Cauchy Filters on Reloids

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Abstract

In this article I consider *low filters* on reloids, generalizing Cauchy filters on uniform spaces. Using low filters, I define Cauchy-complete reloids, generalizing complete uniform spaces.

1 Preface

This is a preliminary partial draft.

To understand this article you need first look into my book [1].

As my book is yet in preprint stage and I may change it, I probably will integrate the content of this article into the book.

http://math.stackexchange.com/questions/401989/what-are-interesting-properties-of-totally-bounded-uniform-spaces

http://ncatlab.org/nlab/show/proximity+space#uniform_spaces for a proof sketch that proximities correspond to totally bounded uniformities.

2 Low filters space

Definition 1. A *lower set*[1] *of proper filters* on U (a set) is a set $\mathscr{C}$ of proper filters on U, such that if $0 \neq \mathcal{G} \sqsubseteq \mathcal{F}$ and $\mathcal{F} \in \mathscr{C}$ then $\mathcal{G} \in \mathscr{C}$. [TODO: Probably should include the improper filter.]

Definition 2. I call *low filters space* a set together with a lower set of proper filters on this set.

Definition 3. $\operatorname{PR}(U; \mathscr{C}) = \mathscr{C}$; $\operatorname{Ob}(U; \mathscr{C}) = U$.[2]

Definition 4. Introduce an order on low filters spaces: $(U; \mathscr{C}) \sqsubseteq (U; \mathscr{D}) \Leftrightarrow \mathscr{C} \sqsubseteq \mathscr{D}$.

3 Cauchy spaces

Definition 5. A *Cauchy space* on a set X is a low filters space $(U; \mathscr{C})$ (element of $\mathscr{C}$ are called *Cauchy filters*) such that:

1. $\forall x \in U: \uparrow^X \{x\} \in \mathscr{C}$;
2. If $\mathcal{F}$, $\mathcal{G}$ are Cauchy filters and $\mathcal{F} \not\asymp \mathcal{G}$ then $\mathcal{F} \sqcup \mathcal{G}$ is a Cauchy filter.

Definition 6. A *completely Cauchy space* on a set X is a low filters space $(U; \mathscr{C})$ (element of $\mathscr{C}$ are called *Cauchy filters*) such that:

1. $\forall x \in X: \uparrow^X \{x\} \in \mathscr{C}$;
2. If S is a nonempty set of Cauchy filters and $\bigsqcap S \neq 0^{\mathfrak{F}(X)}$ then $\bigsqcup S$ is a Cauchy filter.

1. *Remember that our orders on filters is the reverse to set theoretic inclusion. It could be called an* upper *set in other sources.*

2. PR is from English word *profile*.

Obvious 7. Every completely Cauchy space is a Cauchy space.

Proposition 8. $\bigsqcup^{\{\mathcal{X}\in\mathscr{C} \mid \mathcal{X}\sqsupseteq\mathcal{F}\}} S = \bigsqcup S$ for nonempty $S\in\mathscr{P}\{\mathcal{X}\in\mathscr{C} \mid \mathcal{X}\sqsupseteq\mathcal{F}\}$, provided that $\mathcal{F}$ is a fixed Cauchy filter on a completely Cauchy space.

Proof. $\mathcal{F}$ is proper. So for every nonempty $S\in\mathscr{P}\{\mathcal{X}\in\mathscr{C} \mid \mathcal{X}\sqsupseteq\mathcal{F}\}$ we have $\bigsqcap S\sqsupseteq\mathcal{F}\neq 0^{\mathfrak{F}(X)}$. Thus $\bigsqcup S$ is a Cauchy filter and so $\bigsqcup S\in\{\mathcal{X}\in\mathscr{C} \mid \mathcal{X}\sqsupseteq\mathcal{F}\}$. □

Proposition 9. If $\mathcal{F}$ is a fixed Cauchy filter on a completely Cauchy space, then the poset $\{\mathcal{X}\in\mathscr{C} \mid \mathcal{X}\sqsupseteq\mathcal{F}\}$ (with the induced order) is a complete lattice.

