Compact funcoids

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Abstract

Compact funcoids are defined. Under certain conditions it's proved that the reloid corresponding to a compact funcoid is the neighbourdhood of the diagonal of the product funcoid.

Preface

This is a rough partial draft. The proofs are with errors.

In order to understand it, first read my book [2] and this draft article [1].

The rest

Definition 1. A funcoid f is *directly compact* iff

$$\forall \mathcal{F} \in \mathfrak{F}: (\langle f \rangle \mathcal{F} \neq 0 \Rightarrow \operatorname{Cor} \langle f \rangle \mathcal{F} \neq 0).$$

Obvious 2. A funcoid f is directly compact iff $\forall a \in \operatorname{atoms} \operatorname{dom} f: \operatorname{Cor} \langle f \rangle a \neq 0$.

Definition 3. A funcoid f is *reversely compact* iff f^{-1} is directly compact.

Definition 4. A funcoid is *compact* iff it is both directly compact and reversely compact.

Proposition 5. $\prod^{\mathsf{RLD}} a = \uparrow^{\mathsf{RLD}} \prod_{i \in \operatorname{dom} a} (\uparrow^{\mathsf{RLD}})^{-1} a_i$ for every indexed family a of principal filters.

Proof. Because $\prod_{i \in \operatorname{dom} a} (\uparrow^{\mathsf{RLD}})^{-1} a_i \in \operatorname{GR} \prod^{\mathsf{RLD}} a$. [TODO: More detailed proof.] □

Lemma 6. $\prod^{\mathsf{RLD}}_{i \in \operatorname{dom} a} \operatorname{Cor} a_i = \operatorname{Cor} \prod^{\mathsf{RLD}} a$.

Proof. $\operatorname{Cor} \prod^{\mathsf{RLD}} a = \prod \{\uparrow^{\mathsf{RLD}} \prod A \mid A \in \operatorname{up} a\} = \uparrow^{\mathsf{RLD}} \bigcap \{\prod A \mid A \in \operatorname{up} a\} = \uparrow^{\mathsf{RLD}} \bigcap \{\prod A \mid A \in \mathscr{P} \prod \mathfrak{U}, \forall i \in \operatorname{dom} a: A_i \in \operatorname{up} a_i\} = \uparrow^{\mathsf{RLD}} \bigcap \{\prod \bigcap K_i \mid K \in \mathscr{P} \mathscr{P} \prod \mathfrak{U}, \forall i \in \operatorname{dom} a: K_i \in \mathscr{P} \operatorname{up} a_i\} = \uparrow^{\mathsf{RLD}} \bigcap \{\prod (\uparrow^{\mathsf{RLD}})^{-1} \operatorname{Cor} a_i \mid i \in \operatorname{dom} a\} = \uparrow^{\mathsf{RLD}} \prod^{\mathsf{RLD}}_{i \in \operatorname{dom} a} \operatorname{Cor} a_i$. [TODO: Check for little errors.] □

Corollary 7. $\prod^{\mathsf{RLD}}_{i \in n} \langle \operatorname{CoCompl} f_i \rangle \mathcal{X}_i = \left\langle \operatorname{CoCompl} \prod^{(A)} f \right\rangle \prod^{\mathsf{RLD}} \mathcal{X}$ for every n-indexed families f of funcoids and $\mathcal{X}$ of filters on the same set (with $\operatorname{Src} f_i = \operatorname{Base}(\mathcal{X}_i)$ for every $i \in n$).

Proof.

$$
\begin{aligned}
\prod^{\mathsf{RLD}}_{i\in n} \langle \operatorname{CoCompl} f_i\rangle \mathcal{X}_i &= \\
\prod^{\mathsf{RLD}}_{i\in n} \operatorname{Cor} \langle f_i\rangle \mathcal{X}_i &= \\
\operatorname{Cor} \prod^{\mathsf{RLD}}_{i\in n} \langle f_i\rangle \mathcal{X}_i &= (*) \\
\operatorname{Cor} \prod^{\mathsf{RLD}}_{i\in n} \langle f_i\rangle \operatorname{Pr}^{\mathsf{RLD}}_i \left(\prod^{\mathsf{RLD}} \mathcal{X} \right) &= \\
\operatorname{Cor} \left\langle \prod^{(A)} f \right\rangle \prod^{\mathsf{RLD}} \mathcal{X} &= \\
\left\langle \operatorname{CoCompl} \prod^{(A)} f \right\rangle \prod^{\mathsf{RLD}} \mathcal{X}.
\end{aligned}
$$

(*) You should verify the special case when $\mathcal{X}_i = 0^{\mathfrak{F}}$ for some i. □

Theorem 8. Let f be an indexed family of funcoids. [TODO: Reverse theorem (for non-least funcoids).]

