Total boundness of reloids

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June 6, 2013

Abstract

I generalize total boundness of uniform spaces for arbitrary reloids (filters on a cartesian product of sets). For reloids total boundness splits into two different concepts: α-total boundness and β-total boundness.

Notation

I call *reloid* a triple $(A; B; \mathcal{F})$ where A, B are sets and $\mathcal{F}$ is a filter on the cartesian product $A \times B$ of these sets. I will denote $\mathrm{GR}(A; B; \mathcal{F}) = \mathcal{F}$ for a reloid $(A; B; \mathcal{F})$. Reloids are a generalization of uniform spaces. *Source* of a reloid is $\mathrm{Src}(A; B; \mathcal{F}) = A$ and destination $\mathrm{Dst}(A; B; \mathcal{F}) = B$. I also denote $\mathrm{xyGR}(A; B; F) = \{(A; B; f) \mid f \in F\}$.

I will refer to a pair $f = (U; E)$ of a (small) set U and a binary relation $E \subseteq E \times E$ as **Rel**-endomorphism. Furthermore $\mathrm{Ob}\, f = U$ and $\mathrm{GR}\, f = E$.

I denote $\langle E \rangle X = \{y \mid \exists x \in X : x\, E\, y\}$ for a binary relation E and set X and $\langle E \rangle X = \langle \mathrm{GR}\, E \rangle X$ for a **Rel**-endomorphism E.

I define composition of reloids $(B; C; G) \circ (A; B; F) = (A; C; H)$ where H is the filter induced by the filter base $\{g \circ f \mid f \in F, g \in G\}$.

The reverse reloid is defined by the formula $(A; B; F)^{-1} = (B; A; F^{-1})$.

I define partial order on the set of reloids as $(A; B; F) \sqsubseteq (A; B; G) \Leftrightarrow F \supseteq G$. The set of reloids with given source and destination is a complete lattice with join denoted $\sqcup$ and meet denotes $\sqcap$.

See http://www.mathematics21.org/algebraic-general-topology.html for more details.

Thick binary relations

Definition 1. I will call *α-thick* and denote $\mathrm{thick}_\alpha(E)$ a **Rel**-endomorphism E when there exists a finite cover S of $\mathrm{Ob}\, E$ such that $\forall A \in S : A \times A \subseteq \mathrm{GR}\, E$.

Definition 2. $\mathrm{CS}(S) = \bigcup \{A \times A \mid A \in S\}$ for a collection S of sets.

Remark 3. CS means "cartesian squares".

Obvious 4. A **Rel**-endomorphism is *α-thick* iff there exists a finite cover S of $\mathrm{Ob}\, E$ such that $\mathrm{CS}(S) \subseteq \mathrm{GR}\, E$.

Definition 5. I will call *β-thick* and denote $\mathrm{thick}_\beta(E)$ a **Rel**-endomorphism E when iff there exists a finite set B such that $\langle E \rangle B = \mathrm{Ob}\, E$.

Proposition 6. $\mathrm{thick}_\alpha(E) \Rightarrow \mathrm{thick}_\beta(E)$.

Proof. Let $\operatorname{thick}_\alpha(E)$. Then there exists a finite cover S of the set $\operatorname{Ob} E$ such that $\forall A \in S$: $A \times A \subseteq \operatorname{GR} E$. Without loss of generality assume $A \neq \emptyset$ for every $A \in S$. So $A \subseteq \langle E \rangle \{x_A\}$ for some x_A for every $A \in S$. So $\langle E \rangle \{x_A \mid A \in S\} = \bigcup \{\langle E \rangle \{x_A\} \mid A \in S\} = \operatorname{Ob} E$ and thus E is β-thick. □

