A New Kind of Product of Ordinal Number of Relations having Ordinal Numbers of Arguments

BY VICTOR PORTON

Email: porton@narod.ru
Web: http://www.mathematics21.org

Abstract

Infinite associativity is defined for functions taking an ordinal numbers of arguments. As an important example of an infinite associative function I define *ordinated product* and research it's properties. Ordinated product is an infinitely associative function.

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1 Introduction

We will consider some function f which takes an arbitrary ordinal number of arguments. That is f can be taken for arbitary (small, if to be precise) ordinal number of arguments. More formally: Let $x = x_{i \in n}$ is a family indexed by an ordinal n. Then $f(x)$ can be taken. The same function f can take different number of arguments. (See below for the exact definition.)

Some of such functions f are associative in the sense defined below. If a function is associative in the below defined sense, then the binary operation induced by this function is associative in the usual meaning of the word "associativity" as defined in basic algebra.

I also introduce and research an important example of infinitely associative function, which I call *ordinated product*.

Note that my searching about infinite associativity and ordinals in Internet has provided no useful results. As such there is a reason to assume that my research of generalized associativity in terms of ordinals is novel.

2 Used notation

We identify natural numbers with finite Von Neumann's ordinals (further just *ordinals* or *ordinal numbers*).

For simplicity we will deal with small sets (members of a Grothendieck universe). We will denote the Grothendieck universe (aka *universal set*) as $\mho$.

$(\lambda x \in D: f(x)) \overset{\text{def}}{=} \{(x; f(x)) \mid x \in D\}$ for every set D and a form f taking x as argument.

I will denote a tuple of n elements like $[\![a_0; \ldots; a_{n-1}]\!]$. By definition

$$[\![a_0; \ldots; a_{n-1}]\!] = \{(0; a_0), \ldots, (n-1; a_{n-1})\}.$$

Note that an ordered pair $(a; b)$ is not the same as the tuple $[\![a; b]\!]$ of two elements.

Definition 1. An *anchored relation* is a tuple $[\![n; r]\!]$ where n is an index set and r is an n-ary relation.

For an anchored relation arity $[\![n; r]\!] = n$. The *graph*[1] of $[\![n; r]\!]$ is defined as follows: $\mathrm{GR}[\![n; r]\!] = r$.

Definition 2. Pr_i is a small function defined by the formula

$$\mathrm{Pr}_i f = \{x_i \mid x \in f\}$$

1. It is unrelated with graph theory.

for every small n-ary relation f where n is an ordinal number and $i \in n$. Particularly for every n-ary relation f and $i \in n$ where $n \in \mathbb{N}$.

$$\mathrm{Pr}_i f = \{x_i \mid [\![x_0, \ldots, x_{n-1}]\!] \in f\}.$$

Recall that cartesian product is defined as follows:

$$\prod a = \left\{z \in \left(\bigcup \mathrm{im}\, a\right)^{\mathrm{dom}\, a} \mid \forall i \in \mathrm{dom}\, a: z(i) \in a_i\right\}.$$

Obvious 3. If a is a small function, then $\prod a = \{z \in \mho^{\mathrm{dom}\, a} \mid \forall i \in \mathrm{dom}\, a: z(i) \in a_i\}$.

2.1 Currying and uncurrying

2.1.1 The customary definition

Let X, Y, Z are sets.

We will consider variables $x \in X$ and $y \in Y$.

Let a function $f \in Z^{X \times Y}$. Then $\mathrm{curry}(f) \in (Z^Y)^X$ is the function defined by the formula $(\mathrm{curry}(f)x)\, y = f(x; y)$.

Let now $f \in (Z^Y)^X$. Then $\mathrm{uncurry}(f) \in Z^{X \times Y}$ is the function defined by the formula $\mathrm{uncurry}(f)(x; y) = (f x)\, y$.

Obvious 4.

1. $\mathrm{uncurry}(\mathrm{curry}(f)) = f$ for every $f \in Z^{X \times Y}$.
2. $\mathrm{curry}(\mathrm{uncurry}(f)) = f$ for every $f \in (Z^Y)^X$.

2.1.2 Currying and uncurrying with a dependent variable

Let X, Z are sets and Y is a function with the domain X. (Vaguely saying, Y is a variable dependent on X.)