1. $\prod f$ is directly compact if every f_i is directly compact.
2. $\prod f$ is reversely compact if every f_i is reversely compact.
3. $\prod f$ is compact if every f_i is compact.

Proof. It is enough to prove only the first statement.

Let each f_i is directly compact.

Let $\langle \prod f\rangle a \neq 0$. Then $\langle \prod f\rangle a = \left\langle \prod^{(A)} f \right\rangle a = \prod^{\mathsf{RLD}}_{i\in \operatorname{dom} f} \langle f_i\rangle \operatorname{Pr}^{\mathsf{RLD}}_i a$. Thus every $\langle f_i\rangle \operatorname{Pr}^{\mathsf{RLD}}_i a \neq 0$. Consequently by compactness $\operatorname{Cor} \langle f_i\rangle \operatorname{Pr}^{\mathsf{RLD}}_i a \neq 0$; $\prod_{i\in \operatorname{dom} f} \operatorname{Cor} \langle f_i\rangle \operatorname{Pr}^{\mathsf{RLD}}_i a \neq 0$; $\operatorname{Cor} \prod_{i\in \operatorname{dom} f} \langle f_i\rangle \operatorname{Pr}^{\mathsf{RLD}}_i a \neq 0$; $\operatorname{Cor} \langle \prod f\rangle a \neq 0$.

So $\prod f$ is directly compact. □

I will denote Δ the diagonal relation.

Proposition 9. The following expressions are pairwise equal:

1. $\langle f \times f\rangle^{*} \Delta$;
2. $\bigsqcup \{ \langle f \times f\rangle p \mid p \in \operatorname{atoms} \Delta\}$;
3. $\bigsqcup \{ \langle f\rangle x \times^{\mathsf{RLD}} \langle f\rangle x \mid x \in \mathfrak{F}\}$;
4. $(\mathsf{RLD})_{\mathrm{in}} \bigsqcup \{ \langle f\rangle x \times^{\mathsf{FCD}} \langle f\rangle x \mid x \in \mathfrak{F}\}$;
5. $(\mathsf{RLD})_{\mathrm{in}} \bigsqcup \{ f \circ (x \times^{\mathsf{FCD}} x) \circ f^{-1} \mid x \in \mathfrak{F}\}$;
6. $\bigsqcup \{ ((\mathsf{RLD})_{\mathrm{in}} f) \circ (x \times^{\mathsf{RLD}} x) \circ ((\mathsf{RLD})_{\mathrm{in}} f)^{-1} \mid x \in \mathfrak{F}\}$.

7. $(\mathsf{RLD})_{\mathrm{in}}(f^{-1}\circ f)$ [TODO: Use this below.]

Proof. ?? □

Proposition 10. Let g be a reloid and $f=(\mathsf{FCD})g$. Then $\langle f\times f\rangle^{*}\Delta\sqsupseteq g$.

Proof. $\langle f\times f\rangle^{*}\Delta\not\asymp\uparrow^{\mathsf{RLD}}Y\Leftrightarrow\uparrow^{\mathsf{RLD}}\Delta\,[f\times f]\uparrow^{\mathsf{RLD}}Y\Leftrightarrow\uparrow^{\mathsf{FCD}}\Delta\left[f\times^{(C)}f\right]\uparrow^{\mathsf{FCD}}Y\Leftrightarrow f\circ\uparrow^{\mathsf{FCD}}\Delta\circ f^{-1}\not\asymp\uparrow^{\mathsf{FCD}}Y\Leftrightarrow f\circ f^{-1}\not\asymp\uparrow^{\mathsf{FCD}}Y\Leftrightarrow f\not\asymp\uparrow^{\mathsf{FCD}}Y\Leftrightarrow f\sqcap\uparrow^{\mathsf{FCD}}Y\neq 0\Leftarrow(\mathsf{RLD})_{\mathrm{in}}(f\sqcap\uparrow^{\mathsf{FCD}}Y)\neq 0\Leftrightarrow(\mathsf{RLD})_{\mathrm{in}}\,f\sqcap(\mathsf{RLD})_{\mathrm{in}}\uparrow^{\mathsf{FCD}}Y\neq 0\Leftarrow(\mathsf{RLD})_{\mathrm{in}}\,f\sqcap(\mathsf{RLD})_{\mathrm{out}}\uparrow^{\mathsf{FCD}}Y\neq 0\Leftrightarrow(\mathsf{RLD})_{\mathrm{in}}\,f\sqcap\uparrow^{\mathsf{RLD}}Y\neq 0\Leftrightarrow(\mathsf{RLD})_{\mathrm{in}}\,(\mathsf{FCD})g\sqcap\uparrow^{\mathsf{RLD}}Y\neq 0\Leftarrow g\sqcap\uparrow^{\mathsf{RLD}}Y\neq 0\Leftrightarrow g\not\asymp\uparrow^{\mathsf{RLD}}Y$. □

Proposition 11. Let f be a funcoid. Then $V\circ M\circ V^{-1}\in\mathrm{GR}\,\langle f\times f\rangle^{*}M$ for every $V\in\mathrm{GR}\,f$.