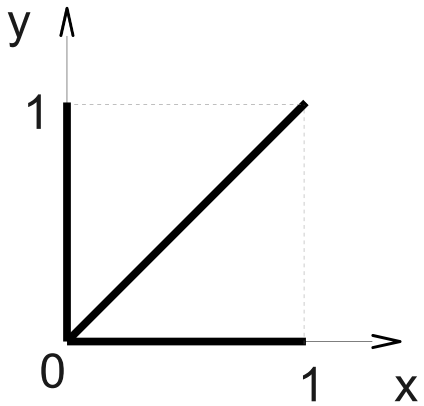
Obvious 7. Let X be a set, A and B are **Rel**-endorelations on X and $B \sqsupseteq A$. Then:

- $\operatorname{thick}_\alpha(A) \Rightarrow \operatorname{thick}_\alpha(B)$;
- $\operatorname{thick}_\beta(A) \Rightarrow \operatorname{thick}_\beta(B)$.

Example 8. There is a β-thick **Rel**-morphism which is not α-thick.

Proof. Consider the **Rel**-morphism on $[0; 1]$ with the below graph:

$$\Gamma = \{(x; x) \mid x \in [0; 1]\} \cup \{(x; 0) \mid x \in [0; 1]\} \cup \{(0; x) \mid x \in [0; 1]\}.$$



Γ is β-chick because $\langle \Gamma \rangle \{0\} = [0; 1]$.

To prove that Γ is not α-thick it's enough to prove that every set A such that $A \times A \subseteq \Gamma$ is finite.

Suppose for the contrary that A is infinite. Then A contains more than one non-zero points y, z ($y \neq z$). Without loss of generality $y < z$. So we have that (y, z) is not of the form $(y; y)$ nor $(0; y)$ nor $(y; 0)$. Therefore $A \times A$ isn't a subset of Γ. □

Totally bounded endo-reloids

The below is a straightforward generalization of the customary definition of totally bounded sets on uniform spaces (it's proved below that for uniform spaces the below definitions are equivalent).

Definition 9. An endoreloid f is *α-totally bounded* ($\operatorname{totBound}_\alpha(f)$) if every $E \in \operatorname{xyGR} f$ is α-thick.

Definition 10. An endoreloid f is *β-totally bounded* ($\operatorname{totBound}_\beta(f)$) if every $E \in \operatorname{xyGR} f$ is β-thick.

Remark 11. We could rewrite the above definitions in a more algebraic way like $\operatorname{xyGR} f \subseteq \operatorname{thick}_\alpha$ (with $\operatorname{thick}_\alpha$ would be defined as a set rather than as a predicate), but we don't really need this simplification.

Proposition 12. If an endoreloid is α-totally bounded then it is β-totally bounded.

Proof. Because $\operatorname{thick}_\alpha(E) \Rightarrow \operatorname{thick}_\beta(E)$. □

Proposition 13. If an endoreloid f is reflexive and $\operatorname{Ob} f$ is finite then f is both α-totally bounded and β-totally bounded.

Proof. It enough to prove that f is α-totally bounded. Really, every $E \in \operatorname{xyGR} f$ is reflexive. Thus $\{x\} \times \{x\} \subseteq E$ for $x \in \operatorname{Ob} f$ and thus $\{\{x\} \mid x \in \operatorname{Ob} f\}$ is a sought for finite cover of $\operatorname{Ob} f$. □

Obvious 14.

- A principal endo-reloid induced by a **Rel**-morphism E is α-totally bounded iff E is α-thick.
- A principal endo-reloid induced by a **Rel**-morphism E is β-totally bounded iff E is β-thick.

Example 15. There is a β-totally bounded endoreloid which is not α-totally bounded.

Proof. It follows from the example above and properties of principal endoreloids. □

Special case of uniform spaces

Definition 16. *Uniform space* is essentially the same as symmetric, reflexive and transitive endoreloid.

Exercise 1. Prove that it is essentially the same as the standard definition of a uniform space (see Wikipedia or PlanetMath).

Theorem 17. Let f is such a endoreloid that $f \circ f^{-1} \sqsubseteq f$. Then f is α-totally bounded iff it is β–totally bounded.

Proof.

⇒. Proved above.