The disjoint union $\coprod Y = \bigcup \{\{i\} \times Y_i \mid i \in \mathrm{dom}\, Y\} = \{(i; x) \mid i \in \mathrm{dom}\, Y, x \in Y_i\}$.

We will consider variables $x \in X$ and $y \in Y_x$.

Let a function $f \in Z^{\coprod_{i \in X} Y_i}$ (or equivalently $f \in Z^{\coprod Y}$). Then $\mathrm{curry}(f) \in \prod_{i \in X} Z^{Y_i}$ is the function defined by the formula $(\mathrm{curry}(f)x)\, y = f(x; y)$.

Let now $f \in \prod_{i \in X} Z^{Y_i}$. Then $\mathrm{uncurry}(f) \in Z^{\coprod_{i \in X} Y_i}$ is the function defined by the formula $\mathrm{uncurry}(f)(x; y) = (f x)\, y$.

Obvious 5.

1. $\mathrm{uncurry}(\mathrm{curry}(f)) = f$ for every $f \in Z^{\coprod_{i \in X} Y_i}$.
2. $\mathrm{curry}(\mathrm{uncurry}(f)) = f$ for every $f \in \prod_{i \in X} Z^{Y_i}$.

Remark 6. There is nothing said anything about currying with dependent variables in Wikipedia. Am I really the first person who formulated this simple generalization of currying and uncurrying?

2.2 Functions with ordinal numbers of arguments

Let Ord is the set of small ordinal numbers.

If X and Y are sets and n is an ordinal number, the set of functions taking n arguments on the set X and returning a value in Y is Y^{X^n}.

The set of all small functions taking ordinal numbers of arguments is $Y^{\bigcup_{n \in \mathrm{Ord}} X^n}$.

I will denote $\mathrm{OrdVar}(X) = \mho^{\bigcup_{n \in \mathrm{Ord}} X^n}$ and call it *ordinal variadic*. ("Var" in this notation is taken from the word *variadic* in the collocation *variadic function* used in computer science.)

3 On sums of ordinals

Let a is an ordinal-indexed family of ordinals.

Proposition 7. $\coprod a$ with lexigraphic order is a well-ordered set.

Proof. Let S is non-empty subset of $\coprod a$.

Take $i_0 = \inf \mathrm{Pr}_0 S$ and $x_0 = \inf a_{i_0}$ (these exist by properties of ordinals). Then $(i_0; x_0)$ is the least element of S. □

Definition 8. $\sum a$ is the unique ordinal order-isomorphic to $\coprod a$.

This ordinal exists and is unique because our set is well-ordered.

Remark 9. An infinite sum of ordinals is not customary defined.

The *structured sum* $\bigoplus a$ of a is an order isomorphism from lexigraphically ordered set $\coprod a$ into $\sum a$.

There exist (for a given a) exactly one structured sum, by properties of well-ordered sets.

Obvious 10. $\sum a = \mathrm{im} \bigoplus a$.

Theorem 11. $(\bigoplus a)(n; x) = \sum_{i \in n} a_i + x$.

Proof. We need to prove that it is an order isomorphism. Let's prove it is an injection that is $m > n \Rightarrow \sum_{i \in m} a_i + x > \sum_{i \in n} a_i + x$ and $y > x \Rightarrow \sum_{i \in n} a_i + y > \sum_{i \in n} a_i + x$.

Really, if $m > n$ then $\sum_{i \in m} a_i + x \geqslant \sum_{i \in n+1} a_i + x > \sum_{i \in n} a_i + x$. The second formula is true by properties of ordinals.

Let's prove that it is a surjection. Let $r \in \sum a$. There exist $n \in \mathrm{dom}\, a$ and $x \in a_n$ such that $r = (\bigoplus a)(n; x)$. Thus $r = (\bigoplus a)(n; 0) + x = \sum_{i \in n} a_i + x$ because $(\bigoplus a)(n; 0) = \sum_{i \in n} a_i$ since $(n; 0)$ has $\sum_{i \in n} a_i$ predecessors. □

4 Ordinated product

4.1 Introduction

Ordinated product defined below is a variation of cartesian product, but is associative unlike cartesian product. However, ordinated product unlike cartesian product is defined not for arbitrary sets, but only for relations having ordinal numbers of arguments.