Proof. If $S\neq\emptyset$ then $\bigsqcup^{\{\mathcal{X}\in\mathscr{C} \mid \mathcal{X}\sqsupseteq\mathcal{F}\}} S = \bigsqcup S$. If $S=\emptyset$ then $\bigsqcup^{\{\mathcal{X}\in\mathscr{C} \mid \mathcal{X}\sqsupseteq\mathcal{F}\}} S = \mathcal{F}$. □

Corollary 10. If $\mathcal{F}$ is a fixed Cauchy filter on a completely Cauchy space, then the poset $\{\mathcal{X}\in\mathscr{C} \mid \mathcal{X}\sqsupseteq\mathcal{F}\}$ (with the induced order) has a maximum.

4 Relationships with symmetric reloids

Definition 11. Denote $(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C}) = \bigsqcup\{\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X} \mid \mathcal{X}\in\mathscr{C}\}$.

Definition 12. $(\mathrm{Low})\nu$ (*low filters* for reloid ν) is a low filters space on U such that

$$\mathrm{PR}\,(\mathrm{Low})\nu = \{\mathcal{X}\in\mathfrak{F}^U\setminus\{0^{\mathfrak{F}}\} \mid \mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq\nu\}.$$

Theorem 13. If $(U;\mathscr{C})$ is a low filters space, then $(U;\mathscr{C}) = (\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$.

Proof. If $\mathcal{X}\in\mathscr{C}$ then $\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$ and thus $\mathcal{X}\in\mathrm{PR}\,(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$. Thus $(U;\mathscr{C})\sqsubseteq(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$.

Let's prove $(U;\mathscr{C})\sqsupseteq(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$.

Let $\mathcal{A}\in\mathrm{PR}\,(\mathrm{Low})(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$. We need to prove $\mathcal{A}\in\mathscr{C}$.

Really $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\sqsubseteq(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$. It is enough to prove that $\exists\mathcal{X}\in\mathscr{C}:\mathcal{A}\sqsubseteq\mathcal{X}$.

Suppose $\nexists\mathcal{X}\in\mathscr{C}:\mathcal{A}\sqsubseteq\mathcal{X}$.

For every $\mathcal{X}\in\mathscr{C}$ obtain $X_{\mathcal{X}}\in\mathcal{X}$ such that $X_{\mathcal{X}}\notin\mathcal{A}$ (if forall $X\in\mathcal{X}$ we have $X_{\mathcal{X}}\in\mathcal{A}$, then $\mathcal{X}\sqsupseteq\mathcal{A}$ what is contrary to our supposition).

It is now enough to prove $\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A}\not\sqsubseteq\bigsqcup\{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}} \mid \mathcal{X}\in\mathscr{C}\}$.

Really, $\bigsqcup\{\uparrow^U X_{\mathcal{X}}\times^{\mathsf{RLD}}\uparrow^U X_{\mathcal{X}} \mid \mathcal{X}\in\mathscr{C}\} = \uparrow^{\mathsf{RLD}(U;U)}\bigcup\{X_{\mathcal{X}}\times X_{\mathcal{X}} \mid \mathcal{X}\in\mathscr{C}\}$. So our claim takes the form $\bigcup\{X_{\mathcal{X}}\times X_{\mathcal{X}} \mid \mathcal{X}\in\mathscr{C}\}\notin\mathrm{GR}(\mathcal{A}\times^{\mathsf{RLD}}\mathcal{A})$ that is $\forall A\in\mathcal{A}:\bigcup\{X_{\mathcal{X}}\times X_{\mathcal{X}} \mid \mathcal{X}\in\mathscr{C}\}\not\supseteq A\times A$ what is true because $X_{\mathcal{X}}\not\supseteq A$ for every $A\in\mathcal{A}$. □

Remark 14. The last theorem does not hold with $\mathcal{X}\times^{\mathsf{FCD}}\mathcal{X}$ instead of $\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}$ (take $\mathscr{C} = \{\{x\} \mid x\in U\}$ for an infinite set U as a counter-example).

Remark 15. Not every symmetric reloid is in the form $(\mathsf{RLD})_{\mathrm{Low}}(U;\mathscr{C})$ for some Cauchy space $(U;\mathscr{C})$. The same Cauchy space can be induced by different uniform spaces. See http://math.stackexchange.com/questions/702182/different-uniform-spaces-having-the-same-set-of-cauchy-filters

[TODO: Is composition of two images of low filter spaces also a low filters space?]