⇐. For every $\varepsilon \in \operatorname{GR} f$ we have that $\langle\varepsilon\rangle\{c_0\}, \ldots, \langle\varepsilon\rangle\{c_n\}$ covers the space. $\langle\varepsilon\rangle\{c_i\} \times \langle\varepsilon\rangle\{c_i\} \subseteq \varepsilon \circ \varepsilon^{-1}$ because for $x \in \langle\varepsilon\rangle\{c_i\}$ (the same as $c_i \in \langle\varepsilon^{-1}\rangle\{x\}$) we have $\langle\langle\varepsilon\rangle\{c_i\} \times \langle\varepsilon\rangle\{c_i\}\rangle\{x\} = \langle\varepsilon\rangle\{c_i\} \subseteq \langle\varepsilon\rangle\langle\varepsilon^{-1}\rangle\{x\} = \langle\varepsilon \circ \varepsilon^{-1}\rangle\{x\}$. There exists $\varepsilon' \in \operatorname{GR} f$ such that $\varepsilon \circ \varepsilon^{-1} \subseteq \varepsilon'$ because $f \circ f^{-1} \sqsubseteq f$. Thus for every ε' we have $\langle\varepsilon\rangle\{c_i\} \times \langle\varepsilon\rangle\{c_i\} \subseteq \varepsilon'$ and so

$$\langle\varepsilon\rangle\{c_0\}, \ldots, \langle\varepsilon\rangle\{c_n\}.$$

is a sought for finite cover. □

Corollary 18. A uniform space is α-totally bounded iff it is β-totally bounded.

Proof. From the theorem and the definition of uniform spaces. □

Relationships with other properties

Theorem 19. Let μ and ν are endoreloids. Let f is a principal $\mathrm{C}'(\mu;\nu)$ continuous, monovalued, surjective reloid. Then if μ is β-totally bounded then ν is also β-totally bounded.

Proof. Let φ is the monovalued, surjective function, which induces the reloid f.

We have $\mu \sqsubseteq f^{-1} \circ \nu \circ f$.

Let $F \in \operatorname{GR} \nu$. Then there exists $E \in \operatorname{GR} \mu$ such that $E \subseteq \varphi^{-1} \circ F \circ \varphi$.

Since μ is β-totally bounded, there exists a finite subset A of $\operatorname{Ob} \mu$ such that $\langle E \rangle A = \operatorname{Ob} \mu$.

We claim $\langle F \rangle \langle \varphi \rangle A = \operatorname{Ob} \nu$.

Indeed let $y \in \operatorname{Ob} \nu$ be an arbitrary point. Since φ is surjective, there exists $x \in \operatorname{Ob} \mu$ such that $\varphi x = y$. Since $\langle E \rangle A = \operatorname{Ob} \mu$ there exists $a \in A$ such that $a \mathrel{E} x$ and thus $a \, (\varphi^{-1} \circ F \circ \varphi) \, x$. So $(\varphi a; y) = (\varphi a; \varphi x) \in F$. Therefore $y \in \langle F \rangle \langle \varphi \rangle A$. □

Theorem 20. Let μ and ν are endoreloids. Let f is a principal $\mathrm{C}''(\mu; \nu)$ continuous, surjective reloid. Then if μ is α-totally bounded then ν is also α-totally bounded.

Proof. Let φ is the surjective binary relation which induces the reloid f.

We have $f \circ \mu \circ f^{-1} \sqsubseteq \nu$.

Let $F \in \operatorname{GR} \nu$. Then there exists $E \in \operatorname{GR} \mu$ such that $\varphi \circ E \circ \varphi^{-1} \subseteq F$.

There exists a finite cover S of $\operatorname{Ob} \mu$ such that

$$\bigcup \{A \times A \mid A \in S\} \subseteq E.$$

Thus $\varphi \circ (\bigcup \{A \times A \mid A \in S\}) \circ \varphi^{-1} \subseteq F$ that is $\bigcup \{\langle \varphi \rangle A \times \langle \varphi \rangle A \mid A \in S\} \subseteq F$.