Let F indexed by an ordinal number is a small family of anchored relations.

4.2 Concatenation

Definition 12. Let z is an indexed by an ordinal number family of functions each taking an ordinal number of arguments. The *concatenation* of z is

$$\mathrm{concat}\, z = \mathrm{uncurry}(z) \circ \left(\bigoplus (\mathrm{dom} \circ z)\right)^{-1}.$$

Obvious 13. If z is a finite family of finitary functions, it is concatenation of $\mathrm{dom}\, z$ tuples in the usual sense (as it is commonly used in computer science).

Proposition 14. If $z \in \prod (\mathrm{GR} \circ F)$ then $\mathrm{concat}\, z = \mathrm{uncurry}(z) \circ (\bigoplus (\mathrm{arity} \circ F))^{-1}$.

Proof. If $z \in \prod (\mathrm{GR} \circ F)$ then $\mathrm{dom}\, z(i) = \mathrm{dom}\, (\mathrm{GR} \circ F)_i = \mathrm{dom}\, F_i = \mathrm{arity}\, F_i$ for every $i \in \mathrm{dom}\, F$. Thus $\mathrm{dom} \circ z = \mathrm{arity} \circ F$. □

Proposition 15. $\mathrm{dom}\, \mathrm{concat}\, z = \sum_{i \in \mathrm{dom}\, z} \mathrm{dom}\, z_i$.

Proof. Because $\mathrm{dom}\, (\bigoplus (\mathrm{dom} \circ z))^{-1} = \sum_{i \in \mathrm{dom}\, F} \mathrm{dom}\, z_i$, it is enough to prove that

$$\mathrm{dom}\, \mathrm{uncurry}(z) = \mathrm{dom} \bigoplus (\mathrm{dom} \circ z).$$

Really,

$\operatorname{dom}\bigoplus(\operatorname{dom}\circ z)=\{(i;x)\mid i\in\operatorname{dom}(\operatorname{dom}\circ z),x\in\operatorname{dom}z_i\}=\{(i;x)\mid i\in\operatorname{dom}z,x\in\operatorname{dom}z_i\}=\coprod z$

and $\operatorname{dom}\operatorname{uncurry}(z)=\coprod_{i\in X}z_i=\coprod z$. □

4.3 Finite example

If F is a finite family (indexed by a natural number $\operatorname{dom}F$) of anchored finitary relations, then by definition $\operatorname{GR}\prod^{(\mathrm{ord})}F=\{[\![a_{0,0};\ldots;a_{0,\operatorname{arity}F_0-1};\ldots;a_{\operatorname{dom}F-1,0};\ldots;a_{\operatorname{dom}F-1,\operatorname{arity}F_{\operatorname{dom}F-1}-1}]\!]\mid[\![a_{0,0};\ldots;a_{0,\operatorname{arity}F_0-1}]\!]\in\operatorname{GR}F_0\wedge\ldots\wedge[\![a_{\operatorname{dom}F-1,\operatorname{arity}F_{\operatorname{dom}F-1}-1}]\!]\in\operatorname{GR}F_{\operatorname{dom}F-1}\}$ and

$$\operatorname{arity}\prod^{(\mathrm{ord})}F=\operatorname{arity}F_0+\ldots+\operatorname{arity}F_{\operatorname{dom}F-1}.$$

The above formula can be shortened to

$$\operatorname{GR}\prod^{(\mathrm{ord})}F=\left\{\operatorname{concat}z\mid z\in\prod(\operatorname{GR}\circ F)\right\}.$$

4.4 The definition

Definition 16. The anchored relation (which I call *ordinated product*) $\prod^{(\mathrm{ord})}F$ is defined by the formulas:

$$\operatorname{arity}\prod^{(\mathrm{ord})}F=\sum(\operatorname{arity}\circ F);$$
$$\operatorname{GR}\prod^{(\mathrm{ord})}F=\left\{\operatorname{concat}z\mid z\in\prod(\operatorname{GR}\circ F)\right\}.$$

Theorem 17. $\prod^{(\mathrm{ord})}F$ is a properly defined anchored relation.

