Certain categories are cartesian closed

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November 25, 2013

Abstract

I prove that the category of continuous maps between endofuncoids is cartesian closed.
Whether the category of continuous maps between endoreloids is cartesian closed is yet an open problem.

This is a rough draft. There are errors!

Cartesian closed categories

Definition 1. A category is cartesian closed iff:

- It has finite products.
- For each objects $A, B$ is given an object $\text{MOR}(A; B)$ (exponentiation) and a morphism $\varepsilon_{A,B}^{\text{Dig}}: \text{MOR}(A; B) \times A \to B$.
- For each morphism $f: Z \times A \to B$ there is given a morphism (exponential transpose) $\sim f: Z \to \text{MOR}(A; B)$.
- $\varepsilon \circ (\sim f \times 1_A) = f$.
- $\sim(\varepsilon \circ (g \times 1_A)) = g$.

Our purpose is to prove (or disprove) that categories $\text{Dig}$, $\text{Fcd}$, and $\text{Rld}$ are cartesian closed. Note that they have finite (and even infinite) products is already proved in http://www.mathematics21.org/binaries/product.pdf

Definitions of our categories

Categories $\text{Dig}$, $\text{Fcd}$, and $\text{Rld}$ are respectively categories of:

1. discretely continuous maps between digraphs;
2. (proximally) continuous maps between endofuncoids;
3. (uniformly) continuous maps between endoreloids.

Definition 2. Digraph is an endomorphism of the category $\text{Rel}$.

Definition 3. Category $\text{Dig}$ of digraphs is the category whose objects are digraphs and morphisms are discretely continuous maps between digraphs. That is morphisms from a digraph $\mu$ to a digraph $\nu$ are functions (or more precisely morphisms of $\text{Set}$) $f$ such that $f \circ \mu \subseteq \nu \circ f$ (or equivalently $\mu \subseteq f^{-1} \circ \nu \circ f$ or equivalently $f \circ \mu \circ f^{-1} \subseteq \nu$).
Remark 4. Category of digraphs is sometimes defined in an other (non equivalent) way, allowing multiple edges between two given vertices.

Definition 5. Category \textbf{Fed} of continuous maps between endofuncoids is the category whose objects are endofuncoids and morphisms are proximally continuous maps between endofuncoids. That is morphisms from an endofuncoid \( \mu \) to an endofuncoid \( \nu \) are functions (or more precisely morphisms of \textbf{Set}) \( f \) such that \( \uparrow^{\text{FCD}} f \circ \mu \subseteq \nu \circ \uparrow^{\text{FCD}} f \) (or equivalently \( \mu \subseteq \uparrow^{\text{FCD}} f^{-1} \circ \nu \circ \uparrow^{\text{FCD}} f \) or equivalently \( \uparrow^{\text{FCD}} f \circ \mu \circ \uparrow^{\text{FCD}} f^{-1} \subseteq \nu \)).

Definition 6. Category \textbf{Rld} of continuous maps between endoreloids is the category whose objects are endoreloids and morphisms are uniformly continuous maps between endoreloids. That is morphisms from an endoreloid \( \mu \) to an endoreloid \( \nu \) are functions (or more precisely morphisms of \textbf{Set}) \( f \) such that \( \uparrow^{\text{RLD}} f \circ \mu \subseteq \nu \circ \uparrow^{\text{RLD}} f \) (or equivalently \( \mu \subseteq \uparrow^{\text{RLD}} f^{-1} \circ \nu \circ \uparrow^{\text{RLD}} f \) or equivalently \( \uparrow^{\text{RLD}} f \circ \mu \circ \uparrow^{\text{RLD}} f^{-1} \subseteq \nu \)).

Category of digraphs is cartesian closed

Category of digraphs is the simplest of our three categories and it is easy to demonstrate that it is cartesian closed. I demonstrate cartesian closedness of \textbf{Dig} mainly with the purpose to show a pattern similarly to which we may probably demonstrate our two other categories are cartesian closed.