5 More on Cauchy filters

Obvious 16. Low filter on an endoreloid ν is a filter $\mathcal{F}$ such that

$$\forall U\in\mathrm{GR}\,f\,\exists A\in\mathcal{F}: A\times A\subseteq U.$$

Remark 17. The above formula is the standard definition of Cauchy filters on uniform spaces.

Proposition 18. If $\nu \sqsupseteq \nu \circ \nu^{-1}$ then every neighborhood filter is a Cauchy filter, that it

$$\nu \sqsupseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\}$$

for every point x.

Proof. $\langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\} = \langle (\mathsf{FCD})\nu \rangle \uparrow^{\mathrm{Ob}\,\nu} \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle \uparrow^{\mathrm{Ob}\,\nu} \{x\} = \nu \circ (\uparrow^{\mathrm{Ob}\,\nu} \{x\} \times^{\mathsf{RLD}} \uparrow^{\mathrm{Ob}\,\nu} \{x\}) \circ \nu^{-1} = \nu \circ \left(\uparrow^{\mathsf{RLD}(\mathrm{Ob}\,\nu; \mathrm{Ob}\,\nu)} \{(x; x)\} \right) \circ \nu^{-1} \sqsubseteq \nu \circ \mathrm{id}^{\mathsf{RLD}(\mathrm{Ob}\,\nu; \mathrm{Ob}\,\nu)} \circ \nu^{-1} = \nu \circ \nu^{-1} \sqsubseteq \nu$. □

Proposition 19. If a filter converges to a point, it is a low filter, provided that every neighborhood filter is a low filter.

Proof. Let $\mathcal{F} \sqsubseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\}$. Then $\mathcal{F} \times^{\mathsf{RLD}} \mathcal{F} \sqsubseteq \langle (\mathsf{FCD})\nu \rangle^* \{x\} \times^{\mathsf{RLD}} \langle (\mathsf{FCD})\nu \rangle^* \{x\} \sqsubseteq \nu$. □

Corollary 20. If a filter converges to a point, it is a low filter, provided that $\nu \sqsupseteq \nu \circ \nu^{-1}$.

6 Maximal Cauchy filters

Lemma 21. Let S be a set of sets with $\bigsqcap \langle \uparrow^{\mathfrak{F}} \rangle S \neq 0^{\mathfrak{F}}$ (in other words, S has finite intersection property). Let $T = \{X \times X \mid X \in S\}$. Then

$$\bigcup T \circ \bigcup T = \bigcup S \times \bigcup S.$$

Proof. Let $x \in \bigcup S$. Then $x \in X$ for some $X \in S$. $\langle \bigcup T \rangle \{x\} \sqsupseteq \uparrow X \supseteq \bigcap S \neq \emptyset$. Thus

$\langle \bigcup T \circ \bigcup T \rangle \{x\} = \langle \bigcup T \rangle \langle \bigcup T \rangle \{x\} \in \langle \uparrow^{\mathsf{FCD}} \bigcup T \rangle \bigsqcap \langle \uparrow^{\mathfrak{F}} \rangle S \sqsupseteq \bigsqcup \{ \langle \uparrow^{\mathsf{FCD}} (X \times X) \rangle \bigsqcap \langle \uparrow^{\mathfrak{F}} \rangle S \mid X \in S\} = \bigsqcup \{ \uparrow^{\mathfrak{F}} X \mid X \in S\} = \bigsqcup \langle \uparrow^{\mathfrak{F}} \rangle S$ that is $\langle \bigcup T \circ \bigcup T \rangle \{x\} \supseteq \bigcup S$. □

Corollary 22. Let S be a set of filters (on some fixed set) with nonempty meet. Let

$$T = \{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \mid \mathcal{X} \in S\}$$

Then

$$\bigsqcup T \circ \bigsqcup T = \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S.$$

Proof. $\bigsqcup T \circ \bigsqcup T = \bigsqcap \{ \uparrow^{\mathfrak{F}} (X \circ X) \mid X \in \bigsqcup T\}$.

If $X \in \bigsqcup T$ then $X = \bigcup_{Q \in T} (P_Q \times P_Q)$ where $P_Q \in Q$. Therefore by the lemma we have

$$\bigcup \{P_Q \times P_Q \mid Q \in T\} \circ \bigcup \{P_Q \times P_Q \mid Q \in T\} = \bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q.$$

Thus $X \circ X = \bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q$.