Proof. $\operatorname{dom}\operatorname{concat}z=\sum_{i\in\operatorname{dom}F}\operatorname{dom}z_i=\sum_{i\in\operatorname{dom}F}\operatorname{arity}F_i=\sum(\operatorname{arity}\circ F)$. □

4.5 Definition with composition for every multiplier

$q(F)_i\overset{\mathrm{def}}{=}(\operatorname{curry}(\bigoplus(\operatorname{arity}\circ F)))i$.

Theorem 18. $\operatorname{GR}\prod^{(\mathrm{ord})}F=\left\{L\in\mho^{\sum(\operatorname{arity}\circ F)}\mid\forall i\in\operatorname{dom}F:L\circ q(F)_i\in\operatorname{GR}F_i\right\}$.

Proof. $\operatorname{GR}\prod^{(\mathrm{ord})}F=\{\operatorname{concat}z\mid z\in\prod(\operatorname{GR}\circ F)\}$;

$\operatorname{GR}\prod^{(\mathrm{ord})}F=\left\{\operatorname{uncurry}(z)\circ(\bigoplus(\operatorname{arity}\circ F))^{-1}\mid z\in\prod_{i\in\operatorname{dom}F}\mho^{\operatorname{arity}F_i},\forall i\in\operatorname{dom}F:z(i)\in\operatorname{GR}F_i\right\}$;

Let $L=\operatorname{uncurry}(z)$. Then $z=\operatorname{curry}(L)$.

$\operatorname{GR}\prod^{(\mathrm{ord})}F=\left\{L\circ(\bigoplus(\operatorname{arity}\circ F))^{-1}\mid\operatorname{curry}(L)\in\prod_{i\in\operatorname{dom}F}\mho^{\operatorname{arity}F_i},\forall i\in\operatorname{dom}F:\operatorname{curry}(L)i\in\operatorname{GR}F_i\right\}$;

$\operatorname{GR}\prod^{(\mathrm{ord})}F=\left\{L\circ(\bigoplus(\operatorname{arity}\circ F))^{-1}\mid L\in\mho^{\coprod_{i\in\operatorname{dom}F}\operatorname{arity}F_i},\forall i\in\operatorname{dom}F:\operatorname{curry}(L)i\in\operatorname{GR}F_i\right\}$;

$\operatorname{GR}\prod^{(\mathrm{ord})}F=\left\{L\in\mho^{\sum(\operatorname{arity}\circ F)}\mid\forall i\in\operatorname{dom}F:\operatorname{curry}(L\circ\bigoplus(\operatorname{arity}\circ F))i\in\operatorname{GR}F_i\right\}$;

$(\operatorname{curry}(L\circ\bigoplus(\operatorname{arity}\circ F))i)x=L((\operatorname{curry}(\bigoplus(\operatorname{arity}\circ F))i)x)=L(q(F)_ix)=(L\circ q(F)_i)x$;

$\operatorname{curry}(L\circ\bigoplus(\operatorname{arity}\circ F))i=L\circ q(F)_i$;

$\operatorname{GR}\prod^{(\mathrm{ord})}F=\left\{L\in\mho^{\sum(\operatorname{arity}\circ F)}\mid\forall i\in\operatorname{dom}F:L\circ q(F)_i\in\operatorname{GR}F_i\right\}$. □

Corollary 19. $\operatorname{GR}\prod^{(\mathrm{ord})}F=\left\{L\in(\bigcup\operatorname{im}(\operatorname{GR}\circ F))^{\sum(\operatorname{arity}\circ F)}\mid\forall i\in\operatorname{dom}F:L\circ q(F)_i\in\operatorname{GR}F_i\right\}$.

Corollary 20. $\operatorname{GR}\prod^{(\mathrm{ord})}F$ is small if F is small.

4.6 Definition with shifting arguments

Let $F_i' = \{L \circ \mathrm{Pr}_1|_{\{i\} \times \operatorname{arity} F_i} \mid L \in \mathrm{GR}\, F_i\}$.

Proposition 21. $F_i' = \{L \circ \mathrm{Pr}_1|_{\{i\} \times \mho} \mid L \in \mathrm{GR}\, F_i\}$.