Proof. $V\circ M\circ V^{-1}\in\mathrm{GR}(f\circ\uparrow M\circ f^{-1})=\mathrm{GR}\left\langle f\times^{(C)}f\right\rangle\uparrow M\supseteq\mathrm{GR}\,\langle f\times f\rangle\uparrow M=\mathrm{GR}\,\langle f\times f\rangle^{*}M$. [FIXME: Wrong direction of $\supseteq$.]

Because

$\uparrow^{\mathsf{FCD}}X\not\asymp\left\langle f\times^{(C)}f\right\rangle\uparrow^{\mathsf{FCD}}M\Leftrightarrow\uparrow^{\mathsf{RLD}}X\not\asymp\langle f\times f\rangle\uparrow^{\mathsf{RLD}}M\Leftrightarrow(\mathsf{FCD})(\uparrow^{\mathsf{RLD}}X\sqcap\langle f\times f\rangle\uparrow^{\mathsf{RLD}}M)\neq 0\Rightarrow(\mathsf{FCD})\uparrow^{\mathsf{RLD}}X\sqcap(\mathsf{FCD})\langle f\times f\rangle\uparrow^{\mathsf{RLD}}M\neq 0\Leftrightarrow(\mathsf{FCD})\uparrow^{\mathsf{RLD}}X\not\asymp(\mathsf{FCD})\langle f\times f\rangle\uparrow^{\mathsf{RLD}}M\Leftrightarrow\uparrow^{\mathsf{FCD}}X\not\asymp(\mathsf{FCD})\langle f\times f\rangle\uparrow^{\mathsf{RLD}}M$;

$\left\langle f\times^{(C)}f\right\rangle\uparrow^{\mathsf{FCD}}M\sqsubseteq(\mathsf{FCD})\langle f\times f\rangle\uparrow^{\mathsf{RLD}}M$

$\mathrm{GR}\left\langle f\times^{(C)}f\right\rangle\uparrow^{\mathsf{FCD}}M\supseteq\mathrm{GR}\,(\mathsf{FCD})\langle f\times f\rangle\uparrow^{\mathsf{RLD}}M\supseteq\mathrm{GR}\,\langle f\times f\rangle\uparrow^{\mathsf{RLD}}M$ □

Proposition 12. $\langle f\times f\rangle^{*}M\sqsubseteq g\circ\uparrow^{\mathsf{RLD}}M\circ g^{-1}$ whenever $(\mathsf{FCD})g=f$ for a reloid g.

Proof. For every $V\in\mathrm{GR}\,g$ we have $V\circ M\circ V^{-1}\in\mathrm{GR}\,\langle f\times f\rangle^{*}M$. Thus $g\circ\uparrow^{\mathsf{RLD}}M\circ g^{-1}=\bigsqcap\{V\circ M\circ V^{-1}\mid V\in\mathrm{GR}\,g\}\sqsupseteq\bigsqcap\mathrm{GR}\,\langle f\times f\rangle^{*}M=\mathrm{GR}\,\langle f\times f\rangle^{*}M$. □

Corollary 13. $\langle f\times f\rangle^{*}M\sqsubseteq\left\langle f\times^{(C)}f\right\rangle^{*}M$.

Corollary 14. $V\circ V^{-1}\in\mathrm{GR}\,\langle f\times f\rangle^{*}\Delta$; $f\circ f^{-1}\sqsupseteq\langle f\times f\rangle^{*}\Delta$.

Proof. ?? □

Lemma 15. $\mathrm{Cor}\,\langle f\times f\rangle^{*}g\sqsubseteq\Delta$ if $(\mathsf{FCD})g=f$ where $(\mathsf{FCD})g=f$ for a T_1-separable reloid g.

Proof. ?? □

Remark 16. I attempted to generalize the below theorem more than the standard general topology theorem about correspondence of compact and uniform spaces, but haven't really succeeded much, as it appears to be needed that the reloid in question is reflexive, symmetric, and transitive, that is just a uniform space as in the standard general topology.