Consequently $\bigsqcup T \circ \bigsqcup T = \bigsqcap \left\{ \uparrow^{\mathfrak{F}} \left(\bigcup_{Q \in T} P_Q \times \bigcup_{Q \in T} P_Q \right) \mid X \in \bigsqcup T \right\} \sqsupseteq \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S$.

$\bigsqcup T \circ \bigsqcup T \sqsubseteq \bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S$ is obvious. □

Definition 23. I call an endoreloid ν *symmetrically transitive* iff for every symmetric endofuncoid $f \in \mathsf{FCD}(\mathrm{Ob}\,\nu; \mathrm{Ob}\,\nu)$ we have $f \sqsubseteq \nu \Rightarrow f \circ f \sqsubseteq \nu$.

Obvious 24. It is symmetrically transitive if at least one of the following holds:

1. $\nu \circ \nu \sqsubseteq \nu$;
2. $\nu \circ \nu^{-1} \sqsubseteq \nu$;
3. $\nu^{-1} \circ \nu \sqsubseteq \nu$.
4. $\nu^{-1} \circ \nu^{-1} \sqsubseteq \nu$.

Corollary 25. Every uniform space is symmetrically transitive.

Proposition 26. $(\mathrm{Low})\nu$ is a completely Cauchy space for every symmetrically transitive endoreloid ν.

Proof. Suppose $S \in \mathscr{P}\{\mathcal{X} \in \mathfrak{F} \setminus \{0^{\mathfrak{F}}\} \mid \mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq \nu\}$ and $S \neq \emptyset$.

$\bigsqcup \{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \mid \mathcal{X} \in S\} \sqsubseteq \nu$; $\bigsqcup \{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \mid \mathcal{X} \in S\} \circ \bigsqcup \{\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \mid \mathcal{X} \in S\} \sqsubseteq \nu$; $\bigsqcup S \times^{\mathsf{RLD}} \bigsqcup S \sqsubseteq \nu$ (taken into account that S has nonempty meet). Thus $\bigsqcup S$ is Cauchy. □

Proposition 27. The neighbourhood filter $\langle(\mathsf{FCD})\nu\rangle^*\{x\}$ of a point $x \in \mathrm{Ob}\,\nu$ is a maximal Cauchy filter, if it is a Cauchy filter and ν is a reflexive reloid. [TODO: Does it holds for all low filters?]

Proof. Let $\mathscr{N} = \langle(\mathsf{FCD})\nu\rangle^*\{x\}$. Let $\mathscr{C} \sqsupseteq \mathscr{N}$ be a Cauchy filter. We need to show $\mathscr{N} \sqsupseteq \mathscr{C}$.

Since $\mathscr{C}$ is Cauchy filter, $\mathscr{C} \times^{\mathsf{RLD}} \mathscr{C} \sqsubseteq \nu$. Since $\mathscr{C} \sqsupseteq \mathscr{N}$ we have $\mathscr{C}$ is a neighborhood of x and thus $\uparrow^{\mathrm{Ob}\,\nu}\{x\} \sqsubseteq \mathscr{C}$ (reflexivity of ν). Thus $\uparrow^{\mathrm{Ob}\,\nu}\{x\} \times^{\mathsf{RLD}} \mathscr{C} \sqsubseteq \mathscr{C} \times^{\mathsf{RLD}} \mathscr{C}$ and hence $\uparrow^{\mathrm{Ob}\,\nu}\{x\} \times^{\mathsf{RLD}} \mathscr{C} \sqsubseteq \nu$;

$$\mathscr{C} \sqsubseteq \operatorname{im}(\nu|_{\uparrow^{\mathrm{Ob}\,\nu}\{x\}}) = \langle(\mathsf{FCD})\nu\rangle^*\{x\} = \mathscr{N}.$$

□

7 Cauchy continuous functions

Definition 28. A function $f: U \rightarrow V$ is *Cauchy continuous* from a low filters space $(U; \mathscr{C})$ to a low filters space $(V; \mathscr{D})$ when $\forall \mathcal{X} \in \mathscr{C}: \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \in \mathscr{D}$.