Proof. If $L \in \mathrm{GR}\, F_i$ then $\operatorname{dom} L = \operatorname{arity} F_i$. Thus

$$L \circ \mathrm{Pr}_1|_{\{i\} \times \operatorname{arity} F_i} = L \circ \mathrm{Pr}_1|_{\{i\} \times \operatorname{dom} L} = L \circ \mathrm{Pr}_1|_{\{i\} \times \mho}. \qquad \square$$

Proposition 22. F_i' is an $(\{i\} \times \operatorname{arity} F_i)$-ary relation.

Proof. We need to prove that $\operatorname{dom}(L \circ \mathrm{Pr}_1|_{\{i\} \times \operatorname{arity} F_i}) = \{i\} \times \operatorname{arity} F_i$ for $L \in \mathrm{GR}\, F_i$, but that's obvious. $\square$

Obvious 23. $\coprod (\operatorname{arity} \circ F) = \bigcup_{i \in \operatorname{dom} F} (\{i\} \times \operatorname{arity} F_i) = \bigcup_{i \in \operatorname{dom} F} \operatorname{dom} F_i'$.

Lemma 24. $P \in \prod_{i \in \operatorname{dom} F} F_i' \Leftrightarrow \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod (\mathrm{GR} \circ F)$ for a $\operatorname{dom} F$ indexed family P where $P_i \in \mho^{\{i\} \times \operatorname{arity} F_i}$ for every $i \in \operatorname{dom} F$, that is for $P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i}$.

Proof. For every $P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i}$ we have:

$P \in \prod_{i \in \operatorname{dom} F} F_i' \Leftrightarrow P \in \{z \in \mho^{\operatorname{dom} F} \mid \forall i \in \operatorname{dom} F\colon z(i) \in F_i'\} \Leftrightarrow P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F\colon P(i) \in F_i' \Leftrightarrow P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F \exists L \in \mathrm{GR}\, F_i\colon P_i = L \circ (\mathrm{Pr}_1|_{\{i\} \times \mho}) \Leftrightarrow P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F \exists L \in \mathrm{GR}\, F_i\colon \big(P_i \in \mho^{\{i\} \times \operatorname{arity} F_i} \wedge \forall x \in \operatorname{arity} F_i\colon P_i(i; x) = L x\big) \Leftrightarrow P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F \exists L \in \mathrm{GR}\, F_i\colon \big(P_i \in \mho^{\{i\} \times \operatorname{arity} F_i} \wedge \operatorname{curry}(P_i) i = L\big) \Leftrightarrow P \in \mho^{\operatorname{dom} F} \wedge \forall i \in \operatorname{dom} F\colon \big(P_i \in \mho^{\{i\} \times \operatorname{arity} F_i} \wedge \operatorname{curry}(P_i) \in \mathrm{GR}\, F_i\big) \Leftrightarrow \forall i \in \operatorname{dom} F \exists Q_i \in (\mho^{\operatorname{arity} F_i})^{\{i\}}\colon (P_i = \operatorname{uncurry}(Q_i) \wedge (Q_i) i \in \mho^{\operatorname{arity} F_i} \wedge Q_i i \in \mathrm{GR}\, F_i) \Leftrightarrow \forall i \in \operatorname{dom} F \exists Q_i \in (\mho^{\operatorname{arity} F_i})^{\{i\}} \big(P_i = \operatorname{uncurry}(Q_i) \wedge \big(\bigcup_{i \in \operatorname{dom} F} Q_i\big) i \in \mathrm{GR}\, F_i\big) \Leftrightarrow \forall i \in \operatorname{dom} F \exists Q_i \in (\mho^{\operatorname{arity} F_i})^{\{i\}} \big(P_i = \operatorname{uncurry}(Q_i) \wedge \bigcup_{i \in \operatorname{dom} F} Q_i \in \prod (\mathrm{GR} \circ F)\big) \Leftrightarrow \forall i \in \operatorname{dom} F\colon \bigcup_{i \in \operatorname{dom} F} \operatorname{curry}(P_i) \in \prod (\mathrm{GR} \circ F) \Leftrightarrow \operatorname{curry}\big(\bigcup_{i \in \operatorname{dom} F} P_i\big) \in \prod (\mathrm{GR} \circ F) \Leftrightarrow \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod (\mathrm{GR} \circ F)$. $\square$

Lemma 25. $\left\{\operatorname{curry}(f) \circ \bigoplus (\operatorname{arity} \circ F) \mid f \in \mathrm{GR} \prod^{(\mathrm{ord})} F\right\} = \prod (\mathrm{GR} \circ F)$.

