Proof. Consider the binary relation $a\sim b\Leftrightarrow a\,(S(F))\,b\wedge b\,(S(F))\,a$. $\sim$ is a symmetric, reflexive, and transitive relation. So all points of A are partitioned into a collection of sets Q. Obviously each component is (strongly) connected. If a set $R\subseteq A$ is greater than one of that connected components A then it contains a point $b\in B$ where B is some other connected component. Consequently R is disconnected. □

Proposition 15. A set is connected (regarding a binary relation) iff it has one connected component.

Proof. Direct implication is obvious. Reverse is proved by contradiction. □

6 Connectedness regarding funcoids and reloids

Definition 16. $S_1^*(\mu)=\bigcap^{\mathscr{F}}\{S_1(M)\mid M\in\operatorname{up}\mu\}$ for a reloid μ.

Definition 17. *Connectivity reloid* for a reloid μ is defined as follows:

$$S^*(\mu)=\bigcap{}^{\mathscr{F}}\{S(M)\mid M\in\operatorname{up}\mu\}.$$

Remark 18. Do not mess the word *connectivity* with the word *connectedness* which means being connected.[1]

Proposition 19. $S^*(\mu)=(=)\cup S_1^*(\mu)$ for any reloid μ.

Proof. Follows from a theorem about distributivity of $\cup$ regarding $\bigcap^{\mathscr{F}}$. □

Proposition 20. $S^*(\mu)=S(\mu)$ if μ is a binary relation.

Proof. $S^*(\mu)=\bigcap^{\mathscr{F}}\{S(\mu)\}=S(\mu)$. □

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[1] In some math literature these two words are used interchangeably.

Definition 21. A filter $\mathcal{A}$ is called *connected* regarding a reloid μ when $S^*(\mu) \supseteq \mathcal{A} \times \mathcal{A}$.

Definition 22. A filter $\mathcal{A}$ is called *connected* regarding a funcoid μ when

$$\forall X, Y \in \mathscr{P}\mho \setminus \{\emptyset\}: (X \cup Y \supseteq \mathcal{A} \Rightarrow X[\mu]Y).$$

Theorem 23. A filter $\mathcal{A}$ is connected regarding a funcoid μ iff

$$\forall \mathcal{X}, \mathcal{Y} \in \mathscr{F} \setminus \{\emptyset\}: (\mathcal{X} \cup \mathcal{Y} \supseteq \mathcal{A} \Rightarrow \mathcal{X}[\mu]\mathcal{Y}).$$

Proof.

Reverse implication. Obvious.

Direct implication. Let $\mathcal{X}$, $\mathcal{Y} \in \mathscr{F}$, $\mathcal{X} \cup \mathcal{Y} \supseteq \mathcal{A}$. Then $X \cup Y \supseteq \mathcal{A}$ for any $X \in \mathrm{up}\mathcal{X}$, $Y \in \mathrm{up}\mathcal{Y}$. Consequently $\forall X \in \mathrm{up}\mathcal{X}, Y \in \mathrm{up}\mathcal{Y}: X[\mu]Y$; $\mathcal{X}[\mu]\mathcal{Y}$. □

Theorem 24. A filter $\mathcal{A}$ is connected regarding a funcoid μ iff $\mathcal{A}$ is connected for every binary relation $F \supseteq \mu$.

Proof.

Direct implication. Obvious.

Reverse implication. Let $\mathcal{A}$ is not connected regarding μ, that is $\neg(X[\mu]Y)$ for some X, $Y \in \mathscr{P}\mho \setminus \{\emptyset\}$ such that $X \cup Y \supseteq \mathcal{A}$. We can assume that $X \cap Y = \emptyset$. (Otherwise replace Y with $Y \setminus X$ or X with $X \setminus Y$; if both $Y \setminus X$ and $X \setminus Y$ are empty then $X = Y = \mathcal{A}$ what is impossible.) Also we can assume $X \cup Y = A$. Consider the relation $F = (X \times X) \cup (Y \times \mathcal{A})$. For any atomic filter a

$$\langle F \rangle a = \left(\begin{cases} X & \text{if } a \subseteq X \\ \mathcal{A} & \text{if } a \nsubseteq X \end{cases} \right) \supseteq \langle \mu \cap \mathcal{A} \times \mathcal{A} \rangle a$$