Proposition 29. Let f is a principal reloid. Then $f \in \mathrm{C}((\mathsf{RLD})_{\mathrm{Low}}\mathscr{C}; (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D})$ iff f is Cauchy continuous.

$$\begin{array}{rl} f \circ (\mathsf{RLD})_{\mathrm{Low}}\mathscr{C} \circ f^{-1} \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} & \Leftrightarrow \\ \bigsqcup \{f \circ (\mathcal{X} \times^{\mathsf{RLD}} \mathcal{X}) \circ f^{-1} \mid \mathcal{X} \in \mathscr{C}\} \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} & \Leftrightarrow \\ \bigsqcup \{\langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \times^{\mathsf{RLD}} \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \mid \mathcal{X} \in \mathscr{C}\} \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} & \Leftrightarrow \\ \forall \mathcal{X} \in \mathscr{C}: \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \times^{\mathsf{RLD}} \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \sqsubseteq (\mathsf{RLD})_{\mathrm{Low}}\mathscr{D} & \Leftrightarrow \\ \forall \mathcal{X} \in \mathscr{C}: \langle\uparrow^{\mathsf{FCD}} f\rangle\mathcal{X} \in \mathscr{D}. & \end{array}$$

Thus we have expressed Cauchy properties through the algebra of reloids.

8 Cauchy-complete reloids

Definition 30. An endoreloid ν is *Cauchy-complete* iff every low filter for this reloid converges to a point.

Remark 31. In my book [1] *complete reloid* means something different. I will always prepend the word "Cauchy" to the word "complete" when meaning is by the last definition.

https://en.wikipedia.org/wiki/Complete_uniform_space#Completeness

9 Totally bounded

http://ncatlab.org/nlab/show/Cauchy+space

Definition 32. Cauchy space is called *totally bounded* when every proper filter contains a Cauchy filter.

Obvious 33. A reloid ν is totally bounded iff

$$\forall X \in \mathscr{P}\mathrm{Ob}\,\nu\, \exists \mathcal{X} \in \mathfrak{F}^{\mathrm{Ob}\,\nu}: (0 \neq \mathcal{X} \sqsubseteq \uparrow^{\mathrm{Ob}\,\nu} X \wedge \mathcal{X} \times^{\mathsf{RLD}} \mathcal{X} \sqsubseteq \nu).$$

Theorem 34. A symmetric transitive reloid is totally bounded iff its Cauchy space is totally bounded.

Proof.

- **⇒.** Let $\mathcal{F}$ be a proper filter on $\operatorname{Ob}\nu$ and let $a\in\operatorname{atoms}\mathcal{F}$. It's enough to prove that a is Cauchy.

  Let $D\in\operatorname{GR}\nu$. Let also $E\in\operatorname{GR}\nu$ is symmetric and $E\circ E\subseteq D$. There existsa finite subset $F\subseteq\operatorname{Ob}\nu$ such that $\langle E\rangle F=\operatorname{Ob}\nu$. Then obviously exists $x\in F$ such that $a\sqsubseteq\uparrow^{\operatorname{Ob}\nu}\langle E\rangle\{x\}$, but $\langle E\rangle\{x\}\times\langle E\rangle\{x\}=E^{-1}\circ(\{x\}\times\{x\})\circ E\subseteq D$, thus $a\times^{\mathsf{RLD}}a\sqsubseteq\uparrow^{\mathsf{RLD}(\operatorname{Ob}\nu;\operatorname{Ob}\nu)}D$.

  Because D was taken arbitrary, we have $a\times^{\mathsf{RLD}}a\sqsubseteq\nu$ that is a is Cauchy.

- **⇐.** Suppose that Cauchy space associated with a reloid ν is totally bounded but the reloid ν isn't totally bounded. So there exists a $D\in\operatorname{GR}\nu$ such that $(\operatorname{Ob}\nu)\setminus\langle D\rangle F\neq\emptyset$ for every finite set F.

  Consider the filter base

  $$S=\{(\operatorname{Ob}\nu)\setminus\langle D\rangle F\mid F\in\mathscr{P}\operatorname{Ob}\nu,F\text{ is finite}\}$$

  and the filter $\mathcal{F}=\bigsqcap\langle\uparrow^{\operatorname{Ob}\nu}\rangle S$ generated by this base. The filter $\mathcal{F}$ is proper because intersection $P\cap Q\in S$ for every $P,Q\in S$ and $\emptyset\notin S$. Thus there exists a Cauchy (for our Cauchy space) filter $\mathcal{X}\sqsubseteq\mathcal{F}$ that is $\mathcal{X}\times^{\mathsf{RLD}}\mathcal{X}\sqsubseteq\nu$.