Proof. First $\mathrm{GR} \prod^{(\mathrm{ord})} F = \{\operatorname{uncurry}(z) \circ (\bigoplus (\operatorname{dom} \circ z))^{-1} \mid z \in \prod (\mathrm{GR} \circ F)\}$, that is

$\left\{f \mid f \in \mathrm{GR} \prod^{(\mathrm{ord})} F\right\} = \{\operatorname{uncurry}(z) \circ (\bigoplus (\operatorname{arity} \circ F))^{-1} \mid z \in \prod (\mathrm{GR} \circ F)\}$.

Since $\bigoplus (\operatorname{arity} \circ F)$ is a bijection, we have

$\left\{f \circ \bigoplus (\operatorname{arity} \circ F) \mid f \in \mathrm{GR} \prod^{(\mathrm{ord})} F\right\} = \{\operatorname{uncurry}(z) \mid z \in \prod (\mathrm{GR} \circ F)\}$ what is equivalent to

$\left\{\operatorname{curry}(f) \circ \bigoplus (\operatorname{arity} \circ F) \mid f \in \mathrm{GR} \prod^{(\mathrm{ord})} F\right\} = \{z \mid z \in \prod (\mathrm{GR} \circ F)\}$ that is

$\left\{\operatorname{curry}(f) \circ \bigoplus (\operatorname{arity} \circ F) \mid f \in \mathrm{GR} \prod^{(\mathrm{ord})} F\right\} = \prod (\mathrm{GR} \circ F)$. $\square$

Lemma 26. $\left\{\bigcup \operatorname{im} P \mid P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i} \wedge \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod (\mathrm{GR} \circ F)\right\} = \left\{L \mid L \in \mho^{\coprod_{i \in X} \operatorname{arity} F_i} \wedge \operatorname{curry}(L) \in \prod (\mathrm{GR} \circ F)\right\}$.

Proof. Let $L' \in \left\{L \mid L \in \mho^{\coprod_{i \in X} \operatorname{arity} F_i} \wedge \operatorname{curry}(L) \in \prod (\mathrm{GR} \circ F)\right\}$. Then $L' \in \mho^{\coprod_{i \in X} \operatorname{arity} F_i}$ and $\operatorname{curry}(L') \in \prod (\mathrm{GR} \circ F)$.

Let $P = \lambda i \in \operatorname{dom} F\colon L'|_{\{i\} \times \operatorname{arity} F_i}$. Then $P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i}$ and $\bigcup \operatorname{im} P = L'$. So $L' \in \left\{\bigcup \operatorname{im} P \mid P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i} \wedge \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod (\mathrm{GR} \circ F)\right\}$.

Let now $L' \in \left\{\bigcup \operatorname{im} P \mid P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i} \wedge \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod (\mathrm{GR} \circ F)\right\}$. Then there exists $P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i}$ such that $L' = \bigcup \operatorname{im} P$ and $\operatorname{curry}(L') \in \prod (\mathrm{GR} \circ F)$. Evidently $L' \in \mho^{\coprod_{i \in X} \operatorname{arity} F_i}$. So $L' \in \left\{\bigcup \operatorname{im} P \mid P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i} \wedge \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod (\mathrm{GR} \circ F)\right\}$. $\square$

Lemma 27. $\left\{ f \circ \bigoplus (\operatorname{arity} \circ F) \mid f \in \operatorname{GR} \prod^{(\mathrm{ord})} F \right\} = \left\{ \bigcup \operatorname{im} P \mid P \in \prod_{i \in \operatorname{dom} F} F'_i \right\}$.