Theorem 17. Let f be a T_1-separable compact reflexive symmetric funcoid and g be a reloid such that

1. $(\mathsf{FCD})g=f$;

2. $g\circ g^{-1}\sqsubseteq g$.

Then $g = \langle f \times f \rangle^\ast \Delta$.

Proof. From the above $\langle f \times f \rangle^\ast \Delta \sqsubseteq g \circ g^{-1} \sqsubseteq g$. [FIXME: Funcoids and reloids are confused.]

It's remainded to prove $g \sqsubseteq \langle f \times f \rangle^\ast \Delta$.

[FIXME: Possible errors.]

Suppose there is $U \in \mathrm{xyGR}\, \langle f \times f \rangle^\ast \Delta$ such that $U \notin \mathrm{GR}\, g$.

Then $\{V \setminus U \mid V \in \mathrm{GR}\, g\} = g \setminus U$ would be a proper filter.

Thus by reflexivity $\langle f \times f \rangle^\ast (g \setminus U) \neq 0$.

By compactness of $f \times f$, $\mathrm{Cor}\, \langle f \times f \rangle^\ast (g \setminus U) \neq 0$.

Suppose $\uparrow \{(x; x)\} \sqsubseteq \langle f \times f \rangle^\ast (g \setminus U)$; then $g \setminus U \not\asymp \langle f^{-1} \times f^{-1} \rangle \{(x; x)\}$; $U \sqsubset \langle f^{-1} \times f^{-1} \rangle \{(x; x)\} \sqsubseteq \langle f^{-1} \times f^{-1} \rangle \Delta$ what is impossible.

Thus there exist $x \neq y$ such that $\{(x; y)\} \sqsubseteq \mathrm{Cor}\, \langle f \times f \rangle^\ast (g \setminus U)$. Thus $\{(x; y)\} \sqsubseteq \langle f \times f \rangle^\ast g$.

Thus by the lemma $\{(x; y)\} \sqsubseteq \Delta$ what is impossible. So $U \in \mathrm{GR}\, g$.

We have $\mathrm{xyGR}\, \langle f \times f \rangle^\ast \Delta \subseteq \mathrm{GR}\, g$; $\langle f \times f \rangle^\ast \Delta \sqsupseteq g$. □

Corollary 18. Let f is a T_1-separable (the same as T_2 for symmetric transitive) compact funcoid and g is a uniform space (reflexive, symmetric, and transitive endoreloid) such that $(\mathsf{FCD}) g = f$. Then $g = \langle f \times f \rangle^\ast \Delta$.

An (incomplete) attempt to prove one more theorem follows:

Theorem 19. Let μ and ν be uniform spaces, $(\mathsf{FCD})\mu$ be a compact funcoid. Then a map f is a continuous map from $(\mathsf{FCD})\mu$ to $(\mathsf{FCD})\nu$ iff f is a (uniformly) continuous map from μ to ν.

Proof. [FIXME: errors in this proof.]

We have $\mu = \langle (\mathsf{FCD})\mu \times (\mathsf{FCD})\mu \rangle \uparrow^{\mathsf{RLD}} \Delta$

$f \in \mathrm{C}_?((\mathsf{FCD})\mu; (\mathsf{FCD})\nu)$. Then

$$f \times f \in \mathrm{C}_?((\mathsf{FCD})(\mu \times \mu); (\mathsf{FCD})(\nu \times \nu))$$

$(f \times f) \circ (\mathsf{FCD})(\mu \times \mu) \sqsubseteq (\mathsf{FCD})(\nu \times \nu) \circ (f \times f)$

For every $V \in \mathrm{GR}(\nu \times \nu)$ we have $\langle g^{-1} \rangle V \in \langle (\mathsf{FCD})(\mu \times \mu) \rangle \{y\}$ for some y.

$\langle g^{-1} \rangle V \in \langle (\mathsf{FCD})\mu \times (\mathsf{FCD})\mu \rangle \uparrow^{\mathsf{RLD}} \Delta = \mathrm{GR}\, \mu$

$\langle g \rangle \langle g^{-1} \rangle V \sqsubseteq V$

We need to prove $f \in \mathrm{C}(\mu; \nu)$ that is $\forall p \in \mathrm{GR}\, \nu \exists q \in \mathrm{GR}\, \mu: \langle f \rangle q \sqsubseteq p$. But this follows from the above. □

Bibliography

[1] Victor Porton. Categorical product of funcoids. At `http://www.mathematics21.org/binaries/product.pdf`.

[2] Victor Porton. *Algebraic General Topology. Volume 1*. 2013.