It remains to prove that $\{\langle \varphi \rangle A \mid A \in S\}$ is a cover of $\operatorname{Ob} \nu$. It is true because φ is a surjection and S is a cover of $\operatorname{Ob} \mu$. □

A stronger statement (principality requirement removed):

Conjecture 21. The image of a uniformly continuous entirely defined monovalued surjective reloid from a (α-, β-)totally bounded endoreloid is also (α-, β-)totally bounded.

Can we remove the requirement to be entirely defined from the above conjecture?

Question 22. Under which conditions it's true that join of (α-, β-) totally bounded reloids is also totally bounded?

Additional predicates

We may consider also the following predicates expressing different kinds of what is intuitively is understood as boundness. Their usefulness is unclear, but I present them for completeness.

- $\forall E \in \operatorname{GR} f \exists n \in \mathbb{N}: \operatorname{thick}_\alpha(E^n)$
- $\forall E \in \operatorname{GR} f \exists n \in \mathbb{N}: \operatorname{thick}_\beta(E^n)$
- $\forall E \in \operatorname{GR} f \exists n \in \mathbb{N}: \operatorname{thick}_\alpha(E^0 \cup \ldots \cup E^n)$
- $\forall E \in \operatorname{GR} f \exists n \in \mathbb{N}: \operatorname{thick}_\beta(E^0 \cup \ldots \cup E^n)$
- $\exists n \in \mathbb{N}: \operatorname{totBound}_\alpha(f^n)$

- $\exists n \in \mathbb{N}\colon \operatorname{totBound}_\beta(f^n)$
- $\exists n \in \mathbb{N}\colon \operatorname{totBound}_\alpha(f^0 \sqcup \ldots \sqcup f^n)$
- $\exists n \in \mathbb{N}\colon \operatorname{totBound}_\beta(f^0 \sqcup \ldots \sqcup f^n)$
- $\operatorname{totBound}_\alpha(f^0 \sqcup f^1 \sqcup f^2 \sqcup \ldots)$
- $\operatorname{totBound}_\beta(f^0 \sqcup f^1 \sqcup f^2 \sqcup \ldots)$

Some of the above defined predicates are equivalent:

Proposition 23.

- $\forall E \in \operatorname{GR} f \, \exists n \in \mathbb{N}\colon \operatorname{thick}_\alpha(E^n) \Leftrightarrow \exists n \in \mathbb{N}\colon \operatorname{totBound}_\alpha(f^n)$.
- $\forall E \in \operatorname{GR} f \, \exists n \in \mathbb{N}\colon \operatorname{thick}_\beta(E^n) \Leftrightarrow \exists n \in \mathbb{N}\colon \operatorname{totBound}_\beta(f^n)$.

Proof. Because every $F \in \operatorname{GR} f^n$ is a superset of E^n for some $E \in \operatorname{GR} f$. □

Proposition 24.

- $\forall E \in \operatorname{GR} f \, \exists n \in \mathbb{N}\colon \operatorname{thick}_\alpha(E^0 \cup \ldots \cup E^n) \Leftrightarrow \exists n \in \mathbb{N}\colon \operatorname{totBound}_\alpha(f^0 \sqcup \ldots \sqcup f^n)$.
- $\forall E \in \operatorname{GR} f \, \exists n \in \mathbb{N}\colon \operatorname{thick}_\beta(E^0 \cup \ldots \cup E^n) \Leftrightarrow \exists n \in \mathbb{N}\colon \operatorname{totBound}_\beta(f^0 \sqcup \ldots \sqcup f^n)$.

Proof. $f^0 \sqcup \ldots \sqcup f^n = f^0 \cap \ldots \cap f^n$. Thus every $F \in \operatorname{GR}(f^0 \cap \ldots \cap f^n)$ we have $F \in f^k$, thus $F \supseteq E_k^k$ for all k for some $E_k \in \operatorname{GR} f$ and so $F \supseteq E^0 \cup \ldots \cup E^n$ where $E = E_0 \cap \ldots \cap E_k \in \operatorname{GR} f$. □

Proposition 25. All predicates in the above list are pairwise equivalent in the case if f is a uniform space.

Proof. Because $f \circ f = f$. □