Proof. $L \in \left\{ \bigcup \operatorname{im} P \mid P \in \prod_{i \in \operatorname{dom} F} F'_i \right\} \Leftrightarrow L \in \left\{ \bigcup \operatorname{im} P \mid P \in \coprod_{i \in \operatorname{dom} F} \mho^{\{i\} \times \operatorname{arity} F_i} \wedge \operatorname{curry}(\bigcup \operatorname{im} P) \in \prod (\operatorname{GR} \circ F) \right\} \Leftrightarrow L \in \mho^{\coprod_{i \in X} \operatorname{arity} F_i} \wedge \operatorname{curry}(L) \in \prod (\operatorname{GR} \circ F) \Leftrightarrow L \in \mho^{\coprod_{i \in X} \operatorname{arity} F_i} \wedge \operatorname{curry}(L) \in \left\{ \operatorname{curry}(f) \circ \bigoplus (\operatorname{arity} \circ F) \mid f \in \operatorname{GR} \prod^{(\mathrm{ord})} F \right\} \Leftrightarrow$ (because $\bigoplus (\operatorname{arity} \circ F)$ is a bijection)$\Leftrightarrow \operatorname{curry}(L) \circ (\bigoplus (\operatorname{arity} \circ F))^{-1} \in \left\{ \operatorname{curry}(f) \mid f \in \operatorname{GR} \prod^{(\mathrm{ord})} F \right\} \Leftrightarrow L \circ (\bigoplus (\operatorname{arity} \circ F))^{-1} \in \left\{ f \mid f \in \operatorname{GR} \prod^{(\mathrm{ord})} F \right\} \Leftrightarrow$ (because $\bigoplus (\operatorname{arity} \circ F)$ is a bijection)$\Leftrightarrow L \in \left\{ f \circ \bigoplus (\operatorname{arity} \circ F) \mid f \in \operatorname{GR} \prod^{(\mathrm{ord})} F \right\}$. □

Theorem 28.

$$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \left(\bigcup \operatorname{im} P \right) \circ \left(\bigoplus (\operatorname{arity} \circ F) \right)^{-1} \mid P \in \prod_{i \in \operatorname{dom} F} F'_i \right\}.$$

Proof. From the lemma, because $\bigoplus (\operatorname{arity} \circ F)$ is a bijection. □

Theorem 29.

$$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \bigcup_{i \in \operatorname{dom} F} \left(P_i \circ \left(\bigoplus (\operatorname{arity} \circ F) \right)^{-1} \right) \mid P \in \prod_{i \in \operatorname{dom} F} F'_i \right\}.$$

Proof. From the previous theorem. □

Theorem 30.

$$\operatorname{GR} \prod^{(\mathrm{ord})} F = \left\{ \bigcup \operatorname{im} P \mid P \in \prod_{i \in \operatorname{dom} F} \left\{ f \circ \left(\bigoplus (\operatorname{arity} \circ F) \right)^{-1} \mid f \in F'_i \right\} \right\}.$$

Proof. From the previous. □

Remark 31. Note that the above formulas contain both $\bigcup_{i \in \operatorname{dom} F} \operatorname{dom} F'_i$ and $\bigcup_{i \in \operatorname{dom} F} F'_i$. These forms are similar but different.

4.7 Associativity of ordinated product

Let f is an ordinal variadic function.

Let S is an ordinal indexed family of functions of ordinal indexed families of functions each taking an ordinal number of arguments in a set X.

I call f *infinite associative* when

1. $f(f \circ S) = f(\operatorname{concat} S)$ for every S;
2. $f(\llbracket x \rrbracket) = x$ for $x \in X$.

4.7.1 Infinite associativity impliess associativity

Proposition 32. Let f is an infinitely associative function taking an ordinal number of arguments in a set X. Define $x \star y = f \llbracket x; y \rrbracket$ for $x, y \in X$. Then the binary operation $\star$ is associative.

Proof. Let $x, y, z \in X$. Then $(x \star y) \star z = f \llbracket f \llbracket x; y \rrbracket; z \rrbracket = f(f \llbracket x; y \rrbracket; f \llbracket z \rrbracket) = f \llbracket x; y; z \rrbracket$. Similarly $x \star (y \star z) = f \llbracket x; y; z \rrbracket$. So $(x \star y) \star z = x \star (y \star z)$. □

4.7.2 Concatenation is associative

First we will prove some lemmas.