Let \( G \) and \( H \) be graphs:

- \( \text{Ob MOR}(G; H) = (\text{Ob H})^{\text{Ob}G} \)
- \( (f; g) \in \text{GR MOR}(G; H) \iff \forall (v; w) \in \text{GR} G: (f(v); g(w)) \in \text{GR} H \) for every \( f, g \in \text{Ob MOR}(G; H) \)

\( \text{GR id}_{\text{MOR}(B; C)} = \text{id}_{\text{Ob MOR}(B; C)} = \text{id}_{(\text{Ob H})^{\text{Ob}G}} \)

Equivalently

\( (f; g) \in \text{GR MOR}(G; H) \iff \forall (v; w) \in \text{GR} G: g \circ \{v; w\} \circ f^{-1} \subseteq \text{GR} H \)

\( (f; g) \in \text{GR MOR}(G; H) \iff g \circ (\text{GR} G) \circ f^{-1} \subseteq \text{GR} H \)

\( (f; g) \in \text{GR MOR}(G; H) \iff \langle f \times^{(G)} g \rangle \text{GR} G \subseteq \text{GR} H \)

The transposition (the isomorphism) is uncurrying.

\( \sim f = \lambda a \in Z \lambda y \in A: f(a; y) \) that is \( (\sim f)(a)(y) = f(a; y) \).

\( (\sim f)(a; y) = f(a)(y) \)

If \( f: A \times B \to C \) then \( \sim f: A \to \text{MOR}(B; C) \)

Proposition 7. Transposition and its inverse are morphisms of \textbf{Dig}.

Proof. It follows from the equivalence \( \sim f: A \to \text{MOR}(B; C) \iff \forall x, y: (x Ay \Rightarrow (\sim f) x (\text{MOR}(B; C)) (\sim f) y) \iff \forall x, y: (x Ay \Rightarrow \forall (v; w) \in B: ((\sim f) x v; (\sim f) y w) \in C) \iff \forall x, y, v, w: (x Ay \land v Bw \Rightarrow ((\sim f) x v; (\sim f) y w) \in C) \iff \forall x, y, v, w: ((x; v) (A \times B) (y; w) \Rightarrow (f(x; v); f(y; w)) \in C) \iff f: A \times B \to C. \)

Evaluation \( \varepsilon: \text{MOR}(G; H) \times G \to H \) is defined by the formula:
Then evaluation is $\epsilon_{B,C} = -(1_{\text{MOR}(B,C)})$.

So $\epsilon_{B,C}(p; q) = (-(1_{\text{MOR}(B,C)}))(p; q) = (1_{\text{MOR}(B,C)})(p)(q) = p(q)$.

**Proposition 8.** Evaluation is a morphism of $\text{Dig}$.

**Proof.** Because $\epsilon_{B,C}(p; q) = -(1_{\text{MOR}(B,C)})$.

It remains to prove: [FIXME: $\epsilon_{X,Y}$. What are $X$ and $Y$?]

- $\epsilon \circ (\sim f \times 1_A) = f$;
- $\sim(\epsilon \circ (g \times 1_A)) = g$.

**Proof.** $\epsilon(\sim f \times 1_A)(a; p) = \epsilon((\sim f)a; p) = (\sim f)ap = f(a; p)$. So $\epsilon \circ (\sim f \times 1_A) = f$.

$\sim(\epsilon \circ (g \times 1_A))(p)(q) = (\epsilon \circ (g \times 1_A))(p; q) = \epsilon(g \times 1_A)(p; q) = \epsilon(gp; q) = g(p)(q)$. So $\sim(\epsilon \circ (g \times 1_A)) = g$.

**Exponentials in category $\text{Fcd}$**

Define $\sim^\text{Fcd} f = \uparrow^{\text{FCD}} \sim^\text{Dig} f$

**Definition 9.** A category is *cartesian closed* iff:

- $\epsilon \circ (\sim f \times 1_A) = f$.
- $\sim(\epsilon \circ (g \times 1_A)) = g$.

But this follows from functoriality of $\uparrow^{\text{FCD}}$.