So $\mu \cap \mathcal{A} \times \mathcal{A} \subseteq F$. Consequently $\mu \cap A \times A \subseteq F$ for some $A \in \mathrm{up}\,\mathcal{A}$. So

$$F_2 = F \cup (\mho \times \mho \setminus A \times A) \supseteq \mu$$

Obviously $\neg(X[F]Y)$ and consequently $\neg(X[F_2]Y)$. So $\mathcal{A}$ is not connected regarding F_2. □

Theorem 25. The following statements are equivalent for a binary relation F and a filter $\mathcal{A}$:

1. $\mathcal{A}$ is connected regarding F as a funcoid.
2. $\mathcal{A}$ is connected regarding F as a reloid.

Proof. From the previous section. □

Theorem 26. A filter $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding every $F \in \mathrm{up}\, f$.

Proof.

Direct implication. Obvious.

Reverse implication. F is connected iff $S(F) = F^0 \cup F^1 \cup F^2 \cup \ldots \supseteq \mathcal{A} \times \mathcal{A}$.
$S^*(f) = \bigcap^{\mathscr{F}} \{S(F) \mid F \in \mathrm{up}\, f\} \supseteq \bigcap^{\mathscr{F}} \{\mathcal{A} \times \mathcal{A} \mid F \in \mathrm{up}\, f\} = \mathcal{A} \times \mathcal{A}$. □

Theorem 27. A filter $\mathcal{A}$ is connected regarding a reloid f iff it is connected regarding the funcoid $(\mathsf{FCD})f$.

Proof. f is connected iff every element of $\mathrm{up}\, f$ is connected. But

$$\mathrm{up}\, f = \{F \mid F \in \mathscr{P}\mho \wedge F \supseteq (\mathsf{FCD})f\}.$$

From the theorem 24 follows that these are equivalent. □

7 Algebraic properties of S and S^*

Theorem 28. $S^*(S^*(f)) = S^*(f)$ for any reloid f.

Proof. $S^*(S^*(f)) = \bigcap^{\mathscr{F}}\{S(R) \mid R \in \operatorname{up} S^*(f)\} \subseteq \bigcap^{\mathscr{F}}\{S(R) \mid R \in \{S(F) \mid F \in \operatorname{up} f\}\} = \bigcap^{\mathscr{F}}\{S(S(F)) \mid F \in \operatorname{up} f\} = \bigcap^{\mathscr{F}}\{S(F) \mid F \in \operatorname{up} f\} = S^*(f)$.

So $S^*(S^*(f)) \subseteq S^*(f)$. $S^*(S^*(f)) \supseteq S^*(f)$ is obvious. □

Corollary 29. $S^*(S(f)) = S(S^*(f)) = S^*(f)$ for any reloid f.

Proof. Obviously $S^*(S(f)) \supseteq S^*(f)$ and $S(S^*(f)) \supseteq S^*(f)$.

But $S^*(S(f)) \subseteq S^*(S^*(f)) = S^*(f)$ and $S(S^*(f)) \subseteq S^*(S^*(f)) = S^*(f)$. □

Conjecture 30. $S(S(f)) = S(f)$ for

1. any reloid f;
2. any funcoid f.

Conjecture 31. For any reloid f

1. $S(f) \circ S(f) = S(f)$;
2. $S^*(f) \circ S^*(f) = S^*(f)$;
3. $S(f) \circ S^*(f) = S^*(f) \circ S(f) = S^*(f)$.

Conjecture 32. $S(f) \circ S(f) = S(f)$ for any funcoid f.

Bibliography

[1] Victor Porton. Funcoids and reloids. At http://www.mathematics21.org/binaries/funcoids-reloids.pdf.