Let a and b are functions on a poset. Let $a\sim b$ iff there exist an order isomorphism f such that $a=b\circ f$. Evidently $\sim$ is an equivalence relation.

Obvious 33. $\operatorname{concat} a=\operatorname{concat} b\Leftrightarrow \operatorname{uncurry}(a)\sim\operatorname{uncurry}(b)$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

Thank to the above, we can reduce properties of concat to properties of uncurry.

Lemma 34. $a\sim b\Rightarrow \operatorname{uncurry} a\sim\operatorname{uncurry} b$ for every ordinal indexed families a and b of functions taking an ordinal number of arguments.

Proof. There exist an order isomorphism f such that $a=b\circ f$.

$\operatorname{uncurry}(a)(x;y)=(ax)y=(bfx)y=\operatorname{uncurry}(b)(fx;y)=\operatorname{uncurry}(b)g(x;y)$ where $g(x;y)=(fx;y)$.

g is an order isomorphism because $g(x_0;y_0)\geqslant g(x_1;y_1)\Leftrightarrow(x_0;y_0)\geqslant(x_1;y_1)$. (Injectivity and surjectivity are obvious.) □

Lemma 35. Let $a_i\sim b_i$ where f_i for every i. Then $\operatorname{uncurry} a\sim\operatorname{uncurry} b$ for every ordinal indexed families a and b of ordinal indexed families of functions taking an ordinal number of arguments.

Proof. Let $a_i=b_i\circ f_i$ where f_i is an order isomorphism for every i.

$\operatorname{uncurry}(a)(i;y)=a_i\,y=b_i\,f_i\,y=\operatorname{uncurry}(b)(i;f_i\,y)=\operatorname{uncurry}(b)g(i;y)=(\operatorname{uncurry}(b)\circ g)(i;y)$ where $g(i;y)=(i;f_i\,y)$.

g is an order isomorphism because $g(i;y_0)\geqslant g(i;y_1)\Leftrightarrow f_i\,y_0\geqslant f_i\,y_1\Leftrightarrow y_0\geqslant y_1$ and $i_0>i_1\Rightarrow g(i_0;y_0)>g(i_1;y_1)$. (Injectivity and surjectivity are obvious.) □

Let now S is an ordinal indexed family of ordinal indexed families of functions taking an ordinal number of arguments.

Lemma 36. $\operatorname{uncurry}(\operatorname{uncurry}\circ S)\sim\operatorname{uncurry}(\operatorname{uncurry} S)$.

Proof. $\operatorname{uncurry}\circ S=\lambda i\in S:\operatorname{uncurry}(S_i)$;

$\operatorname{uncurry}(\operatorname{uncurry}\circ S)(i;(x;y))=(\operatorname{uncurry} S_i)(x;y)=(S_i\,x)y$;

$(\operatorname{uncurry}(\operatorname{uncurry} S))((i;x);y)=((\operatorname{uncurry} S)(i;x))y=(S_i\,x)y$.

Thus $\operatorname{uncurry}(\operatorname{uncurry}\circ S)(i;(x;y))=(\operatorname{uncurry}(\operatorname{uncurry} S))((i;x);y)$ and thus evidently $\operatorname{uncurry}(\operatorname{uncurry}\circ S)\sim\operatorname{uncurry}(\operatorname{uncurry} S)$. □

Theorem 37. concat is an infinitely associative function.

Proof. $\operatorname{concat}(\llbracket x\rrbracket)=x$ for a function x taking an ordinal number of argument is obvious. It is remained to prove

$$\operatorname{concat}(\operatorname{concat}\circ S)=\operatorname{concat}(\operatorname{concat} S);$$

We have, using the lemmas, $\operatorname{concat}(\operatorname{concat}\circ S)\sim\operatorname{uncurry}(\operatorname{concat}\circ S)\sim$ (by lemma 35)$\sim$ $\operatorname{uncurry}(\operatorname{uncurry}\circ S)\sim\operatorname{uncurry}(\operatorname{uncurry} S)\sim\operatorname{uncurry}(\operatorname{concat} S)\sim\operatorname{concat}(\operatorname{concat} S)$. □

Corollary 38. Ordinated product is an infinitely associative function.

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