Embed $\text{Fcd}$ into $\text{Dig}$ by the formulas:

- $A \mapsto \lambda X \in \mathcal{P}\text{Ob} A : \langle A \rangle X$
- $f \mapsto \langle f \rangle$

Obviously this embedding (denote it $T$) is an injective (both on objects and morphisms) functor. $\epsilon^\text{Fcd}_{A,B}(p \times q) = \langle p \rangle q$ [TODO: Should $p$ and $q$ be atomic?]

$\sim^\text{Rld}$ is induced by $\sim^\text{Dig}$.

Due its injectivity and functoriality, it is enough to prove:

1. binary products are preserved
2. $\sim^\text{Dig}_{T A, T B} = T \sim^\text{Fcd}_{A,B}$
3. that $\sim^\text{Dig}_T f = T \sim^\text{Fcd} f$ for every $f : A \to B$

$(T \epsilon^\text{Fcd}_{A,B})(p \times q) = \langle \epsilon^\text{Fcd}_{A,B} \rangle(p \times q) = \langle p \rangle q$

$\epsilon^\text{Dig}_{T A, T B} X = (T B)^{\lambda X \in \mathcal{P}\text{Ob} B : \langle B \rangle Y} \lambda Y \in \mathcal{P}\text{Ob} A : \langle A \rangle X X$
Due its injectivity and functoriality, it is enough to prove:

1. binary products are preserved
2. for every $\varepsilon_{T_A,T_B}$ there exist $\varepsilon_{A,B}^{\text{Fcd}}$ such that $\varepsilon_{T_A,T_B}^{\text{Dig}} = T_\varepsilon_{A,B}^{\text{Fcd}}$
3. for every $f: T_A \to T_B$ there exists $g: A \to B$ that $\sim f = T_\sim g$

Consider $\varepsilon_{T_A,T_B}^{\text{Dig}}$. Then $\varepsilon_{T_A,T_B}^{\text{Dig}} X = (T_B)^{T_A} X = (\lambda X \in \mathcal{P}\text{Ob} B: \langle B \rangle_X) \in (\lambda X \in \mathcal{P}\text{Ob} A: \langle A \rangle_X)$ for as suitable $X$. Thus $\varepsilon_{T_A,T_B}^{\text{Dig}} 0 = 0$ and $\varepsilon_{T_A,T_B}^{\text{Dig}} (I \cup J) = \varepsilon_{T_A,T_B}^{\text{Dig}} I \cup \varepsilon_{T_A,T_B}^{\text{Dig}} J$. Consequently $\varepsilon_{A,B}^{\text{Fcd}}$ exists.

Consider $f: T_A \to T_B$.

??

Then $f \in (T_B)^{T_A}$ and $f \in C(T_A; T_B)$.

$f X = ??$

$(\sim f)(p; q) = f(p)(q) = ??$

Thus ??

??

Binary products are subatomic products and so are compatible with products of graphs.

A try to prove this directly:

**Proposition 10.** Transposition and its inverse are morphisms of $\text{Fcd}$.

**Proof.** ?? [TODO: Use below sets instead of ultrafilters.]

It follows from the equivalence (??is it an equivalence? the last step seems just an implication) $\sim f: A \to \text{MOR}(B; C) \iff \forall x, y \in \text{atoms}^5: (x [A] y \Rightarrow (\sim f)(x) \in \text{MOR}(B; C) \iff \forall x, y \in \text{atoms}^5: (x [A] y \Rightarrow \forall v, w \in \text{atoms}^5: (x [A] y \Rightarrow (\sim f)(xv) \times_{\text{FCD}} (\sim f)(yw) \in \text{atoms}^5: (x [A] y \Rightarrow \forall v, w \in \text{atoms}^5: (x [A] y \Rightarrow (\sim f)(xv) \times_{\text{FCD}} (\sim f)(yw) \in \text{atoms}^5: (x [A] y \Rightarrow \forall v, w \in \text{atoms}^5: (x [A] y \Rightarrow \forall v, w \in \text{atoms}^5: (x [A] y \Rightarrow (f(x; v); f(y; w)) \in C \iff f: A \times B \to C$. $

\square$

**Exponentials in category Rld**

TODO