  Thus there exists $M\in\mathcal{X}$ such that $M\times M\subseteq D$. Let F be a finite subset of $\operatorname{Ob}\nu$. Then $(\operatorname{Ob}\nu)\setminus\langle D\rangle F\in\mathcal{F}\sqsupseteq\mathcal{X}$. Thus $M\not\asymp(\operatorname{Ob}\nu)\setminus\langle D\rangle F$ and so there exists a point $x\in M\cap((\operatorname{Ob}\nu)\setminus\langle D\rangle F)$.

  $\langle M\times M\rangle\{p\}\subseteq\langle D\rangle\{x\}$ for every $p\in M$; thus $M\subseteq\langle D\rangle\{x\}$.

  So $M\subseteq\langle D\rangle(F\cup\{x\})$. But this means that $M\in\mathcal{X}$ does not intersect $(\operatorname{Ob}\nu)\setminus\langle D\rangle(F\cup\{x\})\in\mathcal{F}\sqsupseteq\mathcal{X}$, what is a contradiction (taken into account that $\mathcal{X}$ is proper). □

http://math.stackexchange.com/questions/104696/pre-compactness-total-boundedness-and-cauchy-sequential-compactness

10 Totally bounded funcoids

Definition 35. A funcoid ν is totally bounded iff

$$\forall X\in\operatorname{Ob}\nu\exists\mathcal{X}\in\mathfrak{F}^{\operatorname{Ob}\nu}:(0\neq\mathcal{X}\sqsubseteq\uparrow^{\operatorname{Ob}\nu}X\wedge\mathcal{X}\times^{\mathsf{FCD}}\mathcal{X}\sqsubseteq\nu).$$

This can be rewritten in elementary terms (without using funcoidal product:

$\mathcal{X}\times^{\mathsf{FCD}}\mathcal{X}\sqsubseteq\nu\Leftrightarrow\forall P\in\partial\mathcal{X}:\mathcal{X}\sqsubseteq\langle\nu\rangle P\Leftrightarrow\forall P\in\partial\mathcal{X},Q\in\partial\mathcal{X}:P\,[\nu]^*\,Q\Leftrightarrow\forall P,Q\in\operatorname{Ob}\nu:(\forall E\in\mathcal{X}:(E\cap P\neq\emptyset\wedge E\cap Q\neq\emptyset)\Rightarrow P\,[\nu]^*\,Q)$.

Note that probably I am the first person which has written the above formula (for proximity spaces for instance) explicitly.

11 On principal low filter spaces

Definition 36. A low filter space $(U;\mathscr{C})$ is *principal* when all filters in $\mathscr{C}$ are principal.

Definition 37. A low filter space $(U;\mathscr{C})$ is *reflexive* when $\forall x\in U:\uparrow^U\{x\}\in\mathscr{C}$.

Proposition 38. Having fixed a set U, principal reflexive low filter spaces on U bijectively correspond to principal reflexive symmertic endoreloids on U.

Proof. ??

http://math.stackexchange.com/questions/701684/union-of-cartesian-squares □

12 Rest

https://en.wikipedia.org/wiki/Cauchy_filter#Cauchy_filters

https://en.wikipedia.org/wiki/Uniform_space "Hausdorff completion of a uniform space" here)

http://at.yorku.ca/z/a/a/b/13.htm : the category **Prox** of proximity spaces and proximally continuous maps (i.e. maps preserving nearness between two sets) is isomorphic to the category of totally bounded uniform spaces (and uniformly continuous maps).

https://en.wikipedia.org/wiki/Cauchy_space http://ncatlab.org/nlab/show/Cauchy+space

http://arxiv.org/abs/1309.1748

http://projecteuclid.org/download/pdf_1/euclid.pja/1195521991

http://www.emis.de/journals/HOA/IJMMS/Volume5_3/404620.pdf

~/math/books/Cauchy_spaces.pdf

Bibliography

[1] Victor Porton. *Algebraic General Topology. Volume 1*